Coupling of PDE's and perturbation by transition kernels on a balayage space

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1 Introduction

The main purpose of this paper is to show that coupling of second order linear partial differential equations (each yielding the structure of a harmonic space) can most easily be considered as coupling *within* a balayage space. And then no additional constructions (as e.g. in [CZ96]) are necessary, since the theory of balayage spaces as presented in [BH86] can be directly applied. In particular, this covers the solution of the Dirichlet problem for differential equations $L^n h = 0$, $n \in \mathbb{N}$ and L a linear (elliptic or parabolic) partial differential operator of second order.

Coupling of n PDE's as studied in [CZ96] is achieved by transitions between corresponding points in n copies of the underlying domain, i.e., by very special transitions on the direct sum of n domains. An additional advantage of our method is that it eventually allows us to deal with perturbations given by arbitrary transition kernels within a balayage space (which may or may not be a direct sum of several balayage spaces).

To illustrate our approach let us first discuss a very simple example: Consider two global Kato measures $\mu_1, \mu_2 \geq 0$ on a Green domain D in $\mathbb{R}^d$, $d \geq 1$, (i.e., we have a Green function G_D on D and $G_D^{\mu_j} = \int G_D(\cdot, y)\, \mu_j(dy)$ is a bounded continuous real function on D, $j = 1, 2$) and assume that $\|G_D^{\mu_1}\|_\infty \|G_D^{\mu_2}\|_\infty < 1$. Let U be a regular relatively compact open subset of D and fix continuous real functions φ_1, φ_2 on the boundary ∂U. Suppose we want to solve the coupled Dirichlet problem

$$\Delta h_1 = -h_2 \mu_1 \text{ on } U, \qquad h_1 = \varphi_1 \text{ on } \partial U, \tag{1.1}$$

$$\Delta h_2 = -h_1 \mu_2 \text{ on } U, \qquad h_2 = \varphi_2 \text{ on } \partial U. \tag{1.2}$$

Note that e.g. the biharmonic problem

$$\Delta(\Delta h) = 0 \text{ on } U, \quad h = \varphi_1 \text{ on } \partial U, \quad -\Delta h = \varphi_2 \text{ on } \partial U \tag{1.3}$$

is a special case (take $\mu_1 = \lambda^d$, $\mu_2 = 0$).

Let X be the topological sum of two copies X_1, X_2 of D, each equipped with the harmonic structure given by the Laplacian and let π denote the canonical mapping between these two copies (in section 5 we shall do this more formally). Let U_j be the set U in X_j, $j = 1, 2$. Taking μ on X, h on $\overline{U}_1 \cup \overline{U}_2$, φ on $\partial U_1 \cup \partial U_2$ such that

$$\mu|_{X_j} = \mu_j \ , \ h|_{\overline{U}_j} = h_j \ , \ \varphi|_{\partial U_j} = \varphi_j \qquad (j = 1, 2) \tag{1.4}$$

the equations (1.1) and (1.2) may be rewritten as a single equation

$$\Delta h = -(h \circ \pi)\mu \text{ on } U_1 \cup U_2 \ , \ h = \varphi \text{ on } \partial(U_1 \cup U_2). \tag{1.5}$$

For $j = 1, 2$, let G_{U_j} denote the Green function on U_j and define a kernel $K_{U_j}^\mu$ by

$$K_{U_j}^\mu \psi := G_{U_j}^{\psi\mu} = \int G_{U_j}(\cdot, z)\psi(z)\, d\mu(z).$$

Then $\Delta h = -(h \circ \pi)\mu$ if and only if

$$\Delta \Big(h - K_{U_j}^\mu (h \circ \pi) \Big) = 0 \quad \text{ on } U_j \ , \ j = 1, 2. \tag{1.6}$$

The idea is now the following: Given $j \in \{1, 2\}$ and a regular subset V of X_j, let H_V denote the harmonic kernel of V (i.e., H_V is a kernel on X such that, for every continuous function φ on X, the function $H_V \varphi$ is continuous on X, harmonic on V, and equal to φ on $X \setminus V$) and define a new kernel $\widetilde{H}_V$ on X by

$$\widetilde{H}_V \varphi = H_V \varphi + K_V^\mu (\varphi \circ \pi).$$

The family of all $\widetilde{H}_V$, V regular, $V \subset X_1$ or $V \subset X_2$, yields a balayage space $(X, \widetilde{\mathcal{W}})$ (this requires some proof, see Example 4.3) and then there are corresponding harmonic kernels $\widetilde{H}_U$ for *every* open subset U of X. In particular, $U_1 \cup U_2$ is regular with respect to $(X, \widetilde{\mathcal{W}})$ and then

$$h := \widetilde{H}_{U_1 \cup U_2} \varphi$$

is the solution of (1.5). Indeed, clearly $h = \varphi$ on $\partial(U_1 \cup U_2)$. And, for every $j \in \{1, 2\}$, we have $\widetilde{H}_{U_1 \cup U_2} = \widetilde{H}_{U_j} \widetilde{H}_{U_1 \cup U_2}$, hence

$$h = \widetilde{H}_{U_j} h = H_{U_j} h + K_{U_j}^\mu (h \circ \pi).$$

Since $H_{U_j} h$ is harmonic on U_j, this implies that $\Delta \Big(h - K_{U_j}^\mu (h \circ \pi) \Big) = 0$ on U_j, i.e., (1.6) holds.

2 Balayage spaces

The notion of a balayage space is more general than that of a $\mathcal{P}$–harmonic space as e.g. given by linear elliptic and parabolic partial differential equations of second order. In addition, it covers Riesz potentials as well as Markov chains on discrete spaces. There are various ways of describing a balayage space: By its cone $\mathcal{W}$ of positive hyperharmonic functions, by a family of harmonic kernels, by a corresponding semigroup, by an associated Hunt process (see [BH86, Theorem IV.8.1] or the survey article [Han87]). For our purpose the description using harmonic kernels is very appropriate.

We begin by introducing some notation: Let X be a locally compact space with countable base. For every open set U in X, let $\mathcal{B}(U)$ denote the set of all numerical Borel measurable functions on U. Further, $\mathcal{C}(U)$ will denote the space of all real continuous functions on U and $\mathcal{K}(U)$ ($\mathcal{C}_0(U)$ resp.) the set of all functions in $\mathcal{C}(U)$ having compact support (vanishing at infinity) with respect to U. Occasionally, functions on U will be identified with functions on X which are zero on U^c. Finally, given any set $\mathcal{A}$ of functions let $\mathcal{A}_b$ ($\mathcal{A}^+$ resp.) denote the set of all functions in $\mathcal{A}$ which are bounded (positive resp.)

Let $\mathcal{U}$ be a base of relatively compact open subsets of X and, for every $U \in \mathcal{U}$, let H_U be a kernel on X such that $H_U(x,\cdot) = \varepsilon_x$ for every $x \in U^c$ and $H_U 1_U = 0$. It will be convenient to assume that $\mathcal{U}$ is stable with respect to finite intersections (by [BH86, Remark VII.3.2.4] this is no restriction of generality). Define

$$\mathcal{W} := \{v \mid v : X \to [0,\infty] \text{ l.s.c., } H_U v \le v \text{ for every } U \in \mathcal{U}\} \tag{2.1}$$

and, for every numerical function $f \ge 0$ on X, let

$$R_f := \inf\{v \in \mathcal{W} \;:\; v \ge f\}.$$

A function $s \in \mathcal{C}^+(X)$ is called *strongly* ($\mathcal{W}$-)*superharmonic* if, for every $U \in \mathcal{U}$, $H_U s < s$ on U.

Then $(H_U)_{U\in\mathcal{U}}$ is a family of (regular) harmonic kernels and $(X,\mathcal{W})$ is a balayage space provided the following holds (where $U, V \in \mathcal{U}$):

(H_1) Given $x \in X$, $\lim_{U\downarrow\{x\}} H_U\varphi(x) = \varphi(x)$ for all $\varphi \in \mathcal{K}(X)$ or $R_{1_{\{x\}}}$ is l.s.c. at x.

(H_2') $H_V H_U = H_U$ if $V \subset U$.

(H_3) For every $f \in \mathcal{B}_b(X)$ with compact support, the function $H_U f$ is continuous on U.

(H_4') For every $\varphi \in \mathcal{K}(X)$, the function $H_U\varphi$ is continuous on $\overline{U}$.

(H_5') There exists a strongly superharmonic function $s \in \mathcal{C}^+(X)$.

Remarks 2.1. 1. It will be clear to the specialist how to proceed if we would not assume having a base of regular sets, i.e., if instead of (H_4') we would only suppose that the following property (H_4) holds: For every $x \in U$ there exists a l.s.c. function $w \ge 0$ on U such that $w(x) < \infty$, $H_V w \le w$ if $\overline{V} \subset U$, and $\lim_{\mathcal{F}} w = \infty$ for every non–regular ultrafilter $\mathcal{F}$ on U (see [BH86, p. 94]).

Moreover, properties (H_1) – (H_5') imply the following property (H_5): $\mathcal{W}$ is linearly separating (i.e., for $x, y \in X$, $x \ne y$, and $\lambda \in \mathbb{R}_+$ there exists $v \in \mathcal{W}$ such that $v(x) \ne \lambda v(y)$) and there exists a strictly positive function $s_0 \in \mathcal{W} \cap \mathcal{C}(X)$. Indeed, let $s \in \mathcal{C}^+(X)$ be strongly superharmonic. Then of course $s > 0$ and $s \in \mathcal{W}$. Furthermore, $H_U s \in \mathcal{W}$ for every $U \in \mathcal{U}$: Because of (H_4') the function $H_U s$ is l.s.c. Given $V \in \mathcal{U}$, we have to show that $H_V H_U s \leq H_U s$. Since $H_U s \leq s$ and $H_V s \leq s$, we obtain first that

$$H_V H_U s \leq H_V s \leq s = H_U s \quad \text{ on } U^c.$$

In addition, $H_V H_U s = H_U s$ on V^c. Since $(U \cap V)^c = U^c \cup V^c$, we conclude that

$$H_V H_U s = H_{U \cap V} H_V H_U s \leq H_{U \cap V} H_U s = H_U s.$$

It is now easily seen that $\mathcal{W}$ is linearly separating: Fix $x, y \in X$, $x \neq y$. Choose $U \in \mathcal{U}$ such that $x \in U$, $y \notin U$. For every $\lambda \in \mathbb{R}_+$, $s(x) \neq \lambda s(y)$ or $H_U s(x) \neq \lambda s(y) = \lambda H_U s(y)$.

We finally note that (H_5') holds for every balayage space by [BH86, pp. 17,118].

2. It will be useful to know that $\mathcal{W}$ as defined by (2.1) does not change if we replace $\mathcal{U}$ by a smaller base $\mathcal{U}'$ (see [BH86, Remark III.6.13]).

As for harmonic spaces continuous potentials play an important role. The convex cone $\mathcal{P}(X)$ of all continuous real potentials can be defined and characterized in several ways:

$$\begin{aligned}
\mathcal{P}(X) &= \{p \in \mathcal{W} \cap \mathcal{C}(X) : \inf_{K \text{ compact } \subset X} R_{1_{K^c} p} = 0\} \\
&= \{p \in \mathcal{W} \cap \mathcal{C}(X) : \frac{p}{q} \in \mathcal{C}_0(X) \text{ for some } q \in \mathcal{W} \cap \mathcal{C}(X)\} \\
&= \{p \in \mathcal{W} \cap \mathcal{C}(X) : 0 \leq g \leq p,\, g \in \mathcal{H}^+(X) \Longrightarrow g = 0\}
\end{aligned}$$

where $\mathcal{H}^+(X)$ denotes the set of all positive harmonic functions on X, i.e.,

$$\mathcal{H}^+(X) = \{g \in \mathcal{C}^+(X) : H_U g = g \text{ for every } U \in \mathcal{U}\}.$$

Moreover, we have a Riesz decomposition

$$\mathcal{W}(X) \cap \mathcal{C}(X) = \mathcal{H}^+(X) \oplus \mathcal{P}(X).$$

A function f on X is called *$\mathcal{P}$-bounded* if $|f| \leq p$ for some $p \in \mathcal{P}(X)$.

It is easily seen that we may restrict the balayage space $(X, \mathcal{W})$ on any open subset Y of X defining kernels

$$H_U^Y(x, \cdot) := H_U(x, \cdot)|_Y \qquad (x \in U \in \mathcal{U}\ ,\ \overline{U} \subset Y).$$

Note that the corresponding cone $\mathcal{W}_Y$ contains $\mathcal{W}|_Y$.

It is trivial that finite and countable direct sums of balayage spaces are balayage spaces as well:

Let $(X_i, \mathcal{W}_i)$, $i \in I \subset \mathbb{N}$, be balayage spaces. If $X = \sum_{i \in I} X_i$ denotes the topological sum of all X_i, $i \in I$, and

$$\mathcal{W} = \sum_{i \in I} \mathcal{W}_i = \{v \,\big|\, v : X \to [0, \infty],\, v|_{X_i} \in \mathcal{W}_i \text{ for every } i \in I\}$$

(we identify $v_i \in \mathcal{W}_i$ with a function on X taking $v_i = 0$ on $X \setminus X_i$), then $(X, \mathcal{W})$ is a balayage space. To see this it suffices to take $\mathcal{U} = \bigcup_{i \in I} \mathcal{U}_i$ ($\mathcal{U}_i$ being a base a regular sets for the balayage space $(X_i, \mathcal{W}_i)$) and to extend the harmonic kernels H_U, $U \in \mathcal{U}_i$, defining $H_U(x, \cdot) = \varepsilon_x$ for all $x \in X \setminus X_i$.

Let us note that of course, for every $i \in I$, the restriction of $(X, \mathcal{W})$ on X_i is $(X_i, \mathcal{W}_i)$.

In the following $(X,\mathcal{W})$ will always denote a balayage space associated with a family $(H_U)_{U\in\mathcal{U}}$ of regular harmonic kernels. Moreover, we fix a potential kernel K_X for $(X,\mathcal{W})$, i.e., K_X is a kernel such that

$$(2.2)\qquad K_X f\in\mathcal{P}(X)\cap\mathcal{H}(X\setminus\operatorname{supp}(f))\quad\text{for } f\in\mathcal{B}_b^+(X)\text{ with compact support.}$$

A general minimum principle implies that $v\ge Kf$ whenever $v\in\mathcal{W}$ and $f\in\mathcal{B}^+(X)$ such that $v\ge Kf$ on $\operatorname{supp}(f)$ (see [BH86, ...]).

