Continuity of eigenvalues for Schrödinger operators, L^p-properties of Kato type integral operators

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Abstract

Given an arbitrary relatively compact (finely) open subset of $\mathbb{R}^d$, μ-eigenvalues of $-A(D)+\nu$ are studied where $A(D)$ is the Dirichlet Laplacian on D and μ,ν are measures on $\mathbb{R}^d$ such that G_X^μ is continuous and G_X^ν is bounded for every ball X in $\mathbb{R}^d$ (G_X being Green's function for X). Moreover, it is shown that these eigenvalues depend continuously on D and ν. The results are based on very general compactness and convergence properties of integral operators of Kato type which are developed before.

Key words: Schrödinger operator, eigenvalues, Kato measure, integral operators

1 Introduction

Let μ,ν be (positive) measures on $\mathbb{R}^d$, $d\geq 1$, which are (locally) potentially bounded, i.e., such that the Green potentials G_X^μ, G_X^ν are bounded for every ball X in $\mathbb{R}^d$. Assume that μ is even a (local) Kato measure, i.e., that in addition all G_X^μ are continuous (we recall that for $d\geq 3$ any measure μ having a density V with respect to Lebesgue measure λ which is locally r-integrable for some $r>d/2$ is a Kato measure, but that there are of course many Kato measures, e.g. surface measures, which are singular with respect to λ).

Given an arbitrary bounded open set D or, more generally, an arbitrary bounded finely open set D in $\mathbb{R}^d$, let $A(D)$ denote the Dirichlet Laplacian on D (for unbounded sets see Remark 4.6). A real number α is called *μ-eigenvalue of* $-A(D)+\nu$ if there exists a (quasi-continuous) $u\in W_0^{1,2}(D)\setminus\{0\}$ such that

$$-A(D)u+u\nu=\alpha u\mu, \tag{1.1}$$

i.e., such that

$$\int \nabla u\cdot\nabla v\,d\lambda+\int uv\,d\nu=\alpha\int uv\,d\mu \quad \text{for all (quasi-continuous) } v\in W_0^{1,2}(D)$$

(and then u will of course be called a *μ-eigenfunction* of $-A(D)+\nu$ corresponding to the eigenvalue α).

The main purpose of this paper is the study of these μ-eigenvalues, the corresponding eigenfunctions, and the dependence of the eigenvalues on ν and D. The case $\nu = 0$ and $\mu = \lambda$ has been treated in a different way in [Fug99a] and [Fug99b].

For fixed ν and D we shall establish the following (see Corollary 3.10 and Proposition 3.12):

Theorem 1.1. *The Schrödinger operator $-A(D) + \nu$ has at most countably many μ-eigenvalues, they are strictly positive real numbers and have no finite accumulation point. For each μ-eigenvalue, the corresponding μ-eigenspace is finite dimensional and consists of functions which are bounded and finely continuous. The (orthogonal) sum of all μ-eigenspaces is dense in $L^2(D, \mu)$.*

If D is finely connected, the first eigenvalue is simple and the corresponding (finely continuous) eigenfunction has constant sign on D.

If ν is a Kato measure as well and D is open, the μ-eigenfunctions are even continuous and bounded (differences of continuous bounded potentials on D).

Let

$$\alpha_1(D,\nu) \leq \alpha_2(D,\nu) \leq \cdots \leq \alpha_n(D,\nu) \leq \ldots$$

denote the μ-eigenvalues of $-A(D) + \nu$ repeated according to their multiplicity. We shall obtain the following monotonicity and continuity properties of the mapping

$$(D,\nu) \mapsto \alpha_n(D,\nu)$$

(see Corollary 4.2, Proposition 4.3, and Theorem 4.5):

Theorem 1.2. *If $D \subset D'$ and $\nu' \leq \nu$, then $\alpha_n(D',\nu') \leq \alpha_n(D,\nu)$ for every $n \in \mathbb{N}$.*

Theorem 1.3. *Suppose that D_j, $j \in \mathbb{N} \cup \{\infty\}$, are finely open subsets of D and ν_j, $j \in \mathbb{N} \cup \{\infty\}$, are measures majorized by ν such that the following holds:*

a) The sequence $(D_j)_{j\in\mathbb{N}}$ is increasing to D_∞, or the sequence $(D_j)_{j\in\mathbb{N}}$ is decreasing and D_∞ is the fine interior of $\cap_{j\in\mathbb{N}} D_j$.

b) The sequence $(\nu_j)_{j\in\mathbb{N}}$ is increasing or decreasing to ν_∞.

Then, for every $n \in \mathbb{N}$,

$$\lim_{j\to\infty} \alpha_n(D_j,\nu_j) = \alpha_n(D_\infty,\nu_\infty).$$

We get these results reducing (1.1) to an equation

$$K^\mu_{{}^\nu G_D} u = \alpha^{-1} u \tag{1.2}$$

where ${}^\nu G_D$ is the Green function associated with $-A(D) + \nu$ on D and, given a Borel measurable function $G \geq 0$, we define

$$K^\mu_G u := G^{\,u\mu} := \int G(\cdot, y) u(y)\, \mu(dy).$$

We prove the necessary properties for K_G in the setting of an arbitrary measurable space $(X, \mathcal{B}(X))$. Given a measurable function $G \geq 0$ on $X \times X$, a σ-finite measure μ on X will be called *G-bounded* if G^μ is bounded. We shall say that it is a *G-Kato measure* if, moreover, $(G^{1_{A_n}\mu})$ is uniformly increasing to G^μ whenever $(A_n) \subset \mathcal{B}(X)$ is increasing to X. We define the adjoint function ${}^*G$ by ${}^*G(x,y) := G(y,x)$. The key result which is of independent interest will be the following (see Theorem 2.3, Theorem 2.5):

Theorem 1.4. *Let μ, ν be measures on $(X, \mathcal{B}(X))$ such that μ is a G-Kato measure and ν is ${}^*G$-bounded or that μ is G-bounded and ν is a ${}^*G$-Kato measure. Moreover, let (G_n) be measurable functions on $X \times X$ such that $0 \le G_n \le G$ and the sequence (G_n) converges $\nu \otimes \mu$-a.e. to a function G_∞.*

Then, for every $1 < p < \infty$, the operators $K^\mu_{G_n}$, $n \in \mathbb{N} \cup \{\infty\}$, are compact operators from $L^p(\mu)$ to $L^p(\nu)$ and

$$\lim_{n\to\infty} \|K^\mu_{G_n} - K^\mu_{G_\infty}\|_{L^p(\mu),L^p(\nu)} = 0.$$

Remark 1.5. *By [Her68], it is clear that our method leads to the same results if we replace the Laplacian on $\mathbb{R}^d$ $(d \ge 3)$ with an operator*

$$\mathcal{L} = \sum_{j=1}^d \frac{\partial}{\partial x_j}\Big(\sum_{i=1}^d a_{ij}\frac{\partial}{\partial x_i} + b_i\Big) + \sum_{i=1}^d b_i \frac{\partial}{\partial x_i}$$

such that the functions a_{ij} are measurable, (locally) bounded, the matrix $(a_{ij}(x))$ is (locally) uniformly elliptic, and the functions b_i are (locally) r-integrable for some $r > d/2$.

2 L^p-properties of Kato type integral operators

Let $(X, \mathcal{B}(X))$ be a measurable space and let $\mathcal{M}^+(X)$ denote the set of all σ-additive (positive) measures on $(X, \mathcal{B}(X))$. We shall also use the symbol $\mathcal{B}(X)$ for the set of all measurable numerical functions on X. Given any set $\mathcal{A}$ of functions on X let $\mathcal{A}^+$ ($\mathcal{A}_b$ resp.) be the set of all positive (bounded resp.) functions in $\mathcal{A}$.

Let $G : X \times X \to [0, \infty]$ be measurable. For $\mu \in \mathcal{M}^+(X)$ we define

$$G^\mu := \int G(\cdot, y)\, \mu(dy)$$

and a corresponding kernel K^μ_G on X by

$$K^\mu_G f := G^{f\mu} = \int G(\cdot, y) f(y)\, \mu(dy).$$

(A *kernel* K on X is a numerical function $(x, B) \mapsto K(x, B)$, $x \in X, B \in \mathcal{B}(X)$, such that the functions $x \mapsto K(x, B)$ are Borel measurable and the functions $B \mapsto K(x, B)$ are measures.) Moreover, let ${}^*G$ be the adjoint function defined by

$${}^*G(x, y) := G(y, x) \quad (x, y \in X).$$

Of course,

$${}^*G^\mu := \int {}^*G(\cdot, y)\, \mu(dy) = \int G(y, \cdot)\, \mu(dy).$$

A measure $\mu \in \mathcal{M}^+(X)$ will be called *G-bounded*, if G^μ is bounded. We shall say that a G-bounded $\mu \in \mathcal{M}^+(X)$ is a *G-Kato measure* provided that, for every sequence (A_m) in $\mathcal{B}(X)$ which is increasing to X,

$$G^{1_{A_m}\mu} \uparrow G^\mu \quad \text{uniformly on } X. \tag{2.1}$$

Remarks 2.1. *1. If μ is a finite G-bounded measure and $(\inf(G,n))^\mu$ increases to G^μ uniformly as n tends to infinity, then μ is a G-Kato measure. Indeed, if a sequence (B_m) in $\mathcal{B}(X)$ decreases to the empty set, then the inequality $G^{1_{B_m}\mu} \le G^\mu - (\inf(G,n))^\mu + n\mu(B_m)$ shows that $\lim_{m\to\infty} G^{1_{B_m}\mu} = 0$ uniformly.*

2. The notion of a G-bounded (G-Kato) measure is hereditary: If μ is a G-bounded (a G-Kato measure resp.) and $\tilde\mu \le \mu$, $\tilde G \le G$ ($\tilde\mu \in \mathcal{M}^+(X)$ and $\tilde G$ measurable and positive), then $\tilde\mu$ is $\tilde G$-bounded (a $\tilde G$-Kato measure resp.). Indeed, obviously $\tilde G^{\tilde\mu} \le G^\mu$ and $\tilde G^{\tilde\mu} - \tilde G^{1_A\tilde\mu} = \tilde G^{1_{A^c}\tilde\mu} \le G^{1_{A^c}\mu} = G^\mu - G^{1_A\mu}$.

3. If X is a topological space, $\mathcal{B}(X)$ is its σ-algebra of Borel sets, and $G(\cdot,y)$ is l.s.c. for every $y \in X$, then every $\mu \in \mathcal{M}^+(X)$ such that G^μ is continuous and vanishes at infinity is a G-Kato measure. Indeed, it suffices to note that, by Fatou's lemma, all functions $G^{1_A\mu}$, $A \in \mathcal{B}(X)$, are l.s.c. (in fact, they are continuous because of the equality $G^{1_A\mu} + G^{1_{A^c}\mu} = G^\mu$) and that, consequently, for any sequence (A_n) in $\mathcal{B}(X)$ which is increasing to X the sequence $G^{1_{A_n}\mu}$ which is certainly increasing to G^μ converges uniformly to G^μ.

