Dynamic optimization and Skiba sets in economic examples

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Abstract

We discuss two optimization problems from economics. The first is a model of optimal investment and the second is a model of resource management. In both cases the time horizon is infinite and the optimal control variables are continuous.

Typically, in these optimal control problems multiple steady states and periodic orbits occur. This leads to multiple solutions of the state-costate system each of which relates to a locally optimal strategy but has its own limiting behavior (stationary or periodic).

Initial states that allow different optimal solutions with the same value of the objective function are called Skiba points. The set of Skiba points is of interest, because it provides thresholds for a global change of optimal strategies.

We provide a systematic numerical method for calculating locally optimal solutions and Skiba points via boundary value problems. In parametric or higher dimensional systems Skiba curves (or manifolds) appear and we show how to follow them by a continuation process.

We apply our method to the models above where Skiba sets consist of points or curves and where optimal solutions have different stationary or periodic asymptotic behavior.

Keywords: Skiba sets, multiple steady states, periodic orbits

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1 Introduction

Dynamic optimization has become an important method in economics. Numerous examples have been given where dynamic optimization in economics can lead to multiple steady states and periodic orbits. In earlier growth theory it has been shown that a convex-concave production function leads to multiple steady states. Examples are given in the literature on development economics, see Skiba (1978) [34] and Azariadis & Drazen (1990) [1]. Multiple steady states have further been studied in modern growth theory. In endogenous growth models of Lucas and Romer type multiple steady states or a continuum of steady states may arise, for the Lucas model see Benhabib, Perli & Xie (1994) [3] and for the Romer model see Benhabib & Perli (1994) [2] and Evans, Honkapohja & Romer (1998) [11], see also Santos (1999) [28]. The literature on resource economics and ecological management problems show also numerous examples of models with multiple steady states (see Brock & Starret (1999) [9] and Sieveking & Semmler (1997) [31]). Multiple steady states solutions are also important properties in trade models (see Krugman (1991) [17]), in models of addiction (see Orphanides & Zervos [22, 23]), in labor market search (see, for example, Mortensen (1989) [21]) and in monetary control models (see Benhabib, Schmitt-Grohe & Uribe (1998) [4] and Semmler & Greiner (1999) [29]).

Such systems often lead to multiple solutions of the steady state-costate system each of which relates to a locally optimal strategy but has its own limiting behavior (stationary or periodic). There are initial states that allow different optimal solutions with the same value for the objective function. These are called Skiba points. The set of Skiba points is of interest for economics, since it provides a threshold for a global change of optimal strategies. Our paper develops a methodology of how to numerically study such systems via the Hamiltonian and by solving boundary value problems on finite intervals. We apply our method to two protopye dynamic models, one from modern growth theory and the other from resource eononomics.

In Section 2 we define the class of optimization problems with infinite horizon to which our method applies. It is based on a general technique for approximating so called generalized connecting orbits in continuous dynamical systems, as developed in [24].

In 2.1 we employ a proper version of Pontryagin's maximum principle to define the set of "candidates for optimal solutions" with "stationary or periodic asymptotic behavior". These solutions are of particular interest, because such candidates are bounded in state space and they satisfy the necessary first order conditions for locally optimal solutions. Connecting orbits are located in the space of state and costate variables and they determine optimal controls as well as the system response under this control, if the optimal solutions

are asymptotically stationary or periodic.

The topic of 2.2 is the computation of "Skiba" points with "stationary or periodic asymptotic behavior" by using the approximation method from [24, Chapter 5]. This method is described in some detail in the appendix.

In Section 3 we analyze a model of optimal investment and consumption policy which is presented in [30]. In particular, we approximate two solutions converging to different steady states and we show that there exists a Skiba point.

A model of optimal exploitation of renewable resources and optimal consumption policy, presented in [31], is analyzed in Section 4. Here we detect several periodic solutions and determine appropriate solutions converging to either one or the other.

2 The general setup

In this section we follow [20] to define a class of optimization problems with infinite horizon. In 2.1 and 2.2 we define candidates for optimal solutions and Skiba points with "stationary or periodic asymptotic behavior".

The optimization problem is: Maximize the objective function

$$\hat{U}(x,c) := \int_0^\infty e^{-\theta t} U(x(t),c(t))\mathrm{dt} \tag{2.1}$$

subject to

$$\dot{x}(t) = f\big(x(t),c(t)\big),\ t \in \mathbb{R}_+, \qquad x(0) = x_0, \tag{2.2}$$

where $x(t) \in \mathbb{R}^n$ are the state variables and $c(t) \in \mathbb{R}^n$ are the control variables (in [20] $c(t)$ is more generally in a topological space).

We assume that the control function c is piecewise continuous, solutions $x \in C(\mathbb{R}_+, \mathbb{R}^n)$ are bounded and piecewise continuously differentiable, $U \in C^1(\mathbb{R}^{2n}, \mathbb{R})$ and $f \in C^1(\mathbb{R}^{2n}, \mathbb{R}^n)$.

Definition 2.1 *A pair $\big(x(\cdot), c(\cdot)\big)$ is called an* **admissible trajectory**, *if $x(\cdot)$ is a solution of (2.2) with control $c(\cdot)$, except at the jumps of c, and if the integral (2.1) converges. The set of all admissible trajectories is denoted by $\mathcal{T}_{ad}(x_0)$.*

A pair $\big(\bar{x}(\cdot), \bar{c}(\cdot)\big)$ is called an **optimal solution** *of the problem (2.1), (2.2), if it is admissible and if it maximizes (2.1) over all admissible pairs $\big(x(\cdot), c(\cdot)\big) \in \mathcal{T}_{ad}(x_0)$. The set of all optimal solutions is denoted by $\mathcal{T}_{opt}(x_0)$.*

PROPOSITON 2.2 *If there is an optimal solution* $\big(x(\cdot), c(\cdot)\big)$ *with* $x(0) = x(T)$, $T > 0$, *then the following solutions are optimal:*

A: The solution itself, i. e. $\big(x_A(\cdot), c_A(\cdot)\big) = \big(x(\cdot), c(\cdot)\big)$,

B: The solution starting at $x(T)$ *with control* $c(\cdot + T)$, *i. e.*

$$\big(x_B(\cdot), c_B(\cdot)\big) = \big(x(\cdot + T), c(\cdot + T)\big),$$

C: The T*–periodic solution which switches at each* nT, $n \in \mathbb{N}$ *to* $x(0)$, *i. e.*

$$\big(x_C(\cdot), c_C(\cdot)\big) = \big(x(\cdot \mathrm{mod} T), c(\cdot \mathrm{mod} T)\big).$$

Breaking up the integral in (2.1) and using the geometric series one obtains

$$\hat{U}(x_A, c_A) = e^{-\theta T}\hat{U}(x_B, c_B) + (1 - e^{-\theta T})\hat{U}(x_C, c_C),$$

and hence Proposition 2.2 follows by optimality of solution A.

REMARK: If $\big(x(\cdot), c(\cdot)\big)$ is T–periodic, then the solutions A, B and C coincide.

We will apply the techniques in [24] to approximate "candidates for optimal solutions" and to compare the corresponding values of the objective function (2.1). It is not our intention to prove existence of optimal solutions. Therefore we assume henceforth:

ASSUMPTION 1: There exists at least one optimal solution of the problem (2.1), (2.2).

The technique used here is to apply Pontryagin's maximum principle to detect candidates for optimal solutions. These are admissible trajectories which satisfy the necessary conditions for optimal solutions stated in the forthcoming Theorem 2.3. The theorem is a special version of Pontryagin's maximum principle from [20, Theorem A].

THEOREM 2.3 *A necessary condition for an admissible trajectory* $\big(\bar{x}(\cdot), \bar{c}(\cdot)\big) \in \mathcal{T}_{ad}(x_0)$ *to be an optimal solution of (2.1), (2.2) is, that there exist* $\lambda \in \{0, 1\}$ *and an initial costate value* $y_0 \in \mathbb{R}^n$, *such that*

1. $(\lambda, y_0) \neq 0$.
2. *The costate variable* $y(\cdot)$ *is a solution of the initial value problem*

$$\begin{aligned} \dot{y}(t) &= -\lambda e^{-\theta t}\Big(\tfrac{\partial}{\partial x} U\big(\bar{x}(t), \bar{c}(t)\big)\Big)^{\top} - \Big(\tfrac{\partial}{\partial x} f\big(\bar{x}(t), \bar{c}(t)\big)\Big)^{\top} \cdot y(t), \\ y(0) &= y_0. \end{aligned} \tag{2.3}$$

3. For each $t \in \mathbb{R}_+$ where $\bar{c}(t)$ is continuous, $(\bar{x}(t), \bar{c}(t), y(t), \lambda)$ satisfies

$$H(\bar{x}(t), \bar{c}(t), y(t), \lambda, t) = \max_c H(\bar{x}(t), c, y(t), \lambda, t), \tag{2.4}$$

where H is the Hamiltonian function

$$H(x, c, y, \lambda, t) := \lambda e^{-\theta t} U\big(x, c\big) + f\big(x, c\big)^\top \cdot y.$$

4. The maximum value of the Hamiltonian satisfies for every t

$$M(t) := H(\bar{x}(t), \bar{c}(t), y(t), t) = -q(t), \tag{2.5}$$

where the function $q(t) \in \mathbb{R}$ is the solution of

$$\begin{aligned} &\dot{q} = \theta e^{-\theta t} U\big(\bar{x}(t), \bar{c}(t)\big), \\ &\lim_{t\to\infty} q(t) = 0. \end{aligned} \tag{2.6}$$

PROOF: See [20, Theorem A]. ■

REMARK: The initial costate value y_0 is unknown. If y_0 and $\bar{c}(\cdot)$ were given, then (2.2) and (2.3) constitute an initial value problem for x and y.

For admissible $(\bar{x}, \bar{c})$ the integral (2.1) converges (see Definition 2.1) and hence

$$q(t) = -\theta \int_t^\infty e^{-\theta s} U(\bar{x}(s), \bar{c}(s)) \mathrm{ds} \tag{2.7}$$

is the solution of (2.6).

DEFINITION 2.4 *An admissible trajectory $\big(x(\cdot), c(\cdot)\big) \in \mathcal{T}_{ad}(x_0)$ is called a* **candidate for an optimal solution** *or for short a* **candidate**, *if it satisfies the necessary conditions of Theorem 2.3. The value y_0 is called the* **corresponding costate value** *and the set of all candidates for optimal solutions is denoted by $\mathcal{T}_{can}(x_0)$.*

REMARK: In the literature a solution which satisfies the necessary conditions *1.–3.* is often called an equilibrium (here a **candidate** if *4.* is also satisfied) and a stationary solution is called a steady state-equilibrium (here simply a **steady state**), e. g. [4].

In typical cases $\lambda = 1$, nevertheless this requires a proof in the applications. In this section we will henceforward assume $\lambda = 1$.

In particular, we deal with systems where we can express $\bar{c}(t)$ in terms of $\bar{x}(t)$ and the scaled costate function $z(t) := e^{\theta t} y(t)$ from (2.4), i. e. we can write $\bar{c}(t) = C\big(\bar{x}(t), z(t)\big)$.

Thus we assume:

ASSUMPTION 2: Any candidate for an optimal solution satisfies $\lambda = 1$ and there exists a smooth function C, such that $C(x,z)$ maximizes the scaled Hamiltonian function $\tilde{H}(x,c,z) := U(x,c) + f(x,c)^\top \cdot z$ for all $(x,z) \in \mathbb{R}^{2n}$.