Defining

$$K_U:=K_X-H_UK_X\qquad (U\in\mathcal{U})$$

we obtain a family $(K_U)_{U\in\mathcal{U}}$ of kernels such that

$$(2.3)\qquad K_U(\mathcal{B}_b(X))\subset\mathcal{C}_0(U)\quad\text{and}\quad K_U=K_V+H_VK_U$$

for all $U,V\in\mathcal{U}$ with $V\subset U$ (this is an immediate consequence of (H_2'), (H_3), and (H_4')).

Remarks 2.2. 1. If we have a Green function G_X for X, then $K_Xf=G_X^{f\mu}$ for some measure $\mu\ge 0$ on X and $K_Uf=G_U^{f\mu}$ where $G_U(\cdot,y)=G_X(\cdot,y)-H_UG_X(\cdot,y)$ for $y\in X, U\in\mathcal{U}$.

2. For every $p\in\mathcal{P}(X)$, there exists a unique potential kernel K_X^p such that $K_X^p1=p$ (see [BH86, p. 75]). It is called the *potential kernel associated with p*.

3. If K_X is a potential kernel and $\varphi\in\mathcal{B}^+(X)$ is locally bounded, then $f\mapsto K_X(\varphi f)$ obviously defines a potential kernel.

4. Conversely, for every potential kernel K_X, there exists $p\in\mathcal{P}(X)$ and a strictly positive function $\varphi\in\mathcal{C}^+(X)$ such that

$$K_Xf=K_X^p(\varphi f)\qquad\text{for every } f\in\mathcal{B}^+(X).$$

Indeed, fix a sequence (ψ_n) in $\mathcal{K}^+(X)$ such that $X=\bigcup_{n=1}^\infty\{\psi_n>0\}$. Since $p_n:=K_X\psi_n\in\mathcal{P}(X)$, we may choose reals $\alpha_n>0$, $n\in\mathbb{N}$, such that

$$\psi:=\sum_{n=1}^\infty\alpha_n\psi_n\in\mathcal{C}^+(X),\quad p:=\sum_{n=1}^\infty\alpha_np_n\in\mathcal{P}(X).$$

Obviously, $K_X\psi=p$ and hence

$$K_X^pf=K_X(\psi f)\qquad\text{for every } f\in\mathcal{B}^+(X).$$

So $\varphi:=1/\psi$ has the desired properties.

5. If K_X is a potential kernel on X, then every K_U, $U\in\mathcal{U}$, is a potential kernel on U (this follows easily from the definition of K_U). For the converse, i.e., for the construction of K_X from a compatible family of potential kernels $(K_U)_{U\in\mathcal{U}}$ see the Appendix.

Extending the notion used in [HH88] for harmonic spaces let us say that the balayage space $(X,\mathcal{W})$ is *parabolic*, if for every non-empty compact subset C of X there exists $x\in C$ such that $\liminf_{y\to x}R_{1_C}(y)=0$. For equivalent properties see Theorem 10.2.

3 First coupling within a balayage space

We fix a kernel T on X and assume that, for some sequence (W_n) of open sets increasing to X,

$$T1_{W_n} < \infty, \quad K_X(1_{W_n}T1_{W_n}) \in \mathcal{C}(X) \qquad (n \in \mathbb{N}). \tag{3.1}$$

Such a kernel T will be called an *admissible transition kernel.*

Remarks 3.1. *1. If the sets W_n are relatively compact and the functions $T1_{W_n}$ are bounded on W_n, then (3.1) is already a consequence of (2.2). So every kernel T on X such that $T\varphi$ is locally bounded for every $\varphi \in \mathcal{K}(X)$ is an admissible transition kernel.*

2. It is easily seen that (3.1) implies that

$$K_U(Tf) \in \mathcal{C}_0(U) \quad \text{for all } U \in \mathcal{U} \text{ and } f \in \mathcal{B}_b(X) \text{ with compact support.} \tag{3.2}$$

Indeed, choosing $n \in \mathbb{N}$ such that $\overline{U} \subset W_n$ and supp$(f) \subset W_n$, the lower semi-continuity of the functions $K_X(1_{W_n}Tf^\pm)$, $K_X(1_{W_n}T(\|f\|_\infty 1_{W_n} - f^\pm))$ and the continuity of the sum $\|f\|_\infty K_X(1_{W_n}T(1_{W_n}))$ implies that the functions $K_X(1_{W_n}Tf^\pm)$ are continuous. Thus by (2.3)

$$K_U(Tf) = K_X(Tf) - H_U K_X(Tf) = K_X(1_{W_n}Tf) - H_U K_X(1_{W_n}Tf) \in \mathcal{C}_0(U)$$

(the harmonicity of $K_X(1_{W_n^c}Tf)$ on W_n implies that $H_U K_X(1_{W_n^c}Tf) = K_X(1_{W_n^c}Tf)$).

3. Using lifting of potentials (see Remark 2.1.6) it can be shown that, conversely, (3.2) implies (3.1).

Let $\mathcal{U}^T$ be the set of all $U \in \mathcal{U}$ such that T is a transition from U to the complement of U, i.e.,

$$\boxed{\mathcal{U}^T = \{U \in \mathcal{U} : 1_U T 1_U = 0\}.}$$

In this section we shall assume that

$$\mathcal{U}^T \text{ is a base of } X \tag{3.3}$$

(in Section 9 we shall deal with the general case by approximation). Defining

$$K_U^T := K_U T, \quad H_U^T := H_U + K_U^T \qquad (U \in \mathcal{U}^T)$$

and

$$\mathcal{W}^T := \{v \mid v : X \to [0,\infty] \text{ l.s.c., } H_U^T v \le v \text{ for every } U \in \mathcal{U}^T\}$$

we then know already by Remark 2.1.2 that

$$\mathcal{W}^T \subset \mathcal{W}.$$

Let us check that most of the axioms of a family of harmonic kernels are satisfied by $(H_U^T)_{U \in \mathcal{U}^T}$ without any further assumption: Fix $U, V \in \mathcal{U}^T$, $V \subset U$. Then

$$K_V^T 1_U = K_V T 1_U = K_V(1_V T 1_U) = 0, \tag{3.4}$$

hence (taking $V = U$)

$$H_U^T 1_U = H_U 1_U = 0.$$

Let $f \in \mathcal{B}_b(X)$ with compact support. Then

$$H_U^T f = H_U f = f \quad \text{on } U^c \tag{3.5}$$

showing that $H_U^T(x,\cdot) = \varepsilon_x$ for every $x \in U^c$. Since $K_U^T f \in \mathcal{C}_0(U)$, we obtain by (H_3) that $H_U^T f$ is continuous on U. And if $f \in \mathcal{K}(X)$, then $H_U^T f \in \mathcal{K}(X)$ by (H_4'). Thus the family $(H_U^T)_{U\in\mathcal{U}^T}$ satisfies (H_3) and (H_4').

Moreover, by (3.4) and (3.5), $K_V^T H_U^T f = K_V^T(1_{U^c} H_U^T f) = K_V^T(1_{U^c} f) = K_V^T f$, i.e.,

$$K_V^T H_U^T = K_V^T. \tag{3.6}$$

Since $H_V H_U = H_U$ by (H_2), we obtain by (3.6) and (2.3) that

$$H_V^T H_U^T = H_V(H_U + K_U^T) + K_V^T H_U^T = H_V H_U + H_V K_U^T + K_V^T = H_U + K_U^T = H_U^T.$$

So $(H_U^T)_{U\in\mathcal{U}^T}$ satisfies (H_2) as well.

Given $x \in U$ and $\varphi \in \mathcal{K}^+(X)$, we obtain by (2.3) that $\lim_{V\downarrow\{x\}} K_V^T\varphi(x) = 0$, since $\lim_{V\downarrow\{x\}} H_V K_U(T\varphi)(x) = K_U(T\varphi)(x)$. Hence

$$\lim_{V\downarrow\{x\}} H_V^T\varphi(x) = \varphi(x) \quad \text{if} \quad \lim_{V\downarrow\{x\}} H_V\varphi(x) = \varphi(x).$$

Moreover, defining

$$r := R_{1_{\{x\}}}, \quad r^T := R^T_{1_{\{x\}}} = \inf\{v \in \mathcal{W}^T : v(x) \geq 1\}$$

we have $r^T \geq r$, since $\mathcal{W}^T \subset \mathcal{W}$. Hence $\liminf_{y\to x} r^T(y) \geq \liminf_{y\to x} r(y) = 1$, if r is l.s.c. at x. And then r^T is l.s.c. at x provided there exists $v \in \mathcal{W}^T$ with $v(x) < \infty$ (since then $v/v(x) \geq r^T$, $1 \geq r^T(x)$).

Thus we have the following result:

Theorem 3.2. *If $\mathcal{U}^T$ is a base of X, the following properties are equivalent:*

1. *$(X, \mathcal{W}^T)$ is a balayage space (i.e., $(H_U^T)_{U\in\mathcal{U}^T}$ is a family of harmonic kernels on X).*
2. *There exists a strongly $\mathcal{W}^T$-superharmonic function $s \in \mathcal{C}^+(X)$.*

Remark 3.3. *Let T' be a kernel on X such that $T' \leq T$, $\mathcal{U}^T$ is a base of X, and $(X, \mathcal{W}^T)$ is a balayage space. Then T' is admissible and every $\mathcal{W}^T$-strongly superharmonic function is obviously $\mathcal{W}^{T'}$-strongly superharmonic. So Theorem 3.2 implies that $(X, \mathcal{W}^{T'})$ is a balayage space as well.*

Corollary 3.4. *Suppose that $\mathcal{U}^T$ is a base of X and that there exist $s \in \mathcal{W}$ and $u \in \mathcal{B}^+(X)$ such that*

$$v := s + K_X u \in \mathcal{C}(X), \quad Tv \leq u$$

and, for every $U \in \mathcal{U}^T$,

$$\{H_U s < s\} \cup \{K_U(u - Tv) > 0\} = U.$$

Then $(X, \mathcal{W}^T)$ is a balayage space and v is strongly $\mathcal{W}^T$-superharmonic.

Remarks 3.5. *1. For a version not assuming that $\mathcal{U}^T$ is a base see Theorem 9.3.*

2. If $K_X = K_X^p$ for some strictly superharmonic $p \in \mathcal{P}$, then $TK_X u < u$ implies that taking $s = 0$ we have $K_U(u - Tv) > 0$ on $U \in \mathcal{U}$.

3. For some applications (see e.g. Corollary 4.9) it will be useful to keep in mind that, given any strictly positive locally bounded function $\varphi \in \mathcal{B}(X)$, we may replace the potential kernel K_X by the potential kernel $f \mapsto K_X(\varphi f)$ and the transition kernel T by the transition kernel $f \mapsto T(f)/\varphi$ without changing $(X, \mathcal{W}^T)$.

Proof of Corollary 3.4. It suffices to note that, for every $U \in \mathcal{U}^T$,

$$v - H_U^T v = v - H_U v - K_U(Tv) = s - H_U s + K_U(u - Tv) > 0 \quad \text{on } U.$$

$\square$

Corollary 3.6. *Suppose that $\mathcal{U}^T$ is a base of X, K_X is associated with $p \in \mathcal{P}(X)$, and that for some $s \in \mathcal{W} \cap \mathcal{C}(X)$ the function $v := p + s$ is strongly superharmonic and $Tv < 1$. Then $(X, \mathcal{W}^T)$ is a balayage space and v is strongly $\mathcal{W}^T$-superharmonic.*

Proof. Fix $U \in \mathcal{U}$ and $x \in U$. By assumption, $H_U v(x) < v(x)$. Suppose that $H_U s(x) = s(x)$. Then $H_U p(x) < p(x)$, i.e., $K_U 1(x) > 0$. Since $1 - Tv > 0$, this implies that $K_U(1 - Tv)(x) > 0$. So the statement follows from Corollary 3.4. $\square$

If $(X, \mathcal{W}^T)$ is a balayage space, then, for every $U \in \mathcal{U}^T$, H_U^T is the kernel solving the Dirichlet problem for U with respect to $(X, \mathcal{W}^T)$. We may, however, solve the Dirichlet problem with respect to $(X, \mathcal{W}^T)$ for any $U \in \mathcal{U}$ (if we wanted to we could even solve it for any open set U in X, see [BH86, VII.2]). This leads to the larger family $(H_U^T)_{U \in \mathcal{U}}$ where H_U^T for arbitrary $U \in \mathcal{U}$ can be characterized in the following way:

Proposition 3.7. *Suppose that $(X, \mathcal{W}^T)$ is a balayage space. Then, for every $U \in \mathcal{U}$, the harmonic kernel H_U^T for U with respect to $(X, \mathcal{W}^T)$ has the following property:*

For every $\varphi \in \mathcal{K}^+(X)$, the function $H_U^T \varphi$ is the unique function $h \in \mathcal{K}^+(X)$ such that

$$h - K_U^T h = H_U \varphi.$$

Beweis. 1. Fix $\varphi \in \mathcal{K}^+(X)$ and define $h := H_U^T \varphi$. Then $h \in \mathcal{K}^+(X)$ and hence $K_U^T h \in \mathcal{C}_0(U)$. So

$$g := h - K_U^T h \in \mathcal{K}(X), \qquad g = \varphi \quad \text{on } U^c.$$

For every $V \in \mathcal{U}^T$ with $\overline{V} \subset U$,

$$h = H_V^T h = H_V h + K_V^T h,$$

hence

$$g = h - K_V^T h - H_V K_U^T h = H_V(h - K_U^T h)$$

is harmonic on V. Thus g is harmonic on U, $g = H_U \varphi$.

2. Now let h be any function in $\mathcal{K}^+(X)$ such that

$$h - K_U^T h = H_U \varphi.$$

Then $h = \varphi$ on U^c and, for every $V \in \mathcal{U}^T$ with $\overline{V} \subset U$,

$$H_V^T h = H_V h + K_V^T h = H_V H_U \varphi + H_V K_U^T h + K_V^T h = H_U \varphi + K_U^T h = h.$$

Thus $h = H_U^T \varphi$. $\square$

Remark 3.8. *Assuming that $(X, \mathcal{W}^T)$ is a balayage space we may show in the same way that, for every $\varphi \in \mathcal{K}(X)$, $H_U^T\varphi$ is the unique function $h \in \mathcal{K}(X)$ such that $K_U^T|h| \in \mathcal{C}_0(U)$ and $h - K_U^T h = H_U\varphi$.*

Proposition 3.9. *Let v be a positive numerical function on X. Then $v \in \mathcal{W}^T$ if and only if there exists a function $w \in \mathcal{W}$ such that $v = K_X^T v + w$.*

Proof. Suppose first that $w \in \mathcal{W}$ and $v = K_X^T v + w$. Then v is l.s.c. Fix $U \in \mathcal{U}^T$ and $x \in U$. We have to show that $H_U^T v(x) \le v(x)$. To that end we may assume that $v(x) < \infty$ and hence $H_U K_X^T v(x) \le K_X^T v(x) \le v(x) < \infty$. Then

$$\begin{aligned} H_U^T v(x) &= H_U v(x) + K_U^T v(x) = H_U v(x) - H_U K_X^T v(x) + K_X^T v(x) \\ &= H_U w(x) + K_X^T v(x) \le w(x) + K_X^T v(x) = v(x). \end{aligned}$$

Thus $v \in \mathcal{W}^T$.