Given $1 \le p \le \infty$, we take $1 \le q \le \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Let us note first that for every $1 \le p < \infty$, for every $\mu \in \mathcal{M}^+(X)$, and for all $f \in \mathcal{B}^+(X)$,

$$\big(K^\mu f\big)^p \le (G^\mu)^{\frac{p}{q}}\, K^\mu(f^p). \tag{2.2}$$

Indeed, fixing $x \in X$ and taking

$$u = G(x,\cdot)^{\frac{1}{q}}, \quad v = G(x,\cdot)^{\frac{1}{p}} f$$

we obtain by Hölder's inequality that

$$K_G^\mu f(x) = \int uv\, d\mu \le \Big(\int G(x,\cdot)\, d\mu\Big)^{\frac{1}{q}} \Big(\int G(x,\cdot) f^p\, d\mu\Big)^{\frac{1}{p}} = (G^\mu(x))^{\frac{1}{q}} (K_G^\mu(f^p)(x))^{\frac{1}{p}}.$$

Proposition 2.2. *Let $\mu,\nu \in \mathcal{M}^+(X)$, $1 \le p < \infty$. Then, for every $f \in L^p(\mu)$,*

$$\|K_G^\mu(|f|)\|_{L^p(\nu)} \le \|G^\mu\|_{L^\infty(\nu)}^{\frac{1}{q}}\, \|{}^*G^\nu\|_{L^\infty(\mu)}^{\frac{1}{p}}\, \|f\|_{L^p(\mu)}.$$

Proof. By (2.2),

$$\int \big(K_G^\mu(|f|)\big)^p\, d\nu \le \|G^\mu\|_{L^\infty(\nu)}^{\frac{p}{q}} \int K_G^\mu(|f|^p)\, d\nu$$

where

$$\int K_G^\mu(|f|^p)\, d\nu = \int \Big(\int G(x,y)\,\nu(dx)\Big) |f|^p(y)\,\mu(dy) \le \|{}^*G^\nu\|_{L^\infty(\mu)} \|f\|_{L^p(\mu)}^p.$$

□

Theorem 2.3. *Let $\mu,\nu \in \mathcal{M}^+(X)$ such that μ is a G-Kato measure and ν is ${}^*G$-bounded or that μ is G-bounded and ν is a ${}^*G$-Kato measure. Moreover, let (G_n) be measurable functions on $X \times X$ such that $0 \le G_n \le G$ and the sequence (G_n) converges $\nu \otimes \mu$-a.e. to a function G_∞.*

Then, for every $1 < p < \infty$, the operators $K_{G_n}^\mu$, $n \in \mathbb{N} \cup \{\infty\}$, are bounded operators from $L^p(\mu)$ to $L^p(\nu)$ and

$$\lim_{n\to\infty} \|K_{G_n}^\mu - K_{G_\infty}^\mu\|_{L^p(\mu),L^p(\nu)} = 0.$$

Proof. Modifying G_n, $n \in \mathbb{N} \cup \{\infty\}$, by taking the value 0 on the set where $(G_n)_{n\in\mathbb{N}}$ does not converge to G_∞ the integrals $\int (K^\mu f)^p \, d\nu$, $f \in L^p(\mu)$, do not change. Therefore we may assume without loss of generality that $(G_n)_{n\in\mathbb{N}}$ converges to G_∞ everywhere on $X \times X$.

It follows immediately from Proposition 2.2 that each $K^\mu_{G_n}$, $n \in \mathbb{N} \cup \{\infty\}$, is a bounded operator from $L^p(\mu)$ to $L^p(\nu)$. Given $n \in \mathbb{N}$, let us define

$$\underline{G}_n := \inf_{m\geq n} G_m, \quad \underline{K}^\mu_n := K^\mu_{\underline{G}_n}, \qquad \overline{G}_n := \sup_{m\geq n} G_m, \quad \overline{K}^\mu_n := K^\mu_{\overline{G}_n}.$$

Then, for every $n \in \mathbb{N} \cup \{\infty\}$ and for every positive $f \in L^p(\mu)$,

$$0 \leq \overline{K}^\mu_n f - K^\mu_{G_n} f \leq \overline{K}^\mu_n f - \underline{K}^\mu_n f.$$

So it suffices to show that $\lim_{n\to\infty} \|\overline{K}^\mu_n - \underline{K}^\mu_n\|_{L^p(\mu),L^p(\nu)} = 0$. To that end let

$$\tilde{G}_n := 1_{\underline{G}_n<\infty}(\overline{G}_n - \underline{G}_n) \qquad (n \in \mathbb{N}).$$

Clearly, the sequence $(\tilde{G}_n)$ is decreasing and $\underline{G}_n + \tilde{G}_n = \overline{G}_n$ whence $\underline{K}^\mu_n + K^\mu_{\tilde{G}_n} = \overline{K}_n$ for every $n \in \mathbb{N}$. Hence we have to prove that $\lim_{n\to\infty} \|K^\mu_{\tilde{G}_n}\|_{L^p(\mu),L^p(\nu)} = 0$.

Let us fix $\varepsilon > 0$ and define

$$A_n := \{\tilde{G}^\mu_n \leq \varepsilon\} \cap \{{}^*\tilde{G}^\nu_n \leq \varepsilon\} \qquad (n \in \mathbb{N}). \tag{2.3}$$

Since $\underline{G}_n \uparrow G_\infty$ and $\overline{G}_n \downarrow G_\infty$, we know that $\lim_{n\to\infty} (\overline{G}^\mu_n - \underline{G}^\mu_n) = \lim_{n\to\infty} ({}^*\overline{G}^\nu_n - {}^*\underline{G}^\nu_n) = 0$. Therefore the sequences $(\tilde{G}^\mu_n)$ and $({}^*\tilde{G}^\nu_n)$ are decreasing to 0, i.e., the sequence (A_n) is increasing to X. Let

$$\mu_n := 1_{A_n}\mu, \quad \rho_n := 1_{A_n^c}\mu, \qquad \nu_n := 1_{A_n}\nu, \quad \sigma_n := 1_{A_n^c}\nu,$$

choose $a, b \in \mathbb{R}^+$ such that $G^\mu \leq a$ and ${}^*G^\nu \leq b$, and fix $f \in \mathcal{B}^+(X)$ such that $\int f^p \, d\mu \leq 1$. Since $\tilde{G}^\mu_n \leq \varepsilon$ ν_n-a.e. and ${}^*\tilde{G}^\nu_n \leq \varepsilon$ μ_n-a.e., we obtain by Proposition 2.2 that

$$\int (K^{\mu_n}_{\tilde{G}_n} f)^p \, d\nu \leq a^{\frac{p}{q}} \, \|{}^*\tilde{G}^\nu_n\|_{L^\infty(\mu_n)} \leq a^{\frac{p}{q}}\varepsilon, \qquad \int (K^{\rho_n}_{\tilde{G}_n} f)^p \, d\nu \leq \|G^{\rho_n}\|^{\frac{p}{q}}_{L^\infty(\nu)} \, b$$

and

$$\int (K^\mu_{\tilde{G}_n} f)^p \, d\nu_n \leq \|\tilde{G}^\mu_n\|^{\frac{p}{q}}_{L^\infty(\nu_n)} b \leq \varepsilon^{\frac{p}{q}} b, \qquad \int (K^\mu_{\tilde{G}_n} f)^p \, d\sigma_n \leq a^{\frac{p}{q}} \|{}^*G^{\sigma_n}\|_{L^\infty(\mu)}.$$

By (2.1), $G^{\rho_n} \downarrow 0$ uniformly if μ is a G-Kato measure and ${}^*G^{\sigma_n} \downarrow 0$ uniformly if ν is a ${}^*G$-Kato measure. Obviously, the preceding inequalities imply that in any case $\lim_{n\to\infty} \|K^\mu_{\tilde{G}_n}\|_{L^p(\mu),L^p(\nu)} = 0$ (it suffices to note that $\mu_n + \rho_n = \mu$, $\nu_n + \sigma_n = \nu$, and $(K^{\mu_n+\rho_n}_{\tilde{G}_n} f)^p \leq 2^p \big((K^{\mu_n}_{\tilde{G}_n} f)^p + (K^{\rho_n}_{\tilde{G}_n} f)^p\big)$. $\square$

Corollary 2.4. *Let $\mu, \nu \in \mathcal{M}^+(X)$ such that μ is a G-Kato measure and ν is ${}^*G$-bounded or that μ is G-bounded and ν is a ${}^*G$-Kato measure. Moreover, assume that $(X, \mathcal{B}(X))$ is standard Borel and that K_n, $n \in \mathbb{N} \cup \{\infty\}$, are kernels on $(X, \mathcal{B}(X))$ such that $K_n \leq K^\mu_G$ for every $n \in \mathbb{N} \cup \{\infty\}$ and $(K_n)_{n\in\mathbb{N}}$ is increasing or decreasing to K_∞.*

Then, for every $1 < p < \infty$, the operators K_n, $n \in \mathbb{N} \cup \{\infty\}$, are bounded operators from $L^p(\mu)$ to $L^p(\nu)$ and

$$\lim_{n\to\infty} \|K_n - K_\infty\|_{L^p(\mu),L^p(\nu)} = 0.$$

Proof. For every $x \in X$,

$$K_G^\mu(x,\cdot) = G(x,\cdot)\mu.$$

So by [Dyn72, Lemma 4.1], there exist measurable functions $G_n \geq 0$ on $X \times X$, $n \in \mathbb{N} \cup \{\infty\}$, such that, for every $x \in X$,

$$G_n(x,\cdot)\mu = K_n(x,\cdot).$$

Obviously, we may assume that $G_n \leq G$ for every $n \in \mathbb{N} \cup \{\infty\}$ and that the sequence $(G_n)_{n\in\mathbb{N}}$ is increasing (decreasing resp.) to G_∞. An application of Theorem 2.3 finishes the proof. □

Moreover, Theorem 2.3 has the following consequence:

Theorem 2.5. *Let $1 < p < \infty$ and $\mu, \nu \in \mathcal{M}^+(X)$ such that μ is a G-Kato measure and ν is $^*G$-bounded or that μ is G-bounded and ν is a $^*G$-Kato measure.*

Then K_G^μ is a compact operator from $L^p(\mu)$ to $L^p(\nu)$.

If, in addition, $(X, \mathcal{B}(X))$ is standard Borel, then every kernel K' on $(X, \mathcal{B}(X))$ such that $K' \leq K_G^\mu$ is a compact operator from $L^p(\mu)$ to $L^p(\nu)$.

Proof. Since ν is σ-additive, there exist positive functions $\varphi_n \in L^p(\nu)$, $n \in \mathbb{N}$, such that (φ_n) is increasing to 1. For $n \in \mathbb{N}$ and $x, y \in X$ we define

$$G_n(x,y) := \inf(G(x,y), n), \quad \tilde{G}_n(x,y) = \varphi_n(x)\, G_n(x,y).$$

Then the sequence $(\tilde{G}_n)$ is increasing to G and therefore

$$\lim_{n\to\infty} \|K_{\tilde{G}_n}^\mu - K_G^\mu\|_{L^p(\mu),L^p(\nu)} = 0$$

by Theorem 2.3. To prove that K^μ is compact it thus remains to show that each $K_{\tilde{G}_n}^\mu$ is a compact operator from $L^p(\mu)$ to $L^p(\nu)$.