Both, the state and the costate variables are continuous for an optimal solution and hence the control is continuous by asumption 2. Thus we arive at:

PROPOSITON 2.5 *Assumption 2 implies that the control is continuous.*

Using $z(t) = e^{\theta t} y(t)$, $\bar{c}(t) = C\big(\bar{x}(t), z(t)\big)$, the equations (2.2) and (2.3) imply that $(\bar{x}(t), z(t))$ solves the autonomous dynamical system

$$\dot{x} = f\big(x, C(x,z)\big), \qquad x(0) = x_0, \tag{2.8}$$

$$\dot{z} = \theta z - \Big(\frac{\partial}{\partial x} U\big(x, C(x,z)\big)\Big)^\top - \Big(\frac{\partial}{\partial x} f\big(x, C(x,z)\big)\Big)^\top z, \quad z(0) = y_0 \tag{2.9}$$

if $\big(\bar{x}(\cdot), \bar{c}(\cdot)\big)$ is an optimal solution.

We consider (2.8), (2.9) as initial value problem depending on $y_0 \in \mathbb{R}^n$ and denote its solution by $x(y_0,t), z(y_0,t)$. We abbreviate the solution functions by $x(y_0) := x(y_0,\cdot)$, $z(y_0) := z(y_0,\cdot)$ and the corresponding control by $c(y_0) = c(y_0,\cdot) := C\big(x(y_0,\cdot), z(y_0,t)\big)$.

Suppose that the assumptions 1 and 2 hold, then we reformulate the optimization problem and maximize

$$\bar{U}(y_0) := \int_0^\infty e^{-\theta t} U\big(x(y_0,t), c(y_0,t)\big)\,\mathrm{d}t \tag{2.10}$$

over all $y_0 \in \mathbb{R}^n$, for which $\big(x(y_0), c(y_0)\big)$ is a candidate. The conditions 1.–3. of Theorem 2.3 are satisfied for all $\big(x(y_0), c(y_0)\big)$. Thus, for the set of candidates we include the conditions that the integral (2.10) exists and that the optimality (2.5) holds and get

$$\mathcal{T}_{can}(x_0) = \big\{ \big(x(y_0), c(y_0)\big) \;\big|\; y_0 \in \mathbb{R}^n \text{ and the integral (2.10) exists and (2.5) holds} \big\}.$$

PROPOSITON 2.6 *Suppose that the assumptions 1 and 2 hold and that (x,c) is a candidate with initial costate variable y_0 and with $x(T) = x_0$, $z(y_0,T) \neq y_0$ and $c(T) \neq c(0)$ for some $T > 0$. Then (x,c) cannot be optimal.*

Otherwise the T–periodic solution which jumps at T to (x_0, y_0) (trajectory C in Proposition 2.2) is also optimal (see Proposition 2.2), but its control is not continuous at T. This contradicts Proposition 2.5.

Additional information whether a solution is optimal can be obtained from (2.5) and (2.7). If $(\bar{x}, \bar{c})$ is a candidate, then we have

$$\int_t^\infty e^{-\theta s} U(\bar{x}(s), \bar{c}(s))\mathrm{ds} = \frac{e^{-\theta t}}{\theta}\big(U\big(\bar{x}(t), \bar{c}(t)\big) + f\big(\bar{x}(t), \bar{c}(t)\big)^\top z(t)\big). \tag{2.11}$$

Maximizing this at $t = 0$, i. e.

$$v(x_0, y_0) = \frac{1}{\theta}\big[U\big(x_0, C(x_0, y_0)\big) + f\big(x_0, C(x_0, y_0)\big)^\top y_0\big], \tag{2.12}$$

over all initial values of costate variables $y_0 \in \mathbb{R}^n$ does not solve the original problem, because it is also defined for non-optimal solutions. However, we can test (2.11) for single admissible trajectories by numerical integration to see whether it satisfies (2.5) or not. Moreover, (2.11) is of particular importance for our approximation method, because it allows to determine the value of the objective function in terms of the initial values of an optimal solution.

REMARK: The equation (2.11) always holds true for steady states $\big(\bar{x}, z\big)$ of (2.8), (2.9). If $\big(\bar{x}(t), z(t)\big)$ is an optimal T–periodic solution, then one can show (using periodicity and the geometric series) that (2.11) is equivalent to

$$\int_0^T e^{-\theta s} U(\bar{x}(s+t), \bar{c}(s+t))\mathrm{ds} = \frac{1 - e^{-\theta T}}{\theta}\big(U\big(\bar{x}(t), \bar{c}(t)\big) + f\big(\bar{x}(t), \bar{c}(t)\big)^\top z(t)\big). \tag{2.13}$$

for all $t \in [0, T]$. Hence, testing (2.11) for periodic orbits requires only integration on a finite interval.

2.1 Asymptotic solutions

Using the techniques of [24, Chapter 3] we approximate candidates for optimal solutions with special asymptotic behavior, i. e. those solutions which converge to given solutions η. In particular, we restrict our study to the case where η is either a steady state or a periodic solution.

Henceforward, we assume that $x_0 \in \mathbb{R}^n$ is given and that $\mathcal{Y} = \{\eta_i\}_{i \in I}$, $I = \{1, \ldots, l\}$, $l \in \mathbb{N}$ is a finite number of solutions $\eta_i(t) \in \mathbb{R}^m$, $m = 2n$ of $\binom{\dot{x}}{\dot{z}} = F(x, z)$, where F is defined by (2.8), (2.9). In the applications $\mathcal{Y}$ is either given or computing it is part of the problem.

DEFINITION 2.7 *Suppose that the assumptions 1 and 2 hold.*

A candidate for an optimal solution $\big(x(\cdot), c(\cdot)\big) \in \mathcal{T}_{can}(x_0)$ *with corresponding costate value* y_0 *is called* **asymptotic to** $\boldsymbol{\mathcal{Y}}$, *if* $\xi := \big(x, z(y_0)\big)$, *solves (2.8), (2.9) and is asymptotic to some* $\eta_i \in \mathcal{Y}$, *i. e.* $\lim_{t\to\infty} ||\xi(t) - \eta_i(t)|| = 0$. *The set of these solutions is denoted by* $\mathcal{T}_{asy}(\mathcal{Y})(x_0)$.

A candidate $(\bar{x}, \bar{c}) \in \mathcal{T}_{asy}(\mathcal{Y})(x_0)$ *for an optimal solution which is asymptotic with* $\{\eta_i\}$, $i \in I$ *is called* **nondegenerate** *if there exist neighborhoods* V *of* x_0 *and* W *of* $\bar{x}$, *such that for each* $\tilde{x}_0 \in V$ *there is a unique* $(x, c) \in \mathcal{T}_{asy}(\mathcal{Y})(\tilde{x}_0)$ *converging to* η_i *with* $x \in W$.

We call a candidate $\big(\bar{x}(\cdot), \bar{c}(\cdot)\big) \in \mathcal{T}_{asy}(\mathcal{Y})(x_0)$ *which is asymptotic with* $\mathcal{Y}$ **optimal with respect to** $\boldsymbol{\mathcal{Y}}$, *if it maximizes the objective function over all* $(x, c) \in \mathcal{T}_{asy}(\mathcal{Y})(x_0)$, *i. e.*

$$\hat{U}\big(\bar{x}(\cdot), \bar{c}(\cdot)\big) = \max_{(x,c)\in\mathcal{T}_{asy}(\mathcal{Y})(x_0)} \big(\hat{U}(x, c)\big), \quad \textit{where } \hat{U}(x, c) = \int_0^\infty e^{-\theta t} U(x(t), c(t))\mathrm{dt}.$$

The set of candidates which are optimal with respect to $\boldsymbol{\mathcal{Y}}$ *is denoted by* $\mathcal{T}^{opt}_{asy}(\mathcal{Y})(x_0)$.

REMARK: Definitions 2.1, 2.4 and 2.7 yield the relation

$$\mathcal{T}^{opt}_{asy}(\mathcal{Y})(x_0) \subset \mathcal{T}_{asy}(\mathcal{Y})(x_0) \subset \mathcal{T}_{can}(x_0) \subset \mathcal{T}_{ad}(x_0).$$

Our goal is to compute elements of $\mathcal{T}_{asy}(\mathcal{Y})(x_0)$ in those cases, where the η_i are either steady states or periodic solutions. Moreover we assume

ASSUMPTION 3: For each $\eta_i \in \mathcal{Y}$ there exists a unique nondegenerate candidate for an optimal solution which is asymptotic with $\{\eta_i\}$. The $\eta_i \in \mathcal{Y}$ are

Case 1: hyperbolic steady states with n stable eigenvalues μ, i. e. $\mathrm{Re}(\mu) < 0$ and n unstable eigenvalues ν, i. e. $\mathrm{Re}(\nu) > 0$.

Case 2: hyperbolic periodic orbits of $\binom{\dot{x}}{\dot{z}} = F(x, z)$ with $n-1$ stable Floquet multipliers e^μ, i. e. $|e^\mu| < 1$ and n unstable Floquet multiplier e^ν, i. e. $|e^\nu| > 1$.

REMARK: A steady state is called **hyperbolic**, if its linearization has no eigenvalue μ with $\mathrm{Re}(\mu) = 0$ and a periodic orbit is called **hyperbolic**, if it has a simple Floquet multiplier 1 and no other multiplier on the unit circle.

Assumption 3 implies that the $\eta_i \in \mathcal{Y}$ have stable manifolds of dimension n in both cases. Such an assumption is reasonable if we look for nondegenerate solutions.

Suppose that some $\eta_i \in \mathcal{Y}$ has a stable manifold of dimension less than n, then generically it will not intersect in the state-costate space $\mathbb{R}^{2n}$ the n dimensional affine subspace $\{x_0\} \times \mathbb{R}^n$ of initial values of (2.8), (2.9).

If the dimension of the stable manifold of $\eta_i \in \mathcal{Y}$ is $n+q$, then there exists typically a q–dimensional surface of candidates which are asymptotic with $\{\eta_i\}$. This leads to the problem of maximizing $\bar{U}(y_0)$ on a q–dimensional surface. We can approximate parts of this surface by first fixing q costate variables and applying the forthcoming procedure and then freeing using the q additional parameters for continuation.

For the approximation of nondegenerate candidates, we write $\binom{\xi^1}{\xi^2} := \binom{x}{z} = \xi$ and solve the boundary value problem

$$\dot{\xi}(t) = F\big(\xi(t)\big),\ t \in \mathbb{R}_+ \quad \text{with} \quad \xi^1(0) = x_0 \quad \text{and} \quad \lim_{t\to\infty} |\xi(t) - \eta(t)| = 0. \tag{2.14}$$

The asymptotic boundary condition for $t \to \infty$ will be replaced by a finite boundary condition at $t = T_+$ which specifies that the solution ξ lies in the stabel manifold of the solution η in an approximative sense. For η we distinguish two cases.

