Singular perturbation with Hopf points in the fast dynamics

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Abstract

A singular perturbation problem in ordinary differential equations is investigated without assuming hyperbolicity of the associated slow manifold. More precisely, the slow manifold consists of a branch of stationary points which lose their hyperbolicity at a stationary point with eigenvalues $\pm i$. Thus, it is impossible to reduce the dynamics near the Hopf point to the slow manifold. This situation is examined within a generic one parameter unfolding. It leads to two bifurcating curves of Hopf points and associated to these are two manifolds of periodic orbits and possibly another manifold of invariant tori, all three of which intersect in the central Hopf point. The proof employs a suitable Ansatz resulting in a Hopf bifurcation theorem which determines precisely the bifurcation structure near a certain Hopf point with an additional zero eigenvalue.

1 Introduction

The classical paper of Vasil'eva [26] deals with singular perturbation problems of the form

$$\begin{aligned} \dot{x} &= f(x,y,\epsilon) \in \mathbb{R}^2 \\ \dot{y} &= \epsilon g(x,y,\epsilon) \in \mathbb{R} \end{aligned} \tag{1.1}$$

under the assumption of a hyperbolic structure for the fast x-dynamics at the parameter value $\epsilon = 0$. With this hyperbolicity condition, the fast x-dynamics and the resulting slow y-dynamics for $\epsilon \neq 0$ can be described precisely and, in a certain sense, independently of each other [11]. On the other hand, the underlying theory of normally hyperbolic invariant manifolds [27] collapses completely if the fast x-dynamics loses its hyperbolic character. As a result new phenomena can arise through the coupling of fast and slow dynamics. Examples of this effect are the formation of small periodic orbits near a stationary limit point of the slow manifold $f(x,y,0) = \Theta$ ([4],[5],[23],[25]) and the subsequent transition into global objects such as relaxation orbits or homoclinic orbits [18] over the intermediate stage of duck orbits ([3],[9],[10]). Similarly, invariant tori can arise through a Naimark-Sacker bifurcation near a saddle-node of periodic orbits of the fast x-system $\dot{x} = f(x,y,0)$ [25].

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Now in this paper, the system (1.1) is analyzed when the slow manifold loses its hyperbolicity at a Hopf point; i.e. the fast y-dependent x-system

$$\dot{x} = f(x, y, 0) \in \mathbb{R}^2$$

has a branch of stationary points with a Hopf point at the parameter value $y = 0$. Consequently, the fast x-dynamics loses its hyperbolic structure at $y = 0$, so that the separation of fast and slow dynamics within the full system (1.1) is no longer possible. In ([6], [21]) points of this kind are examined with regard to the take-off behaviour of the slow trajectories which show an unexpected delay after passing the Hopf point. However, this effect only occurs unless the surface $g(x, y, 0) = 0 \in \mathbb{R}$ and the curve $f(x, y, 0) = \Theta \in \mathbb{R}^2$ have intersections very close to the Hopf point. On the other hand, if exactly an intersection of this kind is under consideration (see Fig.2 below), then invariant tori can arise from the Hopf point during the transition from $\epsilon = 0$ to $\epsilon \neq 0$, as suggested in [17] Fig.12 (a) with the help of numerical calculations.

In the following sections, this creation of invariant tori is determined analytically together with the underlying formation of periodic orbits. More precisely, the passage of the surface $g = 0$ through the Hopf point is controlled by a second parameter $\lambda \in \mathbb{R}$, so that a two parameter system of the form

$$\begin{aligned} \dot{x} &= f(x, y, \epsilon, \lambda) \in \mathbb{R}^2 \\ \dot{y} &= \epsilon g(x, y, \epsilon, \lambda) \in \mathbb{R} \end{aligned} \tag{1.2}$$

is considered under the assumptions

$$(f, g)(\Theta) = \Theta, \qquad \{\pm i\} = \mathrm{Sp}[f_x(\Theta)], \qquad \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} (\Theta) \in GL[\mathbb{R}^3],$$

where $\mathrm{Sp}[A]$ and $GL[\mathbb{R}^n]$ denotes the spectrum of a square matrix A and the invertible $(n \times n)$-matrices respectively. Then, a typical bifurcation diagram with respect to the singular perturbation parameter ϵ and the unfolding parameter λ may be depicted as in Figure 1.

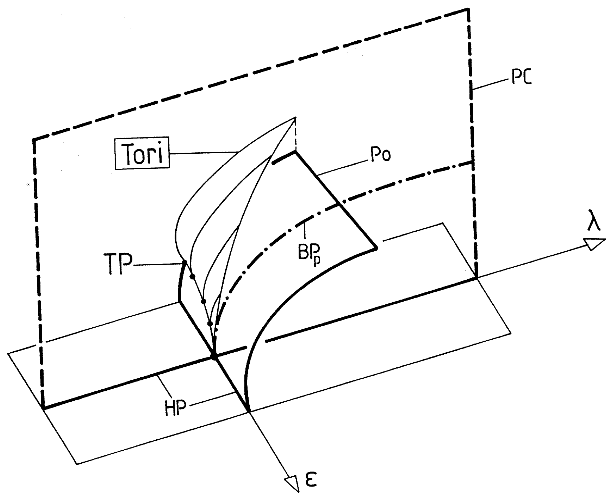

Figure 1: Typical two parameter (ϵ, λ)-bifurcation diagram of the system (1.2). HP=Hopf point, PC=periodic continuum, Po=periodic orbit, BP_p=periodic bifurcation point, TP=torus bifurcation point.

Thus, the unfolded singular perturbation problem (1.2) causes a bifurcation point of two curves of Hopf points HP with associated 2-dimensional manifolds of periodic orbits PC for $\epsilon = 0$ and Po for $\epsilon \neq 0$ which intersect transversally along the curve BP_p of periodic bifurcation points. Moreover, within the manifold Po there exists a curve TP of periodic orbits from which invariant tori branch off through a Naimark-Sacker bifurcation. In the next section, the statements concerning these phenomena are illustrated and given a precise analytical meaning.

2 Results

For the sake of simplicity, the functions f and g in (1.2) are assumed to be of class C^∞. This is also true for all other functions in this paper, so that the degree of differentiability is not stated everywhere. Now, by a standard transformation of the coordinates the system (1.2) may be written as

$$\begin{aligned} \dot{\bar{x}} &= \bar{f}(\bar{x}, \bar{y}, \epsilon, \lambda) \in \mathbb{R}^2 \\ \dot{\bar{y}} &= \epsilon \bar{g}(\bar{x}, \bar{y}, \epsilon, \lambda) \in \mathbb{R} \end{aligned} \tag{2.1}$$

where

$$\begin{gathered} \bar{f}(\Theta, \bar{y}, 0, \lambda) = \Theta, \qquad \bar{g}^0 := \bar{g}(\Theta) = 0, \\ \bar{f}^0_{\bar{x}} := \bar{f}_{\bar{x}}(\Theta) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} =: J_0 \qquad \text{and} \qquad \bar{g}^0_{\bar{y}} \neq 0. \end{gathered} \tag{2.2}$$

Let $\mathrm{tr}(\bar{f}_{\bar{x}})(\bar{x}, \bar{y}, \epsilon, \lambda)$ denote the trace of $\bar{f}_{\bar{x}}(\bar{x}, \bar{y}, \epsilon, \lambda)$ and assume the additional generic conditions

$$\mathrm{tr}_{\bar{y}}(\bar{f}_{\bar{x}})^0 \neq 0 \qquad \text{and} \qquad \bar{g}^0_{\bar{y}} \cdot \mathrm{tr}_\lambda(\bar{f}_{\bar{x}})^0 - \bar{g}^0_\lambda \cdot \mathrm{tr}_{\bar{y}}(\bar{f}_{\bar{x}})^0 \neq 0. \tag{2.3}$$

Then, according to the left inequality, the eigenvalues of $\bar{f}_{\bar{x}}$ on the branch $(\bar{x}, \bar{y}, \epsilon, \lambda) = (\Theta, \bar{y}, 0, 0)$ cross the imaginary axis at $\pm i$ with nonvanishing velocity and the standard Hopf theorem [20] implies a branch of periodic orbits bifurcating from $(\bar{x}, \bar{y}, \epsilon, \lambda) = \Theta$. Moreover, the right inequality in (2.3) ensures that the manifold $\bar{g} = 0$ passes through the Hopf point with nonvanishing velocity with respect to λ. In summary, using the notation HP for Hopf point, SP for stationary point and Po for periodic orbit, the typical situation shown in Fig.2 below arises in $(\bar{x}, \bar{y})$-space for $(\epsilon, \lambda) = \Theta$.

With the surface $\bar{g} = 0$ displaced and the Hopf point HP perturbed, this basic picture remains also valid for $\lambda \neq 0$. Hence at $\epsilon = 0$, the $(\bar{x}, \bar{y})$-space of the system (2.1) contains a continuum of stationary points $\bar{f}(\Theta, \bar{y}, 0, \lambda) = \Theta$ (compare (2.2)) as well as a continuum of periodic orbits. Notice that at $\epsilon = 0$, the $(\bar{x}, \bar{y})$-space can either be interpreted as the phase space of the full system (2.1) or as the phase parameter space of the $\bar{y}$-dependent $\bar{x}$-system. However, during the transition to $\epsilon \neq 0$ these continuum structures generally will be destroyed by the formation of the $\bar{y}$-dynamics. In addition to this destructive effect, new phenomena may arise which will now be stated for the system (2.1) under the assumptions (2.2) and (2.3). The corresponding proofs can be found in a rather extensive form in [25] and in a shorter form in this paper.

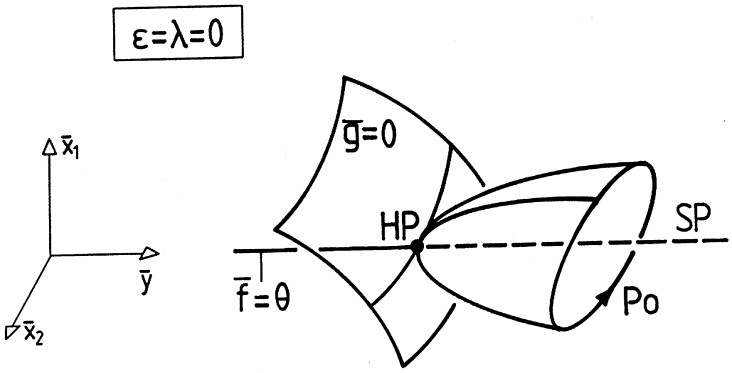

Figure 2: Basic situation in $(\bar{x}, \bar{y})$-space for $(\epsilon, \lambda) = \Theta$. HP=Hopf point, SP=stationary point, Po=periodic orbit.

Lemma 2.1: *Within the $(\bar{x}, \bar{y}, \epsilon, \lambda)$-space there exist two 2-dimensional manifolds of stationary points $(\bar{x}, \bar{y}, \epsilon, \lambda) = (\Theta, \bar{y}, 0, \lambda)$ and $(\bar{x}, \bar{y}, \epsilon, \lambda) = (\bar{x}^s(\epsilon, \lambda), \bar{y}^s(\epsilon, \lambda), \epsilon, \lambda)$ intersecting transversally at $\epsilon = 0$ along the curve of bifurcation points $(\bar{x}, \bar{y}, \epsilon, \lambda) = (\Theta, \bar{y}^s(0, \lambda), 0, \lambda)$. Furthermore, within each of these manifolds there exists exactly one curve of Hopf points $(\bar{x}, \bar{y}, \epsilon, \lambda) = (\Theta, \bar{y}^1(\lambda), 0, \lambda)$ and $(\bar{x}, \bar{y}, \epsilon, \lambda) = (\bar{x}^s(\epsilon, \lambda^2(\epsilon)), \bar{y}^s(\epsilon, \lambda^2(\epsilon)), \epsilon, \lambda^2(\epsilon))$ with associated eigenvalues $i \cdot \omega^1(\lambda)$ and $i \cdot \omega^2(\epsilon)$. These curves of Hopf points and the branch of stationary bifurcation points meet in the central point $(\bar{x}, \bar{y}, \epsilon, \lambda) = \Theta$.*

The $(\epsilon, \lambda, \bar{y})$-bifurcation diagram described by Lemma 2.1 is depicted in Figure 3.

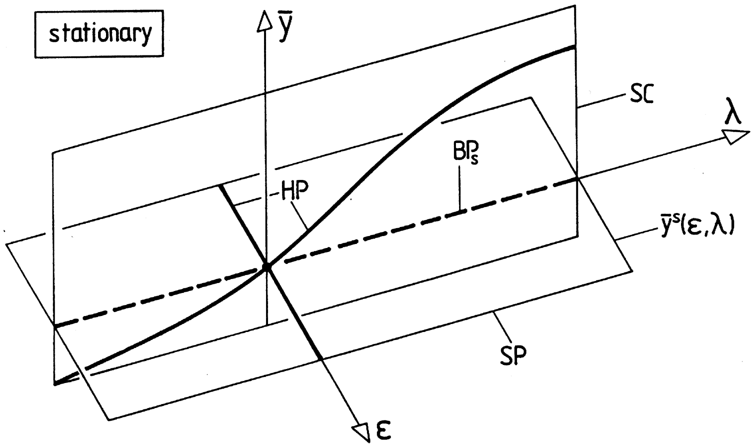

Figure 3: $(\epsilon, \lambda, \bar{y})$-bifurcation diagram of stationary points. HP=Hopf point, SP=stationary point, SC=stationary continuum, BP_s=stationary bifurcation point.

After this characterization of the stationary points with the help of system (2.1), the periodic orbits and the invariant tori are now examined after performing a further (ϵ, λ)-dependent transformation within (2.1). Strictly speaking, after an affine linear coordinate transformation and a time transformation (compare section 4 for details), the resulting system

$$\begin{aligned} \dot{x} &= f(x, y, \epsilon, \lambda) \in \mathbb{R}^2 \\ \dot{y} &= \epsilon g(x, y, \epsilon, \lambda) \in \mathbb{R} \end{aligned} \tag{2.4}$$

with flow $(\Psi,\varphi)(t,x,y,\epsilon,\lambda) \in \mathbb{R}^2 \times \mathbb{R}$ satisfies the following statements. Note, that the notation from section 1 is used again for convenience.