Let us fix $n \in \mathbb{N}$ and let $f_m \in L^p(\mu)$, $m \in \mathbb{N}$, with $\|f_m\|_{L^p(\mu)} \leq 1$. Then there exists a subsequence (f'_m) of (f_m) and $f \in L^p(\mu)$ such that $\lim_{m\to\infty} \int f'_m g\, d\mu = \int fg\, d\mu$ for every $g \in L^q(\mu)$. Let $a \in \mathbb{R}$ such that $G^\mu \leq a$. Clearly, for every $x \in X$, $G_n(x,\cdot) \in L^q(\mu)$, since

$$\int (G_n(x,y))^q\, \mu(dy) \leq n^{q-1} \int G_n(x,y)\, \mu(dy) \leq n^{q-1} a$$

whence

$$\lim_{m\to\infty} K_{G_n}^\mu(f'_m - f)(x) = \lim_{m\to\infty} \int G_n(x,y)(f'_m(y) - f(y))\, \mu(dy) = 0.$$

Moreover, the sequence $(K_{G_n}^\mu(f'_m - f))$ is uniformly bounded since, by (2.2),

$$(K_{G_n}^\mu(|f'_m - f|))^p \leq a^{\frac{p}{q}}\, K_{G_n}^\mu(|f'_m - f|^p) \leq a^{\frac{p}{q}} n^p \int |f'_m - f|^p\, d\mu \leq (2n)^p\, a^{\frac{p}{q}}.$$

Therefore

$$\lim_{m\to\infty} \int (K_{\tilde{G}_n}^\mu f'_m - K_{\tilde{G}_n}^\mu f)^p\, d\nu = \lim_{m\to\infty} \int (K_{G_n}^\mu f'_m - K_{G_n}^\mu f)^p\, \varphi_n^p\, d\nu = 0 \tag{2.4}$$

by Lebesgue's convergence theorem.

Finally, assume in addition that $(X, \mathcal{B}(X))$ is standard Borel and that K' is a kernel on $(X, \mathcal{B}(X))$ such that $K' \leq K_G^\mu$. Using [Dyn72, Lemma 4.1] as in the proof of Corollary 2.4 we obtain a measurable function G' on $X \times X$ such that $0 \leq G' \leq G$ and $K' = K_{G'}^\mu$. By Remarks 2.1 and the preceding part of the proof we conclude that K' is a compact operator from $L^p(\mu)$ to $L^p(\nu)$. □

3 μ-eigenvalues for $-A(D)+\nu$ on finely open sets

As in [Fug99b] where λ-eigenvalues of Δ (λ being Lebesgue measure) on finely open sets are studied we shall work in the Sobolev space $W_0^{1,2} = W_0^{1,2}(\mathbb{R}^d)$, the completion of the space $\mathcal{C}_0^\infty(\mathbb{R}^d)$ of real-valued infinitely differentiable functions of compact support in $\mathbb{R}^d$, $d \ge 1$, in the Sobolev $(1,2)$-norm

$$\|u\|_1 = \Big(\int (u^2 + |\nabla u|^2)\, d\lambda\Big)^{\frac{1}{2}}.$$

By [DL54, p. 353f.], every function $u \in W_0^{1,2}$ can be redefined, or corrected, so as to become finely continuous q.e. (where the term q.e. means quasi-everywhere, i.e., everywhere outside a polar set). In what follows we shall usually assume implicitly that functions from $W_0^{1,2}$ have been corrected in this sense.

For an arbitrary subset A of $\mathbb{R}^d$ one may define

$$W_0^{1,2}(A) := \{u \in W_0^{1,2} : u = 0 \text{ q.e. in } A^c\}$$

(this definition agrees with the standard definition of $W_0^{1,2}(U)$ for open sets U in $\mathbb{R}^d$, see [DL54, p. 359]) and then

$$W_0^{1,2}(A) = W_0^{1,2}(\mathrm{int}_f A)$$

where $\mathrm{int}_f A$ denotes the fine interior of A (see [Fug99b, p. 93]).

Let us fix a bounded finely open set D in $\mathbb{R}^d$. Adding to D the polar set $i(D)$ of all irregular fine boundary points, i.e., of all finely isolated points of D^c, we obtain a finely open set $r(D)$ which is a usual F_σ-set (and regular). Obviously,

$$W_0^{1,2}(D) = W_0^{1,2}(r(D)). \tag{3.1}$$

For every smooth Radon measure $\mu \ge 0$ on $\mathbb{R}^d$, we define

$$L^2(D,\mu) := \{f \in L^2(\mathbb{R}^d,\mu) : f = 0 \ \mu\text{-a.e. on } D^c\}$$

(such a measure does not charge polar sets). By [Fug99b, p. 96], $W_0^{1,2}(D)$ is dense in $L^2(D,\lambda)$ and we may define the negative of the Dirichlet Laplacian on D as the positive self-adjoint operator $-A_D$ in $L^2(D,\lambda)$ associated with the positive definite and closed symmetric bilinear form

$$Q(u,v) := \int \nabla u \cdot \nabla v \, d\lambda \qquad (u,v \in W_0^{1,2}(D)).$$

The domain of $-A(D)$ consists of those $u \in W_0^{1,2}(D)$ for which there exists $f \in L^2(D,\lambda)$ such that

$$Q(u,v) = \int f v\, d\lambda \qquad (v \in W_0^{1,2}(D)); \tag{3.2}$$

and we then have

$$-A(D)u = f. \tag{3.3}$$

In particular, $A(D)u = 0$ implies that $u = 0$ (take $v = u$).

For every Radon measure ρ on $\mathbb{R}^d$ let ρ^{D^c} denote the swept-out of ρ on D^c so that D is a ρ^{D^c}-null set. We recall that $\varepsilon_x^{D^c} = \varepsilon_x$ for every Dirac measure ε_x of a point $x \in r(D)^c$. Moreover, $\rho^{D^c} = \int \varepsilon_x^{D^c}\, \rho(dx)$.

Let us fix a measure μ on $\mathbb{R}^d$. For the moment we shall only assume that μ is (locally) potentially bounded, i.e., that G_X^μ is bounded for every ball X in $\mathbb{R}^d$ (where G_X denote the classical Green function for X, extended by zero to be a function on $\mathbb{R}^d \times \mathbb{R}^d$). We note that such a measure μ is a smooth Radon measure. By Proposition 2.2, every $K_X^\mu := K_{G_X}^\mu$, X a ball in $\mathbb{R}^d$, is a bounded operator on $L^2(\mathbb{R}^d, \mu)$. If the ball X contains the closure $\overline{D}$ of D, then, for every $f \in \mathcal{B}(\mathbb{R}^d)^+$,

$$0 \le G_X^{(f\mu)^{D^c}} \le G_X^{f\mu} = K_X^\mu f \quad \text{on } \mathbb{R}^d, \qquad G_X^{(f\mu)^{D^c}} = G_X^\mu f \quad \text{on } \mathbb{R}^d \setminus r(D). \tag{3.4}$$

Moreover, the difference $G_X^{f\mu}(x) - G_X^{(f\mu)^{D^c}}(x)$ does not depend on the choice of the ball X as long as $G_X^{f\mu}(x) < \infty$. So there is a unique bounded operator K_D^μ on $L^2(\mathbb{R}^d, \mu)$ such that

$$K_D^\mu f = G_X^{f\mu} - G_X^{(f\mu)^{D^c}} \qquad (\mu\text{-a.e.}) \tag{3.5}$$

for every $f \in L^2(\mathbb{R}^d, \mu)$ and every ball X containing $\overline{D}$. The operator K_D^μ lives on $L^2(D, \mu)$: $K_D^\mu f \in L^2(D, \mu)$ for every $f \in L^2(\mathbb{R}^d, \mu)$ by (3.4) and $K_D^\mu f = 0$ if $f = 0$ q.e. on D, since then $(f\mu)^{D^c} = f\mu$.

Since $\mu^{D^c} = \int \varepsilon_x^{D^c} \mu(dx)$, an application of Fubini's theorem yields that

$$K_D^\mu = K_{G_D}^\mu,$$

if we define G_D by

$$G_D(x,y) := \begin{cases} G_X(x,y) - \int G_X(\cdot, y)\, d\varepsilon_x^{D^c}, & x, y \in r(D), x \ne y \text{ or } d = 1, \\ \infty, & x, y \in r(D), x = y \text{ and } d \ge 2, \\ 0, & x, y \in \mathbb{R}^d, x \notin r(D) \text{ or } y \notin r(D). \end{cases}$$

(X being any ball containing $\overline{D}$, see [Fug99b, pp. 94/95] (where "$\Leftrightarrow$" has to be replaced by "$-$")). The function G_D is Borel measurable and symmetric by [Fug83, p. 196].

By Proposition 2.2,

$$\|K_D^\mu\|_{L^2(\mathbb{R}^d,\mu)} \le \|G_D^\mu\|_\infty = \|K_D^\mu 1\|_\infty.$$

Proposition 3.1. *Let $f \in L^2(\mathbb{R}^d, \mu)$. Then $K_D^\mu f \in W_0^{1,2}(D)$ and, for every $v \in W_0^{1,2}(D)$,*

$$\int v^2\, d\mu \le \|K_D^\mu 1\|_\infty \int |\nabla v|^2\, d\lambda, \qquad \int \nabla(K_D^\mu f) \cdot \nabla v\, d\lambda = \int f v\, d\mu.$$

In particular, K_D^μ is a positive (symmetric) operator on $L^2(\mathbb{R}^d, \mu)$:

$$\int f\, K_D^\mu f\, d\mu = \int |\nabla K_D^\mu f|^2\, d\lambda \ge 0 \qquad \textit{for every } f \in L^2(\mathbb{R}^d, \mu). \tag{3.6}$$

Proof. Let us recall that potentials G_X^ρ of measures ρ on X having energy $\int G_X^\rho\, d\rho < \infty$ are contained in $W_0^{1,2}(X)$, that every $v \in W_0^{1,2}(X)$ is ρ-integrable, and that

$$\int \nabla G_X^\rho \cdot \nabla v\, d\lambda = \int v\, d\rho \tag{3.7}$$

(see [Her66, Théorème 10] or [Her68, Théorème 9]). Fix $f \in L^2(X,\mu)$ and assume for the moment that $f \geq 0$. Then $f\mu$ has finite energy on X, since by Proposition 2.2,

$$\int G_X^{f\mu}\, d(f\mu) = \int f\, K_X^\mu f\, d\mu \leq \Big(\int f^2\, d\mu \cdot \int (K_X^\mu f)^2\, d\mu\Big)^{\frac{1}{2}} \leq \|G_X^\mu\|_\infty \int f^2\, d\mu. \tag{3.8}$$

Therefore $G_X^{f\mu} \in W_0^{1,2}(X)$, every $w \in W_0^{1,2}(X)$ is $f\mu$-integrable, and

$$\int \nabla G_X^{f\mu} \cdot \nabla w\, d\lambda = \int fw\, d\mu. \tag{3.9}$$

Of course, we then have $G_X^{f\mu} \in W_0^{1,2}(X)$ and (3.7) for any $f \in L^2(X,\mu)$. Moreover, the $f\mu$-integrability of $w \in W_0^{1,2}(X)$ for every $f \in L^2(X,\mu)$ implies that $w \in L^2(X,\mu)$. And this leads to

$$\int v^2\, d\mu \;\leq\; \|G_X^\mu\|_\infty \int |\nabla v|^2\, d\lambda \tag{3.10}$$

for every $v \in W_0^{1,2}(X)$ (cf. [BS98]). It will be convenient to give the full argument, since we shall need it for the set D as well: Taking $f = w = v$ in (3.9) we obtain by Hölder's inequality that

$$\Big(\int v^2\, d\mu\Big)^2 = \Big(\int \nabla G_X^{v\mu} \cdot \nabla v\, d\lambda\Big)^2 \leq \int |\nabla G_X^{v\mu}|^2\, d\lambda \cdot \int |\nabla v|^2\, d\lambda. \tag{3.11}$$

Choosing $f = v$ and $w = G_X^{v\mu}$ we conclude from (3.9) and (3.8) that

$$\int |\nabla G_X^{v\mu}|^2\, d\lambda = \int v\, G_X^{v\mu}\, d\mu \leq \|G_X^\mu\|_\infty \int v^2\, d\mu. \tag{3.12}$$