Steady state: In this case η satisfies $F(\eta) = 0$ and we compute a matrix $V \in \mathbb{R}^{n,2n}$, where the columns of $V^\top$ span the unstable subspace of $F'(\eta)^\top$, to define an asymptotic boundary condition. We compute such η an V by solving

$$F(\eta) = 0 \ \text{ and } VF'(\eta) = UV, \tag{2.15}$$

where $U \in \mathbb{R}^{n,n}$ with $\mathrm{Re}\big(\sigma(U)\big) > 0$. The **approximation** of a solution $\bar{\xi}$ of (2.14) on a finite interval $[0, T_+]$ is given by the **solution of the boundary value problem**

$$\begin{aligned} \dot{\xi}(t) &= F\big(\xi(t)\big),\ t \in [0, T_+], \\ \xi^1(0) &= x_0, \\ V\big(\xi(T_+) - \eta\big) &= 0. \end{aligned} \tag{2.16}$$

Solving (2.15) and (2.16) simultanously allows continuation of parameters (for the full system see (A.13)).

Periodic solution: Let η be a $\bar{T}$ periodic solution. Then $\nu_0(t) := \eta(t \cdot \bar{T})$ solves

$$\dot{\nu}(t) = \bar{T} F\big(\nu(t)\big),\ t \in \mathbb{R}_+ \quad \text{with} \quad \nu(0) = \nu(1). \tag{2.17}$$

Moreover, each $\nu_\varphi(t) := \eta\big((t+\varphi)\cdot\bar{T}\big)$ solves (2.17). To specify the periodic orbit which corresponds to a phase φ we add a phase fixing condition $\chi(\nu, \varphi) = 0$ to (2.17), e. g. with $\chi(\nu,\varphi) = \dot{\tilde{\eta}}(\varphi\bar{T})\big(\nu(0) - \tilde{\eta}(\varphi\bar{T})\big)$, where $\tilde{\eta}$ is a given approximation of η.

Typically computing an appropriate phase φ is part of the problem.

We can compute an appropriate $V_\varphi \in \mathbb{R}^{n,2n}$, $U_\varphi \in \mathbb{R}^{n,n}$ by solving

$$\dot{V} = -V\bar{T}F'\big(\nu_\varphi(\cdot + T_+)\big) \quad \text{with} \quad V(1) = UV(0), \tag{2.18}$$

where $U \in \mathbb{R}^{n,n}$ satisfies $|\mu| \leq 1$ for all eigenvalues μ of U.

If we assume that we can compute ν_φ and V_φ smoothly in φ, then an approximation of a solution $(\bar{\xi}, \bar{\varphi})$ of the scaled problem

$$\dot{\xi}(t) = \bar{T}F\big(\xi(t)\big),\ t \in \mathbb{R}_+ \quad \text{with} \quad \xi^1(0) = x_0 \quad \text{and} \quad \lim_{t\to\infty} |\xi(t) - \nu_\varphi(t)| = 0 \tag{2.19}$$

on a finite interval $J = [0, T_+]$ is given by the solution (ξ_J, φ_J) of the boundary value problem

$$\begin{aligned} \dot{\xi}(t) &= \bar{T}F\big(\xi(t)\big),\ t \in [0, T_+], \\ \xi^1(0) &= x_0, \\ V_\varphi\big(\xi(T_+) - \nu_\varphi(T_+)\big) &= 0. \end{aligned} \tag{2.20}$$

Note that the approximation ξ_J has been scaled, so that $\xi_J(t\bar{T})$ is the approximation of the original problem (2.14) on $[0, T_+\bar{T}]$.

Typically, computing ν_φ and V_φ smoothly in φ is cumbersome and hence we apply the **boundary corrector method**, i. e. start with an initial approximation φ_0

1. Compute V_{φ_0} as solution of (2.18), at φ_0,

2. Calculate $\big(\xi_J, \varphi_J, \nu_{\varphi_J}\big)$, the solution of

$$G_J^p(\xi, \varphi, \nu) := \begin{pmatrix} \dot{\nu} - \bar{T}F(\nu) \\ \chi(\nu, \varphi) \\ \nu(0) - \nu(1) \\ \dot{\xi} - \bar{T}F(\xi) \\ \xi^1(0) - x_0 \\ V_{\varphi_0}\big(\xi(T_+) - \nu(T_+)\big) \end{pmatrix} = 0, \tag{2.21}$$

3. Repeat once step **1.** with $\varphi_0 = \varphi_J$ and step **2.**

For generalizations of this method and error estimates we refer to the appendix.

Suppose that the assumptions 1–3 hold and let $x_0 \in \mathbb{R}^n$ be a given initial value, then we approximate $\mathcal{T}_{asy}(\mathcal{Y})(x_0)$ and $\mathcal{T}_{asy}^{opt}(\mathcal{Y})(x_0)$ by the following procedure:

Detecting asymptotic candidates

1. For each $i \in I$ solve either (Case 1) the boundary value problem (2.16) or (Case 2) apply the boundary corrector method defined by (2.21), respectively. Let (x_i, z_i) be the corresponding solution and define $c_i = c(x_i, z_i)$.

2. Test (2.11) by numerical integration to decide whether these solutions are candidates or not. This defines an approximation of the set $\mathcal{T}_{asy}(\mathcal{Y})(x_0)$.

3. Choosing those solutions which maximize the objective function (2.1) defines the set of candidates which are optimal with respect to $\mathcal{Y}$.

2.2 Skiba points

Now we define "Skiba" points [34] and derive an approximation procedure based on the results of [24, Chapter 5].

DEFINITION 2.8 *Suppose that assumption 1 holds. We call an initial value $x_0 \in \mathbb{R}^n$ a* **Skiba point**, *if there are multiple optimal solutions.*

Suppose that in addition the assumptions 2 and 3 hold. Then we call an initial value $x_0 \in \mathbb{R}^n$ a **Skiba point with respect to $\mathcal{Y}$**, *if there are multiple candidates which are optimal with respect to $\mathcal{Y}$. We call a Skiba point with respect to $\mathcal{Y}$* **nondegenerate**, *if all candidates are nondegenerate.*

In order to approximate Skiba points with respect to $\mathcal{Y}$ we have to solve two problems of the type in 2.1 and detect initial state values where the values of the objective function are equal.

At first we approximate simultaneously two solutions $\big(x_\pm(t), z_\pm(t)\big) \in \mathbb{R}^{2n}$ which converge to different $\eta_+, \eta_- \in \mathcal{Y}$ and have the same initial state value x_0. The corresponding controls are denoted by $c_\pm(t) = c\big(x_\pm(t), z_\pm(t)\big)$. This is done in an obvious way by doubling the system and possibly scaling the differential equations.

To detect a Skiba point we define one coordinate j of the initial state value $x_0 \in \mathbb{R}^n$ as unknown parameter. Thus we substitute the conditions $x_{+_j}(0) = x_{0_j}$, $x_{-_j}(0) = x_{0_j}$ by $x_{+_j}(0) = x_{-_j}(0)$ and the condition that the values of the objective functions are the same. We abbreviate for $x \in \mathbb{R}^l$, $l \geq n$ the vector of the first n coordinates exept the j–th by $x^* = (x_i)_{\substack{i=1,\dots,n \\ i \neq j}} \in \mathbb{R}^{n-1}$. To define the second new condition we apply (2.12) for both solutions

to achieve a condition which only depends on the initial values. Hence the conditons at $t = 0$ are $x_+^*(0) = x_0^*$, $x_-^*(0) = x_0^*$, $x_{+_j}(0) = x_{-_j}(0)$ and $v(x_+(0), z_+(0)) = v(x_-(0), z_-(0))$.

To detect a Skiba point, where η_- and η_+ are steady states we assume that V_- and V_+ are computed by (2.15) and we solve the boundary value problem

$$G_J^{Sk}(\xi_-, \xi_+) := \begin{pmatrix} \dot{\xi}_- - \tau_+ F(\xi_-) \\ \dot{\xi}_+ - \tau_- F(\xi_+) \\ \xi_-^*(0) - x_0^* \\ \xi_+^*(0) - x_0^* \\ \xi_{+_j}(0) - \xi_{-_j}(0) \\ v\big(\xi_+(0)\big) - v\big(\xi_-(0)\big) \\ V_-\big(\xi_-(T_+) - \eta_-\big) \\ V_+\big(\xi_+(T_+) - \eta_+\big) \end{pmatrix} = 0,$$

Here the $\tau_\pm$ are scaling factors which can be chosen appropriately. This allows us to solve both parts ($\pm$) of the boundary value problem on the same finite interval $[0, T_+]$, while the unscaled solutions are approximated on different intervals $[0, \tau_+ T_+]$ and $[0, \tau_- T_+]$.

To approximate a higher dimensional "Skiba set", we can use the initial state variables (except the x_{0_j} above) as continuation parameters.

Typically the set of Skiba points with respect to $\mathcal{Y}$ is locally a manifold of dimension $n-1$, in particular it is a point if there is one state variable and it is a curve if there are two.

This is seen by following arguments:

If $\bar{x}_0$ is a Skiba point and there are two corresponding nondegenerate optimal candidates with respect to $\mathcal{Y}$ which are asymptotic to η_+ and η_-, then there is a neighborhood V of $\bar{x}_0$, such that there exist unique solutions $\big(x_\pm(x_0), c_\pm(x_0)\big)$ which are asymptotic with $\eta_\pm$ for each $x_0 \in V$.

Let $\hat{U}_\pm(x_0)$ be the value of the objective function at $\big(x_\pm(x_0), c_\pm(x_0)\big)$ for each $x_0 \in V$, then we require $\hat{U}_+(\bar{x}_0) = \hat{U}_-(\bar{x}_0)$ for a Skiba point $\bar{x}_0$.

The sets $W_\pm = \big\{ \big(x_0, \hat{U}_+(x_0)\big) \; | x_0 \in \mathbb{R}^n \big\}$ are manifolds of dimension n in $\mathbb{R}^{n+1}$. Moreover, generically they intersect transversally at $\bar{x}_0$ and hence $T_{\bar{x}_0} W_+ + T_{\bar{x}_0} W_- = \mathbb{R}^{n+1}$. Thus $T_{\bar{x}_0} W_+ \cap T_{\bar{x}_0} W_-$ is of dimension $2n - (n+1) = n - 1$.

3 Optimal investment

We analyze a model of optimal investment and consumption policy which is presented in [30]. There it is shown that a non-explosiveness condition yields a region in state space,

where the borrower remains creditworthy.

The full problem can be split into the problem of optimal investment for maximizing the wealth of the economy and into the problem of optimal consumption policy to maximize the utility of this wealth.

We restrict to the problem of optimal investment. Using Pontryagin's maximum principle yields a 2-dimensional dynamical system in state-costate space. It will be seen that the stable manifolds of stationary points are candidates for optimal solutions.

The model

There are two control variables, the investment $j(t)$ and the consumption $c(t) \geq 0$. The two state equations for the per capita capital stock $k(t) \geq 0$ and the per capita stock of debt $b(t)$ are:

$$\dot{k} = i(k,j), \qquad k(0) = k_0, \tag{3.1}$$

$$\dot{b} = \theta b + c - f(k,j), \quad b(0) = b_0 \tag{3.2}$$

with the net investment function $i(k,j) = j - \sigma k$ (depreciation rate σ), net income function $f(k,j) = c_f k^\alpha - (j + \frac{j^2}{k^\gamma})$, $k > 0$, $f(0,0) = 0$ and the rate of interest θ.

We assume $\sigma \geq 0$, $\theta \geq 0$, $c_f \geq 0$, $\gamma \in (0,1)$, $0 < \alpha < 2 - \gamma$.

