On the absolutely continuous spectrum of Stark Hamiltonians

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Abstract

We analyze the spectral properties of Stark operators perturbed by a non decaying potential. We prove that if the potential has a bounded partial derivative with respect to the electric field and the positive component of this derivative is small at infinity then the spectrum of the corresponding operator is purely absolutely continuous. In the one dimensional case the derivative does not need to be small. We also prove that the tail of the partial derivative of the potential plays a fundamental role in the preservation of the absolute continuity of the spectrum. This is illustrated in an example of smooth potential with bounded partial derivative but the corresponding stark operator has a dense point spectrum.

1 Introduction and main results

1.1. Our purpose in this paper is to analyze the spectral properties of Stark Hamiltonians with a non-decaying potentials. We first describe a class of bounded potentials V for which the corresponding Hamiltonian H has an absolutely continuous spectrum with at most a discrete set as singular spectrum. Basically, our class consists of sufficiently regular potential having a small (at infinity) partial derivative with respect to the electric field. In the one dimensional case the derivative does not need to be small. Afterwards, we prove that the tail of the partial derivative has a fundamental role in the possible occurrence of a rich singular spectrum. Indeed, in the case when the dimension $d \geq 2$ we construct a smooth potential with bounded partial derivative for which the corresponding stark operator has a dense point spectrum.

The operator we are interested with is given by

$$H = H_0 + V$$

acting in $\mathcal{H} = L^2(\mathbb{R}^d)$, where $d \geq 1$ is an integer, and

$$H_0 = -\Delta - <\vec{F}, x>$$

is the free Stark operator and V is the multiplication operator by a real-valued function denoted by the same symbol V. Here above $<\vec{F}, x>$ denotes the scalar product between a given vector $\vec{F} = (F_1, \cdots, F_d) \in \mathbb{R}^d$, which represent the constant electric field, and $x = (x_1, \cdots, x_d)$ with x_i is the multiplication operator by the independent variable x_i.

The operator H describes the motion of a non-relativistic charged particle moving in a constant electric field $\vec{F}$ and submitted to the action of an external force given by the potential V.

Let us remark that one can assume, without loss of generality, that $\vec{F} = F(1, 0, \cdots, 0)$, with $F > 0$, and so H_0 becomes

$$H_0 = -\Delta - F \cdot x_1,$$

which is more convenient for notational reasons.

Let us denote $P_j = -i\partial_{x_j}$ for each $j = 1, \cdots, d$, and consider the unitary operator $U = e^{-iP_1(-P_1^2/3 + P_2^2 + \cdots + P_d^2)}$. A simple commutation shows that $U^{-1} H_0 U = F \cdot x_1$. Then H_0 defines a self-adjoint operator in $\mathcal{H}$ and its spectrum is purely absolutely continuous:

$$\begin{cases} \sigma_{ac}(H_0) = & \sigma_{ac}(H_0) = \mathbb{R} \\ \sigma_{sc}(H_0) = & \sigma_{pp}(H_0) = \emptyset. \end{cases}$$

For simplicity assume that the potential V is bounded (see the remark just after Theorem 3). Then H is self-adjoint operator in $\mathcal{H}$.

It is natural to study the stability of the last spectral structure after a perturbation by a potential V. By stability we mean that the perturbed Hamiltonian H, has an absolutely continuous spectrum, has no singular continuous spectrum, and has at most a discrete set of possible eigenvalues.

These considerations have a long history and there is a large literature about it, for example [16, 17, 2, 9, 14, 11] and references therein. A precise discussion of these references will be given during the rest of this introduction in which we shall describe our main results.

1.2. Let us start by discussing the one dimensional case. It is well known that if V is of class BC^2 (i.e. two times continuously differentiable, bounded with its derivatives) then the spectrum of H is purely absolutely continuous. This can be deduced from Titchmarsh [16] (see also [17]). This result was be done by using the standard techniques of ordinary differential equations, and was be re-obtained by Bentosela and all [3] by using the positive commutator approach.