Combining (3.11) and (3.12), we get that

$$\int v^2\, d\mu \leq \|G_X^\mu\|_\infty \int |\nabla v|^2\, d\lambda.$$

Now fix $v \in W_0^{1,2}(D)$, $f \in L^2(\mathbb{R}^d,\mu)$, $f \geq 0$, and let $p := G_X^{f\mu}$, $\rho := (f\mu)^{D^c}$, and $q := G_X^\rho$. Clearly $\int \nabla p \cdot \nabla v\, d\lambda = \int fv\, d\mu$ by (3.9). Since $\int G_X^\rho\, d\rho \leq \int G_X^{f\mu}\, d(f\mu) < \infty$, we obtain by (3.7) that

$$\int \nabla q \cdot \nabla v\, d\lambda = \int v\, d\rho = 0 \tag{3.13}$$

where the second equality holds, since $v = 0$ q.e. on D^c and D is a ρ-null set. Thus

$$\int \nabla(K_D^\mu f) \cdot \nabla v\, d\lambda = \int fv\, d\mu. \tag{3.14}$$

Arguing as above with D instead of X we finally obtain that

$$\int v^2\, d\mu \leq \|K_D^\mu\|_\infty \int |\nabla v|^2\, d\lambda.$$

□

Corollary 3.2. *The operator K_D^μ is injective on $L^2(D,\mu)$.*

Proof. If $f \in L^2(D,\mu)$ such that $K_D^\mu f = 0$ in $L^2(D,\mu)$, i.e., μ-a.e., then $v := K_D^\mu f \in W_0^{1,2}(D)$ and

$$\int |\nabla v|^2\, d\lambda = \int f v\, d\mu = 0.$$

This implies that v equals a constant q.e. in $\mathbb{R}^d$ whence $v = 0$ q.e. So $K_D^\mu f^+ = K_D^\mu f^-$ everywhere and, by [Fug83, Prop. 2.9], $f^+\mu = f^-\mu$, i.e., $f = 0$ μ-a.e. □

Remark 3.3. The first inequality of Proposition 3.1 implies that the "identity mapping" $j_{L^2(\mathbb{R}^d,\mu)}$ which to the λ-equivalence class of a (quasi-continuous version of) $v \in W_0^{1,2}$ associates the μ-equivalence class of v defines a continuous mapping of $W_0^{1,2}(D)$ into $L^2(D,\mu)$. Its kernel is the space

$$N_0(D,\mu) := \{v \in W_0^{1,2}(D) : v = 0 \ \mu\text{-a.e.}\}.$$

Of course, $N_0(D,\mu) = W_0^{1,2}(D)$ if D is a μ-null set. The converse holds as well, since e.g. $G_D^\lambda \in W_0^{1,2}(D)$ and $G_D^\lambda > 0$ on D.

Moreover, the mapping $j_{L^2(\mathbb{R}^d,\mu)}$ is injective on $W_0^{1,2}(D)$ if and only if the fine support $S_f(\mu)$ of μ contains D ($S_f(\mu)$ exists since μ does not charge polar sets): Suppose first that $D \setminus S_f(\mu) \neq \emptyset$. Then $\tilde{D} := D \setminus S_f(\mu)$ is a non-empty finely open set which is a μ-null set whence $G_{\tilde{D}}^\lambda \in W_0^{1,2}(\tilde{D}) \setminus \{0\} \subset W_0^{1,2}(D) \setminus \{0\}$, but $G_{\tilde{D}}^\lambda = 0$ μ-a.e. Assume now that $D \subset S_f(\mu)$ and let $v \in W_0^{1,2}(D) \setminus \{0\}$. Let P be a (Borel measurable) polar set such that v is finely continuous outside P and $v = 0$ on $D^c \setminus P$. Then the finely open set $U := \{v \neq 0\} \setminus P$ is contained in D. Further, $U \neq \emptyset$, since $\lambda(\{u \neq 0\}) > 0$ and $\lambda(P) = 0$, whence $\mu(U) > 0$.

Corollary 3.4. *For every* $f \in L^2(D,\mu)$,

$$\int |K_D^\mu f|^2\, d\lambda \le \|G_D^\mu\|_\infty \|G_D^\lambda\|_\infty \int f^2\, d\mu, \qquad \int |\nabla K_D^\mu f|^2\, d\lambda \le \|G_D^\mu\|_\infty \int f^2\, d\mu.$$

In particular, the mapping $K_D^\mu : L^2(D,\mu) \to W_0^{1,2}(D)$ *is continuous.*

Proof. The first inequality follows immediately from Proposition 2.2, the second is used in the proof of Proposition 3.1 (and follows again from the equality $\int \nabla(K_D^\mu f) \cdot \nabla v\, d\lambda = \int f v\, d\mu$ for all $v \in W_0^{1,2}(D)$). □

In addition, let us now fix another potentially bounded measure ν on $\mathbb{R}^d$ (let us stress, however, that the case $\nu = 0$ is still of interest as well). In view of (3.2) and (3.3) (and extending the usual definition for the case $\mu = \lambda$) we shall say that a function $u \in W_0^{1,2}(D) \setminus \{0\}$ is a *μ-eigenfunction* of $-A(D) + \nu$ for the *μ-eigenvalue* α and we shall write

$$-A(D)u + u\nu = \alpha u \mu$$

if

$$\int \nabla u \cdot \nabla v\, d\lambda + \int uv\, d\nu = \alpha \int uv\, d\mu \quad \text{for all } v \in W_0^{1,2}(D).$$

Obviously every eigenvalue α of $-A(D) + \nu$ is strictly positive (take $v = u$). By Proposition 3.1, $K_D^\nu u, K_D^\mu u \in W_0^{1,2}(D)$ and

$$\int uv\, d\nu = \int \nabla K_D^\nu u \cdot \nabla v\, d\lambda, \qquad \int uv\, d\mu = \int \nabla K_D^\mu u \cdot \nabla v\, d\lambda.$$

Therefore

$$-A(D)u + u\nu = \alpha u\mu \quad \Leftrightarrow \quad u + K_D^\nu u = \alpha K_D^\mu u \tag{3.15}$$

as functions in $W_0^{1,2}(D)$, i.e., quasi-everywhere, whence μ-a.e.

Equation (3.15) suggests that it will be useful to have a function $\tilde{G}_D$ such that the operator $\tilde{K}_D^\mu := K_{\tilde{G}_D}^\mu$ satisfies $(I + K_D^\nu)\tilde{K}_D^\mu = K_D^\mu$.

Proposition 3.5. *There exists a unique measurable function $\tilde{G}_D \geq 0$ on $\mathbb{R}^d \times \mathbb{R}^d$ such that, for every $y \in \mathbb{R}^d$,*

$$\tilde{G}_D(\cdot, y) + K_D^\nu \tilde{G}_D(\cdot, y) = G_D(\cdot, y) \tag{3.16}$$

and $\tilde{G}_D(y, y) = \infty$ if $y \in r(D)$ and $d \geq 2$. The function $\tilde{G}_D$ is symmetric.

Proof. a) Let us suppose for a moment that ν is even a Kato measure. Then $\Delta - \nu$ leads to a harmonic space and it is well known that, for every ball X in $\mathbb{R}^d$, we have a perturbed Green function $0 \leq \tilde{G}_X \leq G_X$ such that

$$\tilde{G}_X(\cdot, y) + K_X^\nu \tilde{G}_X(\cdot, y) = G_X(\cdot, y) \qquad (y \in \mathbb{R}^d).$$

Denoting by $\tilde{\varepsilon}_x^{D^c}$ the swept-out of ε_x on D^c relative to $\Delta - \nu$ and defining

$$\tilde{G}_D(x, y) := \begin{cases} \tilde{G}_X(x, y) - \int \tilde{G}_X(\cdot, y)\, d\tilde{\varepsilon}_x^{D^c}, & x, y \in r(D), x \neq y \text{ or } d = 1, \\ \infty, & x, y \in r(D), x = y \text{ and } d \geq 2, \\ 0, & x, y \in \mathbb{R}^d, x \notin r(D) \text{ or } y \notin r(D). \end{cases}$$

(see the definition of G_D) we then obtain (3.16).

b) In the general case we choose Kato measures ν_n with $\nu_n \uparrow \nu$. Fix $y \in \mathbb{R}^d$. Using (a) we obtain functions $g_n \geq 0$ such that

$$g_n + K_D^{\nu_n} g_n = G_D(\cdot, y) \qquad (n \in \mathbb{N}). \tag{3.17}$$

This implies that, for every $n \in \mathbb{N}$,

$$(g_n - g_{n+1}) + K_D^{\nu_n}(g_n - g_{n+1}) = K_D^{\nu_{n+1} - \nu_n} g_{n+1}$$

on D, if $d = 1$, on $D \backslash \{y\}$, if $d \geq 2$. Applying Lemma 5.2, we obtain that the sequence (g_n) is decreasing. Define

$$\tilde{G}_D(\cdot, y) := \lim_{n \to \infty} g_n.$$

Fix a ball X containing $\overline{D}$ and let X' denote the concentric ball having double radius. There exists a constant $C > 0$ (depending only on the dimension d) such that

$$K_X^\nu G_X(\cdot, y) \leq C \|G_{X'}^\nu\|_\infty G_X(\cdot, y)$$

(see e.g. [Han99, p. 429]). Taking (3.17) and letting n tend to infinity we thus obtain by Lebesgue's convergence theorem that

$$\tilde{G}_D(\cdot, y) + K_D^\nu \tilde{G}_D(\cdot, y) = G_D(\cdot, y)$$

(for $d \geq 2$ the point y has to be considered separately).

c) The uniqueness of $\tilde{G}_D$ follows easily by Lemma 5.2. For the symmetry of $\tilde{G}_D$ see [Han99, Lemma 15.1] and use the symmetry of G_D. □

Remark 3.6. *The proof of Proposition 3.5 shows that the function $\tilde{G}_D$ decreases if D decreases or ν increases.*

Defining

$$\tilde{K}^\mu_D := K^\mu_{\tilde{G}_D}$$

we then have

$$\tilde{K}^\mu_D + K^\nu_D \tilde{K}^\mu_D = K^\mu_D. \tag{3.18}$$

Using Lemma 5.2 we may now restate the equivalence (3.15) in the following way: For every $u \in W_0^{1,2}(D)$,

$$\boxed{-A(D)u + u\nu = \alpha u\mu \quad \Longleftrightarrow \quad u = \alpha \tilde{K}^\mu_D u \ \text{ in } \ W_0^{1,2}(D).} \tag{3.19}$$

Next let us check that $\tilde{K}^\mu_D$ is related to the positive bilinear form Q_ν corresponding to $-A(D) + \nu$ in quite the same way as K^μ_D is related to Q. We define

$$Q_\nu(u,v) := Q(u,v) + \int uv \, d\nu \qquad (u, v \in W_0^{1,2}(D)).$$

The following proposition generalizes part of Proposition 3.1 and Corollary 3.2:

Proposition 3.7. *Let $f \in L^2(\mathbb{R}^d, \mu)$. Then $\tilde{K}^\mu_D f \in W_0^{1,2}(D)$ and, for every $v \in W_0^{1,2}(D)$,*

$$Q_\nu(\tilde{K}^\mu_D f, v) = \int f v \, d\mu. \tag{3.20}$$

In particular, $\tilde{K}^\mu_D$ is a positive operator on $L^2(\mathbb{R}^d, \mu)$,

$$Q_\nu(\tilde{K}^\mu_D f, \tilde{K}^\mu_D f) = \int f \tilde{K}^\mu_D f \, d\mu,$$

and $\tilde{K}^\mu_D : L^2(D,\mu) \to L^2(D,\mu)$ is injective.