Moreover, the following non-explosiveness condition for the debt is required:

$$\lim_{t\to\infty} e^{-\theta t} b(t) = 0. \tag{3.3}$$

This asymptotic boundary condition implies that the state variables have to stay bounded if the borrower remains creditworthy (see [30]).

To optimize the utilization $\hat{U}(c) := \int_0^\infty e^{-\theta t} U(c(t))\mathrm{dt}$ we have to solve

$$\begin{cases} \max_{c,j} \int_0^\infty e^{-\theta t} U(c(t))\mathrm{dt}, & \\ \dot{k} = i(k,j); & k(0) = k_0, \\ \dot{b} = \theta b + c - f(k,j); & b(0) = b_0, \\ \lim_{t\to\infty} e^{-\theta t} b(t) = 0, & \end{cases} \tag{$P_F(k_0,b_0)$}$$

where U is a strictly monotone increasing utilization function.

This problem can be seperated into two optimization problems.

a) Solve the investment problem for $k_0 \in \mathbb{R}_+$

$$\begin{cases} \max_j \int_0^\infty e^{-\theta s} f(k(s), j(s))\mathrm{ds}, & \\ \dot{k} = i(k,j); & k(0) = k_0. \end{cases} \tag{$P_I(k_0)$}$$

By using an optimal solution $(k^*(t), j^*(t))$ of $(P_I(k_0))$ we define the wealth of the economy at time $t = 0$ by $w^* := \int_0^\infty e^{-\theta s} f(k^*(s), j^*(s))\mathrm{ds} - b_0$.

b) Solve the problem of optimal consumption for given (k, j, b_0), and $w \in \mathbb{R}_+$

$$\begin{cases} \max_{c,\bar{c}\le w} \int_0^\infty U(c(s))e^{-\theta s}\mathrm{ds}, & \\ \dot{b} = \theta b + c - f(k,j), & b(0) = b_0, \\ \lim_{t\to\infty} e^{-\theta t} b(t) = 0, & \end{cases} \qquad (P_C(w,k,j,b_0))$$

where $\bar{c} := \int_0^\infty e^{-\theta s} c(s)\mathrm{ds}$. We denote a solution of $(P_C(w,k,j,b_0))$ by $(b^*(t), c^*(t))$.

It can be seen, that $(\tilde{k}, \tilde{b}, \tilde{j}, \tilde{c})$ is an optimal solution of $(P_F(k_0, b_0))$ if and only if $(\tilde{k}, \tilde{j})$ is an optimal solution of $(P_I(k_0))$ and $(\tilde{b}, \tilde{c})$ is an optimal solution of $(P_C(\tilde{w}, \tilde{k}, \tilde{j}, b_0))$, where $\tilde{w} := \int_0^\infty e^{-\theta s} f(\tilde{k}(s), \tilde{j}(s))\mathrm{ds} - b_0$.

An analytical treatment why and under what conditions such problems can be separated is given in [33]. Henceforward we treat $(P_I(k_0))$. Let us apply Theorem 2.3 to this system.

THEOREM 3.1 *A necessary condition for $(k^*(t), j^*(t))$, $0 \le t < \infty$, to be an optimal solution of $(P_I(k_0))$ is that there exist a real number $\lambda \in \{0, 1\}$ and an initial costate value $x_0 \in \mathbb{R}$, such that*

1. $(\lambda, x_0) \neq 0$.

2. *The costate variable $x(\cdot)$ is the continuous and piecewise differentiable solution of*

$$\begin{cases} \dot{x}(t) = -\lambda e^{-\theta t} \frac{\partial}{\partial k} f(k^*(t), j^*(t)) - x(t) \frac{\partial}{\partial k} i(k^*(t), j^*(t)), \\ x(0) = x_0. \end{cases} \qquad (3.4)$$

3. *The investment $j^*(t)$ maximizes the Hamiltonian function*

$$H(k^*(t), j, x(t), \lambda, t) = \lambda e^{-\theta t} f(k^*(t), j) + x(t)\ i(k^*(t), j),$$

at each t where $j^*(t)$ is continuous.

4. *The maximum of the Hamiltonian satisfies for every t*

$$M(t) := H(k^*(t), j^*(t), x(t), \lambda, t) = -q(t),$$

where the function $q(t)$ is the solution of

$$\begin{cases} \dot{q} = \theta e^{-\theta t} f(k^*(t), j^*(t)), \\ \lim_{t\to\infty} q(t) = 0. \end{cases} \qquad (3.5)$$

Now we consider the specific problem and draw some conclusions from this theorem. Let (k^*, j^*) be an optimal solution of $(P_I(k_0))$ and let λ, x_0, x and q be as in Theorem 3.1.

- If $\lambda = 0$, then *1.* and *2.* imply $x_0 \neq 0$ and $\dot{x}(t) = x(t)\sigma$ (notice $i(k,j) = j - \sigma k$) and hence $x(t) = e^{\sigma t}x_0 \neq 0$. Thus $\frac{\partial}{\partial j}H(k^*(t), j, x(t), a, t) = x(t)\frac{\partial}{\partial j}i(k^*(t), j) = x(t) \neq 0$. This contradicts condition *3.* and hence $\lambda = 1$.

- Condition *3.* implies $j^*(t) = j\big(k^*(t), e^{\theta t}x(t)\big)$ for each $0 \leq t < \infty$, where $j(k,y) = \frac{y-1}{2}k^\gamma$. Continuity of x and k^* imply continuity of j^*. Thus, an optimal solutions must have a continuous control.

- The scaled function $(k(t), y(t)) := (k^*(t), e^{\theta t}x(t))$ is a solution of

$$\dot{k} = \frac{y-1}{2}k^\gamma - \sigma k, \qquad k(0) = k_0 \tag{3.6}$$

$$\dot{y} = (\theta + \sigma)y - \alpha c_f k^{\alpha-1} - \gamma\frac{(y-1)^2}{4}k^{\gamma-1}, \qquad y(0) = x_0. \tag{3.7}$$

- By integration (3.5) implies $q(t) = -\theta\int_t^\infty e^{-\theta s}f(k^*(s), j^*(s))\mathrm{ds}$.

- The definition of y and 4. yields for each t:

$$y(t)\dot{k}^*(t) = \theta\int_t^\infty e^{-\theta(s-t)}f(k^*(s), j^*(s))\mathrm{ds} - f(k^*(t), j^*(t)). \tag{3.8}$$

For the phase portrait of the dynamical system (3.6), (3.7) see Figure 3.1.

The system has two stationary points, (k_u, y_u) is a sink and (k_s, y_s) is a hyperbolic saddle.

We approximate a solution on the stable manifold of (k_s, y_s) by the techniques from [24] and get the solution $z_+(t) = \big(k_+(t), y_+(t)\big)$ drawn in Figure 3.1. Moreover, we define the value of the objective function of this solution by $\hat{F}_+(k_0) := \int_0^\infty e^{-\theta s}f[k_+(s), j(z_+(s))]\mathrm{ds}$. These values are plotted in Figure 3.2 as a function of k_0 (with sign $+$).

For sufficiently small k_0 there is another trajectory $z_-(t) := \big(k_-(t), y_-(t)\big)$ of (3.6), (3.7) which converges to the singular point $(0,1)$. In fact this trajectory reaches $(0,1)$ in finite time $\hat{T}$ and it defines another bounded admissible solution of $(P_I(k_0))$ by

$$\big(k(t), j(t)\big) = \begin{cases} \big(k_-(t), j(k_-(t), y_-(t))\big) & \text{for} \quad t \in [0, \hat{T}), \\ (0,0) & \text{for} \quad t \in [\hat{T}, \infty). \end{cases}$$

The value of its objective function is $\hat{F}_-(k_0) := \int_0^\infty e^{-\theta s}f[k_-(s), j(z_-(s))]\mathrm{ds}$. These values are plotted in Figure 3.2 as a function of k_0 (with sign $\times$).

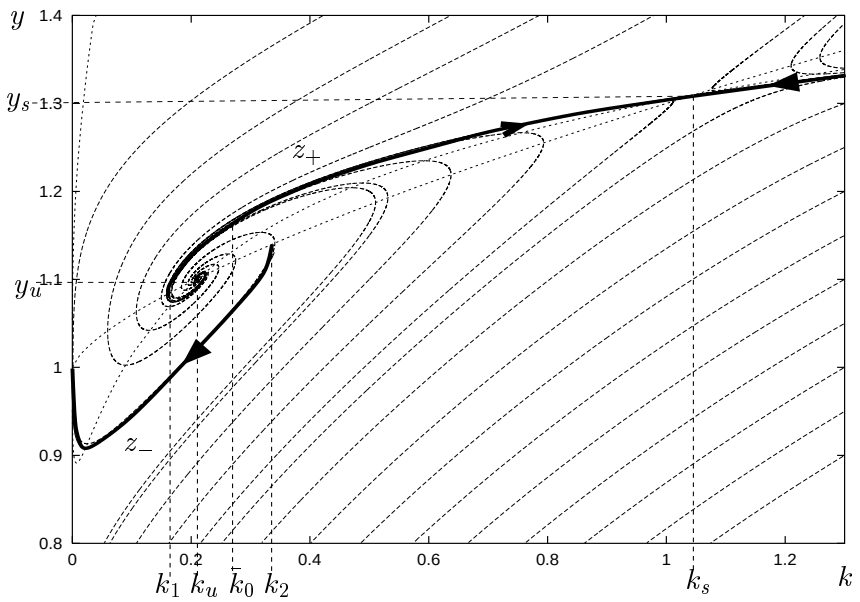

Figure 3.1: k-y phase portrait for (3.6), (3.7) with parameters $\alpha = 1.1$, $c_f = 0.29$, $\sigma = 0.15$, $\theta = 0.1$ and $\gamma = 0.7$.

Since the point $(0, 1)$ is singular, we cannot apply our approximation method directly. Rather we use an anlytical approximation near singularities (compare [10]) as follows

$$\begin{aligned} K(t) &:= 0 + b_1 t^{\epsilon_1} + b_2 t^{\epsilon_2}, \\ Y(t) &:= 1 + a_1 t^{\delta_1} + a_2 t^{\delta_2} \end{aligned}$$

and choose coefficients a_1, a_2, b_1, b_2, $0 < \delta_1 < \delta_2$ and $0 < \epsilon_1 < \epsilon_2$, such that $\big(K(t), Y(t)\big)$ satisfies

$$\begin{pmatrix} \dot{K}(t) \\ \dot{Y}(t) \end{pmatrix} + F(K(t), Y(t)) = \begin{pmatrix} o(t^{\epsilon_2}) \\ o(t^{\delta_2}) \end{pmatrix}$$

where F is defined by (3.6) and (3.7) and $p(t) = o(t^\delta)$ means $\frac{p(t)}{t^\delta} \overset{t \to 0}{\longrightarrow} 0$. In this particular case with $\alpha > 1$, the exponents are $\delta_1 = 1$, $\epsilon_1 = \frac{2}{1-\gamma}$, $\delta_2 = \epsilon_1(\alpha - 1) + 1$ and $\epsilon_2 = \epsilon_1 + 1$. Thus (K, Y) is an approximation for a solution of the time inverted system (3.6), (3.7) with $K(0) = 0$, $Y(0) = 1$. We choose some (small) $\bar{T} > 0$ and solve the following boundary value problem on $[0, 1]$

$$\begin{pmatrix} \dot{k} \\ \dot{y} \end{pmatrix} = T \cdot F(k, y),\ k(0) = k_0,\ k(1) = K(\bar{T}),\ y(1) = Y(\bar{T}),$$

where T is an additional "free" parameter, defining the time needed to reach $\big(K(\bar{T}), Y(\bar{T})\big)$.