This absolute continuity of the spectrum of the Stark operator can be partially or completely destroyed, if the potential is not sufficiently regular. Indeed, Naboko and Pushnitski [12] constructed an example of bounded potential (in fact V even tends to zero at infinity, but its derivative tends to infinity at infinity), and the corresponding Hamiltonian H has a dense set of eigenvalues, i.e. $\sigma_{pp}(H) = \mathbb{R}$. Let us mention that, combining this result with Kiselev's result [11] one can conclude that in this situation the spectrum is mixt. Examples of strongly singular potential V such that the corresponding operator H has no absolutely continuous spectrum can be found in [7] (see also [6]).

This brings us to ask the natural question: what is the minimal regularity of V ensuring the absolute continuity of H?

We have proved in [14], that if V is of class BC^1 (i.e. continuously differentiable, bounded with its derivative), and that V' is Dini continuous, that is

$$\int_0^1 \sup_{x\in\mathbb{R}} |V(x+\varepsilon) - V(x)| \frac{d\varepsilon}{\varepsilon} < \infty. \tag{1.1}$$

then the absolute continuity of the spectrum of H is preserved in the sense explained above. This result was be re obtained partially by Kiselev in [11] by using the asymptotic Gilbert-Pearson method. Here we give slightly more precise result.

Theorem 1 *Assume that V is of class BC^1 with uniformly continuous first derivative. Then the set $\sigma_p(H)$ of eigenvalues of H is discrete.*
If V is smooth in the Zygmund's sense, i.e.

$$\int_0^1 \sup_{x\in\mathbb{R}} |V(x+\varepsilon) - 2V(x) + V(x-\varepsilon)| \frac{d\varepsilon}{\varepsilon} < \infty. \tag{1.2}$$

Then H has no singular continuous spectrum.

Remarks.

1. Let us mention that if V' is Dini continuous then the assumption 1.2 holds, which means that the present result is more precise than that of [14].

2. The fact that there is no restriction on the bound of the potential V nor of its derivatives allows to cover a large class of periodic, quasi-periodic or random potential. As we shall see this is not the case in the multidimensional case (see Theorem 3 and 4).

3. Roughly speaking the last Theorem means that if V is of class BC^{1+0} then H has no singular continuous spectrum. An interesting question concerns the stability of this property only under BC^1 assumption on V. We expect that the answer is positive.

4. A more subtle question concerns whether there exist a potential V that is Hölder continuous of order $0 < \alpha < 1$ for which the singular spectrum can fills (a part of) the real axis. Taking into account the result of [12], one can expect that it is true, in that case it would be interesting to construct such potentials.

Let us mention that concerning the eigenvalues of H Theorem 1 only ensures the absence of the accumulation points of eigenvalues, but does not tells us whether or not some eigenvalues appear. In the following we give an example of smooth potential V bounded with its derivative and the corresponding Stark operator has an eigenvalue.

Theorem 2 *For each real number λ there exist a real-valued potential $V \in C^\infty(\mathbb{R})$, such that V and V' are bounded and λ is an eigenvalue of H.*

To prove such a Theorem we use the ideas of [12]. We mention however that in the last the derivative of the potential is certainly unbounded. We also mention that Theorem 2 holds for any finite number of eigenvalues.

1.3. Now let us study the multidimensional case ($d \geq 1$). In this case there is an important literature on the spectral theory of Stark Hamiltonians. we refer the reader to [2, 9, 10] and references therein. In the most part of these works V or ∇V need to tend to zero at infinity in some direction. This can be explained partially as follows. In one dimensional case the derivative of a bounded regular potential is compact relatively to H_0. This property can be lost in the multi-dimensional case if V or $\partial_{x_1} V$ does not tend to zero at infinity in some direction. Let us mention, however, Jensen's work [10] dealing with some non-decaying potential. But this result is basically one dimensional as it is explained by the author's proofs. Our main result is the following theorem. For a function W we denote by and W_+ its positive part defined by $W_+(x) = \sup(0, W(x))$. Let us set $x = (x_1, x')$ with $x' = (x_2, \cdots, x_d)$.