Lemma 2.2: *The Hopf points from Lemma 2.1 are shifted to the origin of phase space with normalized linearization; that is to say,*

$$f(\Theta,y,0,\lambda)=\Theta, \qquad f_x(\Theta,0,0,\lambda)=J_0 \quad \text{and} \quad (f,g)(\Theta,0,\epsilon,\lambda^2(\epsilon))=\Theta,$$

$$\begin{pmatrix} f_x & f_y \\ \epsilon g_x & \epsilon g_y \end{pmatrix}(\Theta,0,\epsilon,\lambda^2(\epsilon)) = \begin{pmatrix} J_0 & 0 \\ 0 & \epsilon g_y(\Theta,0,\epsilon,\lambda^2(\epsilon)) \end{pmatrix}.$$

Proposition 2.3: *Consider the plane system $\dot{x}=f(x,y,0,\lambda)\in\mathbb{R}^2$ with y and λ taken as parameter. Then, periodic orbits bifurcate in y-direction from the trivial curve of Hopf points $(x,y,\epsilon,\lambda)=(\Theta,0,0,\lambda)$ (compare Lemma 2.2). More precisely, there exists a function $(\tilde{y}^1,\tilde{t}^1)(x_1,\lambda)$, so that*

$$\Psi(\,\cdot\,,x_1,0,\tilde{y}^1(x_1,\lambda)\cdot x_1^2,0,\lambda), \qquad x_1\neq 0$$

is a $(2\pi+\tilde{t}^1(x_1,\lambda)\cdot x_1^2)$-periodic solution of the x-system $\dot{x}=f(x,y,0,\lambda)$, with

$$\begin{pmatrix} \tilde{y}^1 \\ \tilde{t}^1 \end{pmatrix}(0,\lambda) = -\begin{pmatrix} \Psi_{1yx_1} & 0 \\ \Psi_{2yx_1} & -1 \end{pmatrix}(2\pi,\Theta,\lambda)^{-1} \cdot \begin{pmatrix} \frac{1}{6}\Psi_{1x_1x_1x_1} \\ \frac{1}{6}\Psi_{2x_1x_1x_1} \end{pmatrix}(2\pi,\Theta,\lambda).$$

After these preparations, the actually interesting features of system (2.4) are summarized in the next two theorems.

Theorem 2.4: *From the curve of Hopf points $(x,y,\epsilon,\lambda)=(\Theta,0,\epsilon,\lambda^2(\epsilon))$ which exists according to Lemma 2.2 for $\epsilon\neq 0$, periodic orbits bifurcate in λ-direction; i.e. there exists a function $(\tilde{y}^2,\tilde{\lambda}^2,\tilde{t}^2)(x_1,\epsilon)$, so that*

$$(\Psi,\varphi)(\,\cdot\,,x_1,0,\tilde{y}^2(x_1,\epsilon)\cdot x_1^2,\epsilon,\lambda^2(\epsilon)+\tilde{\lambda}^2(x_1,\epsilon)\cdot x_1^2), \qquad x_1\neq 0$$

is a $(2\pi+\tilde{t}^2(x_1,\epsilon)\cdot x_1^2)$-periodic solution of the system (2.4), with

$$\begin{pmatrix} \tilde{y}^2 \\ \tilde{\lambda}^2 \\ \tilde{t}^2 \end{pmatrix}(0,\epsilon) = -\begin{pmatrix} \varphi_y-1 & \varphi_\lambda & 0 \\ \Psi_{1yx_1} & \Psi_{1\lambda x_1} & 0 \\ \Psi_{2yx_1} & \Psi_{2\lambda x_1} & -1 \end{pmatrix}(2\pi,\Theta,\epsilon,\lambda^2(\epsilon))^{-1} \cdot \begin{pmatrix} \frac{1}{2}\varphi_{x_1x_1} \\ \frac{1}{6}\Psi_{1x_1x_1x_1} \\ \frac{1}{6}\Psi_{2x_1x_1x_1} \end{pmatrix}(2\pi,\Theta,\epsilon,\lambda^2(\epsilon))$$

for $\epsilon\neq 0$, and

$$\begin{pmatrix} \tilde{y}^2 \\ \tilde{t}^2 \end{pmatrix}(x_1,0) = \begin{pmatrix} \tilde{y}^1 \\ \tilde{t}^1 \end{pmatrix}(x_1,\tilde{\lambda}^2(x_1,0)x_1^2)$$

as well as

$$\tilde{\lambda}^2(0,0) = \frac{\varphi_{y\epsilon}\cdot\Psi_{1x_1x_1x_1} - 3\varphi_{x_1x_1\epsilon}\cdot\Psi_{1yx_1}}{6\varphi_{\epsilon\lambda}\cdot\Psi_{1yx_1}}(2\pi,\Theta)$$

for $\epsilon=0$.

Consequently, recalling Proposition 2.3, there exist two manifolds of periodic orbits

$$(x_1, y, \epsilon, \lambda) = (x_1, \tilde{y}^1(x_1, \lambda)x_1^2, 0, \lambda) \quad \text{and} \quad (x_1, y, \epsilon, \lambda) = (x_1, \tilde{y}^2(x_1, \epsilon)x_1^2, \epsilon, \lambda^2(\epsilon) + \tilde{\lambda}^2(x_1, \epsilon)x_1^2)$$

intersecting transversally at $\epsilon = 0$ along the curve of bifurcation points of periodic orbits

$$(x_1, y, \epsilon, \lambda) = (x_1, \tilde{y}^2(x_1, 0)x_1^2, 0, \tilde{\lambda}^2(x_1, 0)x_1^2).$$

In summary, an (ϵ, λ, x_1)-bifurcation diagram of periodic orbits arises which is shown in Fig.4 in case of $\tilde{\lambda}^2(0,0) > 0$.

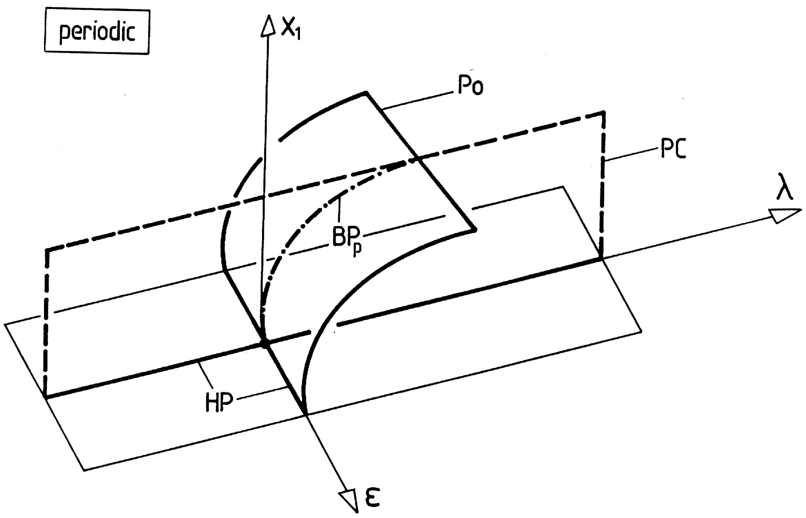

Figure 4: (ϵ, λ, x_1)-bifurcation diagram of periodic orbits. HP=Hopf point, Po=periodic orbit, PC=periodic continuum, BP_p=periodic bifurcation point.

Moreover, the stability of the periodic orbits Po is determined by the following result.

Theorem 2.5: *Depending on the signs of the three constants*

$$\beta_1 := \Psi_{1x_1x_1x_1}(2\pi, \Theta), \qquad \beta_2 := \varphi_{y\epsilon}(2\pi, \Theta), \qquad \beta_3 := (\varphi_{y\epsilon} \cdot \Psi_{1x_1x_1x_1} - 3\varphi_{x_1x_1\epsilon} \cdot \Psi_{1yx_1})(2\pi, \Theta),$$

the periodic orbits from Theorem 2.4, parametrized over the (x_1, ϵ)-plane, cause the stability diagrams depicted in Figure 5. In the case of $\beta_1 \cdot \beta_2 \cdot \beta_3 < 0$ the dotted curves TP represent periodic orbits with a pair of nonreal Floquet multipliers on the unit circle. From these orbits, generically invariant tori bifurcate through a Naimark-Sacker bifurcation.

Therefore, if $\beta_1 \cdot \beta_2 \cdot \beta_3 < 0$ Fig.4 typically can be completed by a manifold of invariant tori, as shown in Fig.1 with the bifurcation direction of the invariant tori drawn arbitrarily. Thus, in view of the bifurcation diagram of stationary points in Fig.3, two curves of Hopf points HP, one curve of stationary bifurcation points BP_s, one curve of periodic bifurcation points BP_p and possibly one curve of torus bifurcation points TP intersect each other in the central Hopf point $(x, y, \epsilon, \lambda) = \Theta$. Finally, the behaviour of the curve of torus bifurcation points is described in more detail in the following proposition.

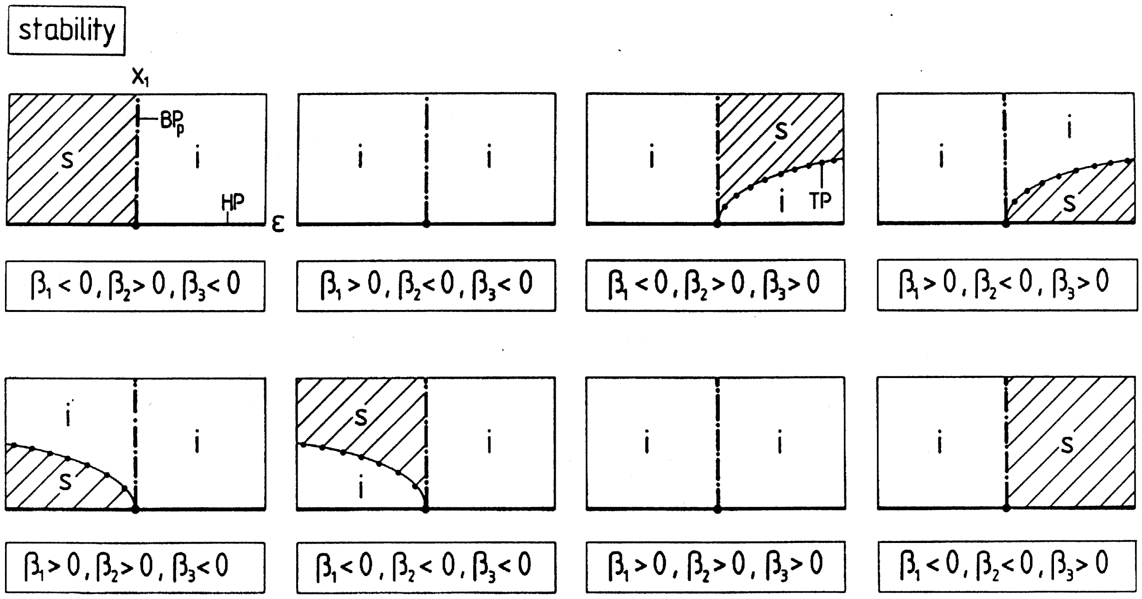

Figure 5: Stability of periodic orbits (s=stable, i=unstable). HP=Hopf point, BP_p=periodic bifurcation point, TP=torus bifurcation point (dotted).

Proposition 2.6: *The* (ϵ, λ)*-portion* $(\epsilon, \lambda) = (\epsilon, \lambda_t(\epsilon))$ *of a curve of TP-points satisfies*

$$\lambda_t{}'(\epsilon) \;\rightarrow\; \lambda^{2'}(0) - 3\tilde{\lambda}^2(0,0)\cdot \beta_2/\beta_1 \quad \textit{as} \quad \epsilon \rightarrow 0.$$

Hence, in case of $\beta_1 \cdot \beta_2 \cdot \beta_3 < 0$ and $\lambda^{2'}(0) - 3\tilde{\lambda}^2(0,0)\cdot \beta_2/\beta_1 \neq 0$, the projection of Fig.1 and Fig.3 onto the (ϵ, λ)-plane typically yields the following structure.

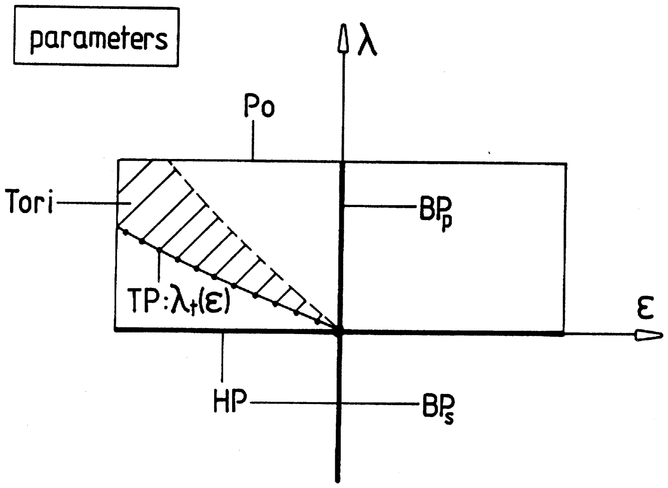

Figure 6: (ϵ, λ)-parameter plane. HP=Hopf point, BP_s=stationary bifurcation point, Po=periodic orbit, BP_p=periodic bifurcation point, TP=torus bifurcation point.