Proof. We know by Proposition 3.1 that, for every $v \in W_0^{1,2}(D)$,

$$Q_\nu(K^\mu_D f, v) = Q(K^\mu_D f, v) + \int (K^\mu_D f) v \, d\nu = \int f v \, d\mu + \int (K^\mu_D f) v \, d\nu.$$

Replacing μ by ν, f by $\tilde{K}^\mu_D f$, and using (3.18) we obtain that

$$Q_\nu(K^\nu_D \tilde{K}^\mu_D f, v) = \int (\tilde{K}^\mu_D f) v \, d\nu + \int (K^\nu_D \tilde{K}^\mu_D f) v \, d\nu = \int (K^\mu_D f) v \, d\nu.$$

Since $Q_\nu(\tilde{K}^\mu_D f, v) = Q_\nu(K^\mu_D f, v) - Q_\nu(K^\nu_D \tilde{K}^\mu_D f, v)$, the desired equation (3.20) follows.
It remains to prove the injectivity of $\tilde{K}^\mu_D : L^2(D,\mu) \to L^2(D,\mu)$. So fix $f \in L^2(D,\mu)$ and suppose that $\tilde{K}^\mu_D f = 0$ μ-a.e. Then

$$0 \le \int (\tilde{K}^\mu_D f)^2 \, d\nu \le Q_\nu(\tilde{K}^\mu_D f, \tilde{K}^\mu_D f) = \int f \tilde{K}^\mu_D f \, d\mu = 0$$

whence $\tilde{K}^\mu_D f = 0$ ν-a.e. This implies that

$$K^\mu_D f = \tilde{K}^\mu_D f + K^\nu \tilde{K}^\mu_D f = 0 \quad \mu-\text{a.e.}$$

and therefore $f = 0$ μ-a.e. by Corollary 3.2. □

Having Proposition 3.7 we are able to improve the reverse implication in (3.19): Let $u \in L^2(D,\mu)\setminus\{0\}$ be an eigenfunction of the operator $\tilde{K}^\mu_D$ on $L^2(D,\mu)$ for an eigenvalue β. Then $\beta \neq 0$, since $\tilde{K}^\mu_D$ is injective. Moreover, having $u \in L^2(D,\mu)$ and $\tilde{K}^\mu_D u = \beta u$ in $L^2(D,\mu)$, i.e. μ-a.e., we obtain that $\tilde{K}^\mu_D u \in W^{1,2}_0(D)$ and

$$\tilde{K}^\mu_D(\tilde{K}^\mu_D u) = \beta \tilde{K}^\mu_D u \quad \text{outside the polar set } \{K^\mu_D|u| < \infty\}.$$

Therefore $\tilde{K}^\mu_D u$ is a μ-eigenfunction of $-A(D)+\nu$ for the eigenvalue β^{-1}. The same is true for the multiple $\beta^{-1}\tilde{K}^\mu_D u$, and this is equal to u in $L^2(D,\mu)$. If v is any function in $W^{1,2}_0(D)$ such that $v = u$ μ-a.e. and $-A(D)v + v\nu = \beta^{-1}v\mu$, then $v = \beta^{-1}\tilde{K}^\mu_D v = \beta^{-1}\tilde{K}^\mu_D u$ q.e.

Thus (equivalence classes of) μ-eigenfunctions for $-A(D)+\nu$ "are" (equivalence classes of) eigenfunctions for the operator $\tilde{K}^\mu_D$ on $L^2(D,\mu)$, the corresponding eigenvalues being the inverse of another. Note that this holds in spite of the fact that the "identity mapping" $j_{L^2(\mathbb{R}^d,\mu)}$ from $W^{1,2}_0(D)$ in $L^2(D,\mu)$ may be far from being injective (cf. Remark 3.3).

From now on let us assume more restrictively that

$$\mu \text{ is a local Kato measure on } \mathbb{R}^d,$$

i.e., that G^μ_X is continuous and real for every ball X in $\mathbb{R}^d$ or – equivalently – that $G^\mu_X \in \mathcal{C}_0(X)$ for every ball X in $\mathbb{R}^d$ (consider a ball X' containing $\overline{X}$ and use the equality $G^\mu_X = G^\mu_{X'} - G^{\mu^{X^c}}_{X'}$). By Remarks 2.1 this implies that μ is a G_D-Kato measure and a $\tilde{G}_D$-Kato measure. Thus, by Theorem 2.5, K^μ_D and $\tilde{K}^\mu_D$ are compact operators on $L^p(\mathbb{R}^d,\mu)$ for every $1 < p < \infty$.

Moreover, K^μ_D is a compact operator on $L^\infty(\mathbb{R}^d,\mu)$ (it is a simple consequence of the fact that the set of all potentials q on X, X a ball containing $\overline{D}$, such that $K^\mu_X 1 - q$ is a potential on X is equicontinuous because of the continuity of $K^\mu_X 1$). This implies that $\tilde{K}^\mu_D : L^\infty(\mathbb{R}^d,\mu) \to L^\infty(\mathbb{R}^d,\mu)$ is compact as well: By Lemma 5.1, the kernel K^ν_D on $\mathbb{R}^d$ satisfies the complete maximum principle. So the operator $I + K^\nu_D$ on $\mathcal{B}_b(\mathbb{R}^d)$ is invertible (and the inverse $(I+K^\nu_D)^{-1} = I - (I+K^\nu_D)^{-1}K^\nu_D$ has norm $\|(I+K^\nu_D)^{-1}\| \le 2$, since $(I+K^\nu_D)^{-1}K^\nu_D f \ge 0$ for every $f \in \mathcal{B}^+_b(\mathbb{R}^d)$ and $0 \le (I+K^\nu_D)^{-1}K^\nu_D 1 \le 1$, see e.g. [BH86, p. 78]). By (3.18),

$$\tilde{K}^\mu_D f = (I + K^\nu_D)^{-1}K^\mu_D f$$

for every $f \in \mathcal{B}_b(\mathbb{R}^d)$. Now the compactness of $\tilde{K}^\mu_D$ on $L^\infty(\mathbb{R}^d,\mu)$ follows immediately.

We note that this compactness has been used in [HH88, Han89, HM90] to study μ-eigenvalues of $-A(D)+\nu$ in the case where D is open and the fine support of μ is all of $\mathbb{R}^d$ and to prove continuity of $D \mapsto \alpha_1(D,\nu)$.

Using Lemma 5.4 and Corollary 5.7 we thus obtain the following:

Theorem 3.8. *The operator $\tilde{K}^\mu_D$ on $L^2(D,\mu)$ has at most countably many eigenvalues, they are strictly positive real numbers and have no accumulation point except (possibly) 0.*

For each eigenvalue, the corresponding eigenspace is finite dimensional and consists of functions which are (μ-a.e.) bounded. The (orthogonal) sum of all eigenspaces is dense in $L^2(D,\mu)$.

Moreover, the spectrum of $\tilde{K}^\mu_D : L^p(D,\mu) \to L^p(D,\mu)$ and the multiplicities of the eigenvalues are independent of $p \in]1,\infty]$.

Remark 3.9. *Note that we did not exclude that $D = \emptyset$ or – more generally – that D is a μ-null set in which case of course $L^2(D,\mu) = \{0\}$ and the operator K^μ_D on $L^2(D,\mu)$ has no μ-eigenvalues at all! If $d = 1$, $D \neq \emptyset$, and $n \in \mathbb{N}$, then we may take n different points $x_1,\dots,x_n \in D$ and $\mu := \sum_{j=1}^n \delta_{x_j}$ leading to $\dim L^2(D,\mu) = n$ and at most n different eigenvalues.*

If, however, $d \geq 2$ and D is not a μ-null set, then $L^2(D,\mu)$ is infinite-dimensional, since μ does not charge points (which are polar), and in this case we have countably many different eigenvalues.

Because of the connection we have between eigenvalues of $\tilde{K}^\mu_D$ and μ-eigenvalues of $-A(D)+\nu$, we immediately have the following consequence:

Corollary 3.10. *The Schrödinger operator $-A(D)+\nu$ has at most countably many μ-eigenvalues, they are strictly positive real numbers and have no finite accumulation point. For each μ-eigenvalue, the corresponding μ-eigenspace is finite dimensional and consists of functions which are bounded (q.e.). The (orthogonal) sum of all μ-eigenspaces is dense in $L^2(D,\mu)$.*

Let

$$\alpha_1(D,\nu) \leq \alpha_2(D,\nu) \leq \dots \leq \alpha_n(D,\nu) \leq \dots (\to \infty)$$

denote the μ-eigenvalues of $-A(D)+\nu$ repeated according to their multiplicity followed by $+\infty$'s if there are only finitely many, and let

$$\beta_1(D,\nu) \geq \beta_2(D,\nu) \geq \dots \geq \beta_n(D,\nu) \geq \dots (\to 0)$$

denote the eigenvalues of $\tilde{K}^\mu_D$ on $L^2(D,\mu)$ repeated according to their multiplicity followed by 0's if there are only finitely many. Then clearly

$$\beta_n(D,\nu) = \alpha_n(D,\nu)^{-1} \qquad (n \in \mathbb{N}). \tag{3.21}$$

The key to the study of the dependence of the eigenvalues on D and ν are Rayleigh-Ritz formulas. For every linear subspace U of $W^{1,2}_0(D)$ let

$$\alpha(D,\nu,U) := \sup\{Q_\nu(f,f) : f \in U, \int f^2\, d\mu = 1\}$$

and for every linear subspace V of $L^2(\mathbb{R}^d,\mu)$ let

$$\beta(D,\nu,V) := \inf\{\int f\tilde{K}^\mu_D f\, d\mu : f \in V, \int f^2\, d\mu = 1\}.$$

Proposition 3.11. *For every $n \in \mathbb{N}$,*

$$\begin{aligned} \alpha_n(D,\nu) &= \inf\{\alpha(D,\nu,U) : U \subset W^{1,2}_0(D),\ \dim j_{L^2(\mathbb{R}^d,\mu)}(U) = n\}, \\ \beta_n(D,\nu) &= \sup\{\beta(D,\nu,V) : V \subset L^2(D,\mu),\ \dim V = n\} \\ &= \sup\{\beta(D,\nu,V) : V \subset L^2(\mathbb{R}^d,\mu),\ \dim V = n\}. \end{aligned}$$

Proof. The first equality for $\beta_n(D,\nu)$ is standard, the second one is a consequence of the fact that $\tilde{K}^\mu_D$ is an operator on $L^2(\mathbb{R}^d,\mu)$ living on $L^2(D,\mu)$. A direct argument runs as follows: If $f \in L^2(\mathbb{R}^d,\mu)$, then $g := 1_{r(D)} f \in L^2(D,\mu)$ and $\int f\tilde{K}^\mu_D f\, d\mu = \int g\tilde{K}^\mu_D g\, d\mu$. Moreover, if V is a subspace of $L^2(\mathbb{R}^d,\mu)$ such that $\dim V = n$, but $\dim\{1_{r(D)} f : f \in V\} < n$, then there exists $f \in V \setminus \{0\}$ such that $1_{r(D)} f = 0$ whence $\int f\tilde{K}^\mu_D f\, d\mu = 0$, $\beta(D,\nu,V) = 0$.