Theorem 3 *Assume that V and $\partial_{x_1} V$ are bounded and that $\|(\partial_1 V)_+\|_\infty < F$. Then H has no eigenvalues.*
If moreover

$$\int_0^1 \sup_{x \in \mathbb{R}^d} |V(x_1 + \varepsilon, x') - 2V(x) + V(x_1 - \varepsilon, x')| \frac{d\varepsilon}{\varepsilon} < \infty \tag{1.3}$$

then H has no singular continuous spectrum.

Remarks

1. The boundedness of the potential V is not needed. Actually, V has only to be such that H defines a self-adjoint operator in $\mathcal{H}$. Let us mention, however, that in Theorem 1 this fact is not true. Indeed, in that case the boundedness of V plays a fundamental role in our argument.

2. It is not difficult to show that the assumption $\|(\partial_1 V)_+\|_\infty < F$ can be replaced by only the smallness at infinity, i.e.

$$\limsup_{|x|\to\infty} (\partial_1 V)_+(x) < F.$$

   In such case, a possible set of eigenvalues of H can appear. But all these eigenvalues are finitely degenerate and form at most a discrete set.

Examples.

1. It is clear that our Theorem covers the case when $\partial_1 V$ tends to zero at infinity, but without asking that the potential V itself tends to zero at infinity as it is usually assumed for example in the papers cited above.

2. In the second example we give another kind of potentials satisfying our assumptions. Let $q : \mathbb{R}^d \longrightarrow \mathbb{R}$ be a bounded function satisfying the regularity assumption of Theorem 1.2. Let us set the potential

$$V_\lambda(x) = q(\lambda x_1, x'),$$

   Where λ is a real number playing the role of a coupling constant. Then there exist a constant $\lambda_0 > 0$ such that for each $|\lambda| \leq \lambda_0$ the operator $H_\lambda = H_0 + V_\lambda$ has a purely absolutely continuous spectrum.

One can think that the derivative $\partial_1 V$ does not need to be small at infinity, and that assumption is only related to the used approach. In the following Theorem we illustrate the importance of the tail of such partial derivative in the preservation of the absolute continuity of the spectrum, or more precisely in the possible occurrence of a rich singular spectrum. Indeed, we show that if the derivative $\partial_1 V$ is only bounded (but not sufficiently small) then a dense point spectrum could appear.

Theorem 4 *Let λ be a fixed real number. Then there exist a real-valued potential $V \in C^\infty(\mathbb{R}^d)$ such that V and $\partial_1 V$ are bounded and $[\lambda, \infty) \subset \sigma_{pp}(H)$.*

The paper is organized as follows. In section 2 we shall describe what we need in our proofs. Section 3 is devoted to prove Theorem 1.1 and 1-3. In section 4 we prove Theorem 2 and 4.

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2 Basic notions

Our proofs are based on the conjugate operator method. It is an abstract theory which proves that an Hamiltonian H has an absolutely continuous spectrum if it has a conjugate operator A, i.e. a self-adjoint operator such that the commutator $[H, iA]$ is strictly positive in an adequate sense. In this section we give a short description of the main points of this theory, which will be used in our proofs. For more details concerning the results of this section we refer the reader to the Amrein, Boutet de Monvel and Georgescu's monograph [1] and the articles [4, 13].

Let H, A be two self-adjoint operators in a Hilbert space $\mathcal{H}$. The C_0-group associated to A will be denoted by $e^{-iAt}, t \in \mathbb{R}$. We shall denote by $R(z) = (H - z)^{-1}$ the resolvent of H for a complex number $z \in \mathbb{C} \setminus \sigma(H)$.