Suppose now conversely that $v \in \mathcal{W}^T$. Then $v \in \mathcal{W}$, so v is finely continuous. Let us choose an increasing sequence (W_n) of relatively compact open sets satisfying (3.1). Defining

$$\varphi_n := 1_{W_n} T(1_{W_n} \inf(v, n)) \qquad (n \in \mathbb{N})$$

we then have $K_X\varphi_n \in \mathcal{P}(X)$ for every $n \in \mathbb{N}$ and

$$K_X\varphi_n \uparrow K_X^T v, \qquad K_U\varphi_n \uparrow K_U^T v$$

for every $U \in \mathcal{U}^T$. Define

$$w_n := v - K_X\varphi_n \qquad (n \in \mathbb{N}).$$

For every $U \in \mathcal{U}^T$,

$$H_U w_n + K_X\varphi_n = H_U v + K_U\varphi_n \le H_U v + K_U^T v = H_U^T v \le v,$$

i.e., $H_U w_n \le w_n$. Since w_n is l.s.c. and $w_n \ge -K_X\varphi_n$, we therefore obtain that $w_n \in \mathcal{W}$. The sequence (w_n) is decreasing and the function w defined by

$$w(x) = \text{f-}\liminf_{y \to x} \inf_n w_n(y), \quad x \in X,$$

is contained in $\mathcal{W}$. Since the functions v and $K_X^T v$ are finely continuous and obviously

$$v = K_X^T v + \inf_n w_n,$$

we finally obtain that $v = K_X^T v + w$. □

4 First applications on direct sums

In this section we shall first consider general transitions between spaces forming a direct sum and then study the important case of direct sums with the same underlying topological space Y and transition between corresponding points in the copies of Y.

Let $I = \{1, 2, \dots, n\}$, $n \in \mathbb{N}$, or $I = \mathbb{N}$ and let $(X, \mathcal{W})$ be the direct sum of balayage spaces $(X_i, \mathcal{W}_i)$, $i \in I \subset \mathbb{N}$ (see Section 2). Let T be an admissible kernel on X satisfying

(4.1) $$T(x, X_i) = 0 \quad \text{for every } i \in I \text{ and } x \in X_i$$

and let K_X be the potential kernel associated with a potential $p \in \mathcal{P}(X)$. Then $\mathcal{U}^T = \mathcal{U} = \bigcup_{i \in I} \mathcal{U}_i$ and we know by Theorem 3.2 that $(X, \mathcal{W}^T)$ is a balayage space provided there exists a $\mathcal{W}^T$-strongly superharmonic function $s \in \mathcal{C}^+(X)$. This may by guaranteed by the existence of a function u on the index set I which is strongly superharmonic with respect to a suitably chosen kernel P.

Let $p_0 \in \mathcal{P}(X)$ such that $\tilde{p} := p + p_0$ is strongly superharmonic and define kernels P and $\tilde{P}$ on I by

$$P(i, \{j\}) := \|1_{X_i} T(1_{X_j} p)\|_\infty = \sup_{x \in X_i} \int_{X_j} p(z)\, T(x, dz), \quad \tilde{P}(i, \{j\}) := \|1_{X_i} T(1_{X_j} \tilde{p})\|_\infty$$

for $i, j \in I$ where of course, $P(i, \{i\}) = \tilde{P}(i, \{i\}) = 0$ by (4.1). Then Theorem 3.2 leads to the following result:

Theorem 4.1. *If there exists a positive real function u on I such that $\tilde{P}u < u$, then $(X, \mathcal{W}^T)$ is a balayage space.*

Remark 4.2. *It is sufficient to know that $Pu < u$ if*

a) p is strongly superharmonic

or

b) I is finite and there exists $w \in \mathcal{W}_b$ such that $w > 0$ and Tw is bounded.

Indeed, in the first case we may take $p_0 = 0$ so that $\tilde{P} = P$. In the second case, there exists $\varepsilon > 0$ such that $Pu + \varepsilon n \|Tw\|_\infty \|u\|_\infty < u$ (n being the number of elements in I) and we may choose a strongly $\mathcal{W}$-superharmonic function $p_0 \in \mathcal{P}(X)$ with $p_0 \le w$. Then $\tilde{p} = p + p_0$ is strongly superharmonic and $\tilde{P}u \le Pu + \varepsilon n \|Tw\|_\infty \|u\|_\infty < u$.

Proof of Theorem 4.1. We define a function $q \in \mathcal{P}(X)$ by

$$q = \sum_{j \in I} 1_{X_j} u(j) \tilde{p}.$$

Fix $i \in I$ and $U \in \mathcal{U}_i$. By definition of $\tilde{P}$, $T(1_{X_j} u(j) \tilde{p}) \le \tilde{P}(i, \{j\}) u(j)$ on U. Moreover, $\tilde{P}u(i) < u(i)$ and $H_U \tilde{p} < \tilde{p}$ on U. Therefore

$$\begin{aligned} K_U^T q &= \sum_{j \in I} K_U(T(1_{X_j} u(j) \tilde{p})) \le \tilde{P}u(i) K_U 1 = \tilde{P}u(i)(p - H_U p) \\ &\le \tilde{P}u(i)(\tilde{p} - H_U \tilde{p}) < u(i)(\tilde{p} - H_U \tilde{p}) = q - H_U q \qquad \text{on } U. \end{aligned}$$

So q is strongly $\mathcal{W}^T$-superharmonic and the proof is finished by an application of Theorem 3.2. □

Example 4.3. Let us consider the example given in the introduction. There we have $I = \{1, 2\}$ and $T(x, \cdot) = \varepsilon_{\pi(x)}$, hence $P(i, \{j\}) = \delta_{ij} \|G_D^{\mu_j}\|_\infty$ so that by assumption $P(1, \{2\}) P(2, \{1\}) < 1$. If $P(1, \{2\}) > 0$, then $Pu < u$ if we take $u(1) = 1$ and $P(2, \{1\}) < u(2) < P(1, \{2\})^{-1}$. Similarly, if $P(2, \{1\}) > 0$. The case $P(1, \{2\}) = P(2, \{1\}) = 0$ (which is of no interest, since we have no coupling at all) can be dealt with taking $u = 1$. Thus $(X, \mathcal{W}^T)$ is a balayage space by Theorem 4.1 and Remark 4.2.

Corollary 4.4. *Suppose that $I = \{1, \ldots, n\}$ and that $T(x, X_j) = 0$ for all $x \in X_i$ and $1 \le j \le i \le n$. Moreover, assume that $p > 0$ and Tp is bounded. Then $(X, \mathcal{W}^T)$ is a balayage space.*

Proof. In view of Theorem 4.1 and Remark 4.2 it suffices to note that we may easily find a positive real function u on I satisfying $Pu < u$: Having $P(i, \{j\}) = 0$ for $1 \leq j \leq i$ and $P(i, \{j\}) < \infty$ for $1 \leq i < j \leq n$ we may take $u(n) = 1$ and choose $u(i) > \sum_{j=i+1}^{n} P(i, \{j\})u(j)$ recursively for $i = n-1, n-2, \ldots, 1$. $\square$

Remark 4.5. *Using the results of [Bou84] it can easily be seen that (strong) biharmonic spaces as introduced by [Smy75, Smy76] (or, more generally, polyharmonic spaces) are a special case. They are balayage spaces if interpreted in the right way.*

Let us now suppose that all X_i, $i \in I$, are copies of a space Y and that we have transitions only between corresponding points in these copies: Let $\mathcal{W}_i$, $i \in I$, be convex cones of l.s.c. positive numerical functions on Y such that every $(Y, \mathcal{W}_i)$ is a balayage space. For every $i \in I$, let p_i be a strongly superharmonic continuous real potential for $(Y, \mathcal{W}_i)$ and let $K_{\mathcal{W}_i}^{p_i}$ denote the corresponding potential kernel. The potentials p_i define a strongly superharmonic continuous real potential p for the direct sum $(X, \mathcal{W})$ and the restriction of K_X^p on the copy of Y corresponding to $(Y, \mathcal{W}_i)$ is the kernel $K_{\mathcal{W}_i}^{p_i}$. Let $g_{ij} \in \mathcal{B}^+(Y)$ describe the transition from points in the i-th copy of Y to the j-th copy of Y, i.e., identifying the i-th copy of Y with $Y \times \{i\}$ we have

$$T((y,i), \cdot) = \sum_{j \in I} g_{ij}(y) \varepsilon_{(y,j)} \qquad (y \in Y, i \in I)$$

where of course $g_{ii} = 0$ by (4.1). We assume that the functions $K_{\mathcal{W}_i}^{p_i}(1_C g_{ij})$ are continuous and real for every compact subset C of Y so that T is admissible.

Then Corollary 3.4 provides the following results (for the case $g_{ii} \neq 0$ see the end of Section 8):

Theorem 4.6. *If there exist functions $u_i \in \mathcal{B}^+(Y)$ such that $K_{\mathcal{W}_i}^{p_i} u_i \in \mathcal{C}(Y)$ and*

$$\sum_{j \in I} g_{ij} K_{\mathcal{W}_j}^{p_j} u_j < u_i$$

for every $i \in I$, then $(X, \mathcal{W}^T)$ is a balayage space.

Corollary 4.7. *Assume that $\mathcal{W}_i = \mathcal{W}_1$ and $p_i = p_1$ for every $i \in I$. Then $(X, \mathcal{W}^T)$ is a balayage space if there exists a strictly positive function $u \in \mathcal{B}^+(Y)$ and strictly positive reals b_i such that $K_{\mathcal{W}_1}^{p_1} u \in \mathcal{C}(Y)$ and, for all $i \in I$,*

$$\sum_{j \in I} g_{ij} b_j < b_i u / K_{\mathcal{W}_1}^{p_1} u. \tag{4.2}$$

Remark 4.8. *Suppose that $I = \{1, \ldots, n\}$, $a_{ij} := \|g_{ij}\|_\infty < \infty$ for all i, j and denote $A := (a_{ij})$. Assume that $u \in \mathcal{B}^+(Y)$ and $\alpha > 0$ such that*

$$\alpha K_{\mathcal{W}_1}^{p_1} u \leq u.$$

Then (4.2) is satisfied if there exists $b \in \mathbb{R}^n$, $b > 0$, such that

$$Ab < \alpha b$$

which in turn holds if and only if the spectral radius of A is strictly less than α.

Corollary 4.9. *Assume that $\mathcal{W}_i = \mathcal{W}_1$ and $p_i = p_1$ for all $i \in I$ and that there exists a strictly positive bounded function in $\mathcal{W}_1$. Then $(X, \mathcal{W}^T)$ is a balayage space if $(Y, \mathcal{W}_1)$ is parabolic and the function $K_{\mathcal{W}_1}^{p_1}(\max_{i\in I} \sum_{j\in I} g_{ij})$ is continuous and bounded.*

Proof. We choose $\varphi_0 \in \mathcal{C}_b(Y)$ such that $\varphi_0 > 0$ and $K_{\mathcal{W}_1}^{p_1}\varphi_0 \in \mathcal{C}_b(Y)$, and define

$$\varphi := \varphi_0 + \max_{i\in I} \sum_{j\in I} g_{ij}, \qquad \tilde{g}_{ij} := g_{ij}/\varphi \quad (i, j \in I)$$

so that $\sum_{j\in I} \tilde{g}_{ij} \le 1$ for every $i \in I$. Moreover, let

$$\tilde{T}((y,i),\cdot) := \sum_{j\in I} \tilde{g}_{ij}\varepsilon_{(y,j)}, \qquad \tilde{K}_1 f := K_{\mathcal{W}_1}^{p_1}(\varphi f) \quad (f \in \mathcal{B}^+(Y)).$$

Then $\tilde{K}_1$ is a potential kernel on $(Y, \mathcal{W}_1)$ such that $\tilde{K}_1 1 \in \mathcal{C}_b(Y)$. For the corresponding kernel $\tilde{K}_X$ on X we obviously have $K_X T = \tilde{K}_X \tilde{T}$. Thus $(X, \mathcal{W}^T)$ is not changed if we replace K_X by $\tilde{K}_X$ and T by $\tilde{T}$.

Our assumption on $\mathcal{W}_1$ implies that there exists a strictly positive bounded function $s \in \mathcal{W}_1$ which is continuous. By Theorem 10.2 and Lemma 10.3, $I - \tilde{K}_1$ is invertible and

$$u := (I - \tilde{K}_1)^{-1}s \in \mathcal{B}_b^+(X).$$

Then $u = \tilde{K}_1 u + s \in \mathcal{C}_b(X)$ and, for all $y \in Y$ and $i \in I$,

$$\sum_{j\in I} \tilde{g}_{ij}(y)\tilde{K}_1 u(y) \le \tilde{K}_1 u(y) = u(y) - s(y) < u(y),$$

By Theorem 4.6 we conclude that $(X, \mathcal{W}^T)$ is a balayage space. □

Proposition 3.7 can be expressed as follows:

Proposition 4.10. *Let $I = \{1, \dots, n\}$. Suppose that $(X, \mathcal{W}^T)$ is a balayage space and that U is a relatively compact open subset of Y which is $\mathcal{W}_i$-regular for every $1 \le i \le n$.*

Then, for any choice of functions $\varphi_1, \dots, \varphi_n \in \mathcal{K}(Y)$, there exist unique functions $h_1, \dots, h_n \in \mathcal{K}(Y)$ such that, for every $1 \le i \le n$,

$$h_i - \sum_{j\in I} K_{\mathcal{W}_j}^{p_j}(g_{ij}h_j) \text{ is } \mathcal{W}_i\text{-harmonic on } U, \qquad h_i = \varphi_i \quad \text{on } U^c.$$

Moreover, the functions $h_1, \dots, h_n$ are positive, if the functions $\varphi_1, \dots, \varphi_n$ are positive.

