Smoothing properties of the heat semigroups associated to Hamiltonians describing point interactions in one and two dimensions

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Abstract

Smoothing properties of the heat semigroups associated to Hamiltonians describing point interactions in on one and two dimensions are investigated. A construction of Hamiltonians describing point interaction on L^p spaces is then derived and a full description of their spectra is given. Particularly we prove the p-independence of their spectra and the exponential growth of the p-norms of such semigroups for large time.

1 Introduction

By the heat semigroup we mean the semigroup associated to the heat equation

$$\frac{\partial \psi}{\partial t} = -H\psi \tag{1}$$

where H is a selfadjoint operator, we will denote it by $\exp(-tH)$. The most important case in applications is when H is a generalized Schrödinger operator on $\mathbb{R}^d$, this is a perturbation of minus the Laplacian by a suitable measure (for example in the generalized Kato class) and especially by a potential. Constructions and investigations of the heat semigroups $\exp(-tH)$ where H is a Schrödinger operator were subject to an extensive literature, rich in its methods (analytic, probabilistic, or related to operator theory) and contents. In the survey paper [19], B.Simon proved L^p-smoothing property of $\exp(-tH)$ where $H = -\triangle + V$ for a large class of potentials: If the negative part of the potential is in the Kato class and its positive part is in the local Kato class then $\exp(-tH)$ maps continuously L^p into L^q, $1 \leq p \leq q \leq \infty$. In the same paper, L^p-growth of these semigroups is also studied. For instance it is proved that for the above described class of potentials $\| \exp(-tH)\|_{p,p}$ has an exponential growth for large t, this is

$$\| \exp(-tH)\|_{p,p} \ \leq \ C\exp(\alpha t) \tag{2}$$

for large t where C is a positive constant and $\alpha \in I\!\!R$. But if $d = 1,\ 2$ and $V < 0$, then because of the existence of negative eigenvalues of H one has $\alpha > 0$. Among other interesting features it is proved [19] that the spectral bound $\inf \sigma(H) = -\lim_{t\to+\infty} t^{-1} \ln \| \exp(-tH)\|_{p,p}$ is p-independent. This result is now a well known fact, for instance R.Hempel and J.Voigt [15],[16] proved that for certain potentials (including those in the Kato class) the spectrum of H is p-independent as well. A more detailed answer to the question asked by B.Simon about p-independence of spectra of Schrödinger operators can be found in [16].

Later on Ph.Blanchard and Z.M.Ma [4], proved also The L^p-smoothing property for the heat semigroups associated to $H = -\triangle + \mu$ on $I\!\!R^d$ $(d \geq 3)$ where μ is a signed measure whose positive part is smooth and negative part is in the generalized Kato class. In an abstract setting, P.Stollmann and J.Voigt [20], investigated the properties of $\exp(-tH\mu)$, $H_\mu = -\triangle + \mu$ on the space $L^2(X, m)$ where m is a measure whose support is X.

Recently (Cf.[13] and references therein) a particular interrest was given to study smoothing properties of $\exp(-tH)$ where $H = -\frac{1}{2}\triangle + V$ on the scale of Bessel potential spaces on $I\!\!R^d$. In [14], the authors proved boundedness of $\exp(-tH)$ from L^p into $L^{p,s}_{\rm loc}$ for potentials $V \in L^{p,s}_{\rm loc}$, while A.Gulisashvili [13] gave a sharp estimate of the $L^p, L^{p,s}_{\rm loc}$ norm of $\exp(-tH)$ for $V \in L^{p,s} \cap K_d$ where K_d is the set of potential in the Kato class.

Untill now operators describing point interactions in $I\!\!R^d$ does not fit into a standard situtation.

What are Hamiltonians describing point interaction or also Schrödinger operators with interactions: These are operators corresponding to perturbations of minus the Laplacian by linear combination of Dirac measures. The mathematical setting and the description of such Hamiltonians are now quite known