To prove the equality for $\alpha_n := \alpha_n(D,\nu)$ we fix functions $f_n \in W^{1,2}_0(D)$, $n = 1,2\dots$, such that

$$\alpha_n \tilde{K}^\mu_D f_n = f_n \ \text{ q.e.}$$

and $f_1, f_2, \ldots$ is an orthonormal base in $L^2(D,\mu)$. Let $\langle\cdot,\cdot\rangle$ denote the scalar product in $L^2(D,\mu)$. Then by Proposition 3.7, for every $n \in \mathbb{N}$ and $f \in W_0^{1,2}(D)$,

$$Q_\nu(f_n, f) = \alpha_n Q_\nu(\tilde{K}_D^\mu f_n, f) = \alpha_n \langle f_n, f\rangle. \tag{3.22}$$

In particular,

$$Q_\nu(f_n, f_m) = \alpha_n \delta_{nm} \tag{3.23}$$

for all $n, m \in \mathbb{N}$. For every $n \in \mathbb{N}$, let U_n denote the linear span of $f_1, \ldots, f_n$ in $W_0^{1,2}(D)$. Then $\dim j_{L^2(\mathbb{R}^d,\mu)} U_n = n$ since $\langle f_k, f_l\rangle = \delta_{kl}$. Fix $f \in U_n$ such that $\langle f, f\rangle = 1$, i.e., $\sum_{k=1}^n \langle f_k, f\rangle^2 = 1$. Then

$$f = \sum_{k=1}^n \langle f_k, f\rangle f_k$$

whence by (3.23)

$$Q_\nu(f,f) = \sum_{k=1}^n \alpha_k \langle f_k, f\rangle^2 \le \alpha_n \sum_{k=1}^n \langle f_k, f\rangle^2 = \alpha_n.$$

Therefore

$$\alpha(D, \nu, U_n) \le \alpha_n.$$

Finally, fix an arbitrary linear subspace U of $W_0^{1,2}(D)$ such that $\dim j_{L^2(\mathbb{R}^d,\mu)}(U) = n$. If P denotes the projection from $L^2(D,\mu)$ on $j_{L^2(\mathbb{R}^d,\mu)}(U_{n-1})$, i.e., the mapping

$$f \mapsto \sum_{k=1}^{n-1} \langle f_k, f\rangle f_k,$$

then the range of P has dimension $n-1$. So there exists a function $f \in U$ such that $j_{L^2(\mathbb{R}^d,\mu)} f \neq 0$, but $P(j_{L^2(\mathbb{R}^d,\mu)} f) = 0$. Of course, we may assume that $\langle f, f\rangle = 1$ so that

$$1 = \sum_{k=1}^\infty \langle f_k, f\rangle^2 = \sum_{k=n}^\infty \langle f_k, f\rangle^2.$$

Let us define

$$g_m = \sum_{k=n}^m \langle f_k, f\rangle f_k \qquad (m \in \mathbb{N}, m \ge n)$$

Then, for every $m \ge n$, by (3.23) and (3.22),

$$Q_\nu(g_m, f) = \sum_{k=n}^m \langle f_k, f\rangle Q_\nu(f_k, f) = \sum_{k=n}^m \alpha_k \langle f_k, f\rangle^2 = \sum_{k=n}^m \langle f_k, f\rangle^2 Q_\nu(f_k, f_k) = Q_\nu(g_m, g_m)$$

whence

$$Q_\nu(g_m, f - g_m) = 0.$$

This implies that

$$\begin{aligned} Q_\nu(f,f) &= Q_\nu(g_m + (f-g_m), g_m + (f-g_m)) \\ &= Q_\nu(g_m, g_m) + Q_\nu(f-g_m, f-g_m) \ge Q_\nu(g_m, g_m) = \sum_{k=n}^m \alpha_k \langle f_k, f\rangle^2. \end{aligned}$$

for every $m \geq n$ and therefore

$$Q_\nu(f,f) \geq \sum_{k=n}^{\infty} \alpha_k \langle f_k, f\rangle^2 \geq \alpha_n \sum_{k=n}^{\infty} \langle f_k, f\rangle^2 = \alpha_n.$$

Thus $\alpha(D,\nu,U) \geq \alpha_n$ finishing the proof. □

Proposition 3.12. *If D is finely connected, the first μ-eigenvalue of $-A(D)+\nu$ on D is simple, and the corresponding eigenfunction has constant sign.*

Proof. a) Let $\beta := \beta_1(D,\nu)$ and fix $f \in L^2(\mathbb{R}^d,\mu)\setminus\{0\}$ such that

$$\tilde{K}^\mu_D f = \beta f \quad \mu\text{-a.e.}$$

Then obviously

$$\tilde{K}^\mu_D |f| \geq |\tilde{K}^\mu_D f| = \beta|f| \quad \mu\text{-a.e.}$$

and, in addition,

$$\int (\tilde{K}^\mu_D|f|)^2\, d\mu \leq \beta^2 \int |f|^2\, d\mu$$

because of $\|\tilde{K}^\mu_D\|_{L^2(\mathbb{R}^d,\mu)} = \beta$. Therefore

$$\tilde{K}^\mu_D|f| = \beta|f| \quad \mu\text{-a.e..}$$

Defining $g := |f| - f$ we obtain that

$$\tilde{K}^\mu_D g = \beta g \quad \mu\text{-a.e..} \tag{3.24}$$

Of course, $g=0$ μ-a.e. or $\mu(\{g>0\})>0$. In the latter case, $K^\mu_D g > 0$ q.e. on D by [Fug72, p. 150] whence $\tilde{K}^\mu_D g > 0$ q.e. on D and $g>0$ μ-a.e. on D by (3.24). (To pass from strict positivity of $K^\mu_D g > 0$ to strict positivity of $\tilde{K}^\mu_D g$ we may use analyticity as in [BHH87, Corollary 2.7] or logarithmic convexity leading to $\tilde{K}^\mu_D g \geq \exp(-K^\nu_D g/g) K^\mu_D g$ as e.g. in [Han99, Section 4].) Consequently,

$$f = |f| \quad \text{or} \quad f = -|f| \quad (\mu\text{-a.e.}). \tag{3.25}$$

b) If $\tilde{f}$ is any function in $L^2(D,\mu)\setminus\{0\}$ such that $K^\mu_D \tilde{f} = \beta\tilde{f}$ (μ-a.e.), then as well $\tilde{f} \geq 0$ or $\tilde{f} \leq 0$ and therefore

$$\left|\int f\tilde{f}\, d\mu\right| = \int |f|\,|\tilde{f}|\, d\mu > 0.$$

This implies that any such function $\tilde{f}$ is a multiple of f (otherwise there would exist $\tilde{f}$ which is orthogonal to f). □

4 Dependence of the μ-eigenvalues of $-A(D)+\nu$ on ν and D

In this section we shall establish monotonicity and continuity properties of

$$(D,\nu) \mapsto (\alpha_n(D,\nu), \beta_n(D,\nu)) \qquad (n \in \mathbb{N}).$$

Let us first note the following: If D, D' are finely open bounded subsets of $\mathbb{R}^d$ such that $D \subset D'$ then, for every $f \in L^2(\mathbb{R}^d, \mu)$,

$$\tilde{K}^{\mu}_{D'} f = \tilde{K}^{\mu}_{D} f + \tilde{G}^{(f\mu)^{D^c}}_{D'}. \tag{4.1}$$

Indeed, if X is a ball containing $\overline{D'}$, then

$$\tilde{K}^{\mu}_{X} f = \tilde{K}^{\mu}_{D} f + \tilde{G}^{(f\mu)^{D^c}}_{X} = \tilde{K}^{\mu}_{D'} f + \tilde{G}^{(f\mu)^{(D')^c}}_{X}$$

by (3.5) where

$$\tilde{G}^{(f\mu)^{D^c}}_{X} = \tilde{G}^{(f\mu)^{D^c}}_{D'} + \tilde{G}^{(f\mu)^{(D')^c}}_{X},$$

since $((f\mu)^{D^c})^{(D')^c} = (f\mu)^{(D')^c}$. Equation (4.1) has the following consequence:

Proposition 4.1. *Let μ, ν be potentially bounded measures on $\mathbb{R}^d$ and let D, D' be bounded finely open sets such that $D \subset D' \subset \mathbb{R}^d$. Then $\tilde{K}^{\mu}_{D'} - \tilde{K}^{\mu}_{D}$ is a positive operator on $L^2(\mathbb{R}^d, \mu)$.*

Proof. Fix $f \in L^2(D', \mu)$. Then $f\mu$ is a finite signed measure. Therefore $\mu' := (f\mu)^{D^c}$ is a finite signed measure as well. We apply Proposition 3.7 to D' and μ'. Since $\tilde{K}^{\mu}_{D} f \in W^{1,2}_0(D) \subset W^{1,2}_0(D')$ and $1 \in L^2(\mathbb{R}^d, \mu')$, we obtain that

$$Q_\nu(\tilde{G}^{\mu'}_{D'}, \tilde{K}^{\mu}_{D} f) = \int \tilde{K}^{\mu}_{D} f \, d\mu' = 0,$$

where the last equality follows from the fact that $\tilde{K}^{\mu}_{D} f = 0$ q.e. outside D' and that D is a μ'-null set. By (4.1) and Proposition 3.7 we thus conclude that

$$\begin{aligned} \int f \, \tilde{K}^{\mu}_{D'} f \, d\mu &= Q_\nu(\tilde{K}^{\mu}_{D'} f, \tilde{K}^{\mu}_{D'} f) \\ &= Q_\nu(\tilde{G}^{\mu'}_{D'}, \tilde{G}^{\mu'}_{D'}) + Q_\nu(\tilde{K}^{\mu}_{D} f, \tilde{K}^{\mu}_{D} f) \geq Q_\nu(\tilde{K}^{\mu}_{D} f, \tilde{K}^{\mu}_{D} f) = \int f \, \tilde{K}^{\mu}_{D} f \, d\mu. \end{aligned}$$

□

Corollary 4.2. *Let μ be a local Kato measure, ν a potentially bounded measure on $\mathbb{R}^d$, and let D, D' be bounded finely open sets such that $D \subset D' \subset \mathbb{R}^d$. Then, for every $n \in \mathbb{N}$,*

$$\alpha_n(D', \nu) \leq \alpha_n(D, \nu), \qquad \beta_n(D', \nu) \geq \beta_n(D, \nu).$$

Proof. By (3.21), it suffices to prove one of the inequalities. Since $W^{1,2}_0(D) \subset W^{1,2}_0(D')$, the first inequality follows immediately from the Rayleigh-Ritz formula for α_n. Alternatively, we can prove the second inequality using Proposition 4.1 and the Rayleigh-Ritz formula for β_n. □

If we want to study the dependence on ν, we have to modify our notation. We replace " $\tilde{}$ " by a left superscript "ν".

Proposition 4.3. *Let μ be a local Kato measure, ν, ν' potentially bounded measures on $\mathbb{R}^d$ such that $\nu \leq \nu'$, and let D be a bounded finely open set in $\mathbb{R}^d$. Then, for every $n \in \mathbb{N}$,*

$$\alpha_n(D, \nu') \geq \alpha_n(D, \nu), \qquad \beta_n(D, \nu') \leq \beta_n(D, \nu).$$

Proof. For every $f \in W_0^{1,2}(D)$,

$$Q_{\nu'}(f,f) - Q_\nu(f,f) = \int f^2\, d(\nu' - \nu) \geq 0$$

whence $\alpha(D,\nu',U) \geq \alpha(D,\nu,U)$ for every subspace U of $W_0^{1,2}(D)$.Therefore the first inequality is an immediate consequence of Proposition 3.11, and then the second inequality follows by (3.21). □

For the remainder of this section let us suppose that μ is a local Kato measure and that ν_0 is a potentially bounded measure on $\mathbb{R}^d$. We assume that we have measures ν_j, $j \in \mathbb{N} \cup \{\infty\}$, such that

$$\nu_j = \varphi_j \nu_0$$

where $\varphi_j : \mathbb{R}^d \to [0,1]$ are Borel measurable functions such that

$$\lim_{j\to\infty} \varphi_j = \varphi_\infty \quad \nu_0\text{-a.e.}.$$

Note that a sequence $(\nu_j)_{j\in\mathbb{N}}$ of measures on $\mathbb{R}^d$ which is majorized by ν_0 and converging increasingly or decreasingly to a measure ν_∞ is a special case.

