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DISCRETIZATIONS OF DYNAMICAL SYSTEMS WITH A SADDLE-NODE HOMOCLINIC ORBIT

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ABSTRACT. We consider a parametrized dynamical system with a homoclinic orbit that connects the center manifold of a saddle node to its strongly stable manifold. This is a codimension 2 homoclinic bifurcation with a well known unfolding. We show that the map obtained by discretizing such a system with a one-step method (the centered Euler scheme), inherits a discrete saddle-node homoclinic orbit. This orbit occurs on the line of saddle nodes and, as the numerical results suggest, there is actually a closed curve of such orbits and almost all of them consist of transversal homoclinic points. Our results complement those of [1], [5] on homoclinic discretization effects in the hyperbolic case.

1. Introduction

We consider a parametrized dynamical system

$$\dot{x} = f(x, \mu), \quad x \in \mathbb{R}^m, \mu \in \mathbb{R} \tag{1.1}$$

which admits a homoclinic orbit at a specific value of the parameter μ. If the equilibrium is hyperbolic and some nondegeneracy condition with respect to the parameter is satisfied (the Melnikov condition, see [8], [14]) then the homoclinic orbit persists under autonomous perturbations in a one-parametric family and it is then said to be of codimension 1 (see [7], [14]).

For numerical computations it is important to know what happens to this homoclinic orbit in a one step discretization of (1.1)

$$x_{n+1} = \Phi(x_n, \mu, \varepsilon), \quad n \in \mathbb{Z}. \tag{1.2}$$

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Here $\varepsilon > 0$ is assumed to be a fixed but small step size and the map $\Phi(\cdot, \mu, \varepsilon)$ is assumed to be a smooth approximation of the ε-flow of equation (1.1) to a certain order in ε. The above question was raised in [1] and it was largely solved in [5].

According to [5], the system (1.2) inherits homoclinic orbits from (1.1) but makes them transversal in a generic sense. Moreover, under analyticity assumptions the domains in the (μ, ε) plane in which transversal homoclinic points occur are shown to be exponentially small. In fact the numerical experiments in [3] suggest that the single continuous homoclinic orbit of the system (1.1) is replaced by a closed loop of discrete homoclinic orbits with at least two homoclinic tangencies.

The purpose of this paper is to begin the analysis of the case in which (1.1) has a homoclinic orbit based at a saddle node. We assume that the homoclinic orbit leaves the equilibrium via the center manifold and returns through the strongly stable manifold. This makes it a codimension-2 phenomenon and the unfolding of which has been analyzed in [4], [9], [11].

Actually we assume that the parameter μ in (1.1) has been chosen to parametrize the saddle node curve in a two-parametric family.

For the special case of the centered Euler scheme we show that the discrete system inherits a saddle node homoclinic orbit from the continuous system. As the step-size tends to zero these orbits converge to their continuous counterparts and the corresponding parameters converge with the order of the method.

As an example we treat the forced Josephson junction equation (cf. [11], [6]). The numerical results suggest that the same behavior as in the hyperbolic case occurs, i.e. a closed curve of discrete saddle-node homoclinic orbits appears.

Our proof of existence employs various techniques from the papers [4], [5], [15]. In [5] it was shown that a one step method of step size ε may be viewed as the 0 to ε flow of a nonautonomous rapidly forced perturbation of the original system (1.1). For this special type of perturbations, center manifolds were then constructed in [15]. This nonautonomous framework is combined with the perturbation analysis from [4] to conclude the existence of discrete saddle-node homoclinic orbits.

2. Assumptions and main results

Consider the dynamical system

$$\dot{x} = f(x, \mu), \quad x \in \mathbb{R}^m, \ \mu \in \mathbb{R}, \tag{2.1}$$

and assume it satisfies the following conditions (compare with [4] section 2).

(1) f and its derivatives up to order $k \geq 4$ are continuous and uniformly bounded in $\mathbb{R}^m$.
(2) $f(0,\mu) = 0$, $\sigma(D_x f(0,\mu)) = \sigma_1(\mu) \cup \{0\}$ and there is a constant $\gamma > 0$, such that for any $\lambda \in \sigma_1(\mu)$, $|\mathrm{Re}\ \lambda| > \gamma$.
(3) 0 is a simple eigenvalue of $D_x f(0,\mu)$ with right and left eigenvectors e_r^μ and e_l^μ.
(4) $e_l^0 D_x^2 f(0,0)(e_r^0, e_r^0) > 0$.
(5) For $\mu = 0$, (2.1) has a homoclinic orbit $\Gamma_0 : x = q(t)$ such that $q(t) \in W_{loc}^c(0;0)$ as $t \to -\infty$ and $q(t) \in W_{loc}^{ss}(0;0)$ as $t \to \infty$, where $W_{loc}^c(0;\mu)$ and $W_{loc}^{ss}(0;\mu)$ are the center and strongly stable manifolds of (2.1) at $(x,\mu) = (0,\mu)$, respectively.
(6) the center stable manifold $W^{cs}(0;0)$ and the center unstable manifold $W^{cu}(0;0)$ intersect transversely along Γ_0.
(7) The homoclinic orbit is non-degenerate which in analytic terms means that $\int_{-\infty}^{+\infty} \psi^* D_\mu f(q(t),0)dt \neq 0$ where ψ solves a certain linearized adjoint equation (cf. Lemma 3.5).

Assumptions (2), (3) and (4) say that equation (2.1) has a family of saddle-node equilibria $(0,\mu)$ for $\mu \in (-\mu_0, \mu_0)$. For an illustration we show the typical change of phase portraits with μ in Fig 2.1. This saddle node line is part of the two-parametric unfolding from [4].

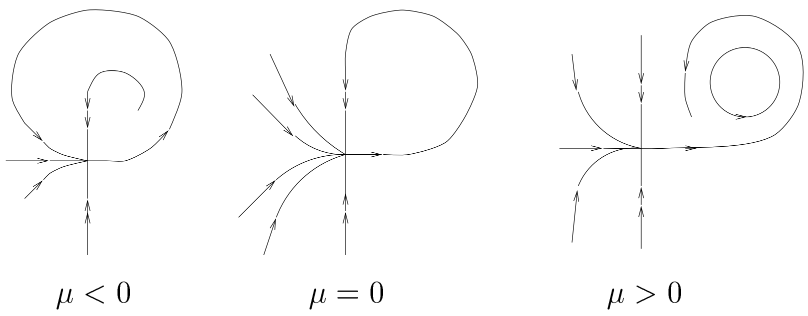

Fig 2.1. The bifurcation diagram while μ varies.

We use the centered Euler scheme with step-size ε to discretize the dynamical system (2.1)

$$\frac{x_{n+1} - x_n}{\varepsilon} = f\left(\frac{x_{n+1} + x_n}{2}, \mu\right) \tag{2.2}$$

which by assumption (1) implicitly determines an iterative map for sufficiently small ε

$$x_{n+1} = \Phi(x_n, \mu, \varepsilon). \tag{2.3}$$

By the symmetric property of (2.2), it is easy to prove that Φ has a family of saddle-node fixed points at $(0, \mu, \varepsilon)$ for all $|\mu| < \mu_0$ and $0 < \varepsilon < \varepsilon_0$.