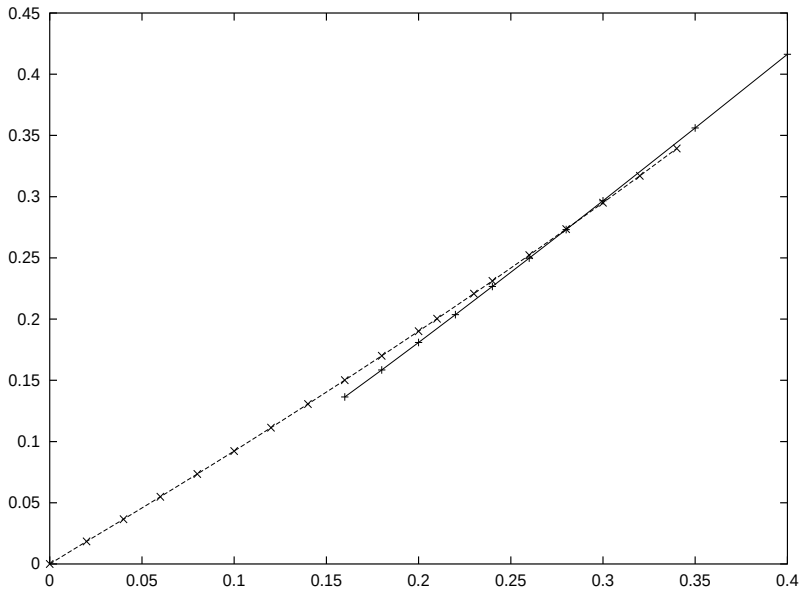

Figure 3.2: Values of the objective functions $\hat{F}_-(k_0)$ for z_- ($\times$- - -$\times$) and $\hat{F}_+(k_0)$ (+—+) for z_+, depending on k_0.

Thus we define an approximation for $z_- = (k_-, y_-)$ by

$$\big(k_-(t), y_-(t)\big) = \begin{cases} \big(k(\frac{t}{T}), y(\frac{t}{T})\big) & : \quad t \in [0, T] \\ \big(K(\bar{T} + T - t), Y(\bar{T} + T - t)\big) & : \quad t \in (T, \bar{T} + T] \\ (0, 1) & : \quad t > \bar{T} + T. \end{cases}$$

In particular, we obtain an approximation of the time $\hat{T} = \bar{T} + T$ which is needed to reach $k = 0$.

All solutions which leave the region $\{(k, y) \mid |y| \leq \sqrt{4c_f k^{\alpha-\gamma} + 1}\}$ where $f(k, \frac{y-1}{2}k^\gamma) \geq 0$ and stay outside do not satisfy (3.8), because its right hand side is negative and its left hand side is positive.

The only solutions which do not leave this region $\{(k, y) \mid f(k, \frac{y-1}{2}k^\gamma) \geq 0\}$ are those converging to $(0, 1)$, (k_u, y_u) or (k_s, y_s).

By Proposition 2.6 and $j(k, y) = \frac{y-1}{2}k^\gamma$ there is no optimal trajectory for $(P_I(k_0))$ with $k(\bar{t}) = k_0$, $y(\bar{t}) \neq y_0$ for some $\bar{t} > 0$. This can be observed by numerical caclulations, too.

Hence we can restrict to those solutions which converge monotonically to one of the steady states.

Let k_1 and k_2 (compare Figure 3.1) be the values where the z_+ and z_-, respectively, intersect the line with $\dot{y} = 0$. If $k(0) \leq k_2$, then the solution defined by z_- which converges to $(0, 1)$ is one candidate, if $k(0) \geq k_1$, then the solution defined by z_+ which converges to (k_s, y_s) is another candidate. At $k(0) = k_u$ the stationary solution $(k_u, \sigma k_u)$ is another candidate, but in a neighborhood of k_u there exists no solution converging to k_u and hence k_u is not nondegenerate. Moreover, the value of its objective function is not optimal (see Figure 3.3).

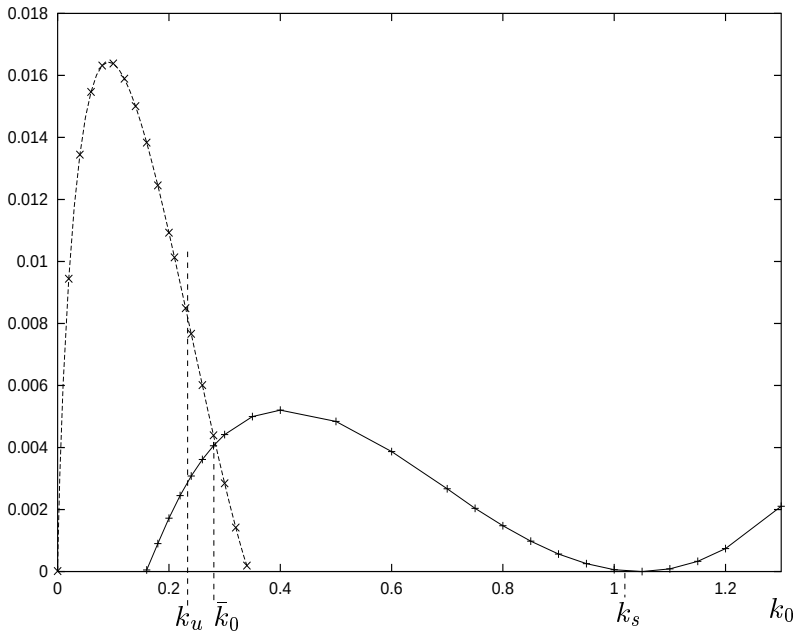

Figure 3.3: Values $\hat{F}_-(k_0) - \frac{f(k_0,\sigma k_0)}{\theta}$ ($\times$- - -$\times$) and $\hat{F}_+(k_0) - \frac{f(k_0,\sigma k_0)}{\theta}$ (+—+) of the objective functions scaled by $\frac{f(k_0,\sigma k_0)}{\theta}$, the value corresponding to $j(t) := \sigma k_0$, for different k_0.

If we compute the values of the objective function we get Figure 3.2. In Figure 3.3 we show the difference of the values of the objective function to $\frac{f(k_0,\sigma k_0)}{\theta}$ which is the value of the objective function of the admissible stationary solution of $(P_I(k_0))$ defined by $(k_0, \sigma k_0)$ (notice that this solution does not solve the differential equation (3.6), (3.7)).

Figure 3.3 shows that there is a Skiba point at a value $\bar{k}_0 \approx 0.29$. Notice that this is not k_u, the value of the unstable steady state.

At the Skiba point the global optimal strategy changes, for an initial capital stock less than $\bar{k}_0$ the optimal strategy is to "sell" the capital stock and for an initial capital stock larger than $\bar{k}_0$ it is optimal to reach the upper steady state.

4 Resource management

In [26] a model of optimization in resource management is presented and the solutions near a steady state are analyzed. The existence of periodic orbits for rather large discount rates is proved in [12] by detecting a Hopf bifurcation.

We analyze a similar model of optimal exploitation of renewable resources and optimal consumption policy due to [31].

We demonstrate in this section the existence of periodic orbits for the latter system and show how to approximate solutions converging to it by using the method from [24].

Using Pontryagin's maximum principle we get a 4-dimensional ordinary differential equation and we detect periodic orbits by parameter continuation and bifurcation theory using CONTENT [19].

The model

The control variables are the rate of extraction of resources $q(t) \geq 0$ and the consumption policy $c(t) \geq 0$. The change of the stock of resource $R(t)$ and the level of countries debt $B(t)$ is given by

$$\dot{R} = g(R) - qR, \qquad R(0) = R_0, \tag{4.1}$$

$$\dot{B} = h(B) - f(qR) + c, \qquad B(0) = B_0, \tag{4.2}$$

where $g(R) = eR(1-R),\ e > 0$ is the natural growth function of resource and the costs of debt are $h(B) = aB^{\sigma},\ a > 0,\ \sigma > 1$. The production function is defined by $f(D) = p((1+D)^{\gamma} - 1),\ 0 < \gamma < 1$, where D is the amount of used resources.

There exists a critical curve $x^*(R)$, such that it is possible to hold debt bounded if $B(t) \leq x^*(R(t))$. In [31] it is described how to compute this curve by using vector field analysis.

We want to optimize the utility stream over an infinite time horizon

$$\int_0^{\infty} U(c(t), R(t)) e^{-\delta t} \mathrm{dt}$$

with a discount rate $\delta > 0$ and a utility function $U(c, R) = dc^{\epsilon} R^{\rho},\ 0 < \epsilon, \rho < 1$.

Using Pontryagin's maximum principle (Theorem 2.3) and scaling the costate variables with $e^{\delta t}$ we get necessary condition for an optimal solution. In particular, we obtain that $\lambda = 1$. Defining scaled costate variables $x,\ y$ we obtain, that an optimal solution

(R, B, x, y) solves the 4-dimensional dynamical system

$$\dot{R} = eR(1-R) - qR; \qquad R(0) = R_0, \tag{4.3}$$

$$\dot{B} = aB^{\sigma} - p((1+qR)^{\gamma} - 1) + c; \qquad B(0) = B_0, \tag{4.4}$$

$$\dot{x} = x\delta - xe(1-2R) - d\rho c^{\epsilon} R^{\rho-1}, \tag{4.5}$$

$$\dot{y} = y\left(\delta - a\sigma B^{\sigma-1}\right) \tag{4.6}$$

with $q = ((-\frac{yp\gamma}{x})^{\frac{1}{1-\gamma}} - 1)/R$ and $c = (-\frac{dR^{\rho}\epsilon}{y})^{\frac{1}{1-\epsilon}}$.

As in [31] we start with parameters $\gamma = 0.76,\ \sigma = 1.01,\ d = 0.19,\ \epsilon = 0.21,\ a = 0.1,$ $e = 0.08,\ \rho = 0.77,\ \delta = 0.1,\ p = 24.85$ and we detect two steady states

$$(R_{E1}, B_{E1}, x_{E1}, y_{E1}) = (0.5762\ ,\ 0.3697\ ,\ 1.1738\ ,\ -0.0624)$$

which leads to $(c_{E1}, q_{E1}) = (0.3315\ ,\ 0.0339)$ and

$$(R_{E2}, B_{E2}, x_{E2}, y_{E2}) = (0.0272\ ,\ 0.3697\ ,\ 4.174,\ ,\ -0.2211)$$

which leads to $(c_{E2}, q_{E2}) = (0.0034\ ,\ 0.0778)$.

For further analysis of the system we use the continuation software CONTENT to detect Hopf bifurcations and hence periodic orbits in this system. CONTENT has been developed by Kuznetsov and V. V. Levitin (see [19]).

We follow the branch of steady states by variation of the parameter δ and reach a limit point. By continuation of the limit point in the 2-dimensional (ϵ, δ)–parameter space we reach a Takens-Bogdanov point at $\epsilon_{TB} = 0.52737,\ \delta_{TB} = 0.10107$.

At a Takens-Bogdanov bifurcation a branch of homoclinic orbits and a branch of Hopf bifurcations emenate (see [18]). For details on the numerical treatment of a Takens-Bogdanov bifurcation and the continuation of the branch of homoclinic orbits see [7].