The hatched region of parameter values leading to invariant tori is again marked arbitrarily, but obviously for $\epsilon = 0$ not any invariant torus can survive within the resulting 2-dimensional system $\dot{x} = f(x, y, 0, \lambda) \in \mathbb{R}^2$. This is a typical feature of singular perturbation problems, where the lower dimensional $\epsilon = 0$ system cannot show the higher dimensional phenomena that may exist within the full system. Moreover, note that the λ-axis in Fig.6 represents one curve of periodic bifurcation points BP_p according to Fig.1 as well as one curve of stationary bifurcation points BP_s and Hopf points HP according to Fig.3.

Naturally, the characteristic constants in Theorem 2.4 and 2.5 which are written in terms of the flow (Ψ, φ) can equally be expressed by derivatives of the functions f and g from (2.4). For example, the constants β_1, β_2 and β_3 may be written as follows (cf. [20], [25])

$$\begin{aligned}\beta_1 = \tfrac{3}{4}\pi(&f_{1x_1x_1x_1} + f_{2x_2x_1x_1} + f_{1x_1x_2x_2} + f_{2x_2x_2x_2} \\ &- f_{1x_1x_2}(f_{1x_1x_1} + f_{1x_2x_2}) + f_{2x_1x_2}(f_{2x_1x_1} + f_{2x_2x_2}) + f_{1x_1x_1}f_{2x_1x_1} - f_{1x_2x_2}f_{2x_2x_2})^0\end{aligned}$$

$$\beta_2 = 2\pi \cdot g_y^0 \quad \text{and} \quad \beta_3 = \beta_1 \cdot \beta_2 - 3\pi(g_{x_1x_1} + g_{x_2x_2})^0 \cdot 2\pi(f_{1x_1y} + f_{2x_2y})^0.$$

A geometric interpretation of these constants also appears in [25].

Furthermore, according to the assumptions (2.2), the full $(\bar{x}, \bar{y})$-system (2.1) obviously exhibits a Hopf point $(\bar{x}, \bar{y}, \epsilon, \lambda) = \Theta$ which has an additional eigenvalue 0, so that the proofs of the above statements can only be accomplished with the help of a Hopf bifurcation theorem that analyzes this degenerate situation with sufficient depth. An appropriate theorem is given in the next section and proved in a very simple and direct way by the help of a suitable Ansatz. Note, that the bifurcation structure near a Hopf point with additional eigenvalue 0 was already examined in many papers, for example ([2], [15],[16],[19]), but typically these papers assume a nontrivial dynamics along the direction of the generalized eigenspace to the eigenvalue 0, while here the case of a continuum of stationary points appears which is not determined by finitely many derivatives. However, in [12], a Hopf point within a continuum of stationary points is examined, but in that situation the assumptions exclude the ordinary singular perturbation form (2.1).

Nevertheless, these papers show that in the proximity of a Hopf point with eigenvalue 0, one should expect invariant tori to emerge through the coupling of different modes [14]. In case of the singular perturbation problem (2.1), these different modes geometrically arise from the fast oscillating $\bar{x}$-dynamics and the slow $\bar{y}$-dynamics leading to invariant tori that grow out of the central Hopf point with $\bar{y}$-frequency starting from 0.

In principle all results of this section can be extended to systems of the form

$$\dot{x} = F(x, y, \epsilon, \lambda) \in \mathbb{R}^n, \qquad \dot{y} = \epsilon g(x, y, \epsilon, \lambda) \in \mathbb{R}$$

with the help of a reduction to an appropriate center manifold [25].

3 Hopf - bifurcation

In this section a general parameter dependent system

$$\begin{aligned}\dot{u} &= F(u, v, \mu) \in \mathbb{R}^2, \quad \mu \in \mathbb{R} \\ \dot{v} &= G(u, v, \mu) \in \mathbb{R}^m, \quad m \geq 1\end{aligned} \qquad \Longleftrightarrow: \qquad \dot{w} = H(w, \mu) \in \mathbb{R}^{2+m} \tag{3.1}$$

is considered with flow $(\chi, \phi)(t, u, v, \mu) \in \mathbb{R}^2 \times \mathbb{R}^m$ and $(\chi, \phi)^0 := (\chi, \phi)(2\pi, \Theta)$. Assume

$$H^0 := (F, G)^0 := (F, G)(\Theta) = \Theta, \qquad H^0_w = \begin{pmatrix} F_u & F_v \\ G_u & G_v \end{pmatrix}^0 = Diag(J_0, G^0_v); \tag{3.2}$$

i.e. a Hopf point at the origin with normalized linearization. Then, subject to the phase condition $u_2 = 0$, the periodic orbits of the system (3.1) can be found as solutions of

$$\begin{pmatrix} \chi_1 \\ \chi_2 \\ \phi \end{pmatrix} (2\pi + t, u_1, 0, v, \mu) - \begin{pmatrix} u_1 \\ 0 \\ v \end{pmatrix} = \Theta.$$

This equation is now examined with the Ansatz

$$\begin{pmatrix} \chi_1 \\ \chi_2 \\ \phi \end{pmatrix} (2\pi + \tilde{t}\cdot u_1^2, u_1, 0, \tilde{v}\cdot u_1^2, \tilde{\mu}\cdot u_1^2) - \begin{pmatrix} u_1 \\ 0 \\ \tilde{v}\cdot u_1^2 \end{pmatrix} = \Theta$$

which is motivated by the possible parametrization

$$(t, u_1, u_2, v, \mu) = (2\pi + \tilde{t}(u_1)\cdot u_1^2, u_1, 0, \tilde{v}(u_1)\cdot u_1^2, \tilde{\mu}(u_1)\cdot u_1^2)$$

of the branch of periodic orbits guaranteed by the standard Hopf theorem [20]. With this Ansatz, a Taylor expansion with respect to u_1 at $u_1 = 0$ directly leads to

$$\begin{aligned} &\begin{pmatrix} \chi_1 \\ \chi_2 \\ \phi \end{pmatrix}^0 + u_1\cdot \{ \begin{pmatrix} \chi_{1u_1} \\ \chi_{2u_1} \\ \phi_{u_1} \end{pmatrix}^0 - \begin{pmatrix} 1 \\ 0 \\ \Theta \end{pmatrix} \} \\ &+ \tfrac{1}{2} u_1^2\cdot \{2 \begin{pmatrix} \chi_{1v} & \chi_{1\mu} & \chi_{1t} \\ \chi_{2v} & \chi_{2\mu} & \chi_{2t} \\ \phi_v & \phi_\mu & \phi_t \end{pmatrix}^0 \cdot \begin{pmatrix} \tilde{v} \\ \tilde{\mu} \\ \tilde{t} \end{pmatrix} + \begin{pmatrix} \chi_{1u_1u_1} \\ \chi_{2u_1u_1} \\ \phi_{u_1u_1} \end{pmatrix}^0 - \begin{pmatrix} 0 \\ 0 \\ 2\tilde{v} \end{pmatrix} \} \\ &+ \tfrac{1}{6} u_1^3\cdot \{6 \begin{pmatrix} \chi_{1vu_1} & \chi_{1\mu u_1} & \chi_{1tu_1} \\ \chi_{2vu_1} & \chi_{2\mu u_1} & \chi_{2tu_1} \\ \phi_{vu_1} & \phi_{\mu u_1} & \phi_{tu_1} \end{pmatrix}^0 \cdot \begin{pmatrix} \tilde{v} \\ \tilde{\mu} \\ \tilde{t} \end{pmatrix} + \begin{pmatrix} \chi_{1u_1u_1u_1} \\ \chi_{2u_1u_1u_1} \\ \phi_{u_1u_1u_1} \end{pmatrix}^0 \} + O(u_1^4) = \Theta, \end{aligned} \tag{3.3}$$

so that the further analysis of this equation depends on certain derivatives of the flow in the Hopf point $(u, v, \mu) = 0$. Now, according to [20] or [25], the solutions of the corresponding variational equations yield the following fundamental identities

$$\begin{pmatrix} \chi_{1u_1} & \chi_{1u_2} & \chi_{1v} \\ \chi_{2u_1} & \chi_{2u_2} & \chi_{2v} \\ \phi_{u_1} & \phi_{u_2} & \phi_v \end{pmatrix} (t, \Theta) = \begin{pmatrix} \cos(t) & \sin(t) & 0 \\ -\sin(t) & \cos(t) & 0 \\ 0 & 0 & \exp(G^0_v\cdot t) \end{pmatrix} \tag{3.4}$$

and

$$\begin{pmatrix} \chi_{1\mu} & \chi_{1u_1u_1} & \chi_{1u_1u_2} \\ \chi_{2\mu} & \chi_{2u_1u_1} & \chi_{2u_1u_2} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.5}$$

which are valid under the assumption of a Hopf point with normalized linearization of the form (3.2). Therefore, after exchanging the χ and ϕ components, the equations (3.3) obviously assume the form

$$\begin{pmatrix} u_1^2 \cdot I_m & & \\ & u_1^3 & \\ & & u_1^3 \end{pmatrix} \cdot \left\{ \begin{pmatrix} \phi_v - I_m & \phi_\mu & \Theta \\ \chi_{1vu_1} & \chi_{1\mu u_1} & 0 \\ \chi_{2vu_1} & \chi_{2\mu u_1} & -1 \end{pmatrix}^0 \cdot \begin{pmatrix} \tilde{v} \\ \tilde{\mu} \\ \tilde{t} \end{pmatrix} + \begin{pmatrix} \frac{1}{2}\phi_{u_1u_1} \\ \frac{1}{6}\chi_{1u_1u_1u_1} \\ \frac{1}{6}\chi_{2u_1u_1u_1} \end{pmatrix}^0 + O(u_1) \right\} = \Theta. \tag{3.6}$$

Then, the implicit function theorem applied to the remainder equation in brackets yields the following Hopf bifurcation result.

Theorem 3.1: *Under the assumptions (3.2) and*

$$B := \begin{pmatrix} \phi_v - I_m & \phi_\mu \\ \chi_{1vu_1} & \chi_{1\mu u_1} \end{pmatrix}^0 \in GL[\mathbb{R}^{m+1}], \tag{3.7}$$

the system (3.1) has a branch of periodic orbits through the Hopf point $(u, v, \mu) = \Theta$. More precisely, there exists a locally unique function $(\tilde{v}, \tilde{\mu}, \tilde{t})(u_1)$, so that

$$(\chi, \phi)(\,\cdot\,, u_1, 0, \tilde{v}(u_1)\cdot u_1^2, \tilde{\mu}(u_1)\cdot u_1^2), \qquad u_1 \neq 0$$

is a $(2\pi + \tilde{t}(u_1)\cdot u_1^2)$-periodic solution. At the Hopf point $(u, v, \mu) = \Theta$, the associated curvature $(\tilde{v}, \tilde{\mu}, \tilde{t})(0)$ of the branch of periodic orbits follows directly from equation (3.6).

Moreover, according to the appendix, a closer inspection of the derivatives in (3.7) shows the equivalences

$$\begin{aligned} & B \in GL[\mathbb{R}^{m+1}] \\ \iff\quad & n^+ i \notin \mathrm{Sp}[G_v^0], \quad n^+ \in \mathbb{N}^+ \quad \wedge \quad C := \begin{pmatrix} H_w & H_\mu \\ \mathrm{tr}_w(F_u) & \mathrm{tr}_\mu(F_u) \end{pmatrix}^0 \in GL[\mathbb{R}^{m+3}] \\ \iff\quad & n^+ i \notin \mathrm{Sp}[G_v^0], \quad n^+ \in \mathbb{N}^+ \quad \wedge \quad \mathrm{rank}(H_w, H_\mu)^0 = 2 + m \quad \wedge \quad \sigma'(0) \neq 0 \end{aligned} \tag{3.8}$$

where $\sigma(\epsilon) + i\cdot\omega(\epsilon)$, $(\sigma, \omega) \in \mathbb{R}^2$ denotes the locally unique smooth continuation of the simple eigenvalue $i = i\cdot\omega(0)$ along the branch of stationary points $(w^s, \mu^s)(\epsilon)$. This branch is well defined because of the maximal rank condition $\mathrm{rank}(H_w, H_\mu)^0 = 2 + m$.

Thus, Theorem 3.1 obviously is an extension of the standard Hopf theorem which can be applied if the Hopf point has an additional eigenvalue 0, for example if it is a Hopf-Bogdanov point. Moreover, at $\mu = 0$ there may be a continuum of stationary points; i.e. a singular perturbation is allowed, as long as the resulting rank deficiency of the linearization H_w^0 is compensated for by the derivative H_μ^0 such that $\mathrm{rank}(H_w, H_\mu)^0 = 2 + m$.