The main result in this paper is

Theorem 2.1. *Let assumptions* (1)–(7) *hold, then there exist constants* $\varepsilon_0 > 0$, $c > 0$ *and a function* $\mu_*(\varepsilon)$ *with* $|\mu_*(\varepsilon)| \leq c\varepsilon^2$, *such that for any* $0 < \varepsilon < \varepsilon_0$, $\mu = \mu_*(\varepsilon)$, *the discrete system* (2.3) *has a homoclinic orbit* $\Gamma_\varepsilon^n, n \in \mathbb{Z}$ *with* $\lim_{n \to \pm\infty} \Gamma_\varepsilon^n = 0$. *Furthermore, this orbit converges to the continuous homoclinic orbit in the following sense*

$$\sup_{n \in \mathbb{Z}} |\Gamma_\varepsilon^n - q(n\varepsilon)| \to 0, \quad \text{as } \varepsilon \to 0.$$

Let us outline the method of proof before the details are provided in the next section. First we construct a non-autonomous system to interpolate the orbits of the discrete system (2.3). This yields a rapidly forced perturbation of the equation (2.1) of order $O(\varepsilon^2)$ as described in [5], [15]. This perturbation is then analyzed with the help of the Liapunov Schmidt reduction method from [4]. In [4] the reduction was utilized to obtain the two-parametric unfolding near the saddle node homoclinic orbit. In our case it leads to a bifurcation function by which we prove the existence of orbits homoclinic to a saddle-node equilibrium for the non-autonomous system. As an application, we get the proof of our main theorem.

3. Proof of main results

In this section, we prove our main results by a sequence of lemmata. First, we introduce the interpolated system which is well studied in [5] and [15] (Theorem 6.2).

Lemma 3.1. *Let the conditions* (1) – (3) *hold. Then there exist* $\varepsilon_0 > 0$, $\mu_1 > 0$ *and a vector field* $g \in C^{k-3}((-\mu_1, \mu_1) \times (-\varepsilon_0, \varepsilon_0) \times \mathbb{R} \times \mathbb{R}^m, \mathbb{R}^m)$ *such that the following properties hold for all* $|\mu| < \mu_1$, $|\varepsilon| < \varepsilon_0$, $\tau \in \mathbb{R}$ *and* $x \in \mathbb{R}^m$.

(i): *The map* $(\mu,\varepsilon,\tau,x)\to\varepsilon^2 g(\mu,\varepsilon,\tau,x)$ *is of class* C^{k-1} *and* 1*-periodic in* τ.

(ii): $|g(\mu,\varepsilon,\tau,x)|+|D_x g(\mu,\varepsilon,\tau,x)|+|D_\mu g(\mu,\varepsilon,\tau,x)|\le C$ *for some* $C>0$.

(iii): $g(\mu,\varepsilon,\tau,0)=0$ *and* $D_x g(\mu,\varepsilon,\tau,0)$ *has the simple eigenvalue* 0 *with right eigenvector* $e_r(\mu)$ *and no other eigenvalues on the imaginary axis.*

(iv): *For* $\varepsilon\neq 0$ *we have* $G(\mu,\varepsilon,\varepsilon,x)=\Phi(\mu,\varepsilon,x)$ *where* Φ *is defined in* (2.2)*, and* $G(\mu,t,\varepsilon,x)$ *is the* t*-flow with* $G(\mu,0,\varepsilon,x)=x$ *of the following nonautonomous system*

$$\dot{x}(t)=f(x(t),\mu)+\varepsilon^2 g(\mu,\varepsilon,t/\varepsilon,x(t)). \tag{3.1}$$

We assume that after a linear μ-dependent change of coordinates the eigenvectors e_r^μ are transformed into $(1,0,\cdots,0)$ and the remaining coordinates form the complementary invariant subspace. We then write $x=(\theta,u)$, $\theta\in\mathbb{R}$, $u\in\mathbb{R}^{m-1}$ and we can state the invariant manifold result for equation (3.1) (cf. [15], Theorem 6.3) as follows.

Lemma 3.2. *Assume that the conditions* (1)-(3) *are satisfied. Then there exist constants* $\mu_2,\varepsilon_1,\delta,c>0$ *and functions*

$$h\in C((-\mu_2,\mu_2)\times(0,\varepsilon_1)\times\mathbb{R}\times(-\delta,\delta),\mathbb{R}^{m-1}),$$
$$\tilde{h}\in C^{k-1}((-\mu_2,\mu_2)\times(-\delta,\delta),\mathbb{R}^{m-1})$$

with the following properties.

(i): $h(\mu,\varepsilon,\tau,\theta)$ *is of class* C^{k-1} *in the variables* (μ,τ,θ)*. Moreover*

$$h(\mu,\varepsilon,\tau,0)=0,\ D_\theta h(\mu,\varepsilon,\tau,0)=0,\ h(\mu,\varepsilon,\tau+1,\theta)=h(\mu,\varepsilon,\tau,\theta).$$

(ii): *The manifold*

$$\{(\mu,\varepsilon,t,\theta,h(\mu,\varepsilon,t/\varepsilon,\theta)):|\mu|<\mu_2,\ 0<\varepsilon<\varepsilon_1,\ t\in\mathbb{R},\ |\theta|<\delta\}$$

is locally invariant for the system (3.1).

(iii): $\tilde{h}(\mu,0)=0$, $D_\theta\tilde{h}(\mu,0)=0$ *and the manifold*

$$\{(\mu,\theta,\tilde{h}(\mu,\theta)):|\mu|<\mu_2,\ |\theta|<\delta\}$$

is locally invariant for the system (2.1).

(iv): *The estimates hold uniformly in* μ, ε, τ *and* θ

$$|h(\mu,\varepsilon,\tau,\theta)-\tilde{h}(\mu,\theta)|+|D_\theta h(\mu,\varepsilon,\tau,\theta)-D_\theta\tilde{h}(\mu,\theta)|$$
$$+|D_\mu h(\mu,\varepsilon,\tau,\theta)-D_\mu\tilde{h}(\mu,\theta)|\le c\varepsilon^2.$$

(v): *The manifolds*

$$M_{\mu,\varepsilon} = \{(\theta, h(\mu,\varepsilon,0,\theta)) : |\theta| < \delta\}, \quad |\mu| < \mu_1, 0 < \varepsilon < \varepsilon_1$$

are locally invariant for the one step map $\Phi(\mu,\varepsilon,\cdot)$.

If we extend h to $\varepsilon = 0$ by setting $h(\mu,0,\tau,\theta) := \tilde{h}(\mu,\theta)$ we obtain from (iv) that the property (i) also holds for the extended function and moreover h, $D_\theta h$, $D_\mu h$ are continuous at $\varepsilon = 0$. From now on, we use the extended center manifold.