Definition 1 *1. We say that H is of class $C^1(A)$ if the map*

$$\mathbb{R} \ni t \longmapsto e^{-iAt} R(z) e^{iAt} \in B(\mathcal{H})$$

is strongly of class C^1, for some (and so for any) complex $z \in \mathbb{C} \setminus \mathbb{R}$

2. We say that H is A-regular if

$$\int_0^1 \| e^{-iA\varepsilon} R(z) e^{iA\varepsilon} - 2R(z) + e^{iA\varepsilon} R(z) e^{-iA\varepsilon} \| \frac{d\varepsilon}{\varepsilon} < \infty.$$

Remark the fact that if H is A-regular then H is of class $C^1(A)$. Assume that H is of class $C^1(A)$. Then the intersection $D(A) \cap D(H)$ is dense in $D(H)$ equipped with the graph topology associated to the norm

$$\|f\|_H = \|f\| + \|Hf\|.$$

Moreover the sesquilinear form defined on $D(A) \cap D(H)$ by

$$< f, [H, A]g >=< Af, Hg > - < Hf, Ag >$$

extends continuously to $D(H)$. Then one can define the open set $\tilde{\mu}^A(H)$ of the real point λ for which there exist a constant $a > 0$, a compact operator K in $\mathcal{H}$ such that

$$E(\Delta)[H, iA]E(\Delta) > aE(\Delta) + K, \tag{2.4}$$

for some open interval Δ containing λ. Here-above E denotes the spectral measure of H.

Proposition 1 *The eigenvalues of H contained in $\tilde{\mu}^A(H)$ are all finitely degenerate and cannot accumulate in $\tilde{\mu}^A(H)$.*

Then in particular the spectrum of H is purely continuous in

$$\mu^A(H) := \tilde{\mu}^A(H) \setminus \sigma_p(H),$$

where $\sigma_p(H)$ denotes the set of eigenvalues of H. One can prove easily that $\mu^A(H)$ is in fact the set of all real point λ for which the Mourre estimate (2.4) holds with $K = 0, i.e$; there exist a constant $a > 0$ and an open interval $\Delta \ni \lambda$ such that

$$E(\Delta)[H, iA]E(\Delta) > aE(\Delta), \tag{2.5}$$

The following theorem shows that if H is A-regular then H has no singular continuous spectrum in $\mu^A(H)$.

Theorem 5 *Assume that e^{iAt} leaves invariant the domain $D(H)$ and that H is A-regular. Then H is purely absolutely continuous in $\mu^A(H)$.*

Remarks.

1. This Theorem is proved in [1], where one can found another version of this theorem in which the invariance of the domain under the action of e^{iAt} is replaced by the fact that H has a gap in its spectrum (which is clearly inadequate for the model considered here). In [13] we have eliminated this condition on the domain without asking the existence of a spectral gap for H, but we ask H to be (locally) slightly more than $A-$regular.

2. Let us mention that if H is of class $C^1(A)$ and that $[H, iA]$ is a bounded operator in $\mathcal{H}$ then e^{iAt} leaves invariant the domain of H (see [8] for a sharper statement).

3 Proof of Theorem 1

(i) By straightforward computations we get

$$[H_0, iA] = F$$

where $A = -id/dx$ is the translation generator. Then H_0 is A-regular (in fact is even of class $C^\infty(A)$ in the sense that the map of Definition 1 is of class C^∞), and

$$\mu^A(H_0) = \mathbb{R}.$$

On the other hand

$$[V, iA] = -V'(x),$$

which is obviously bounded in $\mathcal{H}$ if and only if V' is bounded as function.
To conclude the first part of Theorem 1 we have to prove that the Mourre estimate holds locally on $\mathbb{R}$, i.e. $\tilde{\mu}^A(H) = \mathbb{R}$. For this, and according to

$$E(\Delta)[H, iA]E(\Delta) = FE(\Delta) - E(\Delta)V'(x)E(\Delta), \tag{3.6}$$

it is sufficient to prove that $E(\Delta)V'(x)E(\Delta)$ is a compact operator in $\mathcal{H}$. But this property is a simple consequence of the following assertion which is proved by Bentosela and all in [3] (see also [5]). If a function G is uniformly continuous and