Now, in order to really handle the two parameter singular perturbation problem (2.1), it turns out that a multi-parameter version of Theorem 3.1 has to be derived. At first, a simple calculation shows that the branch of periodic orbits from Theorem 3.1 and the branch of stationary points $(w^s, \mu^s)(\epsilon)$ intersect transversally in the Hopf point $(w, \mu) = \Theta$. Therefore, in contrast to the situation at a stationary bifurcation point, it is tempting to suppose that this transversal intersection is conserved under any parameter perturbation of the system

(3.1) resulting in a manifold of periodic orbits and a manifold of stationary points which intersect transversally along a manifold of Hopf points. Now, for specifying this concept, the following extended system with additional parameter $\nu \in \mathbb{R}^p$ is considered

$$\begin{array}{l} \dot{\bar{u}} = \bar{F}(\bar{u},\bar{v},\nu,\mu) \in \mathbb{R}^2, \quad \nu \in \mathbb{R}^p, p \geq 1, \mu \in \mathbb{R} \\ \dot{\bar{v}} = \bar{G}(\bar{u},\bar{v},\nu,\mu) \in \mathbb{R}^m, \quad m \geq 1 \end{array} \iff: \quad \dot{\bar{w}} = \bar{H}(\bar{w},\nu,\mu) \in \mathbb{R}^{2+m}. \tag{3.9}$$

Let $(\bar{\chi},\bar{\phi})(\bar{t},\bar{u},\bar{v},\nu,\mu) \in \mathbb{R}^2 \times \mathbb{R}^m$ denote its flow and assume

$$\bar{H}(\Theta) = \Theta, \qquad \bar{H}_{\bar{w}}(\Theta) = Diag(J_0, \bar{G}_{\bar{v}}(\Theta)) \tag{3.10}$$

as well as

$$\bar{B} := \begin{pmatrix} \bar{\phi}_{\bar{v}} - I_m & \bar{\phi}_\mu \\ \bar{\chi}_{1\bar{v}\bar{u}_1} & \bar{\chi}_{1\mu\bar{u}_1} \end{pmatrix} (2\pi, \Theta) \in GL[\mathbb{R}^{m+1}]$$

or

$$n^+ i \notin \mathrm{Sp}[\bar{G}_{\bar{v}}(\Theta)], \quad n^+ \in \mathbb{N}^+ \quad \wedge \quad \bar{C}_\mu := \begin{pmatrix} \bar{H}_{\bar{w}} & \bar{H}_\mu \\ \mathrm{tr}_{\bar{w}}(\bar{F}_{\bar{u}}) & \mathrm{tr}_\mu(\bar{F}_{\bar{u}}) \end{pmatrix} (\Theta) \in GL[\mathbb{R}^{m+3}] \tag{3.11}$$

respectively (cf. (3.8)). Then, this regular ν-perturbation naturally embeds the previous branch of stationary points $(\bar{w}^s,\mu^s)(\epsilon)$ into a ν-dependent $(p+1)$-dimensional manifold of stationary points $(\bar{w}^s,\mu^s)(\nu,\epsilon)$ which comprises all stationary points of the system (3.9) in the vicinity of the Hopf point $(\bar{w},\nu,\mu) = \Theta$. In addition, the locally unique continuation $\sigma(\nu,\epsilon)+i\cdot\omega(\nu,\epsilon)$ of the simple eigenvalue $i = i\cdot\omega(0,0)$ on this manifold leads to the equation $\sigma(\nu,\epsilon) = 0$, whose solutions represent further Hopf points within the manifold of stationary points. Now, the last inequality in equation (3.8) implies $\sigma_\epsilon(\Theta,0) \neq 0$ yielding a locally unique function $\epsilon(\nu)$ with $\epsilon(\Theta) = 0$ and $\sigma(\nu,\epsilon(\nu)) = 0$. Thus, the Hopf points of the extended (ν,μ)-system (3.9) with eigenvalues near i are exactly given by $(\bar{w}_h,\mu_h,\omega_h)(\nu) := (\bar{w}^s,\mu^s,\omega)(\nu,\epsilon(\nu))$ leading to the following result.

Lemma 3.2: *The central Hopf point $(\bar{w},\nu,\mu) = \Theta$ with eigenvalue i is embedded into a p-dimensional manifold of Hopf points $(\bar{w}_h,\mu_h,\omega_h)(\nu)$ with no other Hopf points in the neighborhood of $(\bar{w},\nu,\mu,\omega) = (\Theta,1)$.*

The precise geometry of the μ-portion $\mu_h(\nu) \in \mathbb{R}$ of this manifold of Hopf points can essentially be obtained by differentiating the equations

$$\mu_h(\nu) = \mu^s(\nu,\epsilon(\nu)), \qquad \bar{H}(\bar{w}^s(\nu,\epsilon),\nu,\mu^s(\nu,\epsilon)) = \Theta \qquad \text{and} \qquad \sigma(\nu,\epsilon(\nu)) = 0.$$

For example, concerning the first derivatives, one obtains

$$(\mu_h)_{\nu_i}(\Theta) = 0 \in \mathbb{R} \quad \iff \quad \bar{C}_{\nu_i} := \begin{pmatrix} \bar{H}_{\bar{w}} & \bar{H}_{\nu_i} \\ \mathrm{tr}_{\bar{w}}(\bar{F}_{\bar{u}}) & \mathrm{tr}_{\nu_i}(\bar{F}_{\bar{u}}) \end{pmatrix} (\Theta) \notin GL[\mathbb{R}^{m+3}] \tag{3.12}$$

for $i = 1,\ldots,p$ [25]. This equivalence is not needed in the remaining sections, and therefore the proof is omitted. Note, that if the assumption (3.7) is violated in the direction of ν_i, i.e. for μ replaced by ν_i, then the manifold of Hopf points degenerates in this direction in the central Hopf point, as follows from (3.8),(3.11) and (3.12). Further, in this situation either there is a rank deficiency of the extended linearization $(\bar{H}_{\bar{w}}, \bar{H}_{\nu_i})(\Theta)$ resulting in a stationary degeneracy or there is a degeneracy of the velocity of the complex-conjugate pair of eigenvalues in the direction of ν_i.

In the next step, Theorem 3.1 is applied to each of the Hopf points $(\bar{w}_h(\nu), \nu, \mu_h(\nu))$ with corresponding eigenvalues $i\cdot\omega_h(\nu)$. However, Theorem 3.1 is formulated for a Hopf point in the origin of phase space with normalized linearization. Therefore, these requirements will be satisfied first by a transformation of the form

$$\bar{w}\left(\frac{t}{\omega_h(\nu)}\right) = \bar{w}_h(\nu) + S(\nu)\cdot w(t). \tag{3.13}$$

Here, the columns of the matrix $S(\nu) \in \mathbb{R}^{2+m,2+m}$ with $S(\Theta) = I_{2+m}$ are given by the real and imaginary part of an eigenvector to the eigenvalue $i\cdot\omega_h(\nu)$ and the invariant subspace associated with the remaining spectrum of $\bar{H}_{\bar{w}}(\bar{w}_h(\nu), \nu, \mu_h(\nu))$. The basis for this construction of $S(\nu)$ is a theorem about smooth continuation of invariant subspaces which decompose the whole space into a direct sum and which belong to disjoint spectral sets [7],[22],[25]. Then, after performing the transformation (3.13), Theorem 3.1 is applicable to each of the Hopf points implying the following theorem.

Theorem 3.3: *The transformed system*

$$\dot{w} = H(w,\nu,\mu) \in \mathbb{R}^{2+m} \qquad \text{or} \qquad \begin{aligned} \dot{u} &= F(u,v,\nu,\mu) \in \mathbb{R}^2 \\ \dot{v} &= G(u,v,\nu,\mu) \in \mathbb{R}^m \end{aligned} \tag{3.14}$$

with flow $(\chi,\phi)(t,u,v,\nu,\mu) \in \mathbb{R}^2 \times \mathbb{R}^m$ *and* $(\chi,\phi)^0(\nu) := (\chi,\phi)(2\pi,\Theta,\nu,\mu_h(\nu))$ *satisfies:*

i. The Hopf points from Lemma 3.2 are shifted to the origin of phase space with normalized linearization; i.e.

$$H^0(\nu) := H(\Theta,\nu,\mu_h(\nu)) = \Theta \qquad \text{and} \qquad H^0_w(\nu) = Diag(J_0, G^0_v(\nu)).$$

ii. There exists a locally unique function $(\tilde{v},\tilde{\mu},\tilde{t})(u_1,\nu) \in \mathbb{R}^{m+2}$*, so that*

$$(\chi,\phi)(\,\cdot\,,u_1,0,\tilde{v}(u_1,\nu)\cdot u_1^2,\nu,\mu_h(\nu)+\tilde{\mu}(u_1,\nu)\cdot u_1^2), \qquad u_1 \neq 0$$

is a $(2\pi + \tilde{t}(u_1,\nu)\cdot u_1^2)$*-periodic solution.*

iii. At the Hopf points $(w,\nu,\mu) = (\Theta,\nu,\mu_h(\nu))$*, the curvature* $(\tilde{v},\tilde{\mu},\tilde{t})(0,\nu)$ *of the branches of periodic orbits reads*

$$\begin{pmatrix}\tilde{v}\\ \tilde{\mu}\\ \tilde{t}\end{pmatrix}(0,\nu) = -\begin{pmatrix}\phi_v - I_m & \phi_\mu & 0\\ \chi_{1vu_1} & \chi_{1\mu u_1} & 0\\ \chi_{2vu_1} & \chi_{2\mu u_1} & -1\end{pmatrix}^{0}(\nu)^{-1} \cdot \begin{pmatrix}\frac{1}{2}\phi_{u_1u_1}\\ \frac{1}{6}\chi_{1u_1u_1u_1}\\ \frac{1}{6}\chi_{2u_1u_1u_1}\end{pmatrix}^{0}(\nu).$$

Hence, according to ii, the bifurcation diagram of periodic orbits of the transformed system (3.14) is determined by the function

$$\mu(u_1,\nu) := \mu_h(\nu) + \tilde{\mu}(u_1,\nu)\cdot u_1^2, \qquad \mu(\,\cdot\,,\,\cdot\,) : \mathbb{R}^{1+p} \to \mathbb{R},$$

so that the main purpose of Theorem 3.3 is to extract a real function from the full problem. Then, in principle the theorems of singularity theory [13],[14] apply to the function $\mu_h(\nu)$ of the Hopf points as well as to the unifying function $\mu(u_1,\nu)$ of the Hopf points and the periodic orbits. Some results of this kind are represented in [25]. However, in the subsequent analysis of the original problem (2.4), due to its singular perturbation character, the μ-function turns out not to be finitely determined in the central Hopf point, so that the qualitative theorems of singularity theory can not be used and a direct analysis of the function $\mu(\cdot,\cdot)$ is needed. Moreover, note that all the Hopf points in Theorem 3.3 may have the algebraically simple eigenvalue 0 which typically leads to a nontrivial geometry of the manifold of stationary points in (w,ν,μ)-space. Further generalizations of Hopf points with eigenvalue 0 are discussed in [24], where a special case of a Hopf point with eigenvalue 0 and several eigenvectors is considered.

Finally, the proof of the basic Hopf result Theorem 3.1 is accomplished with the sole help of an Ansatz, a Taylor expansion, the use of first and second order variational equations and the implicit function theorem. This goes along the lines of [8]. In particular, no techniques such as Liapunov-Schmidt reduction, center manifold reduction, normal form theory or Poincaré-sections are needed for getting an extended Hopf bifurcation theorem. On the other hand, the disadvantage of such an approach usually is that only solutions compatible with the Ansatz are obtained and no general uniqueness results can be expected. In [25] the above standard techniques from bifurcation theory are used in a more general context.

4 Proof of Theorem 2.4

This section deals with system (2.1) under the assumptions (2.2) and (2.3). Then, the stationary points of this system are determined by the solutions of $\bar{f}(\bar{x},\bar{y},0,\lambda) = \Theta \in \mathbb{R}^2$ for $\epsilon = 0$ and $(\bar{f},\bar{g})(\bar{x},\bar{y},\epsilon,\lambda) = \Theta \in \mathbb{R}^3$ for $\epsilon \neq 0$. Hence, according to (2.2), the solution $(\bar{x},\bar{y},\lambda) = \Theta \in \mathbb{R}^4$ and $(\bar{x},\bar{y},\epsilon,\lambda) = \Theta \in \mathbb{R}^5$ of the first and second equation can be uniquely continued leading directly to the existence of the two manifolds of stationary points from Lemma 2.1 which intersect transversally along a curve of bifurcation points at $\epsilon = 0$. Moreover, the simple eigenvalue i of the central Hopf point $(\bar{x},\bar{y},\epsilon,\lambda) = \Theta$ can be continued smoothly along both of these manifolds resulting in uniquely determined functions $\sigma_1(\bar{y},\lambda) + i\cdot\omega_1(\bar{y},\lambda)$ and $\sigma_2(\epsilon,\lambda) + i\cdot\omega_2(\epsilon,\lambda)$, where a simple calculation together with equation (2.3) shows the inequalities

$$\sigma_{1\bar{y}}(0,0) = \tfrac{1}{2}\,\mathrm{tr}_{\bar{y}}(\bar{f}_{\bar{x}})^0 \neq 0 \qquad \text{and} \qquad \sigma_{2\lambda}(0,0) = \tfrac{1}{2}\,\frac{[\bar{g}_{\bar{y}}\cdot \mathrm{tr}_\lambda(\bar{f}_{\bar{x}}) - \bar{g}_\lambda\cdot \mathrm{tr}_{\bar{y}}(\bar{f}_{\bar{x}})]^0}{\bar{g}_{\bar{y}}^0} \neq 0.$$

This proves the existence of the two curves of Hopf points from Lemma 2.1.

In the next step, the results from section 3 are applied to system (2.1). For that purpose, the (ϵ,λ)-dependent system (2.1) is extended artificially by the parameter $\mu \in \mathbb{R}$ according to

$$\begin{aligned} \dot{\bar{x}} &= \bar{f}(\bar{x},\bar{y},\epsilon,\lambda) \\ \dot{\bar{y}} &= \epsilon\bar{g}(\bar{x},\bar{y},\epsilon,\lambda)+\mu \end{aligned} \quad \Longleftrightarrow: \quad \begin{aligned} \dot{\bar{u}} &= \bar{F}(\bar{u},\bar{v},\nu_1,\nu_2,\mu) \\ \dot{\bar{v}} &= \bar{G}(\bar{u},\bar{v},\nu_1,\nu_2,\mu) \end{aligned} \quad \Longleftrightarrow: \quad \dot{\bar{w}} = \bar{H}(\bar{w},\nu_1,\nu_2,\mu) \tag{4.1}$$

(compare (3.9)). Then, using the notations from section 3, this system obviously meets the requirements (3.10), (3.11) with $p=2$ and $m=1$, and the remaining task is to specify the results of section 3 for $\mu=0$.