To study the saddle node homoclinic orbit, we need to analyze the behavior of the solution of equation (3.1) as $t \to -\infty$ within the center manifolds. By assumption (5) we may assume

$$q(0) = (\theta_0, \tilde{h}(0,\theta_0)) \text{ for some } \theta_0 \in (0,\delta) \tag{3.2}$$

and we let $q(\theta_0,\mu,\varepsilon,t) = (\theta,u)(\theta_0,\mu,\varepsilon,t)$ be the solution of (3.1) with initial value $(\theta_0, h(\mu,\varepsilon,0,\theta_0))$. This is the continuation of the homoclinic solution $q(t)$.

Lemma 3.3. *Let the assumptions* (1) – (4) *from section* 2 *hold. Then there exist constants* μ_0, ε_0, θ_1, $0 < c_1 < c_2$, $c_3 > 0$ *such that the following holds for* $t \le 0$, $|\mu| < \mu_0$, $0 \le \varepsilon < \varepsilon_0$, $0 < \theta \le \theta_1$

(i):

$$|q(\theta_0,\mu,\varepsilon,t)| \le \frac{c_3}{1+|t|}, \tag{3.3}$$

$$\frac{\theta_0}{1 - c_2\theta_0 t} \le \theta(\theta_0,\mu,\varepsilon,t) \le \frac{\theta_0}{1 - c_1\theta_0 t}. \tag{3.4}$$

(ii): $q(\theta_0,\mu,\varepsilon,\cdot)$ *and* $D_\mu q(\theta_0,\mu,\varepsilon,\cdot)$ *are bounded for* $t \le 0$ *and continuous in* $|\mu| < \mu_0$, $0 \le \varepsilon < \varepsilon_0$ *with respect to the uniform norm in* $(-\infty, 0]$.

Proof. Let $J = (\beta, 0]$ be the maximal interval such that the solution $\theta(\theta_0,\mu,\varepsilon,t)$ exists within $(-\delta,\delta)$. In J the solution stays on the center manifold and therefore

$$q(\theta_0,\mu,\varepsilon,t) = (\theta(\theta_0,\mu,\varepsilon,t), h(\mu,\varepsilon,t/\varepsilon,\theta(\theta_0,\mu,\varepsilon,t))). \tag{3.5}$$

Because of Lemma 3.2 (i), (3.3) follows from (3.4) and (3.5).

Now, we use the center manifold to reduce equation (3.1) to the one-dimensional system

$$\dot{\theta} = f_1(\theta, h(\mu,\varepsilon,t/\varepsilon,\theta),\mu) + \varepsilon^2 g_1(\varepsilon, t/\varepsilon, \theta, h(\mu,\varepsilon,t/\varepsilon,\theta)). \tag{3.6}$$

From the assumptions (2) and (3), we know $f_1(0,0,\mu) = D_\theta f_1(0,0,\mu) = D_u f_1(0,0,\mu) = 0$, where $x = (\theta, u)$. This means that $f_1(\theta,u,\mu) = O(\theta^2 + u^2)$

uniformly for $|\mu|$ small enough. From Lemma 3.2 (i), we have $h(\mu, \varepsilon, t/\varepsilon, \theta) = O(\theta^2)$ uniformly for $|\mu|$, $|\theta|$, $\varepsilon > 0$ sufficiently small and $t \in J$ and therefore

$$f_1(\theta, h(\mu, \varepsilon, t/\varepsilon, \theta), \mu) = c_0\theta^2 + O(\theta^3)$$

where $c_0 = e_l^0 D_x^2 f(0,0)(e_r^0, e_r^0) > 0$ by assumption (4).

Similarly, by Lemma 3.1 (i) we have $g(\mu, \varepsilon, t/\varepsilon, \theta, v) = \tilde{c}_0\theta^2 + O(\theta^3)$ uniformly for $|\mu|$, $|\theta|$, $\varepsilon > 0$ sufficiently small and $t \in J$. Then there exist constants μ_0, ε_0, θ_1, $0 < c_1 < c_2$ such that the following holds for $t \in J$, $|\mu| < \mu_0$, $0 < \varepsilon < \varepsilon_0$, $0 < \theta \leq \theta_1$

$$c_1\theta^2 < f_1(\theta, h(\mu, \varepsilon, t/\varepsilon, \theta), \mu) + \varepsilon^2 g_1(\varepsilon, t/\varepsilon, \theta, h(\mu, \varepsilon, t/\varepsilon, \theta)) < c_2\theta^2.$$

The solution of equation $\dot{\theta} = c\theta^2$ with initial value $\theta(0) = \theta_0$ is $\theta(t) = \dfrac{\theta_0}{1 - c\theta_0 t}$. Applying a standard differential inequality (see [13]) we obtain the following estimate uniformly for $|\mu|$, $|\theta_0|$ sufficiently small and $t \in J$

$$\frac{\theta_0}{1 - c_2\theta_0 t} \leq \theta(\theta_0, \mu, \varepsilon, t) \leq \frac{\theta_0}{1 - c_1\theta_0 t}. \tag{3.7}$$

If θ_0 is small this shows that the boundary of $(-\delta, \delta)$ can not be reached by $\theta(\theta_0, \mu, \varepsilon, t)$ in J, hence $\beta = -\infty$ and (3.4) follows.

Next we prove (ii). The differentiability with respect to μ and the continuous dependence on (μ, ε) on compact time intervals are well known. It remains to do the uniform estimate.

Let us write (3.6) as

$$\dot{\theta} = \tilde{f}_1(\theta, \mu, \varepsilon, t), \quad \theta(0) = \theta_0 \tag{3.8}$$

where

(3.9)

$$\begin{aligned} \tilde{f}_1(\theta, \mu, \varepsilon, t) &= f_1(\theta, h(\mu, \varepsilon, t/\varepsilon, \theta), \mu) + \varepsilon^2 g_1(\varepsilon, t/\varepsilon, \theta, h(\mu, \varepsilon, t/\varepsilon, \theta)), \quad &\varepsilon \neq 0 \\ \tilde{f}_1(\theta, \mu, 0, t) &= f_1(\theta, \tilde{h}(\mu, \theta), \mu), &\varepsilon = 0. \end{aligned}$$

By direct computation we get

$$\begin{gathered} \tilde{f}_1(0, \mu, \varepsilon, t) = 0, \quad D_\theta \tilde{f}_1(0, \mu, \varepsilon, \theta) = 0, \quad D_\mu \tilde{f}_1(0, \mu, \varepsilon, t) = 0, \\ D_\theta^2 \tilde{f}_1(0, 0, 0, t) = e_r^0 D_x^2 f(0, 0) e_l^0 e_l^0 = c_0 > 0. \end{gathered} \tag{3.10}$$

Given (μ_1, ε_1), (μ_2, ε_2) and $\rho_0 > 0$, we may choose $T > 0$ by the estimate (3.4) so large that $|\theta(\theta_0, \mu, \varepsilon, t)| \leq \rho_0/2$ holds for $t \leq -T$. Then apply the standard continuity argument to obtain

$$|\theta(\theta_0, \mu_1, \varepsilon_1, t) - \theta(\theta_0, \mu_2, \varepsilon_2, t)| \leq \rho_0/2$$

uniformly for $-T \leq t \leq 0$ and $|\mu_1 - \mu_2|$, $|\varepsilon_1 - \varepsilon_2|$ sufficiently small.