[3] and even a new approach to handel them is given in [5]. For denoting these operators [3] by $H_{\delta,\alpha} = -\triangle + \alpha\delta$, it is easily seen that for $d = 2, 3$ the Dirac measure is not in the Kato class, while for $d = 1$ it is in the Kato class but for negative alpha the theory develloped in [20] does not unclude such operators. However we are some what saved by the explicit knowledge of the kernels associted to $H_{\delta,\alpha}$ [1]. Our aim in this paper is to use these formula to establish p, q smoothing properties of $H_{\delta,\alpha}$ on $L^p(I\!R^d)$ for $d = 1, 2$. As a consequence we get a construction of the operators $H_{\delta,\alpha}$ on L^p-spaces for $1 \leq p < +\infty$ which we denote by $H_{\delta,\alpha,p}$ (for $p = 2$ we omit the subscript p). Then combining techniques used by J.Voigt and P.Stollmann [15],[16] and the new functionnal calculus develloped by E.B.Davies [8] we prove the p-independence of their spectra and derive the exponential growth of the p-norms of $\| \exp(-tH_{\delta,\alpha,p}) \|$.
Such construction was done by S.Albeverio, Z.Brzezniak and L.Dabrowsky [2] using "family of pseudo-resolvent" exploiting thereby the expression of the resolvent kernel of $H_{\delta,\alpha}$. They conclude the construction for $d = 1, p \in [1, +\infty)$ or $d = 2, p \in]1, +\infty[$ or $d = 2, p \in]\frac{3}{2}, 3[$ and the same thing for the C_0-semigroup $\exp(-tH_{\delta,\alpha})$. We will make here the reversed walk: However using the explicit formula of the heat kernel associated to $H_{\delta,\alpha}$ (Cf.[1]) which we denote as in [1] by $P^\alpha(t; x, y)$ we prove that for $d = 1, 2$, this kernel defines also a bounded linear operator from L^p into L^q for $1 \leq p \leq q \leq +\infty$. This is the known "smoothing property". Moreover this kernel defines even a strongly continuous semigroup on L^p for $1 \leq p < +\infty$. Then we denote by $-H_{\delta,\alpha,p}$ the generator of $\exp(-tH_{\delta,\alpha})$ on L^p $(1 \leq p < +\infty)$. Using the integral representation of the resolvent function [9]p.55 (which is the Laplace transform of the semigroup) we get that for k^2 such that $\mathrm{Im}(k) > 0$ and $\mathrm{Re}(k^2) < \min(s(H_{\delta,\alpha}), s(H_{\delta,\alpha,p}))$, $(H_{\delta,\alpha,p} - k^2)^{-1}$ is a kernel operator whose kernel is G_k, where G_k is the kernel of $(H_{\delta,\alpha} - k^2)^{-1}$. Hence for $d = 1, 2$ the construction we propose here includes the one made in [2]. A question arises: why is it so? This is related to the properties of the heat kernel. For instance the heat kernel has better properties than the resolvent kernel (boundedness, sommabilty). A good example is a comparaison between the resolvent and the heat kernel of $-\triangle$.
Unfortunately this method does not work in three dimensions for a reason that we will explain at the end of the paper.
For the notations we will adopt those of [3]. So $H_{\delta,\alpha}$ is Hamiltonian descrabing point interaction placed at the origin in tghe space L^2, where alpha is,

for $d=1$ the coupling constant of the interaction (Cf.[3]p.77) and for $d=2$, $(-2\pi\alpha)^{-1}$ is the scattering lenth of $H_{\delta,\alpha}$ if $\alpha \neq +\infty$ (Cf.[3]p.99), while the free Hamiltonian is denoted by H_0. The space $L^p(I\!R^d)$ is denoted simply L^p for every $1 \leq p \leq +\infty$. For every linear closed operator T the spectrum of T is denoted $\sigma(T)$ whereas its spectral bound is denoted $s(T)$ and is defined by [9]

$$s(T) = \sup\{\mathrm{Re}(\lambda),\ \lambda \in \sigma(T)\} \tag{3}$$

2 Smoothing property in two dimensions

Following the notation adopted in [1], we denote the heat kernel of $\exp(-tH_{\delta,\alpha})$ by $P^\alpha(t;x,y)$ for every $t>0$. It is equal to [1]

$$P(t;x,y) + \frac{1}{2\pi}\int_0^{+\infty} t^{u-1}\frac{e^{(-\alpha u)}}{\Gamma(u)} \int_1^{+\infty}(z-1)^{u-1}z^{-u}e^{(-z\frac{|x|^2+|y|^2}{4t})}K_0(\frac{|x||y|}{2t}z)dzdu \tag{4}$$

where $P(t;x,y) = \frac{1}{2\pi t}\exp(-\frac{|x-y|^2}{2t})$ is the kernel of the free Hamiltonian and K_0 is the Mac-Donald function. We will also denote by $\tilde{P^\alpha}(t;x,y)$ the term $P^\alpha(t;x,y) - P(t;x,y)$.
The way do prove the smoothing property of $\exp(-tH_{\delta,\alpha})$ is as develloped in [19]: first prove the boundedness from L^∞ into L^∞ and boundedness from L^1 into L^∞, then by the use of Ries-Thorin convexity theorem [18] conclude the desired result. The first step is given by this lemma