Moreover, let D_j, $j \in \mathbb{N} \cup \{\infty\}$, be uniformly bounded finely open subsets of $\mathbb{R}^d$ such that the sequence $(D_j)_{n\in\mathbb{N}}$ converges to D_∞ in the sense that

$$\mathrm{int}_f \bigcap_i \bigcup_{j\geq i} D_j = \bigcup_i \mathrm{int}_f \bigcap_{j\geq i} D_j = D_\infty.$$

Proposition 4.4. *The sequence* $({}^{\nu_j}K^\mu_{D_j})_{j\in\mathbb{N}}$ *of operators on* $L^2(\mathbb{R}^d,\mu)$ *converges to* ${}^{\nu_\infty}K^\mu_{D_\infty}$ *as* j *tends to infinity.*

Proof. Let X denote a ball such that $\overline{D}_j \subset X$ for every $j \in \mathbb{N} \cup \{\infty\}$.

a) Fix a potentially bounded measure ν on $\mathbb{R}^d$.

i) Suppose first that the sequence (D_j) is decreasing. Then the sequence $(D_j^c)_{j\in\mathbb{N}}$ is increasing and D_∞^c is the fine closure of $\cup_{j\in\mathbb{N}} D_j^c$. By Remark 3.6,

$$ {}^\nu G_{D_j} \geq {}^\nu G_{D_{j+1}} \geq {}^\nu G_{D_\infty} \tag{4.2}$$

for every $j \in \mathbb{N}$. Moreover, for every $x \in X$,

$$G_X^{\mu^{D_j^c}}(x) = \varepsilon_x^{D_j^c}(G_X^\mu) \uparrow \varepsilon_x^{\cup_{j\in\mathbb{N}} D_j^c}(G_X^\mu) = \varepsilon_x^{D_\infty^c}(G_X^\mu) = G_X^{\mu^{D_\infty^c}}(x)$$

whence

$$G^\mu_{D_j}(x) \downarrow G^\mu_{D_\infty}(x).$$

by (3.5). So, by (4.2) and (3.18),

$$\begin{aligned} {}^\nu G^\mu_{D_\infty} \leq \inf_j {}^\nu G^\mu_{D_j} &= \inf_j (G^\mu_{D_j} - K^\nu_{D_j}\, {}^\nu G^\mu_{D_j}) \\ &\leq \inf_j (G^\mu_{D_j} - K^\nu_{D_\infty}\, {}^\nu G^\mu_{D_\infty}) = G^\mu_{D_\infty} - K^\nu_{D_\infty}\, {}^\nu G^\mu_{D_\infty} = {}^\nu G^\mu_{D_\infty} \end{aligned}$$

and we obtain that

$$ {}^\nu G^\mu_{D_j} \downarrow {}^\nu G^\mu_{D_\infty}. \tag{4.3}$$

ii) Suppose next that the sequence $(D_j)_{j\in\mathbb{N}}$ is increasing to D_∞. Then

$$ {}^{\nu}G_{D_j} \le {}^{\nu}G_{D_{j+1}} \le {}^{\nu}G_{D_\infty} $$

by Remark 3.6. Moreover,

$$ G^{\mu}_{D_j} \uparrow G^{\mu}_{D_\infty} \quad \text{on } \mathbb{R}^d \setminus i(D_\infty). \tag{4.4} $$

Indeed, both sides are zero on $\mathbb{R}^d \setminus (D_\infty \cup i(D_\infty))$. For every $x \in D_\infty$, $\inf_j \varepsilon_x^{D_j^c}(G^{\mu}_X) = \varepsilon_x^{D_\infty^c}(G^{\mu}_X)$ by Lemma 5.3 whence $\sup_j G^{\mu}_{D_j}(x) = G^{\mu}_{D^c_\infty}(x)$ by (3.5). Arguing similarly as in (i) we now conclude that (4.4) holds.

b) Fix a bounded finely open subset D of $\mathbb{R}^d$. If the sequence (ν_n) is increasing (decreasing resp.), then the sequence $({}^{\nu_n}G_D)$ is decreasing (increasing resp.) by Remark 3.6 and the limit satisfies the equality (3.16) characterizing ${}^{\nu_\infty}G_D$. Therefore

$$ \lim_{j\to\infty} {}^{\nu_n}G_D = {}^{\nu_\infty}G_D. $$

In the general case we define

$$ \overline{\nu}_j = (\sup_{i\ge j} \varphi_i)\nu_0, \quad \underline{\nu}_j = (\inf_{i\ge j} \varphi_i)\nu_0. $$

Then the sequence $(\overline{\nu}_j)$ is decreasing to ν_∞, the sequence $(\underline{\nu}_j)$ is increasing to ν_∞, and therefore

$$ \lim_{j\to\infty} {}^{\overline{\nu}_j}G_D = \lim_{j\to\infty} {}^{\underline{\nu}_j}G_D = {}^{\nu_\infty}G_D. $$

Because of the inequalities $\overline{\nu}_j \ge \nu_j \ge \underline{\nu}_j$ we know that, for every $j \in \mathbb{N}$,

$$ {}^{\overline{\nu}_j}G_D \le {}^{\nu_j}G_D \le {}^{\underline{\nu}_j}G_D. $$

Therefore

$$ \lim_{j\to\infty} {}^{\nu_j}G_D = {}^{\nu_\infty}G_D. $$

c) Let us now consider the case where the sequence $(D_j)_{j\in\mathbb{N}}$ is not necessarily monotone. We define

$$ U_j := \operatorname{int}_f \bigcap_{j\ge i} D_j, \quad V_j := \bigcup_{j\ge i} D_j, \qquad (j \in \mathbb{N}). $$

Then (U_j), (V_j) are uniformly bounded sequences of finely open sets such that (U_j) is increasing to D_∞, (V_j) is decreasing and $\operatorname{int}_f \cap_j V_j = D_\infty$.

Combining the arguments in (a) and (b) we conclude that the sequence $({}^{\overline{\nu}_j}G_{U_j})$ is increasing, the sequence $({}^{\underline{\nu}_j}G_{V_j})$ is decreasing,

$$ {}^{\overline{\nu}_j}G_{U_j} \le {}^{\nu_\infty}G_{D_\infty} \le {}^{\underline{\nu}_j}G_{V_j} $$

for every $j \in \mathbb{N}$, and

$$ \lim_{j\to\infty} {}^{\overline{\nu}_j}G^{\mu}_{U_j}(x) = \lim_{j\to\infty} {}^{\underline{\nu}_j}G^{\mu}_{V_j}(x) \quad \text{for every } x \in \mathbb{R}^d \setminus i(D_\infty). $$

Thus we conclude by Theorem 2.3 or Corollary 2.4 that

$$ \lim_{j\to\infty} \| {}^{\overline{\nu}_j}K_{U_j} - {}^{\nu_\infty}K^{\mu}_{D_\infty} \|_{L^2(\mathbb{R}^d,\mu)} = \lim_{j\to\infty} \| {}^{\underline{\nu}_j}K_{V_j} - {}^{\nu_\infty}K^{\mu}_{D_\infty} \|_{L^2(\mathbb{R}^d,\mu)} = 0. $$

Further,

$$ {}^{\overline{\nu}_j}G_{U_j} \le {}^{\nu_j}G_{D_j} \le {}^{\underline{\nu}_j}G_{V_j} $$

for every $j \in \mathbb{N}$ whence

$$ {}^{\overline{\nu}_j}K_{U_j}f \le {}^{\nu_j}K_{D_j}f \le {}^{\underline{\nu}_j}K_{V_j}f \quad \text{for every } f \in L^2(\mathbb{R}^d,\mu), f \ge 0. $$

Thus finally

$$ \lim_{j\to\infty} \|{}^{\nu_j}K^{\mu}_{D_j} - {}^{\nu_\infty}K^{\mu}_{D_\infty}\|_{L^2(\mathbb{R}^d,\mu)} = 0. $$

□

Theorem 4.5. *For every $n \in \mathbb{N}$,*

$$ \lim_{j\to\infty} \alpha_n(D_j,\nu_j) = \alpha_n(D_\infty,\nu_\infty), \qquad \lim_{j\to\infty} \beta_n(D_j,\nu_j) = \beta_n(D_\infty,\nu_\infty). $$

Proof. By Theorem 2.3,

$$ \lim_{j\to\infty} \|{}^{\nu_j}K^{\mu}_{D_j} - {}^{\nu_\infty}K^{\mu}_{D_\infty}\|_{L^2(\mathbb{R}^d,\mu)} = 0. $$

Given an n-dimensional subspace V of $L^2(\mathbb{R}^d,\mu)$, we have

$$ \left|\int f\,({}^{\nu_j}K^{\mu}_{D_j}f)\,d\mu - \int f\,({}^{\nu_\infty}K^{\mu}_{D_\infty}f)\,d\mu\right| \le \|{}^{\nu_j}K^{\mu}_{D_j} - {}^{\nu_\infty}K^{\mu}_{D_\infty}\|_{L^2(\mathbb{R}^d,\mu)} $$

for every $f \in V$ such that $\int f^2\,d\mu = 1$ whence

$$ |\beta(D_j,\nu_j,V) - \beta(D_\infty,\nu_\infty,V)| \le \|{}^{\nu_j}K^{\mu}_{D_j} - {}^{\nu_\infty}K^{\mu}_{D_\infty}\|_{L^2(\mathbb{R}^d,\mu)}, $$

and finally, for every $n \in \mathbb{N}$,

$$ |\beta_n(D_j,\nu_j) - \beta_n(D_\infty,\nu_\infty)| \le \|{}^{\nu_j}K^{\mu}_{D_j} - {}^{\nu_\infty}K^{\mu}_{D_\infty}\|_{L^2(\mathbb{R}^d,\mu)}. $$

The proof is finished using (3.21). □

Remark 4.6. *Finally, let us note that we do not really need that the finely open sets D are bounded. We clearly may drop this assumption if we assume that ν is globally potentially bounded and μ is a global Kato measure, i.e., for $d \ge 3$, that $G^{\nu}_{\mathbb{R}^d}$ is bounded and $G^{\mu}_{\mathbb{R}^d}$ is a continuous real function vanishing at infinity. If $d \le 2$, the sets D have to be contained in an open subset X of $\mathbb{R}^d$ admitting a Green function and the previous assumptions have to be made with respect to X.*

5 Appendix

5.1 Some facts from potential theory

Lemma 5.1. *Let s be a positive finely hyperharmonic function on a finely open bounded subset of $\mathbb{R}^d$ and let ρ be a smooth measure on $\mathbb{R}^d$ with fine support C such that $s \ge G^{\rho}_D$ q.e. on $D \cap C$. Then $s \ge G^{\rho}_D$ on D.*

Proof. Fix a ball X containing $\overline{D}$ and choose Kato measures ρ_n such that $\rho_n \uparrow \rho$ (see e.g. [Fug72] or [GH98, Proposition 9.1]). For every $n \in \mathbb{N}$, the function $f_n := s - G_D^{\rho_n}$ is a finely superharmonic function on $\tilde{D} := D \setminus C$. Obviously $f_n \geq -G_D^{\rho_n} \geq -G_X^{\rho_n}$ where $G_X^{\rho_n}$ is a continuous bounded potential on X. Consider a regular point in the fine boundary $\partial_f \tilde{D}$ of $\tilde{D}$. If $z \in \partial_f D$ then $\text{fine-lim}_{x \in \tilde{D}, x \to z} G_D^{\rho_n} = 0$ and hence $\text{fine-lim}_{x \in \tilde{D}, x \to z} f_n(x) \geq 0$. If $z \in D \cap C$ such that $s(z) \geq G_D^{\rho_n}(z)$ then $\text{fine-lim}_{x \in \tilde{D}, x \to z} f_n(x) = f_n(z) \geq 0$. Therefore $f_n \geq 0$ by [Fug72, Theorem 9.1], i.e., $s \geq G_D^{\rho_n}$. Letting n tend to infinity we finally conclude that $s \geq G_D^{\rho}$. □