The μ-derivative $\tilde{\theta} = D_\mu \theta$ satisfies the variational equation

$$\dot{\tilde{\theta}} = a(t,\mu,\varepsilon)\tilde{\theta} + b(t,\mu,\varepsilon), \quad \tilde{\theta}(0) = 0 \tag{3.11}$$

where

$$(a,b)(t,\mu,\varepsilon) = (D_\theta \tilde{f}_1, D_\mu \tilde{f}_1)(\theta(\theta_0,\mu,\varepsilon,t),\mu,\varepsilon,t).$$

From (3.10) and (3.4) we obtain the estimate

$$0 < a(t,\mu,\varepsilon) \leq \frac{c}{1+|t|}, \quad |b(t,\mu,\varepsilon)| \leq \frac{c}{1+|t|}, \quad t \leq 0.$$

Using this in the solution formula for (3.11) yields the boundedness as well as the continuity of θ with respect to (μ,ε) uniformly in $t \leq 0$.

The corresponding estimate for q then follows from (3.5). ■

Remark 3.4. We notice that estimates similar to (3.4) have been derived in [12] for the autonomous case.

For simplicity, we omit the argument θ_0 and write $q(\mu,\varepsilon,t)$ instead of $q(\theta_0,\mu,\varepsilon,t)$ in the following.

Let γ_1, γ_2 be two real constants and $C^0(\gamma_1,\gamma_2)$ denote the set of continuous functions $x : \mathbb{R} \to \mathbb{R}^m$ such that $|x(t)|e^{-\gamma_1 t}, t \leq 0$ and $|x(t)|e^{-\gamma_2 t}, t \geq 0$ are bounded. The set of such functions form a Banach space with the norm

$$|x|_{C^0(\gamma_1,\gamma_2)} = \sup\{|x(t)|e^{-\gamma_2 t}, |x(-t)|e^{\gamma_1 t}, t \geq 0\}.$$

Let $C^k(\gamma_1,\gamma_2)$ be the Banach space of all C^k functions $x(t)$ such that $D^i x(\cdot) \in C^0(\gamma_1,\gamma_2), i = 1,\cdots,k$ with the norm

$$|x|_{C^k(\gamma_1,\gamma_2)} = \sum_{i=0}^{k} |D^i x|_{C^0(\gamma_1,\gamma_2)}.$$

Let $A(t) = D_x f(q(t),0)$ and define the linear map

$$\mathcal{L} : C^1(\gamma,-\gamma) \to C^0(\gamma,-\gamma) \quad (\mathcal{L}u)(t) = \dot{u}(t) - A(t)u(t), \tag{3.12}$$

where γ comes from assumption (2).

By Lemma 3.2 and 5.1 in [4], we have

Lemma 3.5. *Suppose* (1)–(7) *hold, then*

(i) $\mathcal{L}$ *is a Fredholm operator with* Index $\mathcal{L} = -1$. *Furthermore, the null subspace of* $\mathcal{L}$ *in* $C^1(\gamma,-\gamma)$ *is* $\{0\}$, *the adjoint equation*

$$(\mathcal{L}^* y)(t) = \dot{y}(t) + A^*(t)y(t) = 0$$

has a unique unit solution $\psi(t) \in C^1(-\gamma,\gamma)$;

(ii) *Furthermore, we have* $\psi(\cdot) \in C^1(\gamma, \gamma)$ *and* $\psi(t) \to e_r^0$ *as* $t \to \infty$*, hence* $\psi \in C^1(\gamma, 0)$*. For any* $\phi \in C^1(-\gamma, -\gamma)$*, there holds* $<\psi, \mathcal{L}\phi> = 0$*, where* $<\cdot, \cdot>$ *denotes the standard* L^2*-inner product.*

Assume

$$\int_{-\infty}^{+\infty} \psi^*(t) D_\mu f(q(t), 0) dt > 0. \tag{3.13}$$

This is the nondegeneracy condition for the saddle node homoclinic orbit Γ_0 stated in assumption (7). For more details of this Melnikov type condition see [4] section 2 and [11].

We try to find the saddle node homoclinic orbit of equation (3.1) in the following form

$$x(t) = q(t) + \chi(t)(q(\mu, \varepsilon, t) - q(t)) + z(t), \tag{3.14}$$

where $\chi(t)$ is a C^∞ function with $\chi(t) \equiv 1$ for $t \leq -1$, and $\chi(t) \equiv 0$ for $t \geq 0$, and $z \in C^1(\gamma, -\gamma)$ is unknown.

Now, we set up the equation for z which by (3.1) and (3.14) reads

$$\begin{aligned} & f(q(t), 0) + \frac{d}{dt}(\chi(t)(q(\mu, \varepsilon, t) - q(t))) + \dot{z} \\ = \; & f(q(t) + \chi(t)(q(\mu, \varepsilon, t) - q(t)) + z, \mu) \\ & + \varepsilon^2 g(q(t) + \chi(t)(q(\mu, \varepsilon, t) - q(t)) + z, \mu, \varepsilon, t/\varepsilon). \end{aligned} \tag{3.15}$$

Let

$$\begin{aligned} Q(z, \mu, \varepsilon)(t) \; = \; & f(q(t) + \chi(t)(q(\mu, \varepsilon, t) - q(t)) + z, \mu) - A(t)z \\ & - f(q(t), 0) - \frac{d}{dt}(\chi(t)(q(\mu, \varepsilon, t) - q(t))) \\ & + \varepsilon^2 g(q(t) + \chi(t)(q(\mu, \varepsilon, t) - q(t)) + z, \mu, \varepsilon, t/\varepsilon). \end{aligned} \tag{3.16}$$

Then the equation (3.15) is equivalent to the operator equation

$$\mathcal{L}z - Q(z, \mu, \varepsilon) = 0. \tag{3.17}$$

We want to solve equation (3.17) for z in the Banach space $C^1(\gamma, -\gamma)$.

Lemma 3.6. *Assume that the assumptions* (1) – (7) *hold. Then there exists* $\mu_0 > 0$, $\varepsilon_0 > 0$ *with the following properties*

(i) $Q, Q_\mu : C^0(\gamma, -\gamma) \times (-\mu_0, \mu_0) \times [0, \varepsilon_0) \to C^0(\gamma, -\gamma)$ *are continuous with respect to all variables and* Q *is continuously differentiable with respect to* z.