5 Coupling of partial differential equations

Let D be a domain in $\mathbb{R}^d$, $d \ge 1$, let $n \in \mathbb{N}$, and let L_i, $1 \le i \le n$, be second order (elliptic or parabolic) linear partial differential operators on D leading to harmonic spaces $(D, \mathcal{H}_{L_i})$. (For the definition of harmonic spaces and various sufficient conditions for the differential operators the reader might consult [Her62, CC72, BH86, Kro88, Her68, Bon70]). Moreover, we assume that, for every $1 \le i \le n$, we have a base of L_i-regular sets for D, a Green function G_{L_i} for $(X, \mathcal{H}_{L_i})$, and a Radon measure $\mu_i \ge 0$ on D such that $G_{L_i}^{\mu_i} \in \mathcal{C}_b(D)$ and $(G_{L_i})_V^{\mu_i} > 0$ on V for every (L_i-regular) open subset V of D.

We want to study the coupled system

$$L_i h_i + \sum_{j\neq i} g_{ij} h_j \mu_i = 0 \qquad (1 \le i \le n)$$

where $g_{ij} \in \mathcal{B}^+(D)$ such that $G_{L_i}^{1_A g_{ij}\mu_i} \in \mathcal{C}(D)$ for every compact subset A of D (in Section 8 we shall consider more general systems $L_i h_i + \sum_{j=1}^n g_{ij} h_j \mu_i = 0$). This will be possible by introducing associated transitions on the direct sum of the spaces $(D, \mathcal{H}_{L_i})$. By now it should be intuitively clear how to do it. To get it done in a formally correct way we proceed as follows: For every $1 \le i \le n$, let

$$X_i := D \times \{i\}$$

and let π_i denote the canonical projection from X_i on D. Then the direct sum $(X, \mathcal{H})$ of the spaces $(X_i, \mathcal{H}_{L_i} \circ \pi_i)$, $1 \le i \le n$, is a harmonic space (with the subspace $X = D \times \{1, 2, \dots, n\}$ of $\mathbb{R}^d \times \mathbb{N}$). (If $\mathcal{W}_i$ denotes the convex cone of all positive hyperharmonic functions for $(X_i, \mathcal{H}_{L_i} \circ \pi_i)$ and $\mathcal{W}$ the convex cone of all positive hyperharmonic functions for $(X, \mathcal{H})$, then of course $(X, \mathcal{W})$ is the direct sum of $(X_1, \mathcal{W}_1), \dots, (X_n, \mathcal{W}_n)$.)

Defining $p : X \to \mathbb{R}$ by

$$p|_{X_i} = G_{L_i}^{\mu_i} \circ \pi_i, \qquad 1 \le i \le n,$$

we obtain a continuous real potential on X with a corresponding potential kernel K_X. Finally, we define an admissible transition kernel T on X by

$$T((x,i), \cdot) := \sum_{j \neq i} g_{ij}\, \varepsilon_{(x,j)} \qquad (x \in D, 1 \le i \le n).$$

Suppose for a moment that there exists a strongly $\mathcal{W}^T$-superharmonic function $s \in \mathcal{C}^+(X)$, i.e., that $(X, \mathcal{W}^T)$ is a balayage space. Fix a relatively compact subset U of D and functions $\varphi_1, \dots, \varphi_n \in \mathcal{K}(D)$. For simplicity suppose that U is L_i-regular for every $1 \le i \le n$ (again it will be clear for the specialist how to proceed if this does not hold). Then

$$\tilde{U} := \bigcup_{i=1}^n U \times \{i\}$$

is a regular subset of X. Defining

$$\varphi(x, i) := \varphi_i(x) \qquad (x \in D, 1 \le i \le n)$$

we obtain a function $\varphi \in \mathcal{K}(X)$. By Proposition 3.7, there is a unique function $h \in \mathcal{K}(X)$ such that

$$h - K_{\tilde{U}}^T h = H_{\tilde{U}} \varphi.$$

Of course, $h|_{\tilde{U}}$ depends only on $\varphi|_{\partial \tilde{U}}$, since $T(\tilde{U}) \subset \tilde{U}$ and $H_{\tilde{U}}\varphi$ depends only on $\varphi|_{\partial\tilde{U}}$. Define

$$h_i := h \circ \pi_i^{-1} \qquad (1 \le i \le n)$$

and fix $1 \le i \le n$. Clearly, $h_i \in \mathcal{K}(D)$ and $h_i = \varphi_i$ on $D \setminus U$, since $h = \varphi$ on $X \setminus \tilde{U}$. Furthermore, $L_i((H_{\tilde{U}}\varphi) \circ \pi_i^{-1}) = 0$ on U, since $H_{\tilde{U}}\varphi \in \mathcal{H}(\tilde{U})$ and hence $(H_{\tilde{U}}\varphi) \circ \pi_i^{-1} \in \mathcal{H}_{L_i}(U)$. And

$$(K_{\tilde{U}}^T h) \circ \pi_i^{-1} = K_{\tilde{U}}(Th) \circ \pi_i^{-1} = (G_{L_i})^{(Th) \circ \pi_i^{-1} \mu_i}$$

where, for every $x \in D$, by definition of T

$$(Th) \circ \pi_i^{-1}(x) = Th(x,i) = \sum_{j \neq i} g_{ij} h(x,j) = \sum_{j \neq i} g_{ij} h_j(x).$$

Thus

$$0 = L_i((H_{\tilde{U}}\varphi) \circ \pi_i^{-1}) = L_i[(h - K_{\tilde{U}}^T h) \circ \pi_i^{-1}] = L_i h_i + \sum_{j \neq i} g_{ij} h_j \mu_j$$

and we obtain the following consequence of Proposition 3.7 (see Section 4, Theorem 5.4, and Corollary 5.6 for conditions implying that $(X, \mathcal{W}^T)$ is a balayage space):

Theorem 5.1. *Suppose that $(X, \mathcal{W}^T)$ is a balayage space and let U be a relatively compact subset of D such that U is L_i-regular for every $1 \leq i \leq n$. Then, for every choice of functions $\varphi_1, \dots, \varphi_n \in \mathcal{C}(\partial U)$, there exist unique continuous functions $h_1, \dots, h_n$ on $\overline{U}$ such that, for every $1 \leq i \leq n$,*

$$L_i h_i + \sum_{j \neq i} h_j g_{ij} \mu_i = 0 \quad \text{on } U, \qquad h_i = \varphi_i \quad \text{on } \partial U.$$

Further, the functions $h_1, \dots, h_n$ are positive if the functions $\varphi_1, \dots, \varphi_n$ are positive.

From Corollary 4.4 we get the following:

Corollary 5.2. *Let U be a relatively compact subset of D such that U is L_i-regular for every $1 \leq i \leq n$. Then, for every choice of functions $\varphi_1, \dots, \varphi_n \in \mathcal{C}(\partial U)$, there exist unique continuous functions $h_1, \dots, h_n$ on $\overline{U}$ such that, for every $1 \leq i \leq n$,*

$$L_i h_i + \sum_{j=i+1}^{n} h_j g_{ij} \mu_i = 0 \quad \text{on } U, \qquad h_i = \varphi_i \quad \text{on } \partial U.$$

And the functions $h_i, \dots, h_n$ are positive if the functions $\varphi_1, \dots, \varphi_n$ are positive.

A very special case is the situation where all operators L_i are equal and $g_{ij}\mu_i = \delta_{i+1,j}\lambda$:

Corollary 5.3. *Let D be a bounded domain in $\mathbb{R}^d$, $d \geq 1$, and let L be a second order linear partial differential operator on D leading to a harmonic space $(D, \mathcal{H}_L)$ with Green function G_L such that G_L^λ is continuous and bounded. Let U be a relatively compact $(L-)$regular subset of D, $n \in \mathbb{N}$, and $\varphi_1, \dots, \varphi_n \in \mathcal{C}(\partial U)$. Then there exists a unique function $h \in \mathcal{C}(U)$ such that $Lh, L^2 h, \dots, L^{n-1}h \in \mathcal{C}(U)$,*

$$L^n h = 0 \quad \text{on } U, \quad \lim_{x \to z} (-L)^{i-1} h(x) = \varphi_i(z) \quad \text{for every } 1 \leq i \leq n \text{ and for all } z \in \partial U.$$

And $h, -Lh, L^2 \dots, (-L)^{n-1}h$ are positive, if $\varphi_1, \dots, \varphi_n$ are positive.

Moreover, Theorem 4.6 implies the following result involving μ_i-eigenfunctions for the operators L_i:

Theorem 5.4. *Suppose that there exist strictly positive $\mathcal{P}_{L_i}(D)$-bounded functions $u_i \in \mathcal{C}_b(D)$ and strictly positive real numbers α_i, β_{ij}, $i, j \in \{1, \dots, n\}$, such that*

$$L_i u_i + \alpha_i u_i \mu_i = 0,$$

and

$$u_j \leq \beta_{ij} u_i, \qquad \sum_{j \neq i} \beta_{ij} g_{ij} / \alpha_j < 1.$$

Then $(X, \mathcal{W}^T)$ is a balayage space.

Remark 5.5. *If there exists an L_i-superharmonic function $s_i \geq 1$ on D, then every function $u \in \mathcal{C}_0(D)$ is $\mathcal{P}_{L_i}(D)$-bounded.*

Proof of Theorem 5.4. For every $1 \leq i \leq n$,

$$\alpha_i G_{L_i}^{u_i\mu_i} = u_i,$$

since $u_i - \alpha_i G_{L_i}^{u_i\mu_i}$ is $\mathcal{P}_{L_i}(D)$-bounded and L_i-harmonic on D. Therefore

$$\sum_{j\neq i} g_{ij} G_{L_j}^{u_j\mu_j} = \sum_{j\neq i} g_{ij}\frac{u_j}{\alpha_j} \leq \sum_{j\neq i} g_{ij}\frac{\beta_{ij}}{\alpha_j} u_i < u_i$$

for every $1 \leq i \leq n$. Thus $(X, \mathcal{W}^T)$ is a balayage space by Theorem 4.6. $\square$

Corollary 5.6. *Suppose that $L_1 = \cdots = L_n =: L$. Then $(X, \mathcal{W}^T)$ is a balayage space if one of the following conditions is satisfied:*

1. *$\mu_1 = \cdots = \mu_n =: \mu$ and there exist $\alpha > 0$, a strictly positive $\mathcal{P}_L(D)$-bounded function $u \in \mathcal{C}_b(D)$, and strictly positive real numbers $b_1, \dots, b_n$ such that*

$$Lu + \alpha u\mu = 0 \qquad \text{and} \qquad \sum_{j\neq i} g_{ij} b_j < \alpha b_i \text{ for every } 1 \leq i \leq n.$$

2. *$(D, \mathcal{H}_L)$ is parabolic and the potentials $G_L^{g_{ij}\mu_i}$, $i, j \in \{1, \dots, n\}$, are continuous and bounded.*

Remark 5.7. *Note that the harmonic space associated with the heat equation or a similar parabolic equation is parabolic. Moreover, the last property clearly holds if the functions g_{ij} are bounded.*

Proof of Corollary 5.6. By Theorem 5.4, (1) implies that $(X, \mathcal{W}^T)$ is a balayage space (take $u_i = b_i u$).

So suppose that (2) holds. Since of course $g_{ij}\mu_i = \tilde{g}_{ij}(\mu_1 + \cdots + \mu_n)$ for some Borel function $0 \leq \tilde{g}_{ij} \leq g_{ij}$, we may assume without loss of generality that $\mu_1 = \cdots = \mu_n$. Thus Corollary 4.9 implies that $(X, \mathcal{W}^T)$ is a balayage space. $\square$

6 Perturbation of balayage spaces

In order to get further possibilities for transitions let us briefly discuss perturbation of $(X, \mathcal{W})$. To that end we fix a real function $k \in \mathcal{B}(X)$ such that, for every $U \in \mathcal{U}$,

$$K_U|k| \in \mathcal{C}_0(U).$$

Such a function will be called a *Kato function* (with respect to K_X). Let $M_{k^\pm}$ denote the multiplication operators

$$M_{k^\pm} : f \mapsto k^\pm f$$

so that $K_U M_{k^\pm}$ are the potential kernels associated with $K_U k^\pm$.

Lemma 6.1. *For every $U \in \mathcal{U}$, the mapping $I + K_U M_{k^+}$ is a bijection on $\mathcal{B}_b(U)$ and*

$$0 \leq (I + K_U M_{k^+})^{-1} s \leq s$$

for every $s \in \mathcal{S}_b^+(U)$. Moreover, for every $s \in \mathcal{S}_b^+(U)$, $(I + K_U M_{k^+})^{-1} s > 0$ on $\{s > 0\}$.

Proof. As for harmonic spaces (see [BHH87, p. 104], or [HM90, p. 558]). ☐

In particular, for every $U \in \mathcal{U}$, the operator

$$L_U := (I + K_U M_{k^+})^{-1} K_U M_{k^-}$$

defines a kernel. As for harmonic spaces we obtain (see [HM90]):

Lemma 6.2. *For every $U \in \mathcal{U}$, the following statements are equivalent:*

1. *The operator $I - L_U$ is invertible on $\mathcal{B}_b(U)$ and $(I-L_U)^{-1} f \geq 0$ for every $f \in \mathcal{B}_b^+(U)$.*
2. *$\sum_{n=1}^{\infty} L_U^n 1$ is bounded.*

*If (2) holds, then U is called k-*bounded *and*

$$(I + K_U M_k)^{-1} = \sum_{n=1}^{\infty} L_U^n (I + K_U M_{k^+})^{-1}.$$

Theorem 6.3. $((I + K_U M_{k^+})^{-1} H_U)_{U \in \mathcal{U}}$ *is a family of harmonic kernels on X.*

More generally:

Theorem 6.4. *Suppose that there exist $s \in \mathcal{W}$ and $u \in \mathcal{B}^+(X)$ such that*

$$v := s + K_X u \in \mathcal{C}(X), \qquad 0 \leq u + kv,$$

and, for every $U \in \mathcal{U}$, $\{H_U s < s\} \cup \{K_U(u + kv) > 0\} = U$. Then every $U \in \mathcal{U}$ is k-bounded and defining

$$\tilde{H}_U := (I + K_U M_k)^{-1} H_U \qquad (U \in \mathcal{U}) \tag{6.1}$$

and

$$\widetilde{\mathcal{W}} := \{v \mid v : X \to [0, \infty] \text{ l.s.c., } \tilde{H}_U v \leq v \text{ for every } U \in \mathcal{U}\} \tag{6.2}$$

the family $(\tilde{H}_U)_{U \in \mathcal{U}}$ is a family of harmonic kernels on X, the pair $(X, \widetilde{\mathcal{W}})$ is a balayage space, and v is strongly $\widetilde{\mathcal{W}}$-superharmonic.