First, according to Lemma 3.2, the extended system (4.1) has a 2-dimensional manifold of Hopf points $[\bar{w}_h,\mu_h,\omega_h](\epsilon,\lambda)$ which comprises all Hopf points in the vicinity of $[\bar{w},\epsilon,\lambda,\mu,\omega]=[\Theta,1]$. In particular, the Hopf points from Lemma 2.1 at $\mu=0$ are contained in this manifold which implies

$$\begin{aligned} (\bar{w}_h,\mu_h,\omega_h)(0,\lambda) \quad &= (\bar{u}_h,\bar{v}_h,\mu_h,\omega_h)(0,\lambda) \quad &&= (\Theta,\bar{y}^1(\lambda),0,\omega^1(\lambda)), \\ (\bar{w}_h,\mu_h,\omega_h)(\epsilon,\lambda^2(\epsilon)) \quad &= (\bar{u}_h,\bar{v}_h,\mu_h,\omega_h)(\epsilon,\lambda^2(\epsilon)) \quad &&= ((\bar{x}^s,\bar{y}^s)(\epsilon,\lambda^2(\epsilon)),0,\omega^2(\epsilon)) \end{aligned} \tag{4.2}$$

as well as

$$(\mu_h)_\lambda(\Theta)=0, \qquad (\mu_h)_\epsilon(\Theta)+(\mu_h)_\lambda(\Theta)\cdot\lambda^2_\epsilon(0)=(\mu_h)_\epsilon(\Theta)=0. \tag{4.3}$$

Moreover, according to equation (3.13) and Theorem 3.3 *i*, there exists a coordinate and a time transformation of the form

$$\bar{w}\left(\frac{t}{\omega_h(\epsilon,\lambda)}\right)=\bar{w}_h(\epsilon,\lambda)+S(\epsilon,\lambda)\cdot w(t) \tag{4.4}$$

with $S(\epsilon,\lambda)\in\mathbb{R}^{3,3}$ and $S(\Theta)=I_3$, so that all Hopf points of the transformed system $\dot{w}=H(w,\epsilon,\lambda,\mu)=(F,G)(u,v,\epsilon,\lambda,\mu)$ are located at the origin of phase space with normalized linearization. In addition, a simple calculation [25] shows that the matrix $S(\epsilon,\lambda)$ can be chosen such that $S(0,\lambda)=Diag[S_1(0,\lambda),1]$ with $S_1(0,\lambda)\in\mathbb{R}^{2,2}$ leading to the identities

$$\begin{aligned} F(u,v,0,\lambda,\mu) \quad &= \quad S_1(0,\lambda)^{-1}\cdot\bar{f}(S_1(0,\lambda)u,\bar{y}^1(\lambda)+v,0,\lambda)/\omega_h(0,\lambda), \\ G(u,v,0,\lambda,\mu) \quad &= \quad \mu/\omega_h(0,\lambda) \qquad \text{and} \qquad \phi(t,u,v,0,\lambda,\mu)=v+\mu\cdot t/\omega_h(0,\lambda). \end{aligned} \tag{4.5}$$

Hence, a Taylor expansion with respect to ϵ implies $G(u,v,\epsilon,\lambda,0)=\epsilon g(u,v,\epsilon,\lambda)$, and the $\mu=0$ system $\dot{w}=H(w,\epsilon,\lambda,0)$ takes the singular perturbation form (2.4). Using, (2.2),(4.2),(4.5) and Theorem 3.3 *i*, then proves the assertions of Lemma 2.2. Furthermore, by Theorem 3.3 *ii*, there exists a function $(\tilde{v},\tilde{\mu},\tilde{t})(u_1,\epsilon,\lambda)$ such that

$$(\chi,\phi)(\,\cdot\,,u_1,0,\tilde{v}(u_1,\epsilon,\lambda)\cdot u_1^2,\epsilon,\lambda,\mu_h(\epsilon,\lambda)+\tilde{\mu}(u_1,\epsilon,\lambda)\cdot u_1^2), \qquad u_1\neq 0 \tag{4.6}$$

is a $(2\pi+\tilde{t}(u_1,\epsilon,\lambda)\cdot u_1^2)$-periodic solution of the full μ-dependent system $\dot{w}=H(w,\epsilon,\lambda,\mu)$. Thus, recalling $\mu_h(0,\lambda)=0$, the function

$$(\chi,\phi)(\,\cdot\,,u_1,0,\tilde{v}(u_1,0,\lambda)\cdot u_1^2,0,\lambda,\tilde{\mu}(u_1,0,\lambda)\cdot u_1^2), \qquad u_1\neq 0$$

defines a $(2\pi+\tilde{t}(u_1,0,\lambda)\cdot u_1^2)$-periodic solution of the system $\dot{w}=H(w,0,\lambda,\mu)$. As a consequence, the last identity in (4.5) implies

$$\tilde{\mu}(u_1,0,\lambda)=0. \tag{4.7}$$

yielding Proposition 2.3 under consideration of $(\tilde{y}^1, \tilde{t}^1)(u_1, \lambda) := (\tilde{v}, \tilde{t})(u_1, 0, \lambda)$ and Theorem 3.3 *iii*.

In the following, the nontrivial periodic orbits of the $\mu = 0$ system $\dot{w} = H(w, \epsilon, \lambda, 0),\ \epsilon \neq 0$ are investigated in detail. At first, equation (4.6) shows that for this purpose the zeros of the real function $\mu(u_1, \epsilon, \lambda) := \mu_h(\epsilon, \lambda) + \tilde{\mu}(u_1, \epsilon, \lambda)u_1^2$ have to be determined for $\epsilon \neq 0$. A Taylor expansion with respect to ϵ implies

$$\begin{aligned}
\mu(u_1, \epsilon, \lambda) &= \mu_h(\epsilon, \lambda) + \tilde{\mu}(u_1, \epsilon, \lambda)u_1^2 \\
&= \mu_h(0, \lambda) + \tilde{\mu}(u_1, 0, \lambda)u_1^2 + \epsilon(\mu_h)_\epsilon(0, \lambda) + \ \epsilon\tilde{\mu}_\epsilon(u_1, 0, \lambda)u_1^2 + O(\epsilon^2) \\
&= \epsilon\cdot \left((\mu_h)_\epsilon(0, \lambda) + \tilde{\mu}_\epsilon(u_1, 0, \lambda)u_1^2 + O(\epsilon)\right) =: \epsilon\cdot R(u_1, \epsilon, \lambda)
\end{aligned}$$

(cf. (4.2),(4.7)), where the remainder $R(u_1, \epsilon, \lambda)$ satisfies $R(0, 0, 0) = 0$ and $R_\lambda(0, 0, 0) = (\mu_h)_{\epsilon\lambda}(0, 0)$ according to (4.3). Further, a simple calculation indicated at the end of this section shows that the assumptions (2.2), (2.3) ensure the inequality $(\mu_h)_{\epsilon\lambda}(0, 0) \neq 0$. Therefore, a locally unique function $\lambda(u_1, \epsilon)$ exists with

$$\lambda(0, 0) = 0, \quad \mu_h(\epsilon, \lambda(u_1, \epsilon)) + \tilde{\mu}(u_1, \epsilon, \lambda(u_1, \epsilon))\cdot u_1^2 = 0, \quad \mu_h(\epsilon, \lambda(0, \epsilon)) = 0 \tag{4.8}$$

and $\lambda(0, \epsilon) = \lambda^2(\epsilon)$, as a consequence of uniqueness. Hence, according to (4.6) and (4.8), the function

$$(\chi, \phi)(\ \cdot, u_1, 0, \tilde{v}(u_1, \epsilon, \lambda(u_1, \epsilon))\cdot u_1^2, \epsilon, \lambda(u_1, \epsilon), 0), \qquad u_1 \neq 0 \tag{4.9}$$

is a $(2\pi + \tilde{t}(u_1, \epsilon, \lambda(u_1, \epsilon))\cdot u_1^2)$-periodic solution of the $\mu = 0$ system $\dot{w} = H(w, \epsilon, \lambda, 0)$. This proves the existence part of Theorem 2.4.

The special properties of this manifold of periodic orbits now mainly result from Theorem 3.3 *iii* and from differentiating the second equation in (4.8). But note, that the curvature of the manifold of periodic orbits in Theorem 3.3 *iii* is represented with the help of μ-dependent variables, whereas in Theorem 2.4 only internal $\nu = (\epsilon, \lambda)$-dependent variables appear. Therefore, the μ-dependent variables must be replaced by (ϵ, λ)-dependent variables.

From equation (4.8) and a Taylor expansion with respect to u_1 one concludes

$$\begin{aligned}
0 &= \mu_h(\epsilon, \lambda(u_1, \epsilon)) + \tilde{\mu}(u_1, \epsilon, \lambda(u_1, \epsilon))\cdot u_1^2 \\
&= \mu_h(\epsilon, \lambda(0, \epsilon)) + (\mu_h)_\lambda(\epsilon, \lambda(0, \epsilon))\cdot \lambda_{u_1}(0, \epsilon)\cdot u_1 \\
&+ \tfrac{1}{2}\{(\mu_h)_{\lambda\lambda}(\epsilon, \lambda(0, \epsilon))\cdot \lambda_{u_1}(0, \epsilon)^2 + (\mu_h)_\lambda(\epsilon, \lambda(0, \epsilon))\cdot \lambda_{u_1u_1}(0, \epsilon)\}\cdot u_1^2 \\
&+ \tilde{\mu}(0, \epsilon, \lambda(0, \epsilon))\cdot u_1^2 + O(u_1^3).
\end{aligned}$$

Hence, using $\mu_h(0, \lambda) = 0$ and $\lambda(0, 0) = 0$, comparing coefficients and expanding with respect to ϵ yields

$$\begin{aligned}
0 &= (\mu_h)_\lambda(\epsilon, \lambda(0, \epsilon))\cdot \lambda_{u_1}(0, \epsilon) \\
&= \{(\mu_h)_\lambda(0, 0) + ((\mu_h)_{\lambda\epsilon}(0, 0) + (\mu_h)_{\lambda\lambda}(0, 0)\cdot \lambda_\epsilon(0, 0))\cdot \epsilon + O(\epsilon^2)\}\cdot \lambda_{u_1}(0, \epsilon)\} \\
&= \{(\mu_h)_{\lambda\epsilon}(0, 0) + O(\epsilon)\}\cdot \epsilon\cdot \lambda_{u_1}(0, \epsilon)
\end{aligned}$$

and

$$0 = \tfrac{1}{2}(\mu_h)_{\lambda\lambda}(\epsilon, \lambda(0, \epsilon))\cdot \lambda_{u_1}(0, \epsilon)^2 + \tfrac{1}{2}(\mu_h)_\lambda(\epsilon, \lambda(0, \epsilon))\cdot \lambda_{u_1u_1}(0, \epsilon) + \tilde{\mu}(0, \epsilon, \lambda(0, \epsilon)).$$

But then, the relations $(\mu_h)_{\epsilon\lambda}(0,0) \neq 0$ and $\lambda(0,\epsilon) = \lambda^2(\epsilon)$ imply

$$(\mu_h)_\lambda(\epsilon,\lambda(0,\epsilon)) = \{(\mu_h)_{\lambda\epsilon}(0,0) + O(\epsilon)\}\cdot \epsilon \neq 0 \qquad \text{for} \qquad 0 < |\epsilon| << 1, \tag{4.10}$$

$$\lambda_{u_1}(0,\epsilon) = 0 \qquad \text{for} \qquad |\epsilon| << 1,$$

$$\tfrac{1}{2}\lambda_{u_1u_1}(0,\epsilon) = -\frac{\tilde{\mu}(0,\epsilon,\lambda(0,\epsilon))}{(\mu_h)_\lambda(\epsilon,\lambda(0,\epsilon))} \qquad \text{for} \qquad 0 < |\epsilon| << 1, \tag{4.11}$$

$$\lambda(u_1,\epsilon) = \lambda(0,\epsilon) + (\tfrac{1}{2}\lambda_{u_1u_1}(0,\epsilon) + O(u_1))\cdot u_1^2 =: \lambda^2(\epsilon) + \tilde{\lambda}^2(u_1,\epsilon)\cdot u_1^2 \tag{4.12}$$

and finally an expansion with respect to ϵ of the numerator and the denominator in (4.11) together with $\tilde{\mu}(0,0,\lambda) = 0$ and $\mu_h(0,\lambda) = 0$ yields

$$\tfrac{1}{2}\lambda_{u_1u_1}(0,0) = -\frac{\tilde{\mu}_\epsilon(0,0,0)}{(\mu_h)_{\epsilon\lambda}(0,0)} = \tilde{\lambda}^2(0,0). \tag{4.13}$$

Further, Theorem 3.3 *i* and equation (3.4) imply

$$\begin{pmatrix} \phi \\ \chi 1_{u_1} \\ \chi 2_{u_1} \end{pmatrix} (2\pi,\Theta,\epsilon,\lambda,\mu_h(\epsilon,\lambda)) - \begin{pmatrix} \Theta \\ 1 \\ 0 \end{pmatrix} = \Theta \in \mathbb{R}^3,$$

and by differentiating with respect to λ

$$\begin{pmatrix} \phi_\lambda \\ \chi 1_{u_1\lambda} \\ \chi 2_{u_1\lambda} \end{pmatrix} (2\pi,\Theta,\epsilon,\lambda,\mu_h(\epsilon,\lambda)) + \begin{pmatrix} \phi_\mu \\ \chi 1_{u_1\mu} \\ \chi 2_{u_1\mu} \end{pmatrix} (2\pi,\Theta,\epsilon,\lambda,\mu_h(\epsilon,\lambda))\cdot (\mu_h)_\lambda(\epsilon,\lambda) = \Theta. \tag{4.14}$$

In particular, for $\lambda = \lambda(0,\epsilon), 0 < |\epsilon| << 1$ and in view of equation (4.10) and $\mu_h(\epsilon,\lambda(0,\epsilon)) = 0$, this leads to

$$\begin{pmatrix} \phi_\mu \\ \chi 1_{u_1\mu} \\ \chi 2_{u_1\mu} \end{pmatrix} (2\pi,\Theta,\epsilon,\lambda(0,\epsilon),0) = -\frac{1}{(\mu_h)_\lambda(\epsilon,\lambda(0,\epsilon))} \begin{pmatrix} \phi_\lambda \\ \chi 1_{u_1\lambda} \\ \chi 2_{u_1\lambda} \end{pmatrix} (2\pi,\Theta,\epsilon,\lambda(0,\epsilon),0).$$