(ii) $Q(z, \mu, \varepsilon)$ *is Lipschitz continuous with respect to* z*. The Lipschitz constant is of the form* $\omega(\rho)$ *in the domain* $|\mu|, \varepsilon, |z|_{C^1(\gamma,-\gamma)} < \rho$ *and* $\omega(\rho) \to 0$ *as* $\rho \to 0^+$.

(iii) $|Q(0,0,\varepsilon)|_{C^0(\gamma,-\gamma)} = O(\varepsilon^2)$ *as* $\varepsilon \to 0$.

Proof. First, we consider the case of $t \le -1$. Then by the definition of $q(\mu,\varepsilon,t)$ we have for $z \in C^1(\gamma,-\gamma)$

$$\begin{aligned}
& Q(z,\mu,\varepsilon)(t) \\
= \; & f(q(\mu,\varepsilon,t)+z,\mu) - A(t)z - \frac{d}{dt}q(\mu,\varepsilon,t) + \varepsilon^2 g(q(\mu,\varepsilon,t)+z,\mu,\varepsilon,t/\varepsilon) \\
= \; & f(q(\mu,\varepsilon,t)+z,\mu) - f(q(\mu,\varepsilon,t),\mu) - A(t)z \\
& +\varepsilon^2(g(q(\mu,\varepsilon,t)+z,\mu,\varepsilon,t/\varepsilon) - g(q(\mu,\varepsilon,t),\mu,\varepsilon,t/\varepsilon)) \\
& +f(q(\mu,\varepsilon,t),\mu) + \varepsilon^2 g(q(\mu,\varepsilon,t),\mu,\varepsilon,t/\varepsilon) - \frac{d}{dt}q(\mu,\varepsilon,t) \\
= \; & (\int_0^1 (D_x f(q(\mu,\varepsilon,t)+sz,\mu) - D_x(q(t),0))ds)z \\
& +\varepsilon^2(\int_0^1 D_x g(q(\mu,\varepsilon,t)+sz,\mu,\varepsilon,t/\varepsilon))z \\
& A_-(\mu,\varepsilon,t,z)z. \qquad (3.18)
\end{aligned}$$

For $\varepsilon = 0$ we define A_- by omitting the last term. For the case $t > 0$ we have

$$\begin{aligned}
Q(z,\mu,\varepsilon)(t) &= f(q(t)+z,\mu) - A(t)z - f(q(t),0) + \varepsilon^2 g(q(t)+z,\mu,\varepsilon,t/\varepsilon) \\
&= (\int_0^1 D_x f(q(t)+sz,\mu) - D_x f(q(t),0)ds)z \\
&\quad +\varepsilon^2(\int_0^1 D_x g(s(q(t)+z),\mu,\varepsilon,t/\varepsilon)ds)(q(t)+z) \\
&= A_+(\mu,\varepsilon,t,z)z + A_0(\mu,\varepsilon,t,z)q(t). \qquad (3.19)
\end{aligned}$$

By the assumption (1) and Lemma 3.1 (ii) we know there is a constant $C > 0$ such that

$$|A_-(\mu,\varepsilon,t,z)| + |A_0(\mu,\varepsilon,t,z)| + |A_+(\mu,\varepsilon,t,z)| \le C.$$

Notice that $q(t) = O(e^{-\gamma t})$ as $t \to +\infty$ and this proves that $Q(z,\mu,\varepsilon) \in C^0(\gamma,-\gamma)$.

Moreover, from (3.18), (3.19) and the continuity of $q(\mu,\varepsilon,\cdot)$ in the uniform norm from Lemma 3.3 we obtain that Q is continuous in the given spaces.

A similar argument works for $D_\mu Q$. Notice, for example,

$$
\begin{aligned}
D_\mu A_-(\mu,\varepsilon,\theta,z) \;=\; & \int_0^1 D_\mu D_x f(q(\mu,\varepsilon,t)+sz,\mu) \\
& +D_x^2 f(q(\mu,\varepsilon,t)+sz,\mu)D_\mu q(\mu,\varepsilon,t)ds \\
& +\varepsilon^2\int_0^1 D_\mu D_x g(q(\mu,\varepsilon,t)+sz,\mu,\varepsilon,t/\varepsilon) \\
& +D_x^2 g(q(\mu,\varepsilon,t)+sz,\mu,\varepsilon,t/\varepsilon)D_\mu q(\mu,\varepsilon,t)ds
\end{aligned}
$$

and use Lemma 3.3 to estimate $D_\mu q$.

Finally, the continuous differentiability of Q with respect to z follows directly from the formulas (3.18), (3.19). This completes (i).

By (3.18) and (3.19) we know that $Q(0,0,\varepsilon)$ vanishes for $t\le -1$ and for $t>0$

$$Q(0,0,\varepsilon)=\varepsilon^2\Big(\int_0^1 D_x g(sq(t),0,\varepsilon,t/\varepsilon)ds\Big)q(t)$$

and from this (iii) follows.

Now, let us prove assertion (ii). For any $z_1,z_2\in C^1(\gamma,-\gamma)$, by the definition (3.8) of Q and the mean value theorem we have

$$
\begin{aligned}
& Q(z_1,\mu,\varepsilon)-Q(z_2,\mu,\varepsilon) \\
= \; & \int_0^1 (D_x f(q(t)+\chi(t)(q(\mu,\varepsilon,t)-q(t))+z_2+s(z_1-z_2),\mu)-D_x f(q(t),0))ds \\
& \cdot(z_1-z_2) \\
& +\varepsilon^2\int_0^1 D_x g(q(t)+\chi(t)(q(\mu,\varepsilon,t)-q(t))+z_2+s(z_1-z_2),\mu,\varepsilon,t/\varepsilon)ds \\
& \cdot(z_1-z_2)
\end{aligned}
$$

$$C(\mu,\varepsilon,t,z_1,z_2)(z_1-z_2). \tag{3.20}$$

Define

$$\omega(\rho)=\max\{|C(\mu,\varepsilon,t,z_1,z_2)| : |\mu|,|z_i|_{C^1(\gamma,-\gamma)}<\rho, 0\le\varepsilon<\rho, t\in\mathbb{R}, i=1,2\}. \tag{3.21}$$

and use Lemma 3.3 again to end the proof. ■

Now we follow the usual procedure of Liapunov-Schmidt reduction used in [4]. Let P denote L^2-orthogonal projection on to the co-kernel of $\mathcal{L}$:

$$Pz := < z,\psi > \psi. \tag{3.22}$$

We decompose system (3.17) into

$$(Id - P)(\mathcal{L}z - Q(z, \mu, \varepsilon)) = 0, \tag{3.23}$$

$$P(\mathcal{L}z - Q(z, \mu, \varepsilon)) = -P \cdot Q(z, \mu, \varepsilon) = 0. \tag{3.24}$$

Because of Lemma 3.5 and 3.6 , the system (3.23) can be solved uniquely for $z(\mu, \varepsilon, \cdot) \in C^1(\gamma, -\gamma)$ near $(z, \mu, \varepsilon) = (0, 0, 0)$ with $z(0, 0, \cdot) = 0$, and z as well as $D_\mu z$ are continuous because of Lemma 3.6.