$$\lim_{r\longrightarrow\infty}\sup_{x\in\mathbb{R}}|\frac{1}{r}\int_{x-r}^{x+r}G(y)dy|=0 \tag{3.7}$$

then $E(\Delta)G(x)E(\Delta)$ is a compact operator in $\mathcal{H}$. Clearly if V' is bounded and uniformly continuous, and V is bounded (which are ensured by our assumptions) then $G=V'$ satisfies (3.7). Then $\mu^A(H)=\mathbb{R}$ follows from (3.6) and (3.7), and Proposition 1 finishes the proof of part 1 of Theorem 1.
(ii) Now on let us prove the second part of Theorem 1. According to Theorem 5 and the last part of our proof we have only to show that H is A-regular and that $D(H)$ is invariant under the action of e^{iAt}. According to the obvious property

$$e^{-iA\varepsilon}Ve^{iA\varepsilon}=V(x-\varepsilon).$$

the operator V is A-regular if and only if the function V is smooth in Zygmund's sense, which is exactly our assumption (1.2). Then the operator H is A-regular.
The invariance of the domain $D(H)$ under the action of e^{iAt} follows from the fact that the commutator $[H,iA]=1-V'$ is a bounded operator in $\mathcal{H}$. This finishes the proof of Theorem 1.

4 Proof of Theorem 3

We shall prove that the self-adjoint operator

$$A=-i\partial_{x_1}$$

is strictly conjugate to H on $\mathbb{R}$. Indeed, we have

$$[H_0,iA]=F>0$$

In particular H_0 is of class $C^\infty(A)$ and A is strictly conjugate to H_0 on $\mathbb{R}$, i.e. $\mu^A(H_0)=\mathbb{R}$. On the other hand, we have

$$e^{-i\varepsilon A}Ve^{i\varepsilon A}=V(x_1-\varepsilon,x').$$

Then it is clear that the regularity assumptions on the function V of Theorem 3 ensure that V is A-regular, and so H too (since H_0 is). Moreover, the commutator

$$[H, iA] = F - \partial_{x_1} V$$

is obviously bounded operator in $\mathcal{H}$. And so the domain of H is invariant under the action of e^{iAt}. It is remaining to show that $\mu^A(H) = \mathbb{R}$. But this property follows easily as follows:

$$\begin{aligned}
[H, iA] &= F - \partial_{x_1} V \\
&\geq F - (\partial_{x_1} V)_+(x) \\
&\geq F - \|(\partial_{x_1} V)_+\|_\infty > 0
\end{aligned}$$

which is a global and strict Mourre estimate. This finishes the proof of the second part of Theorem 3.

Remarks

1. Let us remark that when we replace the global condition $\|(\partial_{x_1} V)_+\|_\infty < F$ by

$$\limsup_{|x| \longrightarrow \infty} (\partial_{x_1} V)_+) < F$$

we will have only a local Mourre estimate:

$$E(\Delta)[H, iA]E(\Delta) \geq aE(\Delta) + K,$$

where K is a compact operator, $a = F - \limsup_{|x| \longrightarrow \infty} (\partial_{x_1} V)_+ > 0$ and Δ is any compact interval. Which means that $\tilde{\mu}^A(H) = \mathbb{R}$. Thus according to Proposition 1 there is only a discrete set of possible eigenvalues of H and all these eigenvalues are finitely degenerate.

2. We also mention the fact that our argument ignore completely if the particle is relativistic or non-relativistic. More precisely, our proof still valid for any operator of the form

$$H = h(-i\nabla) - F \cdot x_1 + V(x)$$

where h is a divergent continuous function and F, V are as in Theorem 1.2. An physical interesting situation is when $h(x) = \sqrt{1+|x|^2}$. In such case H becomes

$$H = \sqrt{-\Delta + 1} - F \cdot x_1 + V(x),$$

which describes the motion of a charger relativistic particle in a constant electric field. However, it is not clear whether or not Theorem 1 still valid. To do this, we have to describe the relative compact operator V with respect to $H_0 = h(P) - F \cdot x_1$.