At this stage, the μ-dependent variables have been replaced by (ϵ,λ)-dependent variables, as desired. Next, Theorem 3.3 *iii* shows for $0 < |\epsilon| << 1$

$$\begin{aligned} &\begin{pmatrix} \phi_v - 1 & \phi_\lambda & 0 \\ \chi 1_{vu_1} & \chi 1_{u_1\lambda} & 0 \\ \chi 2_{vu_1} & \chi 2_{u_1\lambda} & -1 \end{pmatrix} (2\pi,\Theta,\epsilon,\lambda(0,\epsilon),0)\cdot \begin{pmatrix} \tilde{v}(0,\epsilon,\lambda(0,\epsilon)) \\ -\dfrac{\tilde{\mu}(0,\epsilon,\lambda(0,\epsilon))}{(\mu_h)_\lambda(\epsilon,\lambda(0,\epsilon))} \\ \tilde{t}(0,\epsilon,\lambda(0,\epsilon)) \end{pmatrix} \\ &= - \begin{pmatrix} \frac{1}{2}\phi_{u_1u_1} \\ \frac{1}{6}\chi 1_{u_1u_1u_1} \\ \frac{1}{6}\chi 2_{u_1u_1u_1} \end{pmatrix} (2\pi,\Theta,\epsilon,\lambda(0,\epsilon),0) \end{aligned} \tag{4.15}$$

where $-\frac{\tilde{\mu}}{(\mu_h)_\lambda}$ can be replaced by $\tilde{\lambda}^2(0,\epsilon)$ according to (4.11) and (4.12). Moreover, with the abbreviation $(\tilde{y}^2,\tilde{t}^2)(u_1,\epsilon) := (\tilde{v},\tilde{t})(u_1,\epsilon,\lambda(u_1,\epsilon))$ and recalling $(\tilde{y}^1,\tilde{t}^1)(u_1,\lambda) = (\tilde{v},\tilde{t})(u_1,0,\lambda)$, (4.12) and $\lambda^2(0) = 0$, the relations

$$(\tilde{y}^2,\tilde{t}^2)(u_1,0) = (\tilde{v},\tilde{t})(u_1,0,\lambda(u_1,0)) = (\tilde{y}^1,\tilde{t}^1)(u_1,\lambda(u_1,0)) = (\tilde{y}^1,\tilde{t}^1)(u_1,\tilde{\lambda}^2(u_1,0)u_1^2)$$

follow. Thus, by (4.9),(4.12) and (4.15), the proof of Theorem 2.4 is complete with the exception of the representation of $\tilde{\lambda}^2(0,0)$. This can be obtained by a simple calculation from (4.13) and by differentiating with respect to ϵ the following identity

$$(\phi_v - 1, \phi_\mu)(2\pi, \Theta, \epsilon, 0, \mu_h(\epsilon, 0)) \cdot \begin{pmatrix} \tilde{v} \\ \tilde{\mu} \end{pmatrix} (0, \epsilon, 0) = -\tfrac{1}{2}\phi_{u_1 u_1}(2\pi, \Theta, \epsilon, 0, \mu_h(\epsilon, 0))$$

(cf. Theorem 3.3 *iii*) and the ϕ-component in (4.14).

Therefore, it remains to prove the above assumption $(\mu_h)_{\epsilon\lambda}(\Theta) \neq 0$. This inequality follows from differentiating the equation $\bar{G}(\bar{w}_h(\epsilon,\lambda), \epsilon, \lambda, \mu_h(\epsilon,\lambda)) = 0$ with respect to ϵ and λ (cf. (4.1) and Lemma 3.2) as well as from differentiating the equation $\mathrm{tr}(\bar{F}_{\bar{u}})(\Theta, \bar{v}_h(0,\lambda), 0, \lambda, 0) = 0$ with respect to λ (cf. (4.1), (4.2)). In combination with (2.3) this finally leads to

$$(\mu_h)_{\epsilon\lambda}(\Theta) = \frac{(\bar{g}_{\bar{y}} \cdot \mathrm{tr}_\lambda(\bar{f}_{\bar{x}}) - \mathrm{tr}_{\bar{y}}(\bar{f}_{\bar{x}}) \cdot \bar{g}_\lambda)(\Theta)}{\mathrm{tr}_{\bar{y}}(\bar{f}_{\bar{x}})(\Theta)} \neq 0.$$

5 Proof of Theorem 2.5

In this final section, the stability of the periodic orbits within the 2-dimensional manifold of periodic orbits from Theorem 2.4 is examined in a generic situation. Introduce the abbreviations

$$\begin{aligned} T(x_1,\epsilon) &:= 2\pi + \tilde{t}^2(x_1,\epsilon)\cdot x_1^2, \qquad y(x_1,\epsilon) := \tilde{y}^2(x_1,\epsilon)\cdot x_1^2, \\ \lambda(x_1,\epsilon) &:= \lambda^2(\epsilon) + \tilde{\lambda}^2(x_1,\epsilon)\cdot x_1^2 \end{aligned} \tag{5.1}$$

and consider the monodromy matrix $V(x_1,\epsilon)$ of the periodic orbits and its spectra

$$\begin{aligned} \mathrm{Sp}[V(x_1,\epsilon)] &= \mathrm{Sp}[\begin{pmatrix} \Psi_x & \Psi_y \\ \varphi_x & \varphi_y \end{pmatrix} (T(x_1,\epsilon), x_1, 0, y(x_1,\epsilon), \epsilon, \lambda(x_1,\epsilon))] \\ &= \{1, \kappa_1, \kappa_2\}, \quad (\kappa_1,\kappa_2) \in \mathbb{C}^2. \end{aligned} \tag{5.2}$$

Here the Floquet multipliers $1, \kappa_1, \kappa_2$ are counted according to their multiplicities. They satisfy

$$\text{either} \qquad (\kappa_1,\kappa_2) \in \mathbb{R}^2 \qquad \text{or} \qquad \kappa_1 = \bar{\kappa_2}, \qquad \kappa_1 \notin \mathbb{R}. \tag{5.3}$$

Now, the crucial point concerning the subsequent calculations is the fact, that the position of the Floquet multipliers κ_1, κ_2 relative to the unit circle, and therefore the stability of the periodic orbits, is determined by the signs of the smooth functions

$$R_1(x_1,\epsilon) := \det[V](x_1,\epsilon) - 1, \tag{5.4}$$

$$R_2(x_1,\epsilon) := \mathrm{tr}[V](x_1,\epsilon) - 2 - \det[V](x_1,\epsilon) \tag{5.5}$$

where $\det[V](x_1,\epsilon)$ denotes the determinant of $V(x_1,\epsilon)$. Strictly speaking, the conclusions

$$\begin{aligned} &R_1(x_1,\epsilon) > 0 \quad \vee \quad (R_1(x_1,\epsilon) < 0, \quad R_2(x_1,\epsilon) > 0) \quad \Longrightarrow \quad |\kappa_1| > 1 \quad \vee \quad |\kappa_2| > 1 \\ &\Longrightarrow \qquad \text{periodic orbit unstable} \end{aligned} \tag{5.6}$$

and

$$\begin{aligned} &R_1(x_1,\epsilon) < 0, \quad R_2(x_1,\epsilon) < 0 \quad \Longrightarrow \quad |\kappa_1| < 1 \quad \wedge \quad |\kappa_2| < 1 \\ &\Longrightarrow \quad \text{periodic orbit stable} \end{aligned} \tag{5.7}$$

are valid, so that the remaining task is to compute the solution curves of $R_1(x_1,\epsilon) = 0$ and $R_2(x_1,\epsilon) = 0$ in the (x_1,ϵ)-plane as well as the signs on different sides of these curves.

In the first step, the criteria (5.6) and (5.7) are shown, with the final stability conclusions in these criteria being well known. Now, Lemma 2.2 together with $\varphi(t,x,y,0,\lambda) = y$ and equation (3.4) imply the following relations at the bifurcation points of periodic orbits and the nontrivial Hopf points respectively

$$V(x_1,0) = \begin{pmatrix} \Psi_x & \Psi_y \\ 0 & 1 \end{pmatrix} (T(x_1,0), x_1, 0, y(x_1,0), 0, \lambda(x_1,0)) \tag{5.8}$$

and

$$V(0,\epsilon) = \begin{pmatrix} \Psi_x & \Psi_y \\ \varphi_x & \varphi_y \end{pmatrix} (2\pi, \Theta, \epsilon, \lambda^2(\epsilon)) = \begin{pmatrix} I_2 & 0 \\ 0 & \exp(2\pi\epsilon\cdot g_y(\Theta,\epsilon,\lambda^2(\epsilon))) \end{pmatrix} \tag{5.9}$$

as well as

$$\begin{aligned} \mathrm{Sp}[V(x_1,0)] &= \mathrm{Sp}[\Psi_x(T(x_1,0), x_1, 0, y(x_1,0), 0, \lambda(x_1,0))] \cup \{1\} \\ &= \{1,\kappa_2\} \cup \{1\} = \{1,1,\kappa_2\}, \\ \mathrm{Sp}[V(0,\epsilon)] &= \{1,1,\kappa_2\} \qquad \text{with} \qquad \kappa_2 = \exp(2\pi\epsilon\cdot g_y(\Theta,\epsilon,\lambda^2(\epsilon))) \end{aligned} \tag{5.10}$$

and at the central Hopf point

$$\mathrm{Sp}[V(0,0)] = \{1,1,1\}. \tag{5.11}$$

In particular, the central Hopf point possesses the threefold Floquet multiplier 1 which generically splits into two Floquet multipliers different from 1 and the remaining simple Floquet multiplier 1. In order to obtain the invariant tori one has to adjust the two Floquet multipliers different from 1 on the unit circle by using the two degrees of freedom on the manifold of periodic orbits. Thus, a curve of periodic orbits on the manifold having two complex-conjugate Floquet multipliers on the unit circle can arise. More precisely, for fixed (x_1,ϵ) and according to (5.4), (5.5), the relations

$$R_1(x_1,\epsilon) \;=\; \kappa_1\cdot\kappa_2 - 1 =: \bar{R}_1(\kappa_1,\kappa_2), \tag{5.12}$$

$$R_2(x_1,\epsilon) \;=\; 1 + \kappa_1 + \kappa_2 - 2 - \kappa_1\cdot\kappa_2 = -(\kappa_1 - 1)(\kappa_2 - 1) =: \bar{R}_2(\kappa_1,\kappa_2) \tag{5.13}$$

hold. In case of $(\kappa_1,\kappa_2) \in \mathbb{R}^2$ (cf. (5.3)), the functions $\bar{R}_1$ and $\bar{R}_2$ obviously show the sign patterns in (κ_1,κ_2)-space represented as in Fig.7 below with the hatched region $0 < \kappa_1, \kappa_2 < 1$ leading to stable periodic orbits.

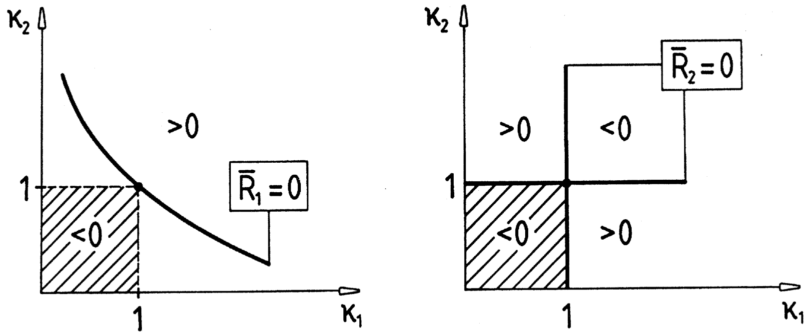

Figure 7: Sign patterns of $\bar{R}_1$ and $\bar{R}_2$.

In the other case $\kappa_1 = \bar{\kappa_2},\ \kappa_1 \notin \mathbb{R}$ one obtains

$$\bar{R}_1(\kappa_1, \kappa_2) = \mathrm{Re}^{\,2}(\kappa_1) + \mathrm{Im}^{\,2}(\kappa_1) - 1 = |\kappa_1|^2 - 1, \tag{5.14}$$

$$\begin{aligned} \bar{R}_2(\kappa_1, \kappa_2) &= 2\mathrm{Re}\,(\kappa_1) - 1 - \mathrm{Re}^{\,2}(\kappa_1) - \mathrm{Im}^{\,2}(\kappa_1) \\ &= -\{\mathrm{Im}^{\,2}(\kappa_1) + (\mathrm{Re}\,(\kappa_1) - 1)^2\} < 0. \end{aligned} \tag{5.15}$$

The conclusions in (5.6),(5.7) thus follow by a comparison of the equations (5.12),(5.13) with Fig.7 in case of $(\kappa_1, \kappa_2) \in \mathbb{R}^2$ and with (5.14), (5.15) in case of $\kappa_1 = \bar{\kappa_2},\ \kappa_1 \notin \mathbb{R}$.