Putting $z = z(\mu, \varepsilon, t)$ in (3.24) and by the definition of P, (3.17) is then equivalent to the scalar equation

$$B(\mu, \varepsilon) = 0, \tag{3.25}$$

where

$$B(\mu, \varepsilon) = \int_{-\infty}^{+\infty} \psi^*(t) Q(z(\mu, \varepsilon, \cdot), \mu, \varepsilon)(t) dt. \tag{3.26}$$

This is called the Melnikov function in [4].

The properties of this function B are summarized in the following lemma.

Lemma 3.7. *Let the assumptions* (1) – (7) *hold. Then there exist constants* $\mu_0 > 0$, $\varepsilon_0 > 0$, $C > 0$ *such that the following holds for* $|\mu| < \mu_0$, $0 \le \varepsilon < \varepsilon_0$

(i): *B and $D_\mu B$ are continuous.*
(ii): $D_\mu B(0, 0) > 0$.
(iii): $|B(0, \varepsilon)| \le C\varepsilon^2$.

Proof. From Lemma 3.5 (ii) we know $\psi \in C^0(\gamma, 0)$, and from Lemma 3.6, there exist constants $\mu_0 > 0$, $\varepsilon_0 > 0$ such that for $|\mu| < \mu_0$, $0 \le \varepsilon < \varepsilon_0$, we have $Q(z(\mu, \varepsilon, \cdot), \mu, \varepsilon) \in C^0(\gamma, -\gamma)$. Therefore, the integral (3.26) converges uniformly for $|\mu| < \mu_0$, $0 \le \varepsilon < \varepsilon_0$. The assertion (i) then follows from Lemma 3.6.

Now, we prove assertion (ii). Let $\varepsilon = 0$ in (3.26), by (3.18) and (3.19) we get

(3.27)
$$\begin{aligned} B(\mu, 0) \;&=\; \int_{-\infty}^{+\infty} \psi^*(f(q(t) + \chi(t)(q(\mu, 0, t) - q(t)) + z(\mu, 0, t), \mu) \\ &\quad -A(t)z(\mu, 0, t) - f(q(t), 0) - \frac{d}{dt}(\chi(t)(q(\mu, 0, t) - q(t)))) dt. \end{aligned}$$

Differentiating (3.27) with respect to μ at $\mu = 0$, we obtain

$$\begin{aligned} B_\mu(0,0) &= \int_{-\infty}^{+\infty} \psi^* D_\mu f(q(t),0)dt \\ &\quad + \int_{-\infty}^{+\infty} \psi^*(D_x f(q(t),0)\chi(t)D_\mu q(0,0,t) - \frac{d}{dt}(\chi(t)D_\mu q(0,0,t))dt \\ &\quad + \int_{-\infty}^{+\infty} \psi^*[D_x f(q(t),0)D_\mu z(0,0,t) - A(t)D_\mu z(0,0,t)]dt \end{aligned} \tag{3.28}$$

Obviously, by the definition of $A(t)$, the third term in (3.28) vanishes. And the second term of (3.28) equals

$$\begin{aligned} &\int_{-\infty}^{+\infty} \psi^*(A(t)\chi(t)D_\mu q(0,0,t) - \frac{d}{dt}(\chi(t)D_\mu q(0,0,t)))dt \\ &= -\int_{-\infty}^{+\infty} \psi^* \mathcal{L}(\chi(t)D_\mu q(0,0,t)dt \\ &= - < \psi, \mathcal{L}(\chi(t)D_\mu q(0,0,t)) > . \end{aligned} \tag{3.29}$$

From Lemma 3.3 we have $\chi(t)D_\mu q(0,0,t) \in C^1(-\gamma,-\gamma)$ and the term in (3.29) vanishes due to Lemma 3.5 (ii). Thus our assertion (ii) follows from assumption (3.13).

Finally, from (3.23) and Lemma 3.6 (iii) we obtain

$$|z(0,\varepsilon,\cdot)|_{C^1(\gamma,-\gamma)} = O(\varepsilon^2) \tag{3.30}$$

and combining this with Lemma 3.6 (ii), (iii) yields

$$\begin{aligned} |Q(z(0,\varepsilon,\cdot),0,\varepsilon)|_{C^0(\gamma,-\gamma)} &\le |Q(z(0,\varepsilon,\cdot),0,\varepsilon) - Q(0,0,\varepsilon)|_{C^0(\gamma,-\gamma)} + O(\varepsilon^2) \\ &\le c|z(0,\varepsilon,\cdot)|_{C^0(\gamma,-\gamma)} + O(\varepsilon^2) = O(\varepsilon^2). \end{aligned}$$

This proves our assertion (iii). ■

Corollary 3.8. *Assume that the assumptions* (1) – (7) *hold. Then there exist constants* $\varepsilon_0 > 0$, $c > 0$ *such that for* $0 \le \varepsilon < \varepsilon_0$, *equation* (3.26) *has a unique solution* $\mu = \mu_*(\varepsilon)$ *in some neighborhood of* 0. *The function* $\mu_*(\varepsilon)$ *is continuous in* ε *and satisfies* $|\mu_*(\varepsilon)| \le c\varepsilon^2$.

The proof follows from Lemma 3.7 and the implicit function theorem. Notice that the continuity of B and $D_\mu B$ is sufficient to get the result.

Proof of Theorem 2.1 Let $z(\mu,\varepsilon,t)$ be the solution of equation (3.23) and let ε_0, $\mu_*(\varepsilon)$ be defined as in Corollary 3.8.

Substituting $z(\mu_*(\varepsilon),\varepsilon,t)$ and $\mu_*(\varepsilon)$ into (3.14) we obtain a function $x_*(\varepsilon,t)$ which is a saddle node homoclinic orbit of equation (3.1) for $0 < \varepsilon < \varepsilon_0$. Define

$$\Gamma_\varepsilon^n = x_*(\varepsilon,n\varepsilon) = q(n\varepsilon) + \chi(n\varepsilon)(q(\mu_*(\varepsilon),\varepsilon,n\varepsilon) - q(n\varepsilon)) + z(\mu_*(\varepsilon),\varepsilon,n\varepsilon). \tag{3.31}$$

Because of Lemma 3.1 (iv), this is a discrete saddle node homoclinic orbit of the scheme (2.3).

Moreover, from (3.30) and Lemma 3.3 (ii) we find the uniform convergence

$$\sup_{n\in\mathbb{Z}} |\Gamma_\varepsilon^n - q(n\varepsilon)| \to 0 \text{ as } \varepsilon \to 0$$

which completes the proof. ■

Remark 3.9. The whole construction started with fixing the phase of the given homoclinic by (3.2) such that the coordinate θ_0 of $q(0)$ is in the center manifold interval $(0, \delta)$. This can of course be done for any phase $\tau \le 0$ and $q(\tau) = (\theta_\tau, \tilde{h}(\tau, \theta_\tau))$. Repeating our proof we then get a curve of saddle node homoclinics, which in principle may be independent of τ, i.e. consist of just one orbit. However, as our computations in the next section show we in fact find a closed curve of saddle node homoclinics.