We follow the branch of Hopf points from $(\epsilon_{TB}, \delta_{TB})$ to $(\epsilon, \delta) = (0.6118, 0.1)$.

At the branch of Hopf bifurcations unstable periodic orbits arise (no stable direction).

We follow the branch of periodic orbits by continuation of the parameter ϵ and for constant $\delta = 0.1$. This yields Figure 4.4, where we draw ϵ versus the period of the orbit.

Starting at the Hopf point H with unstable periodic orbits we reach a period doubling P, such that one stable direction for the periodic orbit arises. At the turning point for periodic orbits L there is also a period doubling and the periodic orbits which turn back have again one stable direction. Therefore, at $\epsilon = 0.6075$ there are two periodic orbits

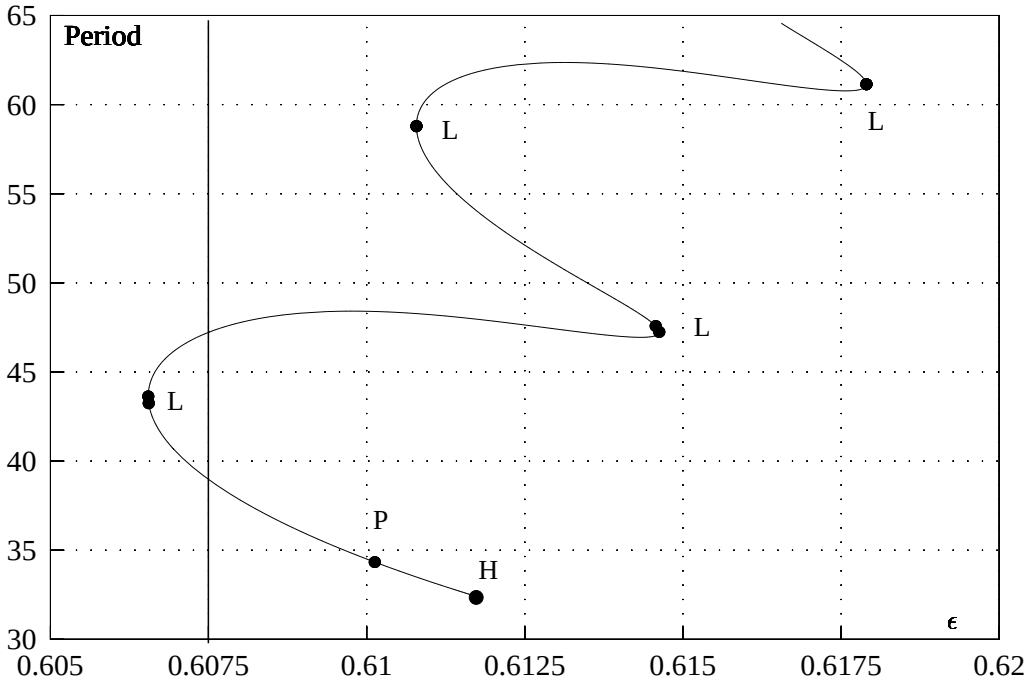

Figure 4.4: Continuation of periodic orbits.

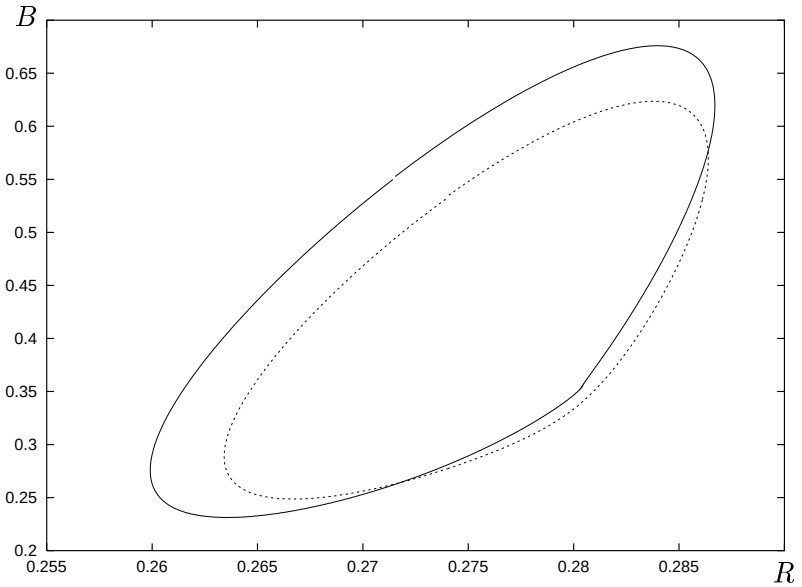

Figure 4.5: Projection of both periodic orbits onto the R–B state space.

for the same parameter value plotted by lines (—) (period $T = 46.9$) and (- -) (period $T = 39.7$) in Figure 4.5.

By numerical integration one observes that (2.13) holds and hence both periodic orbits are candidates if (R_0, B_0) is on the periodic orbit. The value of the objective function depends on the starting point (R_0, B_0, x_0, y_0) on the periodic orbits and is defined by the function v in (2.12).

In Figure 4.6 we plot v for initial points on each of the periodic orbits in the R–B space. In Figure 4.7 and Figure 4.8 we plot its projection onto the $(R, .v)$ and (B, v) plane, respectively.

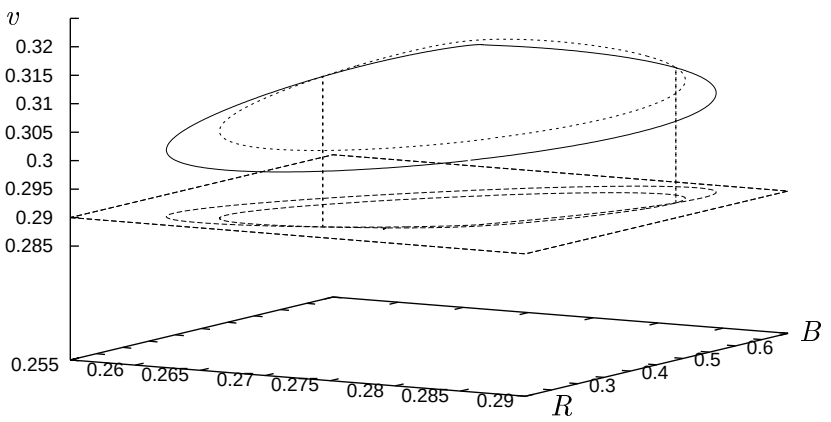

Figure 4.6: Values of the objective function of different initial points on both periodic orbits in the R–B state space.

Conclusion 4.1 *If the projection onto the state space of two periodic orbits intersect and the values of the objective functions at the intersection are equal, then none of the periodic orbits is optimal.*

This can be seen by the following arguments:

If both solutions are optimal, then a solution starting on one periodic orbit and switching at the intersection to the other periodic orbit is admissible and has the same value of the

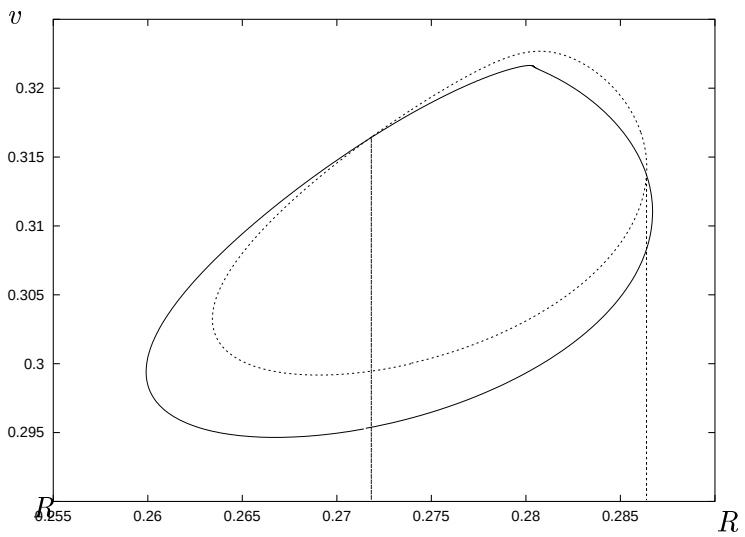

Figure 4.7: Values of the objective function for each of the periodic orbits parametrized by the initial values of R.

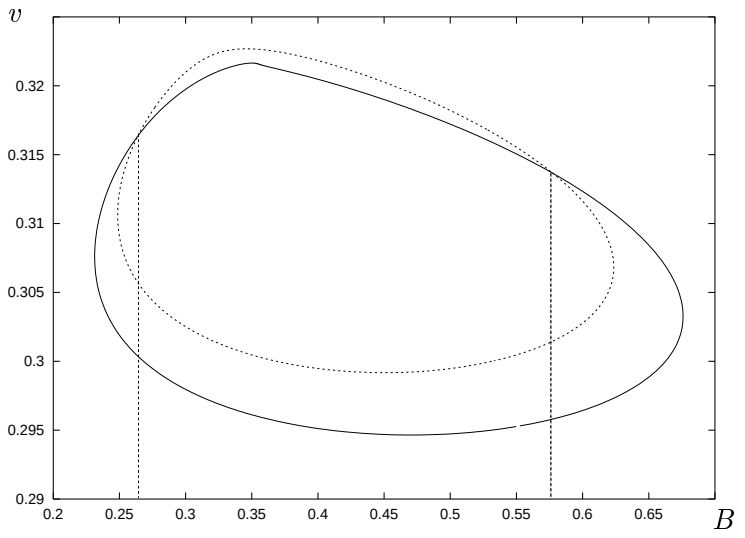

Figure 4.8: Values of the objective function for each of the periodic orbits parametrized by the initial values of B.

objective function and hence is optimal. Thus, this solution is optimal, but its control variable is not continuous. On the other hand continuity of state and costate variables and $q = ((-\frac{yp\gamma}{x})^{\frac{1}{1-\gamma}} - 1)/R$ and $c = (-\frac{dR^{\rho}\epsilon}{y})^{\frac{1}{1-\epsilon}}$ for optimal solutions imply continuity of the control variables.