5 Proof of Theorem 2 and 4

Let us start by proving that Theorem 4 follow easily from Theorem 2 combined with Simon's result [15]. Assume that $d = 2$ and that

$$V(x_1, x_2) = V(x_1) + V(x_2).$$

Then by separation of variable we decompose H as follows

$$H = H_1 \otimes 1 + 1 \otimes H_2 \tag{5.8}$$

acting in $L^2(\mathbb{R}) \otimes L^2(\mathbb{R})$, where

$$H_1 = -\frac{d^2}{dx_1^2} - x_1 + V_1(x_1); \tag{5.9}$$

$$H_2 = -\frac{d^2}{dx_2^2} + V_2(x_2). \tag{5.10}$$

Let λ be a fixed real number. Theorem 2 tell us that there exist a real-valued potential V_1 of class C^∞ such that V_1 and V_1' are bounded and that λ is an eigenvalue of H_1. On the other hand, let $\{\lambda_i\}_i$ be a sequence of positive numbers. Then there exist (cf. Simon [15]) a potential V_2 of class C^∞ such that each λ_i is an eigenvalue of H_2. In particular, if the sequence $\{\lambda_i\}_i$ is dense in $[0, \infty)$ (which we assume from now) then we get that $[0, \infty) \subset \sigma_{pp}(H_1)$. But the decomposition (5.8) of H implies that the numbers $\lambda + \lambda_i$ belong to the set $\sigma_p(H)$ of the eigenvalues of H. It follows then $[\lambda, \infty) \subset \sigma_{pp}(H)$. On the other hand, the potential $V = V_1 + V_2$ is of class C^∞ and $\partial_1 V(x) = V_1'(x_1)$ which is clearly bounded. Finally, an obvious induction allows us to do the same construction for any dimension d.

Proof of Theorem 2.

(1) Without loss of generality one can assume that $\lambda = 0$. We recall that our goal is to construct a real-valued potential V of class $C^\infty(\mathbb{R})$ such that:

(i) V and V' are bounded and

(ii) there exist $u \in L^2(\mathbb{R})$ solution of

$$-u'' - xu + Vu = 0.$$

Let us apply the Liouville transformation by setting for $x \geq 1$

$$\xi = \frac{3}{2} x^{\frac{2}{3}}, \qquad w(\xi) = x(\xi)^{\frac{1}{4}} u(x(\xi)).$$

One can show that (cf. [12]) the problem (ii) is equivalent to:

(ii)' there exist a function $w \in L^2((1, \infty), \xi^{\frac{-2}{3}} d\xi)$ solution of

$$-w'' + q(\xi) w = w; . \tag{5.11}$$

The relation between q and V is

$$q(\xi) = \frac{V(x(\xi))}{x(\xi)} + \frac{5}{36\xi^2}$$

or equivalently

$$V(x) = x(q(\xi(x)) - \frac{5}{36\xi(x)^2}).$$

Moreover, it is not difficult to see that if q of class $C^\infty(1, \infty)$ such that for each integer $m \geq 0$

$$q^{(m)}(\xi) = O(\xi^{-1}), \quad \xi \longrightarrow \infty$$

then V will be of class $C^\infty(\mathbb{R})$, supp $V \subset [1, \infty)$, and is bounded with its derivative (i.e. (i) is satisfied). Let us mention however that $V''(x) = O(\sqrt{x})$ as $x \to \infty$.