Lemma 2.1 *For every $t>0$ we have*

$$\sup_{x\in I\!R^2}\int_{I\!R^2} P^\alpha(t;x,y)dy \ < +\infty \tag{5}$$

Proof: Since $\sup_{x\in I\!R^2}\int_{I\!R^2} P(t;x,y)dy = 1$ we just have to prove that

$$\sup_{x\in I\!R^2}\int_{I\!R^2} \tilde{P^\alpha}(t;x,y)dy \tag{6}$$

is finite. Denote $I(x) = \int_{I\!R^2} \tilde{P^\alpha}(t;x,y)dy$, we have [1]

$$\tilde{P^\alpha}(t;x,y) = \frac{1}{2\pi}\int_0^{+\infty} t^{u-1}\frac{e^{-\alpha u}}{\Gamma(u)} \int_1^{+\infty}(z-1)^{u-1}z^{-u}e^{(-z\frac{|x|^2+|y|^2}{4t})}K_0(\frac{|x||y|}{2t}z)dzdu \qquad (7)$$

Denote $J(x) = \int_{I\!R^2} e^{(-z\frac{|x|^2+|y|^2}{4t})}K_0(\frac{|x||y|}{2t}z)dy$. Then

$$I(x) = \frac{1}{2\pi}\int_0^{+\infty} t^{u-1}\frac{e^{-\alpha u}}{\Gamma(u)}\int_1^{+\infty}(z-1)^{u-1}z^{-u}J(x)dzdu \qquad (8)$$

Let us make a suitable estimate for $J(x)$. Using polar coordinates we get $J(x) = 2\pi e^{(-z\frac{|x|^2}{4t})}\int_0^{+\infty} e^{(-z\frac{r^2}{4t})}K_0(z\frac{r|x|}{2t})rdr$. With the change of variable $s = z\frac{r|x|}{2t}$ one get $J(x) = 2\pi(\frac{2t}{z|x|})^2\int_0^{+\infty} e^{(-\frac{t}{2z|x|^2}s^2)}sK_0(s)ds$ which is equal to (Cf [12]p.717, 3)

$$2\pi e^{(-z\frac{|x|^2}{4t})}\frac{1}{2}(\frac{z|x|^2}{t})^{\frac{1}{2}}e^{(\frac{z|x|^2}{8t})}W_{-\frac{1}{2},0}(\frac{z|x|^2}{4t}) \qquad (9)$$

where $W_{\chi,\mu}$ is the Whittaker function. On the other hand one has (Cf. [17]p.305) $W_{-\frac{1}{2},0}(z) = z^{\frac{1}{2}}e^{(\frac{z}{2})}$ which gives $J(x) = \frac{2\pi}{\sqrt{2}}\frac{t}{z}$. Thus

$$\sup_{x\in I\!R^2} I(x) = \frac{1}{\sqrt{2}}\int_0^{+\infty} t^u \frac{e^{-\alpha u}\Gamma(u)}{\int}{}_1^{+\infty}(z-1)^{u-1}z^{-u-1}dzdu \qquad (10)$$

$$= \frac{1}{\sqrt{2}}\nu(te^{-\alpha}) \qquad (11)$$

where the function ν is as defined in [10]. □
We consequensely get the following

Proposition 2.1 *For every $t > 0$ and evey p such that $1 \le p \le +\infty$ the operator*

$$\exp(-tH_{\delta,\alpha}) : \; L^p \longrightarrow \; L^p \qquad (12)$$

is bounded and defined a strongly continuous semigroup for $1 \le p < +\infty$.

Proof: First let us denote $C(t) = 1 + \sup_{x\in I\!R^2} I(x) = 1 + \dfrac{1}{\sqrt{2}}\nu(te^{-\alpha})$. For $1 < p \le +\infty$, let q be the conjugate of p : $p^{-1} + q^{-1} = 1$. Then by Hölder inequality we have for every $f \in L^p$

$$|\exp(-tH_{\delta,\alpha})f(x)|^p \le \ (C(t))^{\frac{p}{q}}(\int_{I\!R^2} P^\alpha(t;x,y)|f(y)|^p dy)^{\frac{1}{p}} \tag{13}$$

thereby

$$|\int_{I\!R^2} \exp(-tH_{\delta,\alpha}f(x)dx|^p \le \ (C(t))^{\frac{p}{q}} \int_{I\!R^2} |f(y)|^p (\int_{I\!R^2} P^\alpha(t;x,y)dx)dy \tag{14}$$