4. A NUMERICAL EXPERIMENT

In this section we study a numerical example to illustrate the results of the previous sections.

Consider a general discrete dynamical system

$$x_{n+1} = \Phi(x_n, \lambda_1, \lambda_2, \varepsilon), \quad n \in \mathbb{Z} \tag{4.1}$$

where $\Phi : \mathbb{R}^m \times \mathbb{R}^3 \to \mathbb{R}^m$ is sufficiently smooth. In our application Φ will be a one-step map of step size ε which is defined implicitly. Therefore, in our implementation we treat the more general implicit equation

$$F(x_{n+1}, x_n, \lambda_1, \lambda_2, \varepsilon) = 0, \tag{4.2}$$

where $F : \mathbb{R}^{2m} \times \mathbb{R}^3 \to \mathbb{R}^m$.

Assume that there are functions $\alpha(\lambda_2, \varepsilon) \in \mathbb{R}$ and $\xi_i = \xi_i(\lambda_2, \varepsilon) \in \mathbb{R}^m, i = 1, 2$ such that for all (λ_2, ε) the ξ_i are saddle-node fixed points of equation (4.1) or (4.2) at $\lambda_1 = \alpha(\lambda_2, \varepsilon)$, i.e. the following equation hold,

$$\xi_i = \Phi(\xi_i, \alpha(\lambda_2, \varepsilon), \lambda_2, \varepsilon)$$

or

$$F(\xi_i, \xi_i, \alpha(\lambda_2, \varepsilon), \lambda_2, \varepsilon) = 0.$$

Moreover, the matrices $D_x\Phi(\xi_i, \lambda_1, \lambda_2, \varepsilon)$ and $(D_1F + D_2F)(\xi_i, \xi_i, \lambda_1, \lambda_2, \varepsilon)$ for $\lambda_1 = \alpha(\lambda_2, \varepsilon)$ have a simple eigenvalue 1 and 0, respectively, with eigenspace spanned by some unit vector φ. In this situation, the parameter λ_2 plays the role of μ from the previous sections.

If the function $\alpha(\lambda_2, \varepsilon)$ is not known explicitly, we may use the following defining systems to compute it

$$F(\xi_1, \xi_1, \lambda_1, \lambda_2, \varepsilon) = 0, \tag{4.3}$$

$$(D_1F + D_2F)(\xi_1, \xi_1, \lambda_1, \lambda_2, \varepsilon)\varphi_1 = 0, \tag{4.4}$$

$$< \varphi_1, \varphi_1 > -1 = 0, \tag{4.5}$$

$$F(\xi_2, \xi_2, \lambda_1, \lambda_2, \varepsilon) = 0, \tag{4.6}$$

$$(D_1F + D_2F)(\xi_2, \xi_2, \lambda_1, \lambda_2, \varepsilon)\varphi_2 = 0, \tag{4.7}$$

$$< \varphi_2, \varphi_2 > -1 = 0. \tag{4.8}$$

Assume that for some $(\lambda_2^0, \varepsilon^0)$ and with $\lambda_1^0 = \alpha(\lambda_2^0, \varepsilon^0)$ there exists a heteroclinic orbit connecting $\xi_i^0 = \xi_i(\lambda_2^0, \varepsilon^0)$ via the center manifold of ξ_1^0 to the stable manifold of ξ_2. In case $\xi_1 = \xi_2$, we have a saddle node homoclinic orbit. We use the following nonlinear system to approximate the saddle node heteroclinic orbit on a finite set $\{n_-, \ldots, n_+\}$

$$b(x_{n_-}, x_{n_+}, \xi_1, \xi_2, \lambda_1, \lambda_2, \varepsilon) = 0, \tag{4.9}$$

$$F(x_{n+1}, x_n, \lambda_1, \lambda_2, \varepsilon) = 0, \ n = n_-, \cdots, n_+ - 1. \tag{4.10}$$

If we do not know the saddle node function α, we can try to solve equations (4.3) – (4.10) simultaneously.

The function b represents the projection boundary condition and may be chosen as

$$b(x_{n_-}, x_{n_+}, \xi_1, \xi_2, \lambda_1, \lambda_2, \varepsilon) = (b_{cu}(x_{n_-} - \xi_1), b_s(x_{n_+} - \xi_2)). \tag{4.11}$$

where the zero sets of $b_{cu}(x_{n_-} - \xi_1) = 0$ and $b_s(x_{n_+} - \xi_2) = 0$ approximate the center-unstable manifold of ξ_1 and the stable manifold of ξ_2, respectively. For example, we take $b_{cu} \in \mathbb{R}^{m_s, m}$ and $b_s \in \mathbb{R}^{m_{cu}, m}$ such that the rows form a basis of the stable resp. center unstable subspace of $((D_1F)^{-1}D_2F)^T$ at $(\xi_1, \xi_1, \lambda_1, \lambda_2, \varepsilon)$ and $(\xi_2, \xi_2, \lambda_1, \lambda_2, \varepsilon)$ respectively. This is a straightforward generalization of the boundary conditions used for the hyperbolic case in [3].

As an example, we consider the DC Josephson Junction equations (cf [6], [11])

$$\begin{aligned} \dot{x_1} &= f_1(x_1, x_2, \lambda_1, \lambda_2) = x_2, \\ \dot{x_2} &= f_2(x_1, x_2, \lambda_1, \lambda_2) = \frac{1}{\lambda_2}(-x_2 - \sin x_1 + \lambda_1), \end{aligned} \tag{4.12}$$

where $x = (x_1, x_2)$. For this equation, the previous function $\alpha \equiv 1$ and the saddle nodes are $\xi_1 = (\frac{1}{2}\pi, 0)$ and $\xi_2 = (\frac{5}{2}\pi, 0)$ respectively. This system has a saddle node heteroclinic connection at $\lambda_1 = 1$, $\lambda_2 \approx 0.7028165$ (see [6], [11]).

It may be considered as a homoclinic orbit if the phase space for x_1 is taken to be S^1.

We discretize equation (4.12) using the centered Euler method which is determined by the following equation

$$F(x_{n+1}, x_n, \lambda_1, \lambda_2, \varepsilon) = x_{n+1} - x_n - \varepsilon f(\frac{x_{n+1} + x_n}{2}, \lambda_1, \lambda_2). \tag{4.13}$$

We fix the step size at $\varepsilon = 0.5$. At the parameter value $\lambda_2 = 0.68481934278$ with $n_+ - n_- = 450$, we find an approximate saddle node heteroclinic orbit which is shown in Fig. 4.1 together with the center manifold of ξ_1 and the stable manifold of ξ_2. In Fig. 4.2 we show the rather weak transversality of the manifolds $W^c(\xi_1)$ and $W^s(\xi_2)$ at some intermediate points.