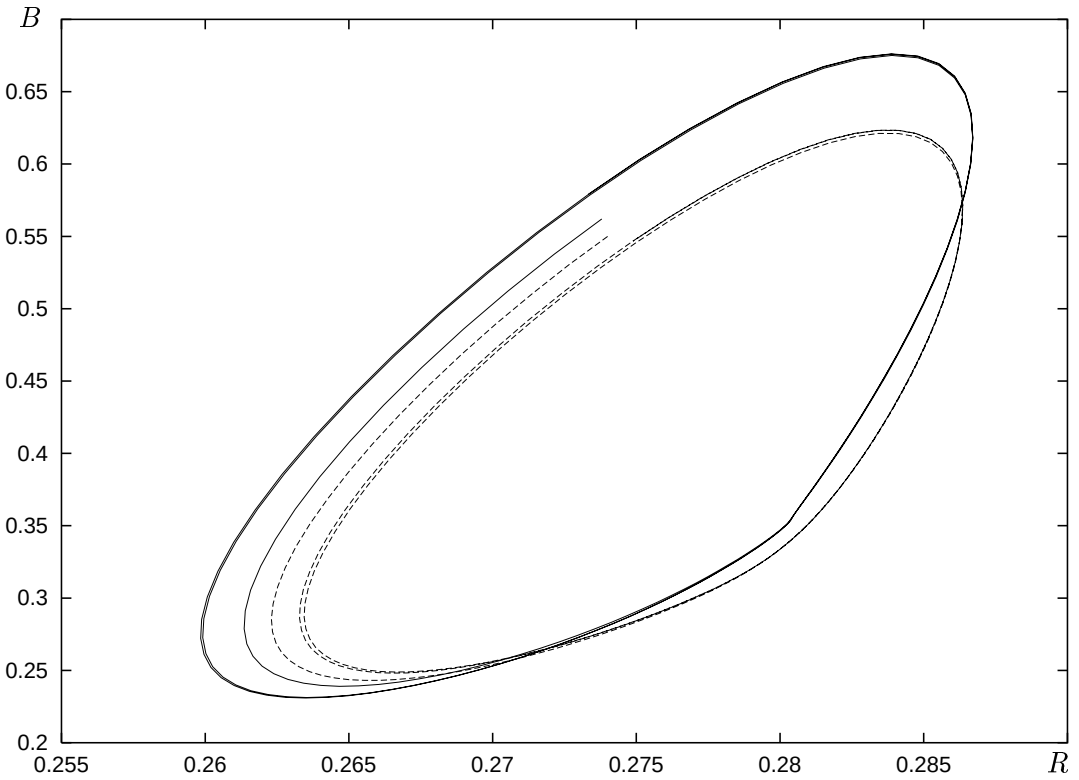

Figure 4.9: Projection of two approximations of solutions converging to different periodic orbits onto the R–B state space.

If the values of the objective function are equal, as suggested by numerical observations, then the periodic orbits are not optimal.

Neverthelss, we approximate two solutions which converges to the periodic orbits. The phase portrait is drawn in Figure 4.9.

The structure of the dynamics near these periodic orbits is complicated and there are several periodic orbits emanating from period doublings nearby. Thus it seems that the non-degeneracy is "weak". We were not able to detect a Skiba point in this case, i. e. initial values with equal state variables different costate variables where the trajectories converge to the different periodic orbits.

5 Conclusions

In this paper we discuss continuous optimization problems with infinite horizon. We develop a numerical tool for approximating candidates for optimal solutions or Skiba points with stationary or periodic asymptotic behaviour. Moreover, this approach allows continuation of parameters to compute parts of higher dimensional Skiba sets. The main results for an error analysis for the numerical method, as presented in [24] (or in a condensed version in [25]), are summerized in the appendix. We apply our method to an optimal investment model, where a Skiba point occurs and to a model of optimal expoitation of renewable resources, where periodic solutions occur, but a Skiba point could not be detected.

This rather general concept is based on first order necessary conditions, hence the question whether the approximated candidates are optimal in a global sense is not resolved by our approach.

A Appendix: The approximation method

In this appendix we summerize theoretical results on our approximation method for solutions of a dynamical system

$$\dot{x} = f(x, \lambda), \quad x(t) \in \mathbb{R}^m, \quad \lambda \in \Lambda \subset \mathbb{R}^p \tag{A.1}$$

with initial value in a given manifold $\mathcal{W}$ and converging either to steady states (Case 1) or to periodic orbits (Case 2) (for details see [24] or [25]).

Thus we assume

P1 $f \in C^k(\mathbb{R}^{m+p}, \mathbb{R}^m)$, $k \geq 2$ and $f^{(k)}$ is locally Lipschitz with respect to x.

Case 1 (Steady States):

P2 $y(\lambda, t) = y(\lambda)$, $t \geq 0$ is a branch of steady states of (A.1), i. e. $f\big(y(\lambda), \lambda\big) = 0$ for all $\lambda \in \Lambda$.

P3 There exists $\alpha < 0$ and $\beta > 0$ such that $\frac{\partial}{\partial x} f\big(y(\lambda), \lambda\big)$ has $m_a \geq 1$ eigenvalues μ with $\mathrm{Re}(\mu) < \alpha < 0$ and $m_b = m - m_a \geq 1$ eigenvalues μ with $\mathrm{Re}(\mu) > \beta$ for each $\lambda \in \Lambda$.

Case 2 (Periodic solutions): Let $\hat{y}(\lambda, \cdot)$ be $T(\lambda)$–periodic solutions of the (original, unscaled) system $\dot{y} = \hat{f}(y, \lambda)$ and let $f(y, \lambda) := T(\lambda)\hat{f}(y, \lambda)$, then $y(\lambda, \cdot) := \hat{y}\big(\lambda, \cdot T(\lambda)\big)$ is a 1–periodic solution of the scaled system $\dot{y} = f(y, \lambda)$.

P2 $y(\lambda,\cdot)$ is a C^k–branch of 1–periodic solution of the scaled system $\dot{y} = f(y,\lambda)$.

P3 There exists $\alpha < 0$ such that all $y(\lambda,\cdot)$ have $m_a \geq 1$ Floquet multipliers μ with $|\mu| < e^{\alpha}$, and $m_b = m - m_a \geq 1$ Floquet multipliers with $|\mu| \geq 1$.

The following lemma implies the existence of smooth stable manifolds (for similiar results see e. g. [16], [15],[32] and for convergence to invariant manifolds [13, 14]).

LEMMA A.1 *Suppose that* **P1**–**P3** *hold, then there exist projectors* $\bar{P}_a(\lambda,t)$ *and* $\bar{P}_b(\lambda,t) = \mathrm{Id} - \bar{P}_a(\lambda,t)$ *onto the stable and (center–) unstable subspaces of the linearization about* $y(\lambda,\cdot)$ *at* t. *Moreover, there exist* (m_a+p)*–dimensional stable manifolds* $\mathcal{M}^\tau \subset \mathbb{R}^{m+p}$ *of* $y(\lambda,\tau)$, $\tau \geq 0$. *These are of type* C^{k-1}.

In [24] the lemma is proved in a more general setup, including the case of strong stable manifolds ($\beta < 0$), in particular.

Now we want to detect **nondegenerate connecting orbits with asymptotic rate $\gamma < 0$ from $\mathcal{W}$ to y**, i. e. solutions $(\bar{x},\bar{\lambda})$ of (A.1) which satisfy

$$\begin{aligned} \dot{x}(t) - f(x(t),\lambda) &= 0, \quad t \geq 0, && \text{(A.2)}\\ (x(0),\lambda) &\in \mathcal{W}, && \text{(A.3)}\\ ||x(t) - y(\lambda,t)|| &\leq Ce^{\gamma t}, \quad t \geq 0, \gamma < 0. && \text{(A.4)} \end{aligned}$$

We assume that $\mathcal{W} = \{(x_0,\mu) \in \mathbb{R}^{m+p} | g(x_0,\mu) = 0\}$ is an m_b–dimensional submanifold of $\mathbb{R}^{m+p}$, where $g \in C^{k-1}(\mathbb{R}^{m+p}, \mathbb{R}^{m_a+p})$ and $g'(x_0,\mu)$ has full rank for all $(x_0,\mu) \in \mathcal{W}$.

Nondegeneratly means that $\mathcal{W}$ and $\mathcal{M}^0$, the stable manifold of $y(\lambda,0)$ intersect transversally at $(\bar{x}(0),\bar{\lambda})$ (for a precise definition see [24]).

We approximate a solution $(\bar{x},\bar{\lambda})$ of (A.2)–(A.4) on a finite interval $J = [0,T_+]$ by solving the following boundary value problem for $\big(x(\cdot),\lambda\big)$:

$$\begin{aligned} \dot{x}(t) - f(x(t),\lambda) &= 0, \quad t \in J, && \text{(A.5)}\\ g(x(0),\lambda) &= 0, && \text{(A.6)}\\ \big(x(T_+),\lambda\big) &\in \big(y(\lambda,T_+),\lambda\big) + T\mathcal{M}^{T_+}, && \text{(A.7)} \end{aligned}$$

where $\mathcal{M}^{T_+}$ is the stable manifold of $y(\lambda,T_+)$ and $T\mathcal{M}^{T_+}$ is its tangent space in $\mathbb{R}^{m+p}$ at $\big(y(\lambda,T_+),\lambda\big)$.

The asymptotic boundary condition (A.7) may be written as

$$\bar{M}(T_+,\lambda)\big(x(T_+) - y(\lambda,T_+)\big) = 0, \qquad \text{(A.8)}$$

where each $\bar{M}(T_+,\lambda) \in \mathbb{R}^{m_b,m}$ is a matrix with

$$\mathcal{N}\big(\bar{M}(T_+,\lambda)\big) = \mathcal{R}\big(\bar{P}_a(\lambda,T_+)\big) = \mathcal{N}\big(\bar{P}_b(\lambda,T_+)\big). \tag{A.9}$$

In the stationary case it is easy to compute $\bar{M}(T_+,\lambda)$ by solving an eigenvalue problem, but in the periodic case it is expensive to compute such $\bar{M}(T_+,\lambda)$ depending on λ, thus we use a simpler appoach, where we only assume

P4 $\hat{M}(T_+,\lambda) \in \mathbb{R}^{m_b,m}$ is a matrix which satisfies

$$\mathcal{N}\big(\hat{M}(T_+,\lambda)\big) \cap \mathcal{R}\big(\bar{P}_b(\lambda,T_+)\big) = \{0\} \tag{A.10}$$

for all $\lambda \in \Lambda$ and $T_+ \in \mathcal{D}$, where $\mathcal{D} \subset \mathbb{R}_+$ is a set of appropriate T_+.

In particular, we may choose $\hat{M}(T_+,\lambda)$ independent of λ.

In A.1 we show how to compute $\bar{M}(T_+,\lambda)$ which satisfy (A.9). In A.2 we set up and analyze an approximation method for connecting orbits with asymptotic rate and in A.3 we develop a boundary corrector method which allows to use $\hat{M}(T_+,\lambda)$ constant in λ and to get estimates of the same equality as in the case where $\bar{M}(T_+,\lambda)$ satisfies (A.9).

A.1 Asymptotic boundary condition and numerical set–up

Now we derive asymptotic boundary conditions and the numerical setup either for steady states or for periodic orbits.

Steady States: To get a smooth matrix valued function $\bar{M}(\lambda)$ (independent of T_+, i. e. we can use $\mathcal{D} = \mathbb{R}_+$) in a neighborhood of $\hat{\lambda}$ we solve for the unstable eigenvalues of $\frac{\partial}{\partial x} f(y(\lambda),\lambda)$ by using the algebraic system

$$V\frac{\partial}{\partial x} f(y(\lambda),\lambda) = U(\lambda)V, \quad \mathrm{Re}\big(\sigma(U)\big) > 0, \quad U \in \mathbb{R}^{m_b,m_b},\ V \in \mathbb{R}^{m_b,m}, \tag{A.11}$$

$$\hat{V}^\top V = \hat{V}^\top \hat{V}, \tag{A.12}$$

where $\hat{V}$, $\hat{U}$ solves (A.11) at some $\hat{\lambda}$ and we define $\bar{M}(\lambda) = V$. Together with (A.5), (A.6) and (A.8) we obtain the operator equation

$$G^e_J(z,\nu,y,V,U) = \begin{pmatrix} f(y,\nu) \\ \dot{z} - f(z,\nu) \\ g\big(z(0),\nu\big) \\ V\big(z(T_+) - y\big) \\ V\frac{\partial}{\partial x} f(y,\lambda) - UV \\ \hat{V}^\top V - \hat{V}^\top \hat{V} \end{pmatrix} = 0 \tag{A.13}$$

to get an approximation (z_J, ν_J) in the case of steady states.