Next, the curve defined by $R_1(x_1, \epsilon) = 0$ is discussed. From (5.4) and (5.9), it follows that

$$R_1(0,0) = 0, \quad R_{1\epsilon}(0,0) = \tfrac{d}{d\epsilon} \exp\left(2\pi\epsilon \cdot g_y(\Theta, \epsilon, \lambda^2(\epsilon))\right)_{|\epsilon=0} = 2\pi \cdot g_y(\Theta) \tag{5.16}$$

or alternatively, in view of $\varphi(t, x, y, 0, \lambda) = y$ and $R_1(0, \epsilon) = \varphi_y(2\pi, \Theta, \epsilon, \lambda^2(\epsilon)) - 1$,

$$R_{1\epsilon}(0,0) = \varphi_{y\epsilon}(2\pi, \Theta) + \varphi_{y\lambda}(2\pi, \Theta) \cdot {\lambda^2}'(0) = \varphi_{y\epsilon}(2\pi, \Theta)$$

yielding together with (2.2)

$$R_{1\epsilon}(0,0) = \varphi_{y\epsilon}(2\pi, \Theta) = 2\pi \cdot g_y(\Theta) = 2\pi \cdot \bar{g}_{\bar{y}}(\Theta) \neq 0, \tag{5.17}$$

provided the identity $g(\Theta, y, 0, 0) = \bar{g}(\Theta, y, 0, 0)$ can be shown. This however follows by a simple calculation from (2.1), (2.4) and the associated transformation (4.4) with $S(0, \lambda) = Diag[S_1(0, \lambda), 1]$. Consequently, by (5.16) and (5.17), a smooth function $\bar{\epsilon}(x_1)$ exists with $\bar{\epsilon}(0) = 0,\ R_1(x_1, \bar{\epsilon}(x_1)) = 0$ such that there are no other zeros of R_1 in a neighborhood of $(x_1, \epsilon) = (0, 0)$. Moreover, differentiation with respect to x_1 implies $R_{1x_1}(x_1, \bar{\epsilon}(x_1)) + R_{1\epsilon}(x_1, \bar{\epsilon}(x_1)) \cdot \bar{\epsilon}'(x_1) = 0$ and recalling (5.1),(5.4),(5.8) as well as (3.5) one gets

$$\begin{aligned} R_{1x_1}(0,0) &= \tfrac{d}{dx_1} \det[\Psi_x](T(x_1, 0), x_1, 0, y(x_1, 0), 0, \lambda(x_1, 0))_{|x_1=0} \\ &= (\Psi_{1x_1x_1}\Psi_{2x_2} + \Psi_{1x_1}\Psi_{2x_2x_1} - \Psi_{2x_1x_1}\Psi_{1x_2} - \Psi_{2x_1}\Psi_{1x_2x_1})(2\pi, \Theta) = 0. \end{aligned}$$

In view of (5.17) this proves the identity $\bar{\epsilon}'(0) = 0$. Summarizing, the behaviour of $\bar{\epsilon}(x_1)$ and the sign of $R_1(x_1, \epsilon)$ are locally determined by

$$\alpha_1 := \bar{\epsilon}''(0) \quad \text{and} \quad \beta_2 := R_{1\epsilon}(0,0) = \varphi_{y\epsilon}(2\pi, \Theta) = 2\pi \cdot g_y(\Theta) = 2\pi \cdot \bar{g}_{\bar{y}}(\Theta) \neq 0, \tag{5.18}$$

if the inequality $\bar{\epsilon}''(0) \neq 0$ is assumed. Then, Fig.8 shows the possible signs of R_1 in (x_1, ϵ)-space depending on the signs of $\alpha_1 \neq 0$ and $\beta_2 \neq 0$.

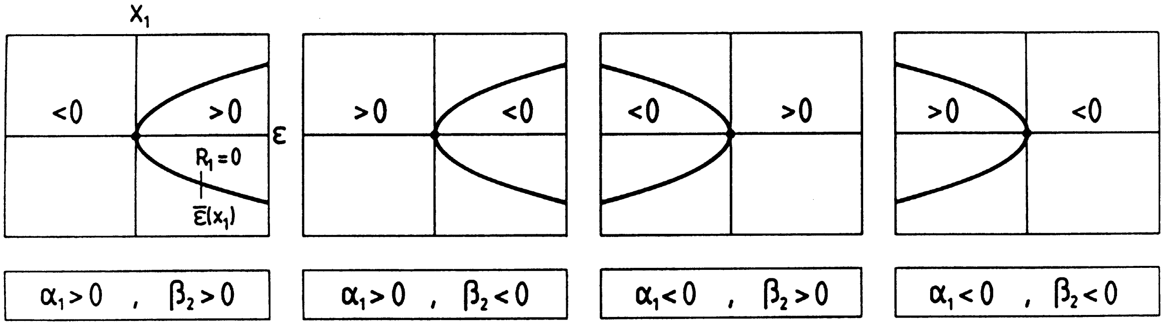

Figure 8: Sign pattern of R_1 in dependence of $\alpha_1 \neq 0$ and $\beta_2 \neq 0$.

In the next step, the curve $R_2(x_1, \epsilon) = 0$ is examined. According to (5.10) and (5.13), the identities $\kappa_1 = 1$ and $R_2(x_1, 0) = R_2(0, \epsilon) = 0$ hold along the curve of bifurcation points of periodic orbits as well as along the curve of nontrivial Hopf points. Moreover, due to the definitions (5.2) and (5.5), one obtains

$$R_{2x_1}(x_1, \epsilon) = \tfrac{d}{dx_1}(\mathrm{tr}[\begin{pmatrix} \Psi_x & \Psi_y \\ \varphi_x & \varphi_y \end{pmatrix}] - \det[\begin{pmatrix} \Psi_x & \Psi_y \\ \varphi_x & \varphi_y \end{pmatrix}])(T(x_1, \epsilon), x_1, 0, y(x_1, \epsilon), \epsilon, \lambda(x_1, \epsilon))$$

and recalling $\lambda(0, \epsilon) = \lambda^2(\epsilon)$ and (5.1)

$$\begin{aligned} R_{2x_1}(0, \epsilon) &= \mathrm{tr}_{x_1}[\begin{pmatrix} \Psi_x & \Psi_y \\ \varphi_x & \varphi_y \end{pmatrix}](2\pi, x_1, \Theta, \epsilon, \lambda^2(\epsilon))_{|x_1=0} - \det{}_{x_1}[\begin{pmatrix} \Psi_x & \Psi_y \\ \varphi_x & \varphi_y \end{pmatrix}](2\pi, x_1, \Theta, \epsilon, \lambda^2(\epsilon))_{|x_1=0} \\ &= \{\Psi_{1x_1x_1} + \Psi_{2x_2x_1} + \varphi_{yx_1} - \varphi_{x_1x_1}\cdot(\Psi_{1x_2}\Psi_{2y} - \Psi_{2x_2}\Psi_{1y}) \\ &\quad - \varphi_{x_1}\cdot(\Psi_{1x_2}\Psi_{2y} - \Psi_{2x_2}\Psi_{1y})_{x_1} + \varphi_{x_2x_1}\cdot(\Psi_{1x_1}\Psi_{2y} - \Psi_{2x_1}\Psi_{1y}) \\ &\quad + \varphi_{x_2}\cdot(\Psi_{1x_1}\Psi_{2y} - \Psi_{2x_1}\Psi_{1y})_{x_1} - \varphi_{yx_1}\cdot(\Psi_{1x_1}\Psi_{2x_2} - \Psi_{2x_1}\Psi_{1x_2}) \\ &\quad - \varphi_y\cdot(\Psi_{1x_1x_1}\Psi_{2x_2} + \Psi_{1x_1}\Psi_{2x_2x_1} - \Psi_{2x_1x_1}\Psi_{1x_2} - \Psi_{2x_1}\Psi_{1x_2x_1})\}(2\pi, \Theta, \epsilon, \lambda^2(\epsilon)). \end{aligned}$$

Thus, by (3.4) and (3.5), the equality $R_{2x_1}(0, \epsilon) = 0$ holds, and because of $R_2(x_1, 0) = 0$, the expansion with respect to ϵ takes the form $R_2(x_1, \epsilon) = \epsilon\cdot(R_{2\epsilon}(x_1, 0) + O(\epsilon)) =: \epsilon\cdot \hat{R}_2(x_1, \epsilon)$ implying together with the previous calculations

$$0 = R_2(0, \epsilon) = \epsilon\cdot \hat{R}_2(0, \epsilon), \quad 0 = R_{2x_1}(0, \epsilon) = \epsilon\cdot \hat{R}_{2x_1}(0, \epsilon), \quad R_{2x_1x_1}(0, \epsilon) = \epsilon\cdot \hat{R}_{2x_1x_1}(0, \epsilon).$$

Now for $\epsilon \neq 0$, the sign of $R_2 = \epsilon\cdot \hat{R}_2$ is determined by the sign of $\hat{R}_2$. Under the additional nondegeneracy condition

$$\alpha_3 := \hat{R}_{2x_1x_1}(0, 0) = R_{2x_1x_1\epsilon}(0, 0) \neq 0$$

this leads to the sign patterns of $R_2(x_1, \epsilon) = \epsilon \hat{R}_2(x_1, \epsilon)$ in the (x_1, ϵ)-space as shown in the following figure.

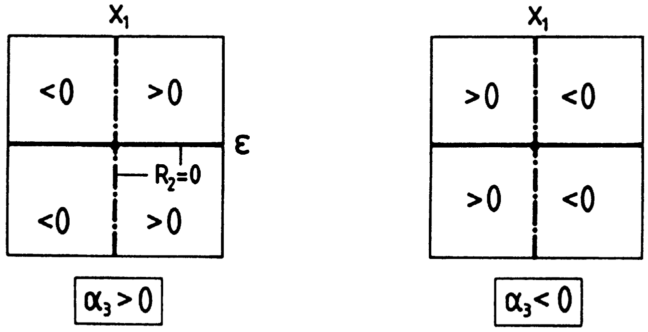

Figure 9: Sign patterns of R_2 in dependence of $\alpha_3 \neq 0$.

With the help of Fig.8, Fig.9 and the criteria (5.6), (5.7), it is now possible to determine the stability conditions on the manifold of periodic orbits in generic situations. For this purpose, it is sufficient to examine the (x_1, ϵ)-space for $x_1 \geq 0$, because the orbits with $x_1 < 0$ do not represent new periodic solutions. Then, according to Fig.8 and Fig.9, this half space $x_1 \geq 0$ is divided by the curves $R_1(x_1, \epsilon) = 0$ and $R_2(x_1, \epsilon) = 0$ into three regions with different signs. For example, in the case of $\alpha_1 > 0$ and in dependence on $\beta_2 \neq 0$ and $\alpha_3 \neq 0$, the following sequence of diagrams results. In each region the two inequalities indicate the sign of R_1 and R_2 respectively, whereas s and i represent the stability according to the criteria (5.6) and (5.7).

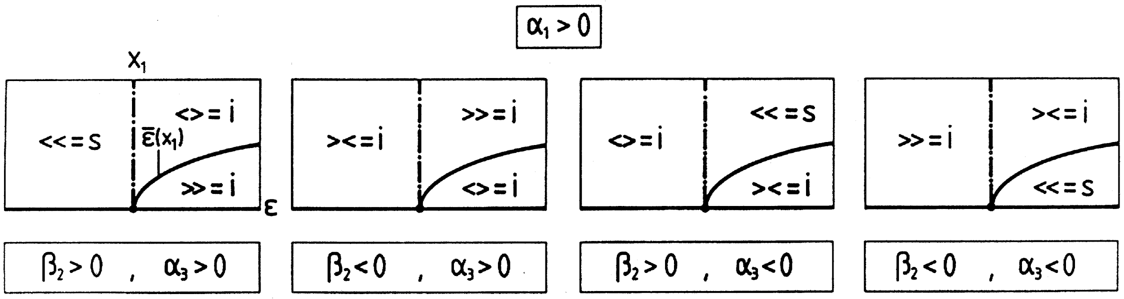

Figure 10: Stability of periodic orbits in dependence on $\alpha_1 > 0$ and $\beta_2 \neq 0, \alpha_3 \neq 0$.

Notice, that the function $\bar{\epsilon}(x_1)$ does not represent a boundary of stability in the two diagrams at the left side of Fig.10, while on the contrary a change of stability takes place in the other two diagrams. Further, the orbits on this border line satisfy $R_1(x_1, \bar{\epsilon}(x_1)) = 0$ and $R_2(x_1, \bar{\epsilon}(x_1)) < 0$ for $x_1 > 0$, so that according to Fig.7 the Floquet multipliers κ_1, κ_2 of these orbits have a nonreal pair on the unit circle. Thus, invariant tori bifurcate from these orbits in the direction of ϵ by a Naimark-Sacker bifurcation, if the Floquet multipliers pass the unit circle with nonvanishing velocity, and if additionally an internal nondegeneracy condition is satisfied [15]. Now, the external condition on the velocity follows directly from $\beta_2 = R_{1\epsilon}(0,0) = \frac{d}{d\epsilon}\det[V](0,0) \neq 0$ (compare (5.4),(5.18)), whereas the internal condition depends on higher derivatives of f and g which were not evaluated in detail. But this nondegeneracy condition is fulfilled generically and determines the bifurcation direction of the invariant tori. On the other hand, the limiting behaviour of the associated ϵ-interval of existence for $x_1 \to 0$ can not be infered from this bifurcation theorem of invariant tori.

The following relations result from a rather lengthy and technical calculation [25] which is not given here

$$\alpha_1 = \bar{\epsilon}''(0) = -\tfrac{2}{3}\cdot\tfrac{\beta_1}{\beta_2} \qquad \text{and} \qquad \alpha_3 = R_{2x_1x_1\epsilon}(0,0) = -\tfrac{2}{3}\cdot\beta_3 \tag{5.19}$$

with the definitions of β_1 and β_3 from Theorem 2.5. Then, the stability assignments in the upper sequence of Fig.5 ($\beta_1\cdot\beta_2 < 0$) follow directly from Fig.10. The corresponding diagrams for $\beta_1\cdot\beta_2 > 0$ in the lower sequence of Fig.5 can be obtained in an analogous fashion.