Proof. Given $U \in \mathcal{U}$, our assumptions imply that

$$\begin{aligned} (I + K_U M_{k^+})(v - L_U v) &= v + K_U M_{k^+} v - K_U M_{k^-} v \\ &= v + K_U(kv) = s + H_U K_X u + K_U(u + kv) \end{aligned}$$

is a strictly positive function in $\mathcal{S}_b^+(U)$ and hence $v - L_U v > 0$ on U by Lemma 6.1. In particular, $v > 0$ on X. Moreover, $L_U v \in \mathcal{C}_0(U)$ and $\inf v(\overline{U}) > 0$. So the function

$$f := v - L_U v$$

satisfies $\inf f(\overline{U}) > 0$. Since by induction

$$v = \sum_{n=0}^{m-1} L_U^n f + L_U^m v$$

for every $m \in \mathbb{N}$, we know that $\sum_{n=0}^{\infty} L_U^n f \le v$. Thus U is k-bounded and we may define a kernel $\tilde{H}_U$ by

$$\tilde{H}_U := (I + K_U M_k)^{-1} H_U = \sum_{n=0}^{\infty} L_U^n (I + K_U M_{k^+})^{-1} H_U. \tag{6.3}$$

Since

$$(I + K_U M_k)(v - \tilde{H}_U v) = v + K_U(kv) - H_U v = (s - H_U s) + K_U(u + kv) =: t$$

is a strictly positive function in $\mathcal{S}_b^+(U)$, we obtain by (6.3) and by Lemma 6.1 that

$$v - \tilde{H}_U v = (I + K_U M_k)^{-1} t \ge (I + K_U M_{k^+})^{-1} t > 0.$$

In particular, $(\tilde{H}_U)_{U \in \mathcal{U}}$ satisfies (H_5').

Obviously, $\tilde{H}_U 1_U = 0$ and $\tilde{H}_U(x, \cdot) = \varepsilon_x$ for all $U \in \mathcal{U}$ and $x \in U^c$. If $f \in \mathcal{B}_b(X)$ with compact support, then $\tilde{H}_U f \in \mathcal{B}_b(X)$, hence $K_U(k\tilde{H}_U f) \in \mathcal{C}_0(U)$. So the equality

$$\tilde{H}_U f + K_U(k\tilde{H}_U f) = H_U f$$

immediately implies that $(\tilde{H}_U)_{U \in \mathcal{U}}$ satisfies (H_3) and (H_4'). Applied to functions in $\mathcal{K}(X)$ we have for all $U, V \in \mathcal{U}$ with $V \subset U$

$$\begin{aligned} (I + K_V M_k)\tilde{H}_U &= \tilde{H}_U + (K_U - H_V K_U) M_k \tilde{H}_U \\ &= H_U - H_V K_U M_k \tilde{H}_U = H_V(H_U - K_U M_k \tilde{H}_U) = H_V \tilde{H}_U, \end{aligned}$$

i.e.,

$$\tilde{H}_U = (I + K_V M_k)^{-1} H_V \tilde{H}_U = \tilde{H}_V \tilde{H}_U.$$

So $(\tilde{H}_U)_{U \in \mathcal{U}}$ satisfies (H_2').

To show that (H_1) holds let us fix $x \in X$ and assume first that $\lim_{U \downarrow \{x\}} H_U \varphi(x) = \varphi(x)$ for every $\varphi \in \mathcal{K}(X)$. Let W be a neighborhood of x. Then, for every $U \in \mathcal{U}$ with $\overline{U} \subset W$,

$$K_U(|k|\tilde{H}_U v) \le K_U(|k| v) \le \sup(v(W)) K_U |k|$$

and $\lim_{U \downarrow \{x\}} \|K_U |k|\|_\infty = 0$. So we conclude that, for every $\varphi \in \mathcal{K}(X)$,

$$\lim_{U \downarrow \{x\}} \tilde{H}_U \varphi(x) = \lim_{U \downarrow \{x\}} H_U \varphi(x) = \varphi(x).$$

By [BH86, Proposition III.2.7], it remains to consider the case where x is ($\mathcal{W}$-)finely isolated. Let

$$\tilde{r} = \inf\{w \in \widetilde{\mathcal{W}} : w(x) \ge 1\}.$$

By Choquet's lemma, there exist $w_n \in \widetilde{\mathcal{W}}$, such that $w_n(x) \ge 1$ for every $n \in \mathbb{N}$ and

$$\hat{\tilde{r}} = \widehat{\inf w_n}.$$

Of course we may assume without loss of generality that $w_{n+1} \le w_n \le v/v(x)$ for every $n \in \mathbb{N}$. Define

$$s_n := w_n + K_U(k^+ w_n) \qquad (n \in \mathbb{N}).$$

Then s_n is l.s.c. and, for every $V \in \mathcal{U}$ with $\overline{V} \subset U$,

$$\begin{aligned} H_V s_n &= \tilde{H}_V w_n + K_V(k\tilde{H}_V w_n) + H_V K_U(k^+ w_n) \\ &\le w_n + K_V(k^+ w_n) + H_V K_U(k^+ w_n) = s_n, \end{aligned}$$

i.e., $s_n \in {}^*\mathcal{H}^+(U)$. Defining $s := \inf s_n$, we hence know that $\hat{s}^{\mathrm{f}} = \hat{s}$ (see [BH86, p. 58]). Let $w = \inf w_n$. Then $s = w + K_U(k^+ w)$ and the continuity of $K_U(k^+ w)$ implies that

$$\hat{w}^{\mathrm{f}} + K_U(k^+ w) = \hat{s}^{\mathrm{f}} = \hat{s} = \hat{w} + K_U(k^+ w),$$

i.e., $\hat{w}^{\mathrm{f}} = \hat{w}$. Since x is finely isolated, we conclude that

$$\hat{\tilde{r}}(x) = \hat{w}(x) = \hat{w}^{\mathrm{f}}(x) = \text{f-}\liminf_{y \to x} w(y) = w(x) = 1 = \tilde{r}(x).$$

Thus $\tilde{r}$ is l.s.c. at x. This finishes the proof of Theorem 6.4. $\square$

Theorem 6.3 is a special case: If $k \geq 0$, then we may take $u = 0$ and any strongly superharmonic $s \in \mathcal{C}^+(X)$. But of course we may as well take the preceding proof and omit its first part noting that, by Lemma 6.1, the operators $(I + K_U M_k)^{-1} H_U$, $U \in \mathcal{U}$, yield kernels $\tilde{H}_U$ and that $\mathcal{W} \subset \tilde{\mathcal{W}}$ if $k \geq 0$.

Moreover we shall need the following:

Proposition 6.5. *If every $U \in \mathcal{U}$ is k-bounded and $(\tilde{H}_U)_{U \in \mathcal{U}}$ is a family of harmonic kernels on X, then there exists a (unique) potential kernel $\tilde{K}_X$ on X with respect to $\widetilde{\mathcal{W}}$ such that*

$$\tilde{K}_X - \tilde{H}_U \tilde{K}_X = (I + K_U M_k)^{-1} K_U \qquad \textit{for every } U \in \mathcal{U}.$$

Proof. Define

$$\tilde{K}_U = (I + K_U M_k)^{-1} K_U \qquad (U \in \mathcal{U}).$$

If $U, V \in \mathcal{U}$ with $V \subset U$, we have $I + K_V M_k = I + K_U M_k - H_V K_U M_k$, hence

$$\begin{aligned} & (I + K_V M_k)(\tilde{K}_V + \tilde{H}_V \tilde{K}_U - \tilde{K}_U) = K_V + H_V \tilde{K}_U - (K_U - H_V K_U M_k \tilde{K}_U) \\ = \; & K_V - K_U + H_V (I + K_U M_k) \tilde{K}_U = K_V - K_U + H_V K_U = 0, \end{aligned}$$

i.e.,

$$\tilde{K}_V = \tilde{K}_U - \tilde{H}_V \tilde{K}_U. \tag{6.4}$$

By Remark 2.2,6, it therefore suffices to show that every $\tilde{K}_U$ is a potential kernel on U with respect to $\widetilde{\mathcal{W}}$.

So fix $U \in \mathcal{U}$ and $f \in \mathcal{B}_b^+(U)$. If $V \in \mathcal{U}$ with $\overline{V} \subset U$, then (6.4) implies that $\tilde{H}_V \tilde{K}_U f \leq \tilde{K}_U f$ with equality if $f = 0$ on V. If $0 \leq h \leq \tilde{K}_U f$ such that h is harmonic on U with respect to $(\tilde{H}_V)_{V \in \mathcal{U}}$, then $g := h + K_U(kh)$ is harmonic on U and $0 \leq g \leq K_U f$, hence $g = 0$, $h = 0$. $\square$

7 Coupling and perturbation in a balayage space

We shall now combine assumptions of Section 3 and Section 6: Let us assume that k is a Kato function on X (with respect to K_X) and that T is an admissible transition kernel on the balayage space $(X, \mathcal{W})$ such that $\mathcal{U}^T$ is a base of X (in Section 9 we shall get rid of the last assumption).

For every k-bounded $U \in \mathcal{U}^T$ we define a kernel $\tilde{H}_U^T$ by

$$\tilde{H}_U^T = (I + K_U M_k)^{-1}(H_U + K_U T). \tag{7.1}$$

If every $U \in \mathcal{U}^T$ is k-bounded, we define

$$\widetilde{\mathcal{W}}^T := \{v \mid v : X \to [0, \infty] \text{ l.s.c.}, \tilde{H}_U^T v \leq v \text{ for every } U \in \mathcal{U}^T\}. \tag{7.2}$$

The following result generalizes Corollary 3.4:

Theorem 7.1. *Suppose that there exist $s \in \mathcal{W}$ and $u \in \mathcal{B}^+(X)$ such that*

$$v := s + K_X u \in \mathcal{C}(X), \qquad Tv \le u + kv,$$

and, for every $U \in \mathcal{U}$,

$$\{H_U s < s\} \cup \{K_U(u + kv - Tv) > 0\} = U.$$

Then every $U \in \mathcal{U}$ is k-bounded, $(\tilde{H}_U^T)_{U \in \mathcal{U}^T}$ is a family of harmonic kernels on X, $(X, \widetilde{\mathcal{W}^T})$ is a balayage space, and v is strongly $\widetilde{\mathcal{W}^T}$-superharmonic.

Proof. By Theorem 6.4, every $U \in \mathcal{U}$ is k-bounded and $\tilde{H}_U := (I + K_U M_k)^{-1} H_U$, $U \in \mathcal{U}$, defines a family of harmonic kernels on X. By Proposition 6.5, there exists a potential kernel $\tilde{K}_X$ with respect to $(\tilde{H}_U)_{U \in \mathcal{U}}$ such that, for every $U \in \mathcal{U}$,

$$\tilde{K}_U := \tilde{K}_X - \tilde{H}_U \tilde{K}_X = (I + K_U M_k)^{-1} K_U.$$

Fix $U \in \mathcal{U}$ and let

$$f := v - \tilde{H}_U^T v = v - (I + K_U M_k)^{-1}(H_U v + K_U(Tv)).$$

Then

$$t := (I + K_U M_k) f = v + K_U(kv) - H_U v - K_U(Tv) = s - H_U s + K_U(u + kv - Tv)$$

is a positive superharmonic function on U, hence $f \ge 0$. By assumption $t > 0$ and therefore $f > 0$. The proof is finished by an application of Theorem 3.2. □

Corollary 7.2. *Assume that, for every $U \in \mathcal{U}$, the function $K_U 1$ is strictly positive on U. Then the following holds:*

1. *If $1 \in \mathcal{W}$ and $k > T1$, then the assumptions of Theorem 7.1 are satisfied and 1 is strongly $\widetilde{\mathcal{W}^T}$-superharmonic.*
2. *If $u \in \mathcal{B}^+(X)$ such that $q := K_X u \in \mathcal{C}(X)$ and $Tq < u + kq$, then the assumptions of Theorem 7.1 are satisfied and q is strongly $\widetilde{\mathcal{W}^T}$-superharmonic.*

Proposition 7.3. *Suppose that $(X, \widetilde{\mathcal{W}^T})$ is a balayage space. Then, for every $U \in \mathcal{U}$, the harmonic kernel $\tilde{H}_U^T$ for U with respect to $(X, \widetilde{\mathcal{W}^T})$ has the following property: For every $\varphi \in \mathcal{K}^+(X)$, the function $\tilde{H}_U^T \varphi$ is the unique function $h \in \mathcal{K}^+(X)$ such that*

$$h + K_U(kh - Th) = H_U \varphi.$$

Proof (see the proof of Proposition 3.7). 1. Fix $\varphi \in \mathcal{K}^+(X)$ and define $h := \tilde{H}_U^T \varphi$. Then $h \in \mathcal{K}^+(X)$, hence $K_U(kh - Th) \in \mathcal{C}_0(U)$. So

$$g := h + K_U(kh - Th) \in \mathcal{K}(X), \qquad g = \varphi \quad \text{on } U^c.$$

For every $V \in \mathcal{U}^T$ with $\overline{V} \subset U$,

$$h = \tilde{H}_V^T h = (I + K_V M_k)^{-1}(H_V \varphi + K_V(T\varphi))$$

and therefore

$$\begin{aligned} g &= h + K_V(kh) + H_V K_U(kh) - K_U(Th) \\ &= H_V \varphi + K_V(T\varphi) + H_V K_U(kh) - K_U(Th) = H_V(\varphi + K_U(kh - Th)) \end{aligned}$$

is harmonic on V (note that $\varphi = h$ on U^c implies that $T\varphi = Th$ on V, since $1_V T 1_V = 0$). Thus g is harmonic on U, $g = H_U\varphi$.