Now by the symmetry property of the heat kernel we get

$$\int_{I\!R^2} |\exp(-tH_{\delta,\alpha})f(x)|^p dx \le \ (C(t))^p \int_{I\!R^2} |f|^p dy \tag{15}$$

To prove that it defines a strongly continuous semigroup, lets denote $S(t)$ the operators whose kernel is $\tilde{P}^\alpha$ then we have $\|S(t)\|_{p,p} \le \ \frac{1}{\sqrt{2}}\nu(te^{-\alpha})$. By the properties of the funcion ν [10]p.219, we have $\lim_{t\to 0}\nu(t) = 0$ we then get

$$\lim_{t\to 0} \exp(-tH_{\delta,\alpha}) = I \tag{16}$$

Now defining $T_\alpha(t)$ by: $T_\alpha(t) = \exp(-tH_{\delta,\alpha})$ for $t > 0$ and $T_\alpha(0) = I$ we get a strongly continuous semigroup, which we always denote by $\exp(-tH_{\delta,\alpha})$. For $p = 1$ the proof is straightforward. $\square$

We are going now to prove boundedness from L^1 into L^∞.

Lemma 2.2 *For every $t > 0$ we have*

$$\sup_{x,y\in I\!R^2} P^\alpha(t;x,y) < +\infty \tag{17}$$

Proof: For this goal we are going to use the construction of the Hamiltonian $H_{\delta,\alpha}$ via Dirichlet forms as done in [1] which we recall here. We omit the case $\alpha = \infty$, which corresponds to the free Hamiltonian. Let φ_α be the following function

$$\varphi_\alpha(x) = H_0^{(1)}(2ie^{-2\pi\alpha+\Psi(1)}|x|), \ x \in I\!R^2 \setminus \{0\} \tag{18}$$

where $H_0^{(1)}$ is the Hankel function. Denote H_{φ_α} the operator associated to the local positive Dirichlet form

$$\mathcal{E}_{\varphi_\alpha} : \mathcal{D}(\mathcal{E}_{\varphi_\alpha}) \subset \ L^2(\varphi_{\alpha^2}), \ \mathcal{E}_{\varphi_\alpha}(f,g) = \int_{I\!R^2} \nabla f \bar{\nabla} g \varphi_\alpha^2 dx. \tag{19}$$

Then the Hamiltonian $H_{\delta,\alpha}$ is related to this Dirichlet form as follow [3]

$$H_{\delta,\alpha} = \varphi_\alpha [H_{\varphi_\alpha} - \beta I] {\varphi_\alpha}^{-1} \tag{20}$$

where $\beta = 4e^{2(-2\pi\alpha+\Psi(1))}$. Clearly $\exp(-tH_{\delta,\alpha}) = e^{\beta t}\varphi_\alpha \exp(-tH_{\varphi_\alpha}){\varphi_\alpha}^{-1}$. Now if we denote $q^\alpha(t;x,y)$ the heat kernel of the operator $\exp(-tH_{\varphi_\alpha})$, then $P^\alpha(t;x,y) = e^{\beta t}\varphi_\alpha(x){\varphi_\alpha}^{-1}(y)q^\alpha(t;x,y)$. Now since the Dirichlet form $\mathcal{E}_{\varphi_\alpha}$ is local and positive, then the kernel $q^\alpha(t;x,y)$ is Markovian [11] hence $0 < q^\alpha(t;x,y) \leq 1$. Now we have on the diagonal the following estimate

$$P^\alpha(t;x,x) \ \leq \ e^{\beta t} \tag{21}$$

Using the Chapman-Kolmogorov equation:

$$P^\alpha(t+s;x,y) = \int_{I\!R^2} P^\alpha(s;x,z)P^\alpha(t;z,y)dz \tag{22}$$

we get $P^\alpha(t;x,y) \leq \ e^{\beta t}$ which completes the proof. □

We now achieve

Theorem 2.1 *For every $t > 0$ and every $1 \leq p \leq q \leq \infty$ the operator*

$$\exp(-tH_{\delta,\alpha}) : \ L^p \longrightarrow \ L^q \tag{23}$$

is bounded.

Theorem(2.1) gives a variety of properties of the heat semigroup $\exp(-tH_{\delta,\alpha})$ known for a wide class of Schrödinger operators at least for operators $H = -\triangle + V$ where V is in the Kato class.