The following lemma is a generalization of [Han99, Lemma 4.1] to finely superharmonic functions on finely open sets (we shall only need the case $g = 0$):

Lemma 5.2. *Let D be a bounded finely open subset of $\mathbb{R}^d$, $s \geq 0$ a finely superharmonic function on D, $g : D \to [0, \infty]$, P a polar set, and ρ a smooth measure on $\mathbb{R}^d$. Let $f : \mathbb{R}^d \to \overline{\mathbb{R}}$ be Borel measurable such that f and $K_D^{\rho}|f|$ are real outside P and $f + K_D^{\rho} f + g = s$ on $\mathbb{R}^d \setminus P$. Then $f + g \geq 0$ on $D \setminus P$.*

Proof. We choose an increasing sequence (C_n) of compact subsets of $\{f^+ > 0\}$ such that $\rho(\{f^+ > 0\} \setminus \cup_{n=1}^{\infty} C_n) = 0$, and define

$$\rho_n := 1_{C_n} \rho \qquad (n \in \mathbb{N}).$$

Then, for every $n \in \mathbb{N}$,

$$s + K_D^{\rho} f^- = f^+ + K_D^{\rho} f^+ + g \geq K_D^{\rho_n} f^+ = G_D^{f^+ \rho_n} \qquad \text{on } C_n \setminus P$$

whence

$$s + K_D^{\rho} f^- \geq K_D^{\rho_n} f^+ \quad \text{on } D$$

by Lemma 5.1. Therefore $s + K_D^{\rho} f^- \geq K_D^{\rho} f^+$ on D and finally

$$f + g = s + K_D^{\rho} f^- - K_D^{\rho} f^+ \geq 0 \quad \text{on } D \setminus P.$$

□

Lemma 5.3. *Let (D_n) be a sequence of finely open sets which is increasing to a bounded subset D of $\mathbb{R}^d$. Then, for every $x \in D$, the sequence $(\varepsilon_x^{D_n^c})$ converges weakly to $\varepsilon_x^{D^c}$.*

Proof. Fix a ball X containing $\overline{D}$ and let p be a continuous real potential on X. Then, for every $x \in X$,

$$p(x) \geq \varepsilon_x^{D_n^c}(p) \geq \varepsilon_x^{D_{n+1}^c}(p) \geq \varepsilon_x^{D^c}(p) \qquad (n \in \mathbb{N}) \tag{5.1}$$

and the functions $x \mapsto \varepsilon_x^{D_n^c}(p)$ are finely harmonic on D_n. We define

$$v(x) := \inf_n \varepsilon_x^{D_n^c}(p) \qquad (x \in D).$$

Then v is finely subharmonic on D and, for every $z \in \partial_f D$,

$$\text{fine-limsup}_{x \in D, x \to z} v(x) \leq \text{fine-limsup}_{x \in D, x \to z} p(x) = p(z).$$

By [Fug72, Theorem 14.6] we conclude that $v(x) \leq \varepsilon_x^{D^c}(p)$ for every $x \in D$. Therefore

$$\lim_{n \to \infty} \varepsilon_x^{D_n^c}(p) = \varepsilon_x^{D^c}(p) \qquad (x \in D).$$

The proof is finished by approximation with differences of continuous real potentials. □

5.2 Spectral invariance

Lemma 5.4. *Let K_j be compact operators on complex Banach spaces B_j, $j = 1, 2$, and suppose that $B_1 \cap B_2$ is dense in B_1 and B_2 and that $K_1 = K_2$ on $B_1 \cap B_2$. Then any eigenvalue $\alpha \neq 0$ of K_1 is an eigenvalue of K_2 and the algebraic multiplicities coincide.*

Proof. Fix $\alpha \in \mathbb{C} \setminus \{0\}$, $0 < r < \operatorname{dist}(\alpha, (\sigma(K_1) \cup \sigma K_2)) \setminus \{\alpha\})$ and let $\Gamma(t) = \alpha + re^{it}$, $0 \le t \le 2\pi$. The spectral projection E_j corresponding to α relative to K_j is given by

$$E_j = \frac{1}{2\pi i} \int_\Gamma (\xi I - K_j)^{-1} \, d\xi \qquad (j = 1, 2).$$

Since $K_1 = K_2$ on $B_1 \cap B_2$, we easily obtain that $E_1 = E_2$ on $B_1 \cap B_2$ (if $|\xi| > \max(\|K_1\|, \|K_2\|)$, then $(\xi I - K_j)^{-1} = \xi^{-1} \sum_{n=0}^{\infty} (\xi^{-1} K_j)^n$ whence $(\xi I - K_1)^{-1} = (\xi I - K_2)^{-1})$ on $B_1 \cap B_2$; by analyticity, this holds for every $\xi \in (\sigma(K_1) \cup \sigma(K_2))^c)$.

Given $j \in \{1, 2\}$, we know that the range $E_j(B_j)$ is finite dimensional and that the dimension of $E_j(B_j)$ is the algebraic multiplicity $d_j(\alpha)$ of α with respect to K_j (non-zero iff α is an eigenvalue of K_j) (see e.g. [Dow78, p. 53]). In particular, $E_j(B_j)$ is a closed subspace of B_j. So the density of $B_1 \cap B_2$ in B_j implies that

$$E_j(B_j) = E_j(B_1 \cap B_2).$$

Since $E_1 = E_2$ on $B_1 \cap B_2$, we conclude that $E_1(B_1) = E_2(B_2)$ whence $d_1(\alpha) = d_2(\alpha)$. □

5.3 A general remark on eigenvectors

Let K be a bounded self-adjoint operator on a complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$, the corresponding norm will be denoted by $\| \cdot \|_2$. Moreover, we assume that we have a complex Banach space $(B, \| \cdot \|)$ such that B is dense in H, the inclusion map is continuous $(\| \cdot \|_2 \le c \| \cdot \|)$ and the restriction of K on B is a compact operator on $(B, \| \cdot \|)$. In the application we have in mind we shall have $H = L^2(D, \mu)$ and $B = L^\infty(D, \mu)$, $D \subset \mathbb{R}^d$ bounded and finely open, μ a local Kato measure on $\mathbb{R}^d$.

Let $S(B)$ denote the set of all bounded operators A on $(B, \| \cdot \|)$ such that $\langle Ax, y \rangle = \langle x, Ay \rangle$ for all $x, y \in B$. We recall that every $A \in S(B)$ admits a unique (self-adjoint) extension $\tilde{A}$ on $(H, \langle \cdot, \cdot \rangle)$ (see e.g. [Jör70, p. 41]). Indeed, if $x \in B$, $\|x\|_2 \le 1$, then for every $n \in \mathbb{N}$,

$$\|A^n x\|_2 = \langle A^n x, A^n x \rangle^{\frac{1}{2}} = \langle A^{2n} x, x \rangle^{\frac{1}{2}} \le \|A^{2n} x\|_2^{\frac{1}{2}}$$

and therefore

$$\|Ax\|_2 \le \|A^{2^n} x\|_2^{2^{-n}} \le (c \, \|A^{2^n} x\|)^{2^{-n}} \le (c \, \|x\|)^{2^{-n}} \|A\|$$

whence $\|Ax\|_2 \le \|A\|$, $\|\tilde{A}\|_2 \le \|A\|$.

Moreover, it is easily seen that, for all $A_1, A_2 \in S(B)$,

$$\widetilde{A_1 A_2} = \tilde{A}_1 \tilde{A}_2.$$

A first consequence it the following:

Lemma 5.5. *Every real $\alpha \in \sigma(K) \setminus \{0\}$ is an eigenvalue of $K|_B$.*

Proof. Suppose that $\alpha \in \mathbb{R}\setminus\{0\}$ is not an eigenvalue of $K|_B$. Then there exists a bounded operator A on $(B, \|\cdot\|)$ such that

$$A(\alpha I - K|_B) = (\alpha I - K|_B)A = I \quad \text{ on } B.$$

Given $x, y \in B$, we take $u = Ax$, $v = Ay$, and obtain that

$$\langle Ax, y\rangle = \langle u, \alpha v - Kv\rangle = \langle \alpha u - Ku, v\rangle = \langle x, Ay\rangle,$$

i.e., $A \in S(B)$. So we have a bounded operator $\tilde{A}$ on H and

$$\tilde{A}(\alpha I - K) = (\alpha I - K)\tilde{A} = I \quad \text{ on } H,$$

i.e., $\alpha \notin \sigma(K)$. □

Proposition 5.6. *Every real $\alpha \in \sigma(K) \setminus \{0\}$ is an eigenvalue of K and the eigenspace*

$$E_\alpha := \{x \in H : Kx = \alpha x\}$$

is a (finite dimensional) subspace of B.

Proof. Fix $\alpha \in \mathbb{R} \cap (\sigma(K) \setminus \{0\})$ and let

$$F_\alpha := B \cap E_\alpha = \{x \in B : Kx = \alpha x\}.$$

Then F_α is finite dimensional ($F_\alpha \neq \{0\}$ by Lemma 5.5) and we have to show that $F_\alpha = E_\alpha$.

Let $P : H \to F_\alpha$ denote the orthogonal projection and define

$$H_1 := F_\alpha^\perp = \{x \in H : \langle x, y\rangle = 0 \text{ for all } y \in F_\alpha\}, \quad B_1 := B \cap H_1.$$

Then B_1 is dense in H_1: Given $x \in H_1$, there exists a sequence (x_n) in B such that $x_n \to x$ in H. Clearly, for every $n \in \mathbb{N}$,

$$\tilde{x}_n := x_n - Px_n \in B_1$$

and $Px_n \to Px = 0$ in H whence $\tilde{x}_n \to x$ in H. Further, B_1 is closed in B: If $x_n \in B_1$ and $x \in B$ such that $\lim_{n\to\infty} \|x - x_n\| = 0$, then $\lim_{n\to\infty} \|x - x_n\|_2 = 0$ and, for every $y \in F_\alpha$

$$|\langle x, y\rangle| = |\langle x - x_n, y\rangle| \leq \|x - x_n\|_2 \|y\|_2$$

whence $\langle x, y\rangle = 0$. Finally, if $x \in H_1$, then $\langle Kx, y\rangle = \langle x, Ky\rangle = \langle x, \alpha y\rangle = \alpha\langle x, y\rangle = 0$ for every $y \in F_\alpha$, i.e., $K(H_1) \subset H_1$ and $K(B_1) \subset B_1$. Clearly, the definition of B_1 implies that α is not an eigenvalue of $K|_{B_1}$. If $F_\alpha \neq E_\alpha$, then α would be an eigenvalue of $K|_{H_1}$. This is impossible by Lemma 5.5 (applied to H_1, B_1 and $K|_{H_1}$). Thus $F_\alpha = E_\alpha$ and the proof is finished. □

Corollary 5.7. *Every eigenvector of K corresponding to an eigenvalue $\alpha \neq 0$ is contained in B.*

Proof. This follows immediately from Proposition 5.6, since all eigenvalues of the self-adjoint operator K on H are real. □

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