2. Now let h be any function in $\mathcal{K}^+(X)$ such that

$$h + K_U(kh - Th) = H_U\varphi.$$

Then $h = \varphi$ on U^c and, for every $V \in \mathcal{U}^T$ with $\overline{V} \subset U$,

$$\begin{aligned}(I + K_V M_k)\tilde{H}_V^T h &= H_V h + K_V^T h = H_V H_U \varphi - H_V K_U(kh - Th) + K_V^T h \\ &= H_U\varphi + K_U(Th) - H_V K_U(kh) = h + K_V(kh),\end{aligned}$$

i.e., $\tilde{H}_V^T h = h$. Thus $h = \tilde{H}_U^T \varphi$. □

To close this section let us briefly consider the situation discussed at the end of Section 4: Let $(X, \mathcal{W})$ be the direct sum of balayage spaces $(Y, \mathcal{W}_i)$, $i \in I$. Let p_i be strongly superharmonic continuous real potentials for $(Y, \mathcal{W}_i)$, $i \in I$, and let K_X be the potential kernel on X composed from the potential kernels $K_{\mathcal{W}_i}^{p_i}$ on the copies of $Y \times \{i\}$ of Y. Let $g_{ij} \ge 0$ be Kato functions on Y with respect to $K_{\mathcal{W}_i}^{p_i}$, $i, j \in I$, $i \ne j$, and

$$T((y,i), \cdot) = \sum_{j \in I \setminus \{i\}} g_{ij}(y)\varepsilon_{(y,i)} \qquad (y \in Y, i \in I).$$

In addition, we now take a Kato function k with respect to K_X and define

$$g_{ii}(y) := -k(y,i) \qquad (y \in Y, i \in I).$$

Replacing Corollary 3.4 by Theorem 7.1 we of course obtain the same results as at the end of Section 4 replacing $\mathcal{W}^T$ by $\widetilde{\mathcal{W}}^T$:

Theorem 7.4. *If there exist functions $u_i \in \mathcal{B}^+(Y)$ such that $K_{\mathcal{W}_i}^{p_i} u_i \in \mathcal{C}(Y)$ and*

$$\sum_{j \in I} g_{ij} K_{\mathcal{W}_j}^{p_j} u_j < u_i$$

for every $i \in I$, then $(X, \widetilde{\mathcal{W}}^T)$ is a balayage space.

Corollary 7.5. *Assume that $\mathcal{W}_i = \mathcal{W}_1$ and $p_i = p_1$ for every $i \in I$. Then $(X, \widetilde{\mathcal{W}}^T)$ is a balayage space if there exists a strictly positive function $u \in \mathcal{B}^+(Y)$ and strictly positive reals b_i such that $K_{\mathcal{W}_1}^{p_1} u \in \mathcal{C}(Y)$ and, for all $i \in I$,*

$$\sum_{j \in I} g_{ij} b_j < b_i u / K_{\mathcal{W}_1}^{p_1} u. \tag{7.3}$$

Remark 7.6. *Suppose that $I = \{1, \dots, n\}$, $a_{ij} := \|g_{ij}\|_\infty < \infty$ for all i, j and denote $A := (a_{ij})$. Assume that $u \in \mathcal{B}^+(Y)$ and $\alpha > 0$ such that*

$$\alpha K_{\mathcal{W}_1}^{p_1} u \le u.$$

Then (7.3) is satisfied if there exists $b \in \mathbb{R}^n$, $b > 0$, such that

$$Ab < \alpha b$$

which in turn holds if and only if the spectral radius of A is strictly less than α.

Corollary 7.7. *Assume that $\mathcal{W}_i = \mathcal{W}_1$ and $p_i = p_1$ for all $i \in I$ and that there exists a strictly positive bounded function in $\mathcal{W}_1$. Then $(X, \widetilde{\mathcal{W}}^T)$ is a balayage space if $(Y, \mathcal{W}_1)$ is parabolic and the function $K^{p_1}_{\mathcal{W}_1}(\max_{i\in I} \sum_{j\in I} g_{ij})$ is continuous and bounded.*

Proposition 7.8. *Let $I = \{1, \dots, n\}$. Suppose that $(X, \widetilde{\mathcal{W}}^T)$ is a balayage space and that U is a relatively compact open subset of Y which is $\mathcal{W}_i$-regular for every $1 \le i \le n$.*

Then, for any choice of functions $\varphi_1, \dots, \varphi_n \in \mathcal{K}(Y)$, there exist unique functions $h_1, \dots, h_n \in \mathcal{K}(Y)$ such that, for every $1 \le i \le n$,

$$h_i - \sum_{j\in I} K^{p_j}_{\mathcal{W}_j}(g_{ij}h_j) \text{ is } \mathcal{W}_i\text{-harmonic on } U, \qquad h_i = \varphi_i \quad \text{on } U^c.$$

Moreover, the functions $h_1, \dots, h_n$ are positive, if the functions $\varphi_1, \dots, \varphi_n$ are positive.

8 Further applications on PDE's

Again let D be a domain in $\mathbb{R}^d$ and $L_1, \dots, L_n$ second order linear partial differential operators on D leading to harmonic spaces $(D, \mathcal{H}_{L_i})$ (having a base of regular sets) with Green functions G_{L_i}. For every $1 \le i \le n$, let μ_i be a (positive) Radon measure on D such that $G^{\mu_i}_{L_i} \in \mathcal{C}_b(D)$ and $(G_{L_i})^{\mu_i}_V > 0$ on V for every (L_i-regular) open subset V of D.

We want to study the coupled system

$$L_i h_i + \sum_{j=1}^n g_{ij} h_j \mu_i = 0 \qquad (1 \le i \le n)$$

where $g_{ij} \in \mathcal{B}(D)$ such that $g_{ij} \ge 0$ for $i \ne j$ and $G^{1_A|g_{ij}|\mu_i}_{L_i} \in \mathcal{C}(D)$ for every compact subset A of D and all $i, j \in \{1, \dots, n\}$.

Using $X_i = D \times \{i\}$ and the canonical projections $\pi_i : X_i \to D$ the direct sum $(X, \mathcal{H})$ of the spaces $(X, \mathcal{H}_{L_i} \circ \pi_i)$, $1 \le i \le n$, is a harmonic space as before. We define a continuous bounded potential p, a kernel T and a function $k \ge 0$ on X by

$$p(x,i) = G^{\mu_i}_{L_i}(x), \quad T((x,i),\cdot) = \sum_{j\ne i} g_{ij}\, \varepsilon_{(x,j)}, \quad k(x,i) = -g_{ii}(x), \qquad (x \in D, 1 \le i \le n).$$

Then k is a Kato function, T is admissible with respect to K^p_X, and the results of the preceding section can be applied. In particular, we have a convex cone $\widetilde{\mathcal{W}}^T$ of functions on X.

Arguing as in Section 5 or applying Proposition 7.8 we obtain the following generalization of Theorem 5.1:

Theorem 8.1. *Assume that $(X, \widetilde{\mathcal{W}}^T)$ is a balayage space. Let U be a relatively compact open subset of D which is L_i-regular for every $1 \le i \le n$ and $\varphi_1, \dots, \varphi_n \in \mathcal{C}(\partial U)$. Then there exist unique functions $h_1, \dots, h_n \in \mathcal{C}(\overline{U})$ such that*

$$L_i h_i + \sum_{j=1}^n h_j g_{ij} \mu_i = 0 \quad \text{on } U, \quad h_i|_{\partial U} = \varphi_i \qquad (1 \le i \le n).$$

Further, if $\varphi_1, \dots, \varphi_n$ are positive, then $h_1, \dots, h_n$ are positive.

Combined with Theorem 8.1 the following result is similar to [CZ96, Theorem ...] (where $\mu_i = \lambda$ and all L_i are uniformly elliptic):

Theorem 8.2. *Suppose that exists a strictly positive real function s on D such that, for every $1 \le i \le n$, one of the following conditions is satisfied:*

1. $\sum_{j=1}^n g_{ij} \le 0$ *and s is strongly L_i-superharmonic.*
2. $\sum_{j=1}^n g_{ij} < 0$ *and s is L_i-superharmonic.*

Then $(X, \widetilde{\mathcal{W}^T})$ is a balayage space.

Proof. Define $s \in \mathcal{W}$ by $s(x,i) = s(x)$ and fix $1 \le i \le n$. Then, for every $x \in D$,

$$(ks - Ts)(x,i) = -g_{ii}(x) - \sum_{j \ne i} g_{ij}(x) \ge 0,$$

So $(X, \widetilde{\mathcal{W}^T})$ is a balayage space by Theorem 7.1 (taking $u = 0$). □

Among various other possible criteria for getting a balayage space $(X, \widetilde{\mathcal{W}^T})$ let us mention just one, a generalization of Corollary 5.6:

Theorem 8.3. *Suppose that $L_1 = \dots = L_n =: L$. Then $(X, \widetilde{\mathcal{W}^T})$ is a balayage space if one of the following conditions is satisfied:*

1. $\mu_1 = \dots = \mu_n =: \mu$ *and there exist $\alpha > 0$, a strictly positive $\mathcal{P}_L(D)$-bounded function $u \in \mathcal{C}_b(D)$, and strictly positive real numbers $b_1, \dots, b_n$ such that*

$$Lu + \alpha u \mu = 0 \quad \text{and} \quad \sum_{j \ne i} g_{ij} b_j < \alpha b_i \text{ for every } 1 \le i \le n.$$

2. $(D, \mathcal{H}_L)$ *is parabolic and the functions $G_L^{g_{ij}\mu_i}$, $i, j \in \{1, \dots, n\}$, are continuous and bounded on D.*

Remark 8.4. *Note that in Theorem 8.2 we necessarily have $g_{ii} \le 0$, whereas Theorem 8.3 leaves some range for positive values of g_{ii}.*

9 General coupling and perturbation in a balayage space

As in Section 7 we shall assume that k is a (not necessarily positive) Kato function on X and that T is an admissible transition kernel (both with respect to the given potential kernel K_X). The essential difference will be that we shall no longer assume that $\mathcal{U}^T$ (as defined in (3.3)) is a base of X. So our result will be new even if there is no perturbation at all, i.e., if $k = 0$.

We shall need the following stability result with respect to increasing limits which is of interest in itself:

Proposition 9.1. *Let $\mathcal{U}$ be a base of relatively compact open sets in X and, for every $n \in \mathbb{N}$, let $(H_U^n)_{U \in \mathcal{U}}$ be a family of (regular) harmonic kernels on X. Suppose that, for every $U \in \mathcal{U}$, the sequence $(H_U^n)_{n \in \mathbb{N}}$ is increasing to a kernel H_U^∞. Then the following are equivalent:*

1. $(H_U^\infty)_{U \in \mathcal{U}}$ *is a family of harmonic kernels on U.*

2. *There exists $s \in \mathcal{C}^+(X)$ such that, for every $U \in \mathcal{U}$, the function $H_U^\infty s$ is continuous on X and $H_U^\infty s < s$ on U.*

Proof. (1) $\Longrightarrow$ (2): By general properties of a family of harmonic kernels (see [BH86]).

(2) $\Longrightarrow$ (1): For every $n \in \mathbb{N} \cup \{\infty\}$, define

$$\mathcal{W}^n := \{v \mid v : X \to [0, \infty],\ v \text{ l.s.c.},\ H_U^n v \le v \text{ for every } U \in \mathcal{U}\}.$$

Then

$$\mathcal{W}^\infty = \bigcap_{n=1}^{\infty} \mathcal{W}^n.$$

By assumption (2), the function s is strongly $\mathcal{W}^\infty$-superharmonic.

If $U, V \in \mathcal{U}$ and $V \subset U$, then $H_V^n H_U^n = H_U^n$ for every $n \in \mathbb{N}$, and hence

$$H_V^\infty H_U^\infty = H_U^\infty.$$

Fix a sequence (ψ_m) in $\mathcal{K}^+(X)$ which is increasing to 1, fix $U \in \mathcal{U}$ and $f \in \mathcal{B}_b^+(X)$ with compact support. Choose $\alpha \in \mathbb{R}_+$ such that $f \le \alpha s$. Then, for every $n \in \mathbb{N}$, the function $H_U^n f$ is continuous on U and the function $H_U^n(\alpha s - f) = \sup_m H_U^n(\psi_m(\alpha s - f))$ is l.s.c. on U. So the increasing limits $H_U^\infty f$ and $H_U^\infty(\alpha s - f)$ are l.s.c. on U. Knowing that their sum $H_U^\infty(\alpha s) = \alpha H_U^\infty s$ is continuous on U we obtain continuity of $H_U^\infty f$ and $H_U^\infty(\alpha s - f)$ on U.

Now suppose that f is even continuous, i.e., that $f \in \mathcal{K}^+(X)$. Then we have the corresponding continuity properties on X. In particular, we see that $H_U^\infty f \in \mathcal{K}(X)$.

So we already know that $(H_U^\infty)_{U \in \mathcal{U}}$ has the properties (H_5'), (H_2), (H_3), and (H_4').

It remains to show that (H_1) is satisfied. So fix $x \in X$. Assume first that, for every $\varphi \in \mathcal{K}(X)$,

$$\lim_{V \downarrow \{x\}} H_V^1 \varphi(x) = \varphi(x).$$

Fix $\varphi_1 \in \mathcal{K}^+(X)$ and choose $\alpha \in \mathbb{R}_+$, $\varphi_2 \in \mathcal{K}^+(X)$ such that $\varphi_1 + \varphi_2 \le \alpha s$, $(\varphi_1 + \varphi_2)(x) = \alpha s(x)$. Then

$$\liminf_{V \downarrow \{x\}} H_V^\infty \varphi_j(x) \ge \lim_{V \downarrow \{x\}} H_V^1 \varphi_j(x) = \varphi_j(x), \qquad j = 1, 2$$

and, for every $x \in V \in \mathcal{U}$,

$$H_V^\infty \varphi_1(x) + H_V^\infty \varphi_2(x) \le H_V^\infty(\alpha s)(x) \le \alpha s(x) = \varphi_1(x) + \varphi_2(x).$$

Therefore

$$\lim_{V \downarrow \{x\}} H_V^\infty \varphi_j(x) = \varphi_j(x), \qquad j = 1, 2.$$

Finally, define

$$r_1 = \inf\{v \in \mathcal{W}^1 : v(x) \ge 1\}, \quad r_\infty = \inf\{v \in \mathcal{W}^\infty : v(x) \ge 1\},$$

and suppose that r_1 is l.s.c. at x. Since $\mathcal{W}^\infty$ is contained in $\mathcal{W}^1$, we have $r_1 \le r_\infty$. Moreover, obviously $r_\infty \le s/s(x)$. Therefore

$$1 = \liminf_{y \to x} r_1(y) \le \liminf_{y \to x} r_\infty(y) \le \liminf_{y \to x} s(y)/s(x) = 1 = r_\infty(x),$$

i.e., r_∞ is l.s.c. at x. $\square$

Suppose next that T is an admissible kernel on X such that $T(x,\{x\})=0$ for every $x\in X$. Moreover, assume that there exists $s\in\mathcal{C}^+(X)$ such that, for every $U\in\mathcal{U}$, $H_Us+K_U^Ts<s$ on U.