For every $1 \leq p < +\infty$, let us denote $-H_{\delta,\alpha,p}$ the generator of $\exp(-tH_{\delta,\alpha})$ on L^p which will be denoted $\exp(-tH_{\delta,\alpha,p})$. For $p = 2$ we omit the index p. Interpretation of point interaction on L^p spaces as extention of $-\triangle$ on $D_0 = \{f \in C_0^\infty, \ f(0) = 0\}$ was done in [6]. There the authors show that point interaction can be defined on L^p for $d = 1,2$ and $1 < p < +\infty$ or $d = 3$ and $\frac{3}{2} < p < 3$ as negative generators of analytic semigroups. Their

construction is essentially based on estimate of the resolvent of $H_{\delta,\alpha}$ with $\alpha = 1$. We here note that for $d = 1, 2$ we have less restrictions on p than in [2] or in [6].
Now the question of the p-independence of the spectra of $H_{\delta,\alpha}$ arises. Let us note that for $p = +\infty$ one can not hope to get the inclusion $\sigma(H_{\delta,\alpha}) \subset \sigma(H_{\delta,\alpha,p})$. For the only eigenfuction of $H_{\delta,\alpha}$ which is equal to $\frac{1}{..}H_0^{(1)}(2ie^{\beta}|x|)$ is unbounded.

Proposition 2.2 *For every $t > 0$ and p such that $1 \leq p < +\infty$ we have:*
i) The spectrum of $H_{\delta,\alpha,p}$ is p-independent.
ii) Every eigenvalue of $H_{\delta,\alpha}$ of algebraic nultiplicity m is an eigenvalue of $H_{\delta,\alpha,p}$ with the same multiplicity and conversely.
iii) Te spectral bound of $H_{\delta,\alpha,p}$ satisfies

$$-\lim_{t\to\infty} t^{-1} \ln \| \exp(-tH_{\delta,\alpha,p})\|_{p,p} = -4\exp(2(2\alpha + \Psi(1))) \tag{24}$$

hence it is also p-independent.

Proof: The proof of assertion (ii) is as in [15], (iii) follows from (i) and the characterization of the spectral bound of generators associated to strongly continuous semigroups [9]p.299. So the improttant point is to prove (i). Following R.Hempel and J.Voigt [15] we are going to prove first that

$$\sigma(H_{\delta,\alpha}) \subset \sigma(H_{\delta,\alpha,p}) \tag{25}$$

A crucial argument to prove(25) is that for every $\xi \in \rho(H_{\delta,\alpha}) \cap \rho(H_{\delta,\alpha,p})$ we have: for every $f \in L^p \cap L^q$,

$$R_\xi(H_{\delta,\alpha})f = R_\xi(H_{\delta,\alpha,p})f \tag{26}$$

In fact: Let $f \in L^p \cap L^q$ then clearly $\exp(-tH_{\delta,\alpha})f = \exp(-tH_{\delta,\alpha,p})f$, now by the integral representation of the resolvent function [9] we have for every ξ such that $\mathrm{Re}(\xi) < \min(s(H_{\delta,\alpha}), s(H_{\delta,\alpha,p}))$, $R_\xi(H_{\delta,\alpha})f = R_\xi(H_{\delta,\alpha,p})f$. Finally the analiticity of the resolvent functions gives the equality on $\rho(H_{\delta,\alpha}) \cap \rho(H_{\delta,\alpha,p})$. Once (26) is proved one can continue the proof of (25) as in [20].
Let us prove now the reversed inclusion. We may suppose that $2 \leq p \leq +\infty$, and then conclude by duality. Let $\xi \in \rho(H_{\delta,\alpha})$, $f \in L^p \cap L^q$, then $(H_{\delta,\alpha} - \xi I)^{-1}(H_{\delta,\alpha} - \xi I)f = f$ which implies that $(H_{\delta,\alpha} - \xi I)^{-1}(H_{\delta,\alpha} - \xi I)\exp(-tH_{\delta,\alpha,p})f = \exp(-tH_{\delta,\alpha})f$. On the other hand we have (Cf [16])