Finally, the proof of Proposition 2.6 is accomplished. First of all, the torus bifurcation points TP are parametrized by x_1 in (ϵ,λ)-space according to

$$(\epsilon,\lambda) = (\bar{\epsilon}(x_1), \lambda(x_1,\bar{\epsilon}(x_1))) = (\bar{\epsilon}(x_1), \lambda^2(\bar{\epsilon}(x_1)) + \tilde{\lambda}^2(x_1,\bar{\epsilon}(x_1))\cdot x_1^2) \tag{5.20}$$

(cf. (5.1)). In addition, under the assumption $\alpha_1 = \bar{\epsilon}''(0) \neq 0$, the inverse function $\bar{x}_1(\epsilon)$ of $\bar{\epsilon}(x_1)$ exists for $x_1 \geq 0$, and therefore the parameter curve (5.20) can also be parametrized by ϵ according to $(\epsilon,\lambda) = (\epsilon, \lambda^2(\epsilon) + \tilde{\lambda}^2(\bar{x}_1(\epsilon),\epsilon)\cdot\bar{x}_1(\epsilon)^2) =: (\epsilon, \lambda_t(\epsilon))$ where

$$\lambda_t'(\epsilon) = \lambda^{2\prime}(\epsilon) + (\tilde{\lambda}^2_{x_1}(\bar{x}_1(\epsilon),\epsilon)\cdot\bar{x}_1'(\epsilon) + \tilde{\lambda}^2_{\epsilon}(\bar{x}_1(\epsilon),\epsilon))\cdot\bar{x}_1(\epsilon)^2 + \tilde{\lambda}^2(\bar{x}_1(\epsilon),\epsilon)\cdot 2\bar{x}_1(\epsilon)\cdot\bar{x}_1'(\epsilon) \tag{5.21}$$

and one has to take $\epsilon > 0$ for $\alpha_1 = \bar{\epsilon}''(0) > 0$ and $\epsilon < 0$ for $\alpha_1 = \bar{\epsilon}''(0) < 0$. Then, differentiation of $\epsilon = \bar{\epsilon}(\bar{x}_1(\epsilon))$ leads to $\bar{x}_1'(\epsilon) = 1/\bar{\epsilon}'(\bar{x}_1(\epsilon))$, and the rule of l'Hospital combined with equation (5.19) yields

$$\lim_{\epsilon\to 0}\ \bar{x}_1(\epsilon)\cdot\bar{x}_1'(\epsilon) = \lim_{\epsilon\to 0}\ \frac{\bar{x}_1'(\epsilon)}{\bar{\epsilon}''(\bar{x}_1(\epsilon))\cdot\bar{x}_1'(\epsilon)} = \frac{1}{\bar{\epsilon}''(0)} = \frac{1}{\alpha_1} = -\frac{3}{2}\frac{\beta_2}{\beta_1}.$$

In view of (5.21) this proves the assertion

$$\lambda_t{}'(\epsilon) \ \to\ \lambda^{2\prime}(0) + \tilde{\lambda}^2(0,0)\cdot 2(-\tfrac{3}{2}\tfrac{\beta_2}{\beta_1}) \qquad \text{for} \qquad \epsilon \to 0$$

from Proposition 2.6.

6 Appendix

Consider system (3.1) with flow $(\chi,\phi)(t,u,v,\mu) \in \mathbb{R}^2\times\mathbb{R}^m$ and $(\chi,\phi)^0 = (\chi,\phi)(2\pi,\Theta)$ and let the assumptions (3.2) hold. The aim of this appendix is to prove the two equivalences in (3.8). First, differentiation of $(\dot{\chi},\dot{\phi})(t,u,v,\mu) = (F,G)((\chi,\phi)(t,u,v,\mu),\mu)$ at $(t,u,v,\mu) = (t,\Theta)$ yields the $(m+2)\times(m+1)$-matrix equation

$$\begin{pmatrix} (\phi_v \dot{-} I_m) & \dot{\phi}_\mu \\ \dot{\chi}_{u_1v} & \dot{\chi}_{u_1\mu} \end{pmatrix} = \begin{pmatrix} G_v^0(\phi_v - I_m) + G_v^0 & G_v^0\phi_\mu + G_\mu^0 \\ J_0\chi_{u_1v} + M_1\phi_v & J_0\chi_{u_1\mu} + M_1\phi_\mu + M_2\chi_\mu + M_3 \end{pmatrix},$$

$$\begin{pmatrix} \phi_v - I_m & \phi_\mu \\ \chi_{u_1v} & \chi_{u_1\mu} \end{pmatrix}_{|t=0} = 0 \in \mathbb{R}^{m+2,m+1}$$

where

$$M_1 := F^0_{u_1v}\cdot\chi_{1u_1} + F^0_{u_2v}\cdot\chi_{2u_1}, \qquad M_2 := F^0_{u_1u}\cdot\chi_{1u_1} + F^0_{u_2u}\cdot\chi_{2u_1},$$

$$M_3 := F^0_{u_1\mu}\cdot\chi_{1u_1} + F^0_{u_2\mu}\cdot\chi_{2u_1}.$$

Therefore, the variation of constants formula implies

$$\begin{pmatrix} \phi_v - I_m & \phi_\mu \\ \chi_{u_1 v} & \chi_{u_1 \mu} \end{pmatrix}(t,\Theta) =$$

$$\begin{pmatrix} \int_0^t \exp((t-s)G_v^0)ds\cdot G_v^0 & \int_0^t \exp((t-s)G_v^0)ds\cdot G_\mu^0 \\ \int_0^t \exp((t-s)J_0)M_1(s,\Theta)\phi_v(s,\Theta)ds & \int_0^t \exp((t-s)J_0)(M_1\phi_\mu + M_2\chi_\mu + M_3)(s,\Theta)ds \end{pmatrix} =:$$

$$\begin{pmatrix} A(t)\cdot G_v^0 & A(t)\cdot G_\mu^0 \\ \int_0^t \exp((t-s)J_0)M_1(s,\Theta)(A(s)G_v^0 + I_m)ds & \int_0^t \exp((t-s)J_0)(M_1 A(s)G_\mu^0 + M_2\chi_\mu + M_3)(s,\Theta)ds \end{pmatrix}$$

or

$$\begin{pmatrix} \phi_v - I_m & \phi_\mu \\ \chi_{u_1 v} & \chi_{u_1 \mu} \end{pmatrix}(t,\Theta) =: \begin{pmatrix} A(t)\cdot G_v^0 & A(t)\cdot G_\mu^0 \\ I^1(t)\cdot G_v^0 + I^2(t) & I^1(t)\cdot G_\mu^0 + I^3(t) \end{pmatrix} \tag{6.1}$$

with the obvious definition of $(I^1, I^2, I^3)(t)$. Then, the square matrix $B \in \mathbb{R}^{m+1,m+1}$ from (3.7) equals the first $(m+1)$-rows of the solution matrix (6.1) at time $t = 2\pi$. Thus, with the notation $I_1^i(t), i = 1,2,3$ for the first row of $I^i(t)$, the following equivalence is valid

$$B = \begin{pmatrix} A(2\pi) & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} G_v^0 & G_\mu^0 \\ I_1^1(2\pi)\cdot G_v^0 + I_1^2(2\pi) & I_1^1(2\pi)\cdot G_\mu^0 + I_1^3(2\pi) \end{pmatrix} \in GL[\mathbb{R}^{m+1}]$$

$$\iff \quad A(2\pi) \in GL[\mathbb{R}^m] \quad \wedge \quad \begin{pmatrix} G_v^0 & G_\mu^0 \\ I_1^2(2\pi) & I_1^3(2\pi) \end{pmatrix} \in GL[\mathbb{R}^{m+1}]$$

and consequently, the first equivalence in (3.8) is shown if the relations

$$A(2\pi) \in GL[\mathbb{R}^m] \quad \iff \quad n^+ i \notin \mathrm{Sp}[G_v^0], \quad n^+ \in \mathbb{N}^+ \tag{6.2}$$

and

$$\begin{pmatrix} G_v^0 & G_\mu^0 \\ I_1^2(2\pi) & I_1^3(2\pi) \end{pmatrix} \in GL[\mathbb{R}^{m+1}] \quad \iff \quad C = \begin{pmatrix} H_w & H_\mu \\ \mathrm{tr}_w(F_u) & \mathrm{tr}_\mu(F_u) \end{pmatrix}^0 \in GL[\mathbb{R}^{m+3}] \tag{6.3}$$

hold true. The equivalence (6.2) follows essentially from $\phi_v(t,\Theta) - I_m = A(t)\cdot G_v^0 = \exp(G_v^0 t) - I_m$ and some spectral properties of $\exp(G_v^0 t)$ infered from $n^+ i \notin \mathrm{Sp}[G_v^0]$, $n^+ \in \mathbb{N}^+$, whereas the equivalence (6.3) is a result of a direct calculation.

Finally, the second equivalence in (3.8) is an obvious consequence of

$$C \in GL[\mathbb{R}^{m+3}] \quad \iff \quad \mathrm{rank}(H_w, H_\mu)^0 = 2 + m \quad \wedge \quad \sigma'(0) \neq 0$$

which follows from

$$2\sigma'(0) = (\mathrm{tr}_w(F_u), \mathrm{tr}_\mu(F_u))^0 \cdot n \quad \text{with} \quad N[(H_w, H_\mu)^0] = \mathrm{span}\{n\}.$$

Here N is used to denote the kernel of a matrix. The above representation of the velocity $\sigma'(0)$ results, according to a technique in [1], by differentiating the eigenvector equation

$$H_w((w^s,\mu^s)(\epsilon))\cdot(v_1+i\cdot v_2)(\epsilon)=(\sigma+i\cdot\omega)(\epsilon)\cdot(v_1+i\cdot v_2)(\epsilon), \qquad v_1,v_2\in\mathbb{R}^{2+m}$$

and by applying the scalar product $< e_1+i\cdot e_2 \mid \cdot >$ to both sides where e_1, e_2 denote the first and second unit vector in $\mathbb{R}^{2+m}$. The details of these calculations can be found in [25].

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References

[1] Amann H. (1983): Gewöhnliche Differentialgleichungen. de Gruyter.

[2] Armbruster D., Dangelmayr G., Güttinger W. (1985): Imperfection sensitivity of interacting Hopf and steady-state bifurcations and their classification. Physica **16D**, 99–123.

[3] Arnold V. I., Afrajmovich V. S., Il'yashenko Y. S., Shil'nikov L. P. (1994): Dynamical Systems V: Bifurcation Theory. Springer, Berlin, New York.

[4] Baer S. M., Erneux T. (1986): Singular hopf bifurcation to relaxation oscillations. SIAM J. Appl. Math. **46**, 5, 721–739.

[5] Baer S. M., Erneux T. (1992): Singular hopf bifurcation to relaxation oscillations II. SIAM J. Appl. Math. **52**, 6, 1651–1664.

[6] Baer S. M., Erneux T., Rinzel J. (1989): The slow passage through a Hopf bifurcation: Delay, memory effects, and resonance SIAM J. Appl. Math. **49**, 1, 55–71.

[7] Beyn W.–J. (1990): The Numerical Computation of Connecting Orbits in Dynamical Systems. IMA J. Numer. Anal. **10**, 379-405.

[8] Descloux J., (1984): On Hopf and subharmonic Bifurcations. In: Numerical Methods for Bifurcation Problems. Birkhäuser, Basel, Boston.

[9] Diener M. (1981): Étude générique des canards. IRMA, Strasbourg.

[10] Eckhaus W. (1983): Relaxation oscillations including a standard chase on french ducks. In: Lecture notes in Mathematics 985, 449-494, Springer, Berlin, New York.

[11] Fenichel N. (1979): Geometric singular perturbation theory for ordinary differential equations. J. Differential Equations **31**, 53–98.

[12] Fenichel N. (1983): Oscillatory Bifurcations in Singular Perturbation Theory, II. Fast Oscillations. SIAM J. Math. Anal. **14**, 5, 868–874.

[13] Gibson C. G. (1979): Singular points of smooth mappings. Pitman.

[14] Golubitsky M., Stewart I., Schaeffer D. G. (1988): Singularities and Groups in Bifurcation Theory II. Springer, Berlin, New York.

[15] Guckenheimer J., Holmes P. J. (1983): Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields. Springer, Berlin, New York.

[16] Huseyin K., Yu P. (1989): On the Dynamics and Stability of a Codimansion-3 autonomous System. Int. J. Engng Sci. **27**, 5, 583–604.

[17] Koper M. (1995): Bifurcations of mixed-mode oscillations in a three-variable autonomous Van der Pol-Duffing model with a cross-shaped phase diagram. Physica **D80**, 72–94.

[18] Kuznetsov Y. A. (1995): Elements of Applied Bifurcation Theory. Appl. Math. Sci. **112**, Springer, Berlin, New York.

[19] Langford W. F. (1983): A review of interactions of Hopf and steady-state bifurcations. In: Barenblatt G., Iooss G., Joseph D. D., eds., Nonlinear Dynamics and Turbulence. 215–237, Pitman.

[20] Marsden J. E., McCracken M. (1976): The Hopf Bifurcation and Its Applications. Appl. Math. Sci. **19**, Springer, Berlin, New York.

[21] Neishtadt A. I. (1987): Persistence of stability loss for dynamical bifurcations I. . Differential Equations **23**, 1385–1391.

[22] Schropp J. (1992): Langzeitverhalten dynamischer Systeme am Beispiel eines biologischen Abbauprozesses. Dissertation, Hartung-Gorre, Konstanz.

[23] Stiefenhofer M. (1993): Periodische Lösungen bei singulär gestörten Grenzpunkten. ZAMM **73**, 6.

[24] Stiefenhofer M. (1994): Periodische Lösungen bei singulär gestörten Hopfpunkten. ZAMM **74**, 6.

[25] Stiefenhofer M. (1995): Singuläre Störung und Verzweigung bei Schleimpilzen. Dissertation, Hartung-Gorre, Konstanz.

[26] Vasil'eva A. B. (1963): Asymptotic behaviour of solutions to certain problems involving nonlinear differential equations containing a small parameter multiplying the highest derivatives. Russian Math. Surveys **18**, 13–84.

[27] Wiggins S. (1994): Normally Hyperbolic Invariant Manifolds in Dynamical Systems. Springer, Berlin, New York.