Let ρ be a metric for X and define kernels T_n, T'_n on X by

$$T_n(x,\cdot)=1_{B(x,1/n)^c}\,T(x,\cdot),\quad T'_n(x,\cdot)=1_{B(x,1/n)}\,T(x,\cdot)\qquad (n\in\mathbb{N}, x\in X)$$

(where of course $B(x,1/n)=\{y\in X:\rho(x,y)<1/n\}$). Then, for every $n\in\mathbb{N}$, the set $\mathcal{U}^{T_n}=\{U\in\mathcal{U}:1_UT_n1_U=0\}$ is a base of X and we have kernels

$$K_U^{T_n}=K_UT_n,\quad H_U^{T_n}=H_U+K_U^{T_n}\qquad (U\in\mathcal{U}^{T_n}).$$

Since obviously, for every $V\in\mathcal{U}^{T_n}$,

$$H_V^{T_n}s=H_Vs+K_V^{T_n}s\le H_Vs+K_V^Ts<s\quad \text{ on } V,$$

the function s is strongly $\mathcal{W}^{T_n}$-superharmonic and we conclude by Theorem 3.2 that $(H_U^{T_n})_{U\in\mathcal{U}^{T_n}}$ is a family of harmonic kernels and that $(X,\mathcal{W}^{T_n})$ is a balayage space. In particular, for every $n\in\mathbb{N}$ and for *every* $U\in\mathcal{U}$, we have a harmonic kernel $H_U^{T_n}$ solving the Dirichlet problem with respect to $(X,\mathcal{W}^{T_n})$ (see [BH86, Chapter VII]).

Clearly, $\mathcal{U}^{T_{n+1}}\subset\mathcal{U}^{T_n}$ and $H_U^{T_n}\le H_U^{T_{n+1}}$ for every $U\in\mathcal{U}^{T_{n+1}}$. We claim that in fact

$$H_U^{T_n}\le H_U^{T_{n+1}}\qquad \text{for every } U\in\mathcal{U}. \tag{9.1}$$

Indeed, fix $U\in\mathcal{U}$, $\varphi\in\mathcal{K}^+(X)$, and define

$$t:=H_U^{T_{n+1}}\varphi.$$

Then, for every $V\in\mathcal{U}^{T_{n+1}}$ with $\overline{V}\subset U$,

$$H_V^{T_n}t\le H_V^{T_{n+1}}t=t,$$

hence t is superharmonic on U with respect to $(X,\mathcal{W}^{T_n})$. Moreover, $t\in\mathcal{K}^+(X)$ and $t=\varphi$ on U^c. Therefore

$$H_U^{T_n}\varphi\le t$$

proving (9.1). In particular, the sequence $(\mathcal{W}^{T_n})$ is decreasing and defining

$$H_U^T:=\sup_n H_U^{T_n}$$

we have

$$\mathcal{W}^T:=\{v\mid v:X\to[0,\infty]\text{ l.s.c.}, H_U^Tv\le v\text{ for every }U\in\mathcal{U}\}=\bigcap_{n\in\mathbb{N}}\mathcal{W}^{T_n}.$$

We now obtain the following extension of Theorem 3.2 (see also Remark 9.4):

Theorem 9.2. *Let T be an admissible kernel such that $T(x,\{x\})=0$ for every $x\in X$. Suppose that there exists $s\in\mathcal{C}^+(X)$ such that, for every $U\in\mathcal{U}$, K_U^Ts is continuous on U and $H_Us+K_U^Ts<s$ on U. Then the following holds:*

1. *$(X,\mathcal{W}^T)$ is a balayage space and s is strongly $\mathcal{W}^T$-superharmonic.*
2. *For every $U\in\mathcal{U}$ and for every $\varphi\in\mathcal{K}^+(X)$, the Dirichlet solution $H_U^T\varphi$ is the unique function $h\in\mathcal{K}^+(X)$ such that $h-K_U^Th=H_U\varphi$.*

3. If v is any positive numerical function on X, then $v \in \mathcal{W}^T$ if and only if there exists a function $w \in \mathcal{W}$ such that

$$v = K_X^T v + w.$$

Proof. 1. Fix $U \in \mathcal{U}$. By Proposition 9.1 it suffices to show that $H_U^T s$ is continuous on X and $H_U^T s < s$ on U. Let us note first that obviously $s \in \mathcal{W} \cap \mathcal{C}(X)$ and hence $H_U s \in \mathcal{C}(X)$ and $s - H_U s \in \mathcal{C}_0(X)$. Given $n \in \mathbb{N}$, we have $s \in \mathcal{W}^{T_n}$. So

$$h_n := H_U^{T_n} s \le s.$$

and, by Proposition 3.7,

$$h_n = H_U s + K_U^{T_n} h_n.$$

Letting n tend to infinity we obtain that

$$h := H_U^T s = \lim_{n \to \infty} h_n = H_U s + K_U^T h \le s$$

and hence

$$h \le H_U s + K_U^T s < s \quad \text{ on } U.$$

Moreover, $K_U^T h \in \mathcal{C}(U)$, since $0 \le h \le s$ and $K_U^T s$ is continuous on U by assumption. Since $0 \le K_U^T h \le K_U^T s \le s - H_U s$, we know that $K_U^T h$ tends to zero at the boundary of U. Thus $K_U^T h \in \mathcal{C}_0(U)$ and $h = H_U s + K_U^T h \in \mathcal{C}(X)$.

2. Fix $\varphi \in \mathcal{K}^+(X)$. Since by Proposition 3.7

$$H_U^{T_n} \varphi - K_U^{T_n} H_U^{T_n} \varphi = H_U \varphi,$$

we immediately obtain that

$$H_U^T \varphi - K_U^T H_U^T \varphi = H_U \varphi. \tag{9.2}$$

Conversely, let h be any function in $\mathcal{K}^+(X)$ such that

$$h - K_U^T h = H_U \varphi. \tag{9.3}$$

Let C be the support of h. By (3.2), $K_U^T 1_C \in \mathcal{C}_0(U)$. Given $x \in U$, the functions

$$K_V^T 1_C = K_U^T 1_C - H_V K_U^T 1_C, \qquad x \in V, \overline{V} \subset U$$

are uniformly decreasing to zero as V decreases to $\{x\}$. So we may choose $V_x \in \mathcal{U}$ such that $x \in V_x, \overline{V}_x \subset U$ and $K_V^T 1_C \le \gamma$ for some real $\gamma < 1$. Fix $V \in \mathcal{U}$ such that $x \in V \subset V_x$ and define a positive operator N on $\mathcal{B}_b(X)$ by $Nf := K_V^T(1_C f)$. Then the operator $I - N$ is invertible.

Applying H_V on both sides of (9.3) we obtain that

$$H_V h - H_V K_U^T h = H_V H_U \varphi = H_U \varphi = h - K_U^T h,$$

and therefore

$$H_V h = h - K_U^T h + H_V K_U^T h = h - K_V^T h = (I - N) h.$$

On the other hand,

$$H_V h = H_V^T h - K_V^T H_V^T h = (I - N) H_V^T h$$

(using (9.2) for h instead of φ and V instead of U). Since $I-N$ is invertible, we conclude that

$$h = H_V^T h.$$

By [BH86, Proposition III.4.4], this shows that h is harmonic on U with respect to $(X, \mathcal{W}^T)$. Thus $h = H_U^T \varphi$.

3. Suppose that $w \in \mathcal{W}$ such that $v = K_X^T v + w$. Then, for every $n \in \mathbb{N}$,

$$v = K_X^{T_n} v + K_X^{T_n'} v + w$$

where $K_X^{T_n'} v + w \in \mathcal{W}$. Thus Proposition 3.9 implies that

$$v \in \bigcap_{n=1}^{\infty} \mathcal{W}^{T_n} = \mathcal{W}^T.$$

Assume conversely that $v \in \mathcal{W}^T$. Then, for every $n \in \mathbb{N}$, there exists a function $w_n \in \mathcal{W}^{T_n}$ such that

$$K_X^{T_n} v + w_n = v.$$

Defining $w \in \mathcal{W}$ by

$$w(x) = \text{f-}\liminf_{y \to x} \inf_n w_n(y)$$

we finally get that $K_X^T v + w = v$. □

We now obtain the results of Theorem 7.1 and Proposition 7.3 not assuming any more that $\mathcal{U}^T$ is a base of X.

Theorem 9.3. *Let T be an admissible transition kernel and let k be a Kato function (with respect to K_X). Suppose that there exist $s \in \mathcal{W}$ and $u \in \mathcal{B}^+(X)$ such that*

$$v := s + K_X u \in \mathcal{C}(X), \qquad Tv \le u + kv,$$

and, for every $U \in \mathcal{U}$, $\{H_U s < s\} \cup \{K_U(u + kv - Tv) > 0\} = U$.

Then, for every $U \in \mathcal{U}$ and for every $\varphi \in \mathcal{K}^+(X)$, there exists a unique function $h = \tilde{H}_U^T \varphi \in \mathcal{K}^+(X)$, such that

$$h + K_U(kh - Th) = H_U \varphi.$$

Moreover, $(\tilde{H}_U^T)_{U \in \mathcal{U}}$ is a family of harmonic kernels on X for which v is strongly superharmonic.

Remark 9.4. *Note that taking $k = 0$ we obtain the statements of Theorem 9.2 without the assumption that $T(x, \{x\}) = 0$ for $x \in X$.*

Proof of Theorem 9.3. Replacing T by the kernel $x \mapsto T(x, \cdot) - T(x, \{x\})\varepsilon_x$, k by the function $x \mapsto k(x) - T(x, \{x\})$ we may assume that $T(x, \{x\}) = 0$ for every $x \in X$.

We now proceed as in the proof of Theorem 7.1: By Theorem 6.4, every $U \in \mathcal{U}$ is k-bounded and defining $\tilde{H}_U$, $U \in \mathcal{U}$, by (6.1) and $\widetilde{\mathcal{W}}$ by (6.2) we obtain a family $(\tilde{H}_U)_{U \in \mathcal{U}}$ of harmonic kernels and a balayage space $(X, \widetilde{\mathcal{W}})$ such that v is strongly $\widetilde{\mathcal{W}}$-superharmonic. Moreover, by Proposition 6.5, there exists a potential kernel $\tilde{K}_X$ such that, for every $U \in \mathcal{U}$,

$$\tilde{K}_U := \tilde{K}_X - \tilde{H}_U \tilde{K}_X = (I + K_U M_k)^{-1} K_U. \tag{9.4}$$

We claim that, for every $U \in \mathcal{U}$,

$$\tilde{H}_U v + \tilde{K}_U^T v < v \quad \text{ on } U.$$

Indeed, defining $f := v - \tilde{H}_U v - \tilde{K}_U^T v$ we obtain that

$$(I + K_U M_k) f = v + K_U(kv) - H_U v - K_U(Tv) = s - H_U s + K_U(u + kv - Tv)$$

is a strictly positive superharmonic function on U and hence $f > 0$ on U. Clearly, $K_X u \in \mathcal{C}(X)$ and hence $K_U u \in \mathcal{C}_0(U)$. Since $|kv| \le \sup v(U)\,|k|$ on U, we know that $K_U|kv| \in \mathcal{C}_0(U)$. Therefore the inequality $0 \le Tv \le u + kv$ implies that $K_U^T v \in \mathcal{C}_0(U)$ and hence $\tilde{K}_U^T v \in \mathcal{C}_0(U)$.

Replacing $(H_U)_{U\in\mathcal{U}}$ by $(\tilde{H}_U)_{U\in\mathcal{U}}$ and $(K_U)_{U\in\mathcal{U}}$ by $(\tilde{K}_U)_{U\in\mathcal{U}}$ we get a balayage space $(X, \widetilde{\mathcal{W}^T})$ such that v is strongly $\widetilde{\mathcal{W}^T}$-superharmonic.

Moreover, for every $\varphi \in \mathcal{K}^+(X)$, the function

$$\tilde{H}_U^T \varphi = \lim_{n\to\infty} \tilde{H}_U^{T_n} \varphi$$

is the unique function $h \in \mathcal{K}^+(X)$ such that

$$h - \tilde{K}_U^T h = \tilde{H}_U \varphi.$$

By (6.1) and (9.4), the last equation is equivalent to

$$h + K_U(kh - Th) = H_U \varphi,$$

and the proof is finished. □

10 Appendix

In this section we shall first characterize parabolic balayage spaces and then construct a potential kernel corresponding to a compatible family of potential kernels $(K_U)_{U\in\mathcal{U}}$ (see Remark 2.2,5).

We shall need the following result on compactness of operators K_X^q which is of independent interest:

Lemma 10.1. *Suppose that there exists a strictly positive bounded function in $\mathcal{W}$ and let $p \in \mathcal{P}(X)$ such that p is harmonic outside a compact set C. Then K_X^p is a compact operator on $\mathcal{B}_b(X)$.*

Proof(cf. also [Han81, p. 504]). Let $K := K_X^p$ and let us fix $w \in \mathcal{W}$ such that $0 < w \le 1$. There exists $\alpha > 0$ such that $p \le \alpha w$ on C and hence $p \le aw$ on X. So p is bounded. We intend to show first that the subset $\{Kf : f \in \mathcal{B}(X), 0 \le f \le 1\}$ of $\mathcal{P}_b(X)$ is equicontinuous. Fix $x \in X$, $\varepsilon > 0$, and let L be a compact neighborhood of x. By Dini's theorem, there exists an open neighborhood U of x in L such that $K1_{U\setminus\{x\}} < \varepsilon$ on L. For every $f \in \mathcal{B}(X)$ such that $0 \le f \le 1$,

$$Kf = f(x)K1_{\{x\}} + K(1_{U\setminus\{x\}}f) + K(1_{U^c}f)$$

where $K1_{\{x\}}$ is continuous (it vanishes if $\{x\}$ is semi-polar), $0 \le K(1_{U\setminus\{x\}}f) < \varepsilon$ on C, and the functions $K(1_{U^c}f)$ are equicontinuous, since they are harmonic on U and bounded by p. So there exists a neighborhood V of x in U such that, for every $f \in \mathcal{B}(X)$ with $0 \le f \le 1$,

$$|Kf - Kf(x)| < 3\varepsilon \quad \text{on } V.$$

Fix a sequence (f_n) in $\mathcal{B}(X)$ such that $0 \le f_n \le 1$ for every $n \in \mathbb{N}$. By our preceding considerations, there exist a subsequence (g_n) of (f_n) such that the sequence (Kg_n) is locally convergent on X. Fix $\delta > 0$. There exists a natural n_0 such that, for all $n, m \ge n_0$,

$$|Kg_n - Kg_m| < \delta w \quad \text{on } C.$$

Fix $n, m \ge n_0$. Having $Kg_n \le \delta s + Kg_m$ on C and knowing that Kg_n is harmonic outside C, we conclude that $Kg_n \le \delta s + Kg_m$ on X. Similarly, $Kg_m \le \delta s + Kg_n$ on X. Thus

$$|Kg_n - Kg_m| \le \delta s \le \delta \quad \text{on } X.$$

□

Theorem 10.2. *Suppose that there exists a strictly positive bounded function in $\mathcal{W}$ and let $p \in \mathcal{P}(X)$ be strongly superharmonic. Then the following statements are equivalent:*

1. *$(X, \mathcal{W})$ is parabolic, i.e., for every non-empty compact subset C of X, there exists $x \in C$ such that* $\liminf_{y \to x} R_{1_C}(y) = 0$.