Periodic Solutions:

We choose $\hat{M}(T_+, \lambda)$ 1–periodic with respect to $T_+ \in \mathcal{D} := \cup_{N\in\mathbb{N}}[N-\tau_-, N+\tau_+]$ for some $\tau_-, \tau_+ \in [0,1]$ such that $\mathcal{N}\big(\hat{M}(T_+, \bar{\lambda})\big) \cap \mathcal{R}\big(\bar{P}_b(T_+)\big) = \{0\}$ holds on the interval $[-\tau_-, \tau_+]$.

A good choice for $\hat{M}(T_+, \lambda) \in \mathbb{R}^{m_b, m}$ is $\hat{M}(T_+, \lambda) = V(0, \lambda)$ where V is a solution of the adjoint variational equation

$$\begin{aligned} \dot{V} &= -V\frac{\partial}{\partial x} f(y(\lambda, T_+ + \cdot), \lambda), && \text{(A.14)} \\ V(1, \lambda) &= U(\lambda)V(0, \lambda) && \text{(A.15)} \end{aligned}$$

for some $U(\lambda) \in \mathbb{R}^{m_b, m_b}$ with $|\mu| \leq 1$ for all eigenvalues μ of $U(\lambda)$ (see [8],[24]).

Adding these equations to (A.5), (A.6), (A.8) will blow up the dimension of the boundary value problem from $2m$ to $(2 + m_b)m$. Computing it smoothly in terms of λ as in the steady state case requires in each evaluation step to solve an $(m_b m)$–dimensional boundary value problem.

To avoid this we develop the **boundary corrector method** (see also Section A.3),

1. Compute $\hat{M}(T_+, \mu_0)$ as a solution of (A.14), (A.15) at some initial approximation μ_0,
2. Calculate the solution $\big(x_J(\cdot), \mu_J, y(\mu_J, \cdot), T(\mu_J)\big)$ of

$$G_J^p\big(z(\cdot), \nu, y(\cdot), T\big) = \begin{pmatrix} \dot{y} - T\hat{f}(y, \nu) \\ \chi(y, \nu) \\ y(0) - y(1) \\ \dot{z} - T\hat{f}(z, \nu) \\ g\big(z(0), \nu\big) \\ \hat{M}(T_+, \mu_0)\big(z(T_+) - y(T_+)\big) \end{pmatrix} = 0, \qquad \text{(A.16)}$$

3. Repeat step **1.** with $\mu_0 = \mu_J$ and step 2 once.

We apply the approximation result to the scaled system. If $(\hat{x}, \bar{\lambda})$ is a connecting orbit with asymptotic rate of the unscaled system. Then $(\bar{x}, \bar{\lambda})$, $\bar{x}(t) := \hat{x}(t \cdot \bar{T})$ is a connecting orbit with asymptotic rate of the scaled system. Furthermore let (x_J, λ_J) be a solution of (A.16), i. e. (x_J, λ_J) is an approximation of $(\bar{x}, \bar{\lambda})$ in the scaled system. Then $(\hat{x}_J, \lambda_J)$, where $\hat{x}_J(t) := x_J(\frac{t}{T(\lambda_J)})$ is an approximation of $(\hat{x}, \bar{\lambda})$ in the unscaled system and one can show the estimate

$$||\hat{x}_J - \hat{x}||_\infty \leq ||x_J - \bar{x}||_\infty + C\frac{T'(\bar{\lambda})}{T(\bar{\lambda})} \cdot T_+ \cdot |\lambda_J - \bar{\lambda}| \quad \text{where } C \approx M_f \qquad \text{(A.17)}$$

(see [24]). With the forthcoming error estimates in $|\lambda_J - \bar{\lambda}|$ and $|x_J - \bar{x}|$ in the scaled system carried over to the unscaled system except for the factor T_+.

A.2 Approximation theorem

In this section we assume that $y(\lambda)$ is known and that **P1**–**P4** hold. Furthermore, let $(\bar{x}, \bar{\lambda})$ be a nondegenerate connecting orbit with asymptotic rate γ from $\mathcal{W}$ to y, where $\alpha < \gamma < 0$.

For $T_+ > 0$, $J := [0, T_+]$ we define Banach spaces $\mathcal{Y} = C^1(J, \mathbb{R}^m) \times \mathbb{R}^p$ with

$$||(x,\lambda)||_{\mathcal{Y}} = ||x||^*_{\mathcal{Y}} + |\lambda|, \quad ||x||^*_{\mathcal{Y}} = \sup_{t\in J} \big(|x(t)|e^{-\beta t}\big) \tag{A.18}$$

and $\mathcal{Z} = C^0(J, \mathbb{R}^m) \times \mathbb{R}^{m_a+p} \times \mathbb{R}^{m_b}$ with

$$\begin{aligned} ||(v, r_0, r_+)||_{\mathcal{Z}} &= ||v||^*_{\mathcal{Z}} + |r_0| + e^{-\beta T_+}|r_+|, \\ ||v||^*_{\mathcal{Z}} &= ||v||_\beta + \int_0^{T_+} |v(s)|e^{-\beta s}\mathrm{ds}, \quad ||v||_\beta = \sup(|v(t)|e^{-\beta t}) \end{aligned} \tag{A.19}$$

THEOREM A.2 *Suppose that* **P1**–**P4** *hold and let* $(\bar{x}, \bar{\lambda})$ *be a nondegenerate connecting orbit with asymptotic rate* γ *from* $\mathcal{W}$ *to* y *with* $\alpha < \gamma < 0$.

Then there exist $\delta > 0$, $C > 0$ *and* $\bar{T}_+ > 0$ *sufficiently large, such that for* $T_+ \in \mathcal{D}$, $T_+ \geq \bar{T}_+$ *the operator equation*

$$G_J(z,\nu) = \begin{pmatrix} \dot{z} - f(z,\nu) \\ g\big(z(0),\nu\big) \\ \hat{M}(T_+,\nu)\big(z(T_+) - y(\nu, T_+)\big) \end{pmatrix} = 0, \tag{A.20}$$

has a unique solution (z_J, ν_J) *in the ball* $\mathcal{B}_{\frac{\delta}{r(\beta,T_+)}}(\bar{x}|_J, \bar{\lambda})$ *in* $\mathcal{Y}$ *where*

$$r(\beta, T_+) := \sup_{t\in J}\Big(e^{|\beta|t}\Big) + \int_0^{T_+} e^{|\beta|t}\mathrm{dt} = \begin{cases} e^{|\beta|T_+} + \frac{1}{|\beta|}\left(e^{|\beta|T_+} - 1\right) & : \beta \neq 0 \\ 1 + T_+ & : \beta = 0. \end{cases}$$

Furthermore, the following estimates hold with $d = 1$

$$|\nu_J - \bar{\lambda}| \leq Ce^{(d\gamma-\beta)T_+} \tag{A.21}$$

$$|z_J(t) - \bar{x}|_J(t)| \leq Ce^{d\gamma T_+} \cdot e^{-\beta(T_+ - t)},\ 0 \leq t \leq T_+. \tag{A.22}$$

If $\hat{M}(T_+, \lambda)$ *satisfies*

$$|\hat{M}(T_+, \bar{\lambda})\big(\bar{x}(T_+) - y(\bar{\lambda}, T_+)\big)| \leq \bar{c}|\bar{x}(T_+) - y(\bar{\lambda}, T_+)|^2 \tag{A.23}$$

for a constant $\bar{c} > 0$ *and sufficiently large* $T_+ \in \mathcal{D}$ *then the estimates (A.21), (A.22) hold for* $d = 2$.

REMARK: If $\mathcal{N}\big(\hat{M}(T_+,\bar{\lambda})\big) = \mathcal{R}\big(\bar{P}_a(\bar{\lambda})(T_+)\big)$ and $k \geq 3$ then (A.23) is satisfied for T_+ sufficiently large (see [24]) and we get exponential estimates.

As a direct result of Theorem A.2 we obtain a super-convergence phenomenon for the parameter similiar in sup–norm as in the connecting orbit case [5], [6], [27].

COROLLARY A.3 *Under the assumptions of Theorem A.2 we have the* sup*–norm estimates*

$$|\nu_J - \bar{\lambda}| \leq Ce^{(d\gamma-\beta)T_+}, \tag{A.24}$$

$$||z_J - \bar{x}|_J||_\infty \leq Ce^{d\gamma T_+}. \tag{A.25}$$

A.3 Boundary corrector method

The asymptotic boundary conditions with (A.9) yields "quadratic" estimates for $k \geq 3$ (see Theorem A.2 and its Remark). On the other hand, e. g. in the periodic case, it can be cumbersome to compute $\bar{M}(T_+,\lambda)$ at $\bar{\lambda}$ since $\bar{\lambda}$ is unknown. But it is realistic to assume

P5 There exists a neighborhood Λ_0 of $\bar{\lambda}$ and a matrix valued function $\bar{M}(T_+,\cdot) \in C^1\big(\Lambda_0, \mathbb{R}^{m_b,m}\big)\big)$ such that (A.9) holds for each $\lambda \in \Lambda_0$.

With the boundary corrector method developed in this section we have to compute such $\bar{M}(T_+,\lambda)$ only twice.

Boundary Corrector Method

1. Start with some $\mu_0 \in \Lambda_0$, i=0,
2. Calculate (z_{i+1}, μ_{i+1}), the solution of

$$G_J^{\mu_i}(z,\nu) = \begin{pmatrix} \dot{z} - f(z,\nu) \\ g\big(z(0),\nu\big) \\ \bar{M}(T_+,\mu_i)\big(z(T_+) - y(\nu,T_+)\big) \end{pmatrix} = 0, \tag{A.26}$$

3. Repeat with **2.** and "$i = i+1$" once.

The following proposition states that the error of the solution of the boundary corrector method has the same exponential rate $(2\gamma-\beta)$ as the solution of Theorem A.2 with $d = 2$.

PROPOSITON A.4 *Let the assumptions of Theorem A.2 and* **P5** *hold and $\bar{T}_+$ sufficiently large. Then there exists some $\tilde{c} > 0$ such that for $T_+ \in \mathcal{D}$, $T_+ \geq \bar{T}_+$ and $\mu_0 \in \Lambda_0$ the result (z_J, ν_J) of the* **boundary corrector method** *can be estimated by*

$$||(z_J,\nu_J) - (\bar{x}|_J, \bar{\lambda})||_{\mathcal{Y}} \leq \tilde{c}e^{(2\gamma-\beta)T_+}. \tag{A.27}$$

A.4 Skiba points

To detect Skiba points we have to solve problems of the type

$$\begin{aligned} \dot{x}_\pm(t) - f_\pm(x_\pm(t), \lambda) &= 0, \quad t \geq 0, \\ (x_+(0), x_-(0), \lambda) &\in \mathcal{W}, \\ ||x_\pm(t) - y_\pm(\lambda)(t)|| &\leq Ce^{\gamma_\pm t}, \quad t \geq 0, \gamma_\pm < 0, \end{aligned}$$

where the equality of values can be expressed in terms of initial values by (2.11).

For this kind of problem results similiar to those above can be obtained ([24, Chapter 4,5]).

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