$(H_{\delta,\alpha}-\xi I)\exp(-tH_{\delta,\alpha,p})f = \exp(-tH_{\delta,\alpha})(H_{\delta,\alpha,p}-\xi I)$ we then get $(H_{\delta,\alpha}-\xi I)^{-1}\exp(-tH_{\delta,\alpha})(H_{\delta,\alpha,p}-\xi I)f = \exp(-tH_{\delta,\alpha,p})f$, taking the limit as t tends to 0 we get

$$(H_{\delta,\alpha}-\xi I)^{-1}(H_{\delta,\alpha,p}-\xi I)f = f \tag{27}$$

It is now sufficient to prove that $T(\xi) = (H_{\delta,\alpha}-\xi I)^{-1}$ is bounded as an operator on L^p. Indeed, from the kernel formula of $(H_{\delta,\alpha}-\xi I)^{-1}$ [3] we get that $T(\xi)$ is a closed operator in L^∞ whose domain is the whole space L^∞, hence by Banach theorem we conclude that $T(\xi)$ is bounded on L^∞. On the other hand it is bounded on L^2, thus by the Riesz-Thorin convexity theorem we get the boundedness of $T(\xi)$ on L^p for every $2 \le p \le +\infty$ which gives the result.□

Remark 2.1 *From proposition(2.2) we get*

$$\|\exp(-tH_{\delta,\alpha,p})\|_{p,p} \sim \exp(4t\exp(2\alpha+\Psi(1))) \tag{28}$$

for large t which expresses the exponential growtht of the p-norm of the operator $\exp(-tH_{\delta,\alpha,p})$*, while for small t we have*

$$\|\exp(-tH_{\delta,\alpha,p})\|_{p,p} \le \ 1 + \frac{C}{|\log(t)|} \tag{29}$$

In [4] it is proved that for $1 \le p < +\infty$ and every $f \in L^p$ the function $\exp(-tH)$ tends to zero at infinity. The same phenomenon occurs in our situatuion.

Proposition 2.3 *For $1 \le p < +\infty$ and $f \in L^p$ we have*

$$\lim_{|x|\to 0} |\exp(-tH_{\delta,\alpha,p})f(x)| = 0 \tag{30}$$

Proof: We give the proof for $p=2$, for $p \neq 2$ the proof is substantially the same.
The proof is based on the alternative formula of the heat kernel [1]Eq.(3.16) which we recall here:

$$\begin{aligned} P^\alpha(t;x,y) &= P(t;x,y) + \frac{e^{-\frac{A}{t}}}{(4\pi t|x||y|)^{\frac{1}{2}}}\int_0^{+\infty}\frac{t^u e^{-\alpha u}}{\Gamma(u)} \\ & \int_0^{+\infty}\frac{r^{u-1}}{(r+1)^{u+\frac{1}{2}}}e^{-\frac{A}{t}r}\widetilde{K_0}(\frac{|x||y|}{2t}(r+1))drdu \end{aligned} \tag{31}$$

where $A = \frac{(|x|+|y|)^2}{4}$ and $\widetilde{K_0}(z) = \sqrt{\frac{2z}{\pi}}\exp(z)K_0(z)$. Let's recall some properties of the function $\widetilde{K_0}$ established in [1].

$$\sup_{r\geq 0}|\widetilde{K_0}(r)| \;=\; M < +\infty \tag{32}$$

Taking into account that $\exp(-tH_0)f \to 0$, we should just prove that

$$\begin{aligned}\frac{1}{(4\pi t|x|^{\frac{1}{2}})} &\int_0^{+\infty}\frac{t^u e^{-\alpha u}}{\Gamma(u)}\int_0^{+\infty}\frac{r^{u-1}}{(r+1)^{u+\frac{1}{2}}}\\ &\int_{I\!R^2}\frac{e^{-\frac{A}{t}}}{(4\pi t|y|)}e^{-\frac{A}{t}r}\widetilde{K_0}(\frac{|x||y|}{2t}(r+1))f(y)dydrdu\end{aligned} \tag{33}$$

tends to zero for $|x|\to\infty$. We denote $A(x)$ the last term in Eq.(33), then $A(x) \leq\; M\int_{I\!R^2}\frac{1}{|y|^{\frac{1}{2}}}e^{-\frac{(|x|+|y|^2)}{4t}}|f(y)|dy$ which gives by Hölder inequality

$$A(x) \leq\; 2\pi^{\frac{1}{2}}M\|f\|_{L^2}(\int_0^{+\infty}e^{-\frac{s^2}{2t}}ds)^{\frac{1}{2}} \tag{34}$$