2. *For every $q \in \mathcal{P}(X)$ and for every non-empty compact subset C of X, there exists $x \in C$ such that $K_X^q 1_C(x) = 0$.*

$2'$. *For every non-empty compact subset C of X, there exists $x \in C$ such that $K_X^p 1_C(x) = 0$.*

3. *For every $q \in \mathcal{P}_b(X)$ such that K_X^q is a compact operator on $\mathcal{B}_b(X)$, the operator $I - K_X^q$ is invertible.*

$3'$. *For every compact subset C of X and for every $\alpha > 0$, the operator $I - \alpha K_X^p M_{1_C}$ on $\mathcal{B}_b(X)$ is invertible.*

Proof. (1) $\Longrightarrow$ (2): Fix $q \in \mathcal{P}(X)$ and a non-empty compact C subset of X. There exists $\alpha > 0$ such that $\alpha q \le 1$ on C and hence $\alpha K_X^q 1_C \le R_{1_C}$. By (1), there exists $x \in C$ such that $\liminf_{y \to x} R_{1_C}(y) = 0$ and therefore

$$\alpha K_X^q 1_C(x) = \lim_{y \to x} \alpha K_X^q 1_C(y) \le \liminf_{y \to x} R_{1_C}(y) = 0$$

whence $K_X^q 1_C(x) = 0$.

(2) $\Longrightarrow$ (2′): Trivial.

(2′) $\Longrightarrow$ (1): Suppose that there is a non-empty compact C subset of X such that $\liminf_{y \to x} R_{1_C}(y) > 0$ for every $x \in C$. Then there exists a compact neighborhood C' of C such that $R_{1_{C'}} > 0$ on C'. Define $q' := K_X^p 1_{C'}$. Since p is strongly superharmonic, we know that $q' > 0$ on the interior of C' whence $\beta q' \ge 1$ on C for some $\beta > 0$. This implies that $\beta q' \ge R_{1_C}$. In particular, $q' > 0$ on C'.

(2) $\Longrightarrow$ (3): Fix $q \in \mathcal{P}_b(X)$ such that K_X^q is a compact operator on $\mathcal{B}_b(X)$. Assume that, for some $\alpha > 0$, the operator $I - \alpha K_X^q$ is not invertible and let $K = \alpha K_X^q$. Then there exists a function $f \in \mathcal{B}_b(X) \setminus \{0\}$ such that $f = Kf$, and we may assume without

loss of generality that $|f| \le 1$ and $\{f > 0\} \neq \emptyset$. Since the kernel K is a compact operator on $\mathcal{B}_b(X)$, there exists a real $\varepsilon > 0$ and a compact subset C of $\{f \ge \varepsilon\}$ such that

$$K1_{\{0<f<\varepsilon\}} < 1/2 \quad \text{and} \quad K1_{\{f\ge\varepsilon\}\setminus C} < \varepsilon/2.$$

By (2), there exists $x \in C$ such that $K1_C(x) = 0$ and therefore

$$\varepsilon \le f(x) = Kf(x) \le K(f1_{\{f>0\}}(x) \le \varepsilon K1_{\{0<f<\varepsilon\}}(x) + K1_{\{f\ge\varepsilon\}\setminus C}(x) < \varepsilon.$$

This contradiction shows that $I - K$ is invertible.

(3) $\Longrightarrow$ (3′): Trivial, since, for every compact subset C of X, $K_X^p M_{1_C}$ is the operator K_X^q for $q := K_X^p 1_C \in \mathcal{P}_b(X)$ (see Remark 2.2,2) and K_X^q is compact by Lemma 10.1.

(3′) $\Longrightarrow$ (2′): Suppose that there exists a non-empty compact subset C of X such that $K_X^p 1_C > 0$ on C. Then there exists a real $\gamma > 0$ such that $\gamma K_X^p 1_C \ge 1$ on C. Defining $q := \gamma K_X^p 1_C$ we already noted before that $K_X^q = \gamma K_X^p M_{1_C}$. In particular, $K_X^q 1 = q \ge 1$ on C and $K_X^q 1_{C^c} = 0$. Therefore $(K_X^q)^n 1 \ge 1$ on C whence $\sum_{n=0}^\infty (K_X^q)^n 1 = \infty$ on C. Thus the following lemma implies that (3) does not hold. □

Lemma 10.3. *Let K be a bounded kernel on X and $\gamma > 0$ such that $I - \alpha K$ is invertible for every $0 < \alpha \le \gamma$. Then $(I - \gamma K)^{-1} = \sum_{n=0}^\infty (\gamma K)^n$.*

Proof. Let

$$\beta := \sup\{\alpha \in [0, \gamma] : (I - \alpha K)^{-1} f \ge 0 \text{ for every } f \in \mathcal{B}_b^+(X)\}.$$

By continuity, $(I - \beta K)^{-1} f \ge 0$ for every $f \in \mathcal{B}_b^+(X)$. So

$$(I - \beta K)^{-1} = \sum_{n=0}^\infty (\beta K)^n$$

by [HH88, Lemma 1.3]. If $\beta < \gamma$, then by continuity again, there exists $\beta < \beta' \le \gamma$ such that

$$(I - \beta' K)^{-1} = \sum_{n=0}^\infty (\beta' K)^n$$

and therfore $(I - \beta' K)^{-1} f \ge 0$ for every $f \in \mathcal{B}_b^+(X)$. This contradicts the definition of β. Thus $\beta = \gamma$ and the proof is finished. □

Now assume that, for every $U \in \mathcal{U}$, we have a potential kernel K_U on U such that $K_U = K_V + H_V K_U$ whenever $U, V \in \mathcal{U}$ with $V \subset U$ (such a family $(K_U)_{U\in\mathcal{U}}$ is called *compatible*). To construct a corresponding potential kernel K_X we shall need the following *lifting property*:

Theorem 10.4. *Let U be an open subset of X and q a continuous real potential on U which is harmonic outside a compact subset C of U. Then there exists a unique $p \in \mathcal{P}(X)$ such that p is harmonic outside C and $p - q$ is harmonic on U.*

For harmonic spaces the proof is already fairly technical (see [Her62, Theorem 13.2]), for balayage spaces it is even more delicate:

Proof of Theorem 10.4 (cf. [Alb95]). The uniqueness of p is easily established. Indeed, if p and p' have the desired properties, then $p-p'$ is harmonic on U and harmonic outside C. Therefore $p-p'$ is harmonic on X. Since $p-p'$ is of course $\mathcal{P}(X)$-bounded, we conclude that $p = p'$.

To prove the existence let us define

$$\mathcal{F} := \{p \in \mathcal{P} : p - q \in \mathcal{S}^+(U)\}.$$

We intend to show that there is a smallest element in $\mathcal{F}$ and that this function $\inf \mathcal{F}$ has the desired properties.

1. First we claim that the set $\mathcal{F}$ is non-empty: We choose an open set V and a compact set L such that $C \subset V \subset L \subset U$. By a general approximation property (see [BH86, I.1.2]) there exist $q_1, q_2 \in \mathcal{P}(X)$ such that

$$q_2 - q_1 \geq q \quad \text{on } V, \qquad q_1 = q_2 \quad \text{on } L^c.$$

Then

$$p_0 := \inf(q + q_1, q_2) \in \mathcal{S}^+(U).$$

Moreover, $p_0 \in \mathcal{S}^+(L^c)$. Thus $p_0 \in \mathcal{W}$. Since $p_0 \in \mathcal{C}(X)$ and $p_0 \leq q_2$ we obtain that in fact $p_0 \in \mathcal{P}(X)$.

Obviously $p_0 \geq q$ on V and therefore on U, since q is harmonic outside the subset C of U. In addition, $p_0 - q = q_1$ on V and $p_0 - q \leq q_1$ whence $p_0 - q \in \mathcal{S}^+(V)$. Further, obviously $p_0 - q \in \mathcal{S}^+(U \setminus C)$ and $p_0 - q \leq q_1$. So $p_0 - q \in \mathcal{S}^+(U)$, $p_0 \in \mathcal{F}$.

2. Obviously $\mathcal{F}$ is stable with respect to finite infima, since both $\mathcal{P}(X)$ and $\mathcal{S}^+(U)$ are.

3. Next we show that $\inf F$ is harmonic outside C: Let us fix an open neighborhood W of C in U. Clearly it suffices to show that $\inf \mathcal{F}$ is harmonic outside the closure of W. For the present fix $p \in \mathcal{F}$. Then $K_X^p 1_W - q = (p-q) - K_X^p 1_{W^c} \in \mathcal{S}(W)$ and $K_X^p 1_W - q \in \mathcal{S}(U \setminus C)$, hence $K_X^p 1_W - q \in \mathcal{S}(U)$. Since $q \in \mathcal{P}(U)$, we obtain that $K_X^p 1_W - q \geq 0$. Therefore $K_X^p 1_W \in \mathcal{F}$, i.e.

$$\inf \mathcal{F} = \inf\{K_X^p 1_W : p \in \mathcal{F}\}.$$

Since $\mathcal{F}$ is stable with respect to finite infima, the set of all $K_X^p 1_W$, $p \in \mathcal{F}$, is decreasingly filtered and therefore contains a decreasing sequence (p_n) converging to $\inf \mathcal{F}$. Since all functions $K_X^p 1_W$, $p \in \mathcal{F}$, are harmonic outside $\overline{W}$, we conclude in particular that $\inf \mathcal{F}$ is harmonic outside $\overline{W}$ as well.

4. Moreover, $\inf \mathcal{F} - q$ is harmonic on U: Fix $p \in \mathcal{F}$, a compact neighborhood L of C in U and an open neighborhood W of C such that $\overline{W}$ is contained in the interior of L. Choose $\varphi \in \mathcal{C}(X)$ such that $0 \leq \varphi \leq 1$, $\varphi = 1$ on L^c, and $\varphi = 0$ on W. Define

$$p' := \inf(R_{\varphi p} + q, p).$$

Then $p' = p$ on L^c, so p' is continuous on L^c. Further, the continuity of the functions $R_{\varphi p}$, q, and p on U implies that p' is continuous on U. Therefore p' is continuous on X.

Clearly, $p' \in \mathcal{S}^+(U)$. Moreover, $p' \in \mathcal{S}^+(L^c)$, since $p' = p$ on L^c and $p' \leq p$. Therefore $p' \in \mathcal{W}$ and even $p' \in \mathcal{P}(X)$, since p' is continuous and $p' \leq p$. Since $p - q \in \mathcal{S}^+(U)$, we obtain that $p' - q = \inf(R_{\varphi p}, p - q) \in \mathcal{S}^+(U)$. Thus $p' \in \mathcal{F}$.

Further, $R_{\varphi p} \leq R_{1_{W^c} p} = H_W p$ whence $p' - q \leq H_W p$. So, for every $n \in \mathbb{N}$ and for every $V \in \mathcal{U}$ with $\overline{V} \subset W$, we obtain that

$$p_n - q \geq H_V(p_n - q) \geq H_W(p_n - q) = H_W p_n - H_W q \geq p_n' - q - H_W q.$$

Since obviously $\inf \mathcal{F} = \inf p_n = \inf p'_n$, we conclude that

$$\inf \mathcal{F} - q \geq H_V(\inf \mathcal{F} - q) \geq \inf \mathcal{F} - q - H_W q.$$

Since $\lim_{W \uparrow U} H_W q = 0$, this implies that

$$\inf \mathcal{F} - q = H_V(\inf \mathcal{F} - q)$$

for all $V \in \mathcal{U}$ with $\overline{V} \subset U$. Thus $\inf \mathcal{F} - q$ is harmonic on U.

Knowing that $\inf \mathcal{F} - q$ is harmonic on U and $\inf \mathcal{F}$ is harmonic on C^c we see immediately that $\inf \mathcal{F}$ is continuous on X. Thus $\inf \mathcal{F} \in \mathcal{P}(X)$, and the proof is finished. □

Proposition 10.5. *Let $(K_U)_{U \in \mathcal{U}}$ be a compatible family of potential kernels. Then there exists a unique potential kernel K_X on X such that $K_U = K_X - H_U K_X$ for every $U \in \mathcal{U}$.*

Proof. Indeed, if $f \in \mathcal{B}_b^+(X)$ with compact support in some $U \in \mathcal{U}$, then $K_X f$ has to be the lifting of $K_U f$. So we have uniqueness of K_X.

To prove its existence we may choose a locally finite covering of X by a sequence (U_n) in $\mathcal{U}$ and continuous functions $\varphi_n \geq 0$ on X with compact support in U_n, $n \in \mathbb{N}$, such that $\sum_{n=1}^{\infty} \varphi_n = 1$. For every $n \in \mathbb{N}$, let p_n be the lifting of $K_{U_n} \varphi_n$ on X so that

$$K_X^{p_n} - H_{U_n} K_X^{p_n} = K_{U_n} M_{\varphi_n}. \tag{10.1}$$

Define

$$K_X := \sum_{n=1}^{\infty} K_X^{p_n}.$$

Clearly, K_X is a potential kernel on X. Fix $U \in \mathcal{U}$, $n \in \mathbb{N}$, and $f \in \mathcal{B}_b^+(X)$ with compact support in U. Then $\varphi_n f$ has compact support in $U_n \cap U$ and our compatibility assumption implies that $K_U(\varphi_n f)$ is the lifting of $K_{U_n \cap U}(\varphi_n f)$ on U and $K_{U_n}(\varphi_n f)$ is the lifting of $K_{U_n \cap U}(\varphi_n f)$ on U_n. By (10.1), $K_X^{p_n} f$ is the lifting of $K_{U_n}(\varphi_n f)$ on X. Therefore

$$K_X^{p_n} f - H_U K_X^{p_n} f = K_U(\varphi_n f).$$

Taking the sum over all $n \in \mathbb{N}$ we finally conclude that $K_X - H_U K_X = K_U$. □

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