The transversal saddle node heteroclinic orbit for maps will persist under small perturbation of λ_2 and this suggests to do the continuation with respect to this parameter. In Fig 4.3 we present the continuation results in a parameter-amplitude plane where the amplitude of the heteroclinic orbit is defined as

$$\max_{n_- \le i \le n_+} \min_{j=1,2} \operatorname{dist}(x_i, \xi_j)$$

We obtain a smooth curve of saddle node heteroclinic orbits in the complete space $(x_{n_-}, \ldots, x_{n_+}, \lambda_2)$. The two corners in Fig. 4.3 are caused by the non-smoothness of the *max-min* functions which we used for the amplitude.

For each value of λ_2 in a small parameter range we have two distinct saddle node heteroclinic orbits and at the two turning points in Fig. 4.3 a saddle node heteroclinic tangency occurs. Moreover, after one complete turn with the continuation method we find the same heteroclinic orbit but shifted by an index 1. Therefore, if we plot $(x_n(s), \lambda_2(s)), n = n_-, \ldots, n_+$ ($s =$ arclength parameter) in a common x_1, x_2, λ_2 diagram we obtain a single curve shown in Fig. 4.4. Any horizontal section in this picture produces two heteroclinic orbits for this particular value of the parameter. The special orbit which we started with in Fig. 4.1 is marked with symbols in Fig. 4.4. The whole phenomenon is quite analogous to the behavior found in the hyperbolic case [3].

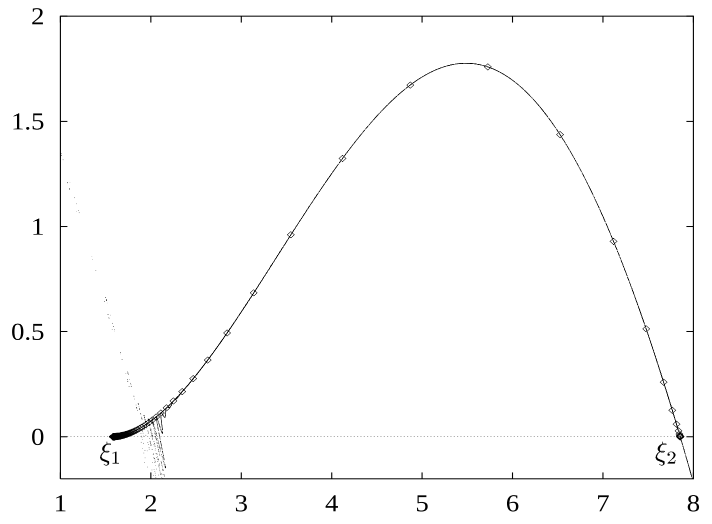

Fig 4.1. A saddle node heteroclinic orbit together with center and stable manifold.

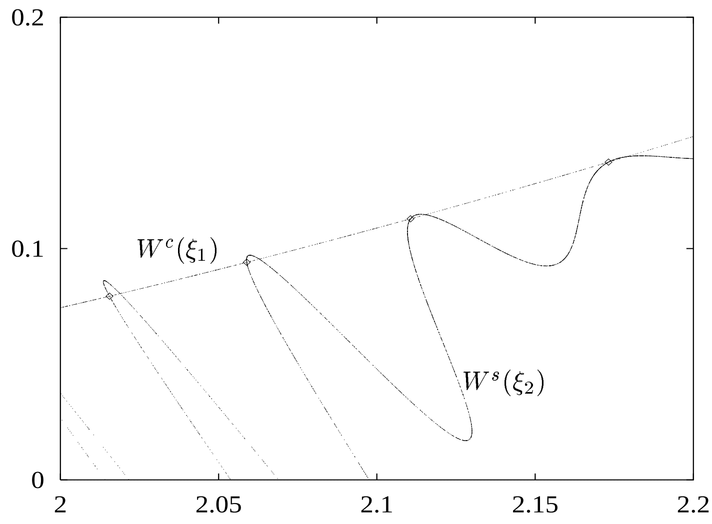

Fig 4.2. The transversal intersection of center and stable manifold at heteroclinic orbit.

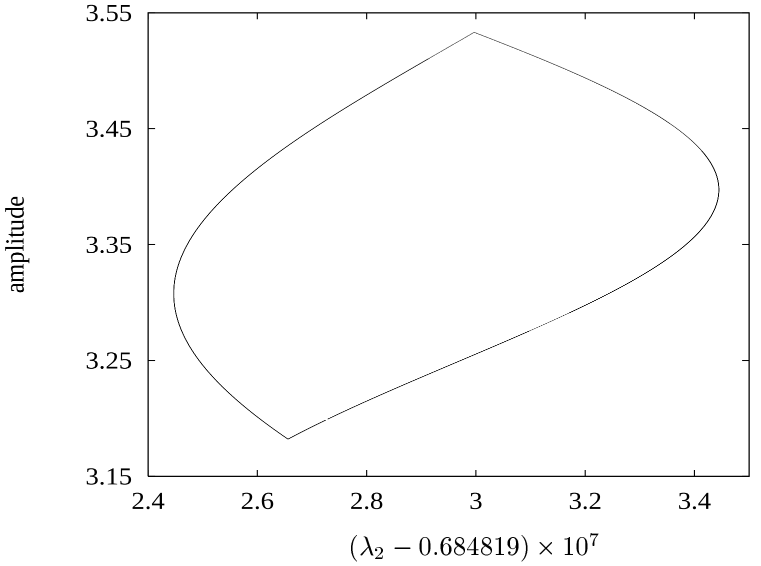

Fig 4.3. The continuation of saddle node heteroclinic orbits.

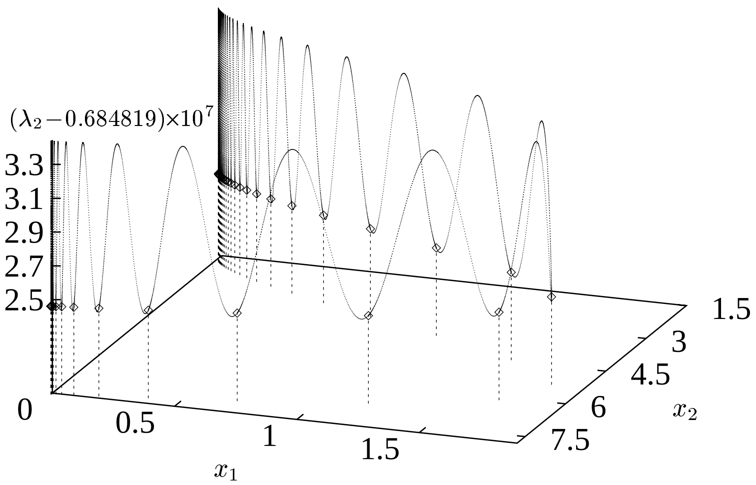

Fig 4.4. The loop of saddle node heteroclinic orbits in (x_1, x_2, λ_2)-plane.

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