This yields

$$|\exp(-tH_{\delta,\alpha,p})f(x)| \leq\; |\exp(-tH_0)f(x)| + C\frac{\nu(te^{-\alpha})}{|x|^{\frac{1}{2}}}\|f\|_{L^2} \tag{35}$$

where $C > 0$, and this completes the proof. □

3 Smoothing property in one dimension

In one dimension the Hamiltonian $H_{\delta,\alpha}$ corresponds for $\alpha = 0$ to the free Hamiltonian, so along this section we will omit the case $\alpha = 0$. For $\alpha \neq 0$ the heat kernel of $H_{\delta,\alpha}$ is given by [3]

$$\begin{aligned}P^\alpha(t;x,y) \;&=\; P(t;x,y) + \frac{sgn(xy)}{(2\pi t)^{\frac{1}{2}}}e^{-\frac{(|x|+|y|)^2}{2t}}\\ &+2\frac{sgn(xy)}{\alpha}\int_0^{+\infty}e^{-\frac{2}{|\alpha|}u}e^{-\frac{(|x|+|y|+u)^2}{2t}}du\end{aligned} \tag{36}$$

where $P(t;x,y) = \frac{1}{(2\pi t)^{\frac{1}{2}}}e^{-\frac{|x-y|^2}{2t}}$ is the heat kernel of the free Hamiltonian. The arguments used to prove p,q-smoothing property of the heat semigroup in one dimension are quite similar to those used in the last section. So we have

Lemma 3.1 *For every $t > 0$ we have*

$$\sup_{x \in I\!R} \int_{I\!R} |P^\alpha(t; x, y)| < +\infty \tag{37}$$

Proof: A direct computation shows that $\sup_{x \in I\!R} \int_{I\!R} |P^\alpha(t; x, y)| \leq \; 3 + 2\sqrt{2\pi t} = C_1(t)$. □

Lemma 3.2 *For every $t > 0$ we have*

$$\sup_{x,y \in I\!R} |P^\alpha(t; x, y)| \leq \; C' + Ct^{-\frac{1}{2}} \tag{38}$$

where C, C' are two positive constants.

The proof is quite easy to do.
Now using lemma(3.1), (3.2) and the Riesz-Thorin theorem one can esily establish the following

Theorem 3.1 *For every $t > 0$ and every $1 \leq p \leq q \leq \infty$ the operator*

$$\exp(-tH_{\delta,\alpha}) : \; L^p \longrightarrow \; L^q \tag{39}$$

is bounded and for $p = q < +\infty$ it defines a strongly continuous semigroup.

Now as in the first section we denote $-H_{\delta,\alpha,p}$ the generator of the oparator $\exp(-tH_{\delta,\alpha})$ on the space L^p and for $d = 2$ we will omit the subscript p. The spectral properties of the operators $H_{\delta,\alpha,p}$ can be easily investigated using the smoothing properties of their semigroups.

Proposition 3.1 *For every $1 \leq p < +\infty$ we have:*
i) $\sigma(H_{\delta,\alpha,p})$ is p-independent.
ii) Every eigenvalue of $H_{\delta,\alpha}$ of algebraic multiplicity m is an eigenvalue of $H_{\delta,\alpha,p}$ with the same multiplicity and conversely.
iii)

$$\lim_{t \to +\infty} t^{-1} \ln \| \exp(-tH_{\delta,\alpha,p}) \|_{p,p} = \begin{cases} \frac{\alpha^2}{4} & ; \quad \alpha < 0 \\ 0 & ; \quad \alpha > 0 \end{cases}$$

We shall prove only assertion(i). To this end we use a new method whish relies on the new functionnal calculus introduced by E.B.Davies [8]. The application of this calculus requires an estmate for the resolvent function and the the spectra of $H_{\delta,\alpha,p}$ must be real. Thanks to the explicit formula of the kernel of the oparator $(H_{\delta,\alpha}-k^2)^{-1}$ one can prove the suitable estimate for the resolvent functions. Let us first recall [1] that for $k^2 \in \rho(H_{\delta,\alpha})$, $\mathrm{Im}k > 0$ the kernel of $(H_{\delta,\alpha} - k^2)^{-1}$ is given by [1]p.77

$$G_k(x,y) = \frac{1}{2k} e^{ik|x-y|} + \frac{\alpha}{2k(i\alpha + 2k)} e^{ik(|x|+|y|)} \tag{40}$$

It is abvious that $G_k(x,y)$ is also the kernel of $(H_{\delta,\alpha,p} - k^2)^{-1}$ for every $k^2 \in \rho(H_{\delta,\alpha}) \cap \rho(H_{\delta,\alpha,p})$ with $\mathrm{Im}(k) > 0$.