(2) We shall now construct q. For this let us apply the Prüfer transformation by setting

$$\begin{cases} w = R\cos\phi \\ w' = R\sin\phi, \end{cases}$$

Straightforward computations give the equations

$$\begin{cases} \frac{R'}{R} = & \frac{1}{2}q\sin 2\phi \\ \phi' = & -1 - q\cos^2\phi, \end{cases} \tag{5.12}$$

or equivalently

$$\begin{cases} R^2 = & C\exp\int_1^\xi q\sin 2\phi dt \\ \phi' = & -1 - q\cos^2\phi, \end{cases} \tag{5.13}$$

The potential q will have the form

$$q(x) = \sum_{k\geq 1} q_k j(\frac{\xi - \xi_k}{\Delta})$$

where ξ_k and q_k are two adequate sequences we have to construct, while $\Delta > 0$ will be choosed sufficiently small and $j \in C^\infty((0,1)), j(x) \geq 0, \int j(x) = 1$. Assume that

$$q\sin 2\phi \leq -\frac{1}{2}q,$$

which is ensured if $\phi(x) \approx 3\pi/4$ on the interval $[\xi_k, \xi_k + \Delta]$. Then (remark that $q = 0$ on $[\xi_k + \Delta, , \xi_{k+1}]$

$$\begin{aligned}
\int_1^\infty R^2 \frac{d\xi}{\xi^{2/3}} &= C\int_1^\infty [\exp\int_1^\xi q\sin 2\phi dt]\frac{d\xi}{\xi^{2/3}} \\
&\leq C_1 + C_2\sum_{k>0}\int_{\xi_k}^{\xi_{k+1}} \exp(\int_1^\xi q\sin 2\phi dt)\frac{d\xi}{\xi^{2/3}} \\
&\leq C_1 + C_2\sum_{k>0}\int_{\xi_k}^{\xi_{k+1}} \exp(-\frac{1}{2}\int_1^\xi qdt)\frac{d\xi}{\xi^{2/3}} \\
&\leq C_1 + C_2\sum_{k>0}[\xi_{k+1} - \xi_k]\exp(-\frac{1}{2}\int_1^{\xi_k} qdt) \\
&\leq C_1 + C_2\sum_{k>0}[\xi_{k+1} - \xi_k]\exp -\frac{\Delta}{2}\sum_{j=1}^{j=k} q_j.
\end{aligned}$$

Then if

$$\begin{cases} \xi_{k+1} - \xi_k = O(1) \\ q_k = \frac{C}{\xi_k} \end{cases}, \tag{5.14}$$

where C is sufficiently large constant, then the right member of the last inequality is convergent, which means that w lies in $L^2((1,\infty), \xi^{\frac{-2}{3}} d\xi)$. Moreover, by construction the potential q satisfies

$$q^{(m)} = (1/\xi), \quad \text{as } x \longrightarrow \infty, \quad \forall m \geq 0.$$

Thus, it is sufficient to construct by induction a sequence ξ_k such that: $\phi(\xi_k) = 3\pi/4(\text{mod } \pi)$ and $q(\xi) = q_k j(\frac{\xi - \xi_k}{\Delta})$ on the interval $[\xi_k, \xi_k + \Delta]$; and $q(\xi) = 0$ on the interval $[\xi_k + \Delta, \xi_{k+1}]$.
Assume that ξ_k is constructed, and let us set $q(\xi) = 0$ for $\xi > \xi_k + \Delta$. Integrating the equation (5.13) between $\xi_k + \Delta < \xi$ we get

$$\phi(\xi) = -\xi + \xi_k + \Delta + \phi(\xi_k + \Delta).$$

Let us choose ξ_{k+1} as the nearest point on the right of ξ_k such that $\phi(\xi_{k+1}) = 3\pi/4(\text{mod } \pi)$. We also have,

$$\xi_{k+1} - \xi_k = \phi(\xi_k + \Delta) - \phi(\xi_{k+1}) - \Delta = O(1).$$

Let us integrate (5.13) between ξ_k and $\xi \leq \xi_k + \Delta$ to get

$$| \phi(\xi) - \phi(\xi_k) | \leq \Delta + \int_{\xi_k}^{\xi} q dt \leq \Delta + \Delta q_k.$$

For Δ sufficiently small we ensure the fact that $\phi(\xi) \approx 3\pi/4(\text{mod } \pi)$ on $[\xi_k, \xi_k + \Delta]$. the construction is now completed.

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