Lemma 3.3 *For every* $k^2 \in \rho(H_{\delta,\alpha})$ *with* $\mathrm{Im}k > 0$ *we have*

$$\sup_{x \in \mathbb{R}} \int_{\mathbb{R}} |G_k(x,y)| dy \leq \frac{4}{|\mathrm{Im}(k^2)|} \tag{41}$$

Proof: We have

$$|G_k(x,y)| \leq \frac{1}{|2k|} e^{-\mathrm{Im}(k)|x-y|} + \left|\frac{\alpha}{2k(i\alpha + 2k)}\right| e^{-\mathrm{Im}(k)(|x|+|y|)} \tag{42}$$

this yields

$$\int_{\mathbb{R}} |G_k(x,y)| dy \leq \frac{1}{|k|\mathrm{Im}(k)} + \left|\frac{\alpha}{2k(i\alpha + 2k)}\right| \frac{e^{-\mathrm{Im}(k)|x|}}{\mathrm{Im}(k)} \leq 2\frac{1}{|k|\mathrm{Im}(k)} \tag{43}$$

Observing that $|k|\mathrm{Im}(k) \geq |\mathrm{Re}(k)|\mathrm{Im}(k) = \frac{1}{2}|\mathrm{Im}(k^2)|$ we get the desired estimate. □

By the estimate(41) we conclude that for every $k^2 \in \rho(H_{\delta,\alpha})$ with $\mathrm{Im}k > 0$ the operator whose kernel is G_k defines a bounded operator on L^p for $1 \leq r \leq +\infty$. This operator is nothing but $(H_{\delta,\alpha,p} - k^2)^{-1}$, giving thereby that

$$\mathbf{C} \setminus \mathbb{R} \subset \rho(H_{\delta,\alpha,p}) \tag{44}$$

Thus $\sigma(H_{\delta,\alpha,p}) \subset \mathbb{R}$. Now the operators $H_{\delta,\alpha,p}$ fullfil all hypotheses (especially H_1) required by the functionnal calculus [8].

Proof(of proposition(3.1)): Using lemma(4) in [7] and the fact that for every $\xi \in \rho(H_{\delta,\alpha}) \cap \rho(H_{\delta,\alpha,p})$ and every $f \in L^p \cap L^q$ we have $R_\xi(H_{\delta,\alpha})f = R_\xi(H_{\delta,\alpha,p})f$ we get the spectral p-independence. □

Remark 3.1 *From proposition(3.1-(iii)) we observe that for* $\alpha < 0$ *the* L^p *norm of* $\exp(-tH_{\delta,\alpha,p})$ *has an exponential growth for large* t*, this is however similar to case of perturbations of the Laplacian by a negative potentials. While for* $\alpha > 0$*,* $\|\exp(-tH_{\delta,\alpha,p})\|_{p,p}$ *behavies like* $\|\exp(-tH)\|_{p,p}$ *for large* t*. So that the operator* $\exp(-tH_{\delta,\alpha,p})$ *in one dimension behaves somewhat different then in two dimensions where the behavior does not depends on the sign of* α*.*

We close this section by

Proposition 3.2 *For* $1 \leq p < +\infty$ *and* $f \in L^p$ *we have*

$$\lim_{|x|\to 0} |\exp(-tH_{\delta,\alpha,p})f(x)| = 0 \tag{45}$$

The method we give here does not work for $d = 3$ for the simple reason that for each $\alpha \neq +\infty$ we have $\sup_{x\in \mathbb{R}^3} \int_{\mathbb{R}^3} \tilde{P}^\alpha(t; x, y)dy = +\infty$. In fact for $\alpha \leq 0$ we have [1]

$$P^\alpha(t; x, y) \geq \frac{2t}{|x||y|} P(t; |x| + |y|) \tag{46}$$

where $P(t; |x| + |y|) = \frac{1}{(4\pi t)^{\frac{3}{2}}} e^{-\frac{(|x|+|y|)^2}{4t}}$. Hence

$$\int_{\mathbb{R}^3} \tilde{P}^\alpha(t; x, y)dy \geq \frac{4\pi^2 t}{|x|} D_{-2}\left(\frac{|x|}{\sqrt{t}}\right) e^{-\frac{|x|^2}{2t}} \tag{47}$$

where D_ν is the cylindrical hypergeometric function [17]. This yields that $\lim_{x\to 0} \int_{\mathbb{R}^3} \tilde{P}^\alpha(t; x, y)dy = +\infty$. Similarly one can prove rhe same thing for positive α.

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