Critical phenomena and strange attractors for some infinite-dimensional physical systems

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Abstract

In this paper we consider an infinite-dimensional system of ordinary differential equations which is applied to some popular and important physical problems and in which infinitely many degrees of freedom play an essential role. Bifurcations of spatially homogeneous solutions are described and infinite-dimensional strange attractors that lead to spacetime chaos are constructed.

0. Introduction

The main aim of this paper is to explain the critical phenomena and the emergence of stable chaos in some nonconservative physical systems consisting of very many (practically infinitely many) number of particles. Typical (and very popular) example of such system is the sandpile under the influence of external forces. In this example the forces between the particles are based on the low of the friction and therefore the differential equations describing the motion of particles contain the first (and not the second as in the conservative systems of classical mechanics) derivatives. Let us describe the main physical problems related to this system. We assume that the sand is under the action of sufficiently smooth external forces depending on some parameters: for example, the sand from external source is pouring on the surface of the pile. The the form of the pile principally does not change for all moments of the time for which the mass of the sand smaller or equal than a some threshold playing the role of a critical value but from this critical value of the mass the form of the pile is sharply and strongly changed. In other words here we observe a bifurcation or in the words of statistical physics a phase transition. In addition, the original form of the pile can be very complicated and chaotic, and nevertheless this form can be stable in the sense that under small perturbations of it the reconstruction of the original form takes place. Similar problems originate in many other systems and some theories created at the present time of their explanation: bifurcation theory and catastrophie theory [1]. The main peculiarity of the sandpile model (and as of many other models) which makes the application of these theories difficult is the very high dimension (practically infinity) of the system of differential equations describing these physical systems.

In the present paper one undertake the attempt to resolve these problems for a broad class of systems consisting of infinitely many number of particles $P_{\vec{n}}$ numbered by all r-dimensional integer vectors $\vec{n} = (n_1, \ldots, n_r) \in \mathbf{Z}^r$ and are moving in r-dimensional real space $\mathbf{R}^r$. Here r is a natural number and $\mathbf{Z}^r$ is the integer lattice in $\mathbf{R}^r$. We suppose that each particle $P_{\vec{n}}$ interacts only with the neighboring particles along each direction in $\mathbf{Z}^r$ in accordance with some law generalizing the law of elastic spring. We also assume that each particle is under the action of external field of force along each coordinate axis in $\mathbf{R}^r$. The point $P_{\vec{n}}$ is given by r-dimensional vector $x_{\vec{n}} = (x_{\vec{n}}^{(1)}, \ldots, x_{\vec{n}}^{(r)}) \in \mathbf{R}^r$ and formal potential of the interaction U has the form

$$U = \sum_{\vec{n}\in\mathbf{Z}^r} \sum_{S=1}^{r} G^{(S)}(x_{\vec{n}}^{(S)}) + \chi \sum_{(\vec{n}',\vec{n}'')\in\Gamma} \mid x_{\vec{n}'} - x_{\vec{n}''} - \vec{a} \mid^2 \ , \tag{1}$$

where

$$\begin{aligned} x_{\vec{n}} &= (x_{\vec{n}}^{(1)}, \ldots, x_{\vec{n}}^{(r)}) \in \mathbf{R}^r, \vec{n}' = (n_1', \ldots, n_r') \in \mathbf{Z}^r, \\ \vec{n}'' &= (n_1'', \ldots, n_r'') \in \mathbf{Z}^r, x_{\vec{n}'} = (x_{\vec{n}'}^{(1)}, \ldots, x_{\vec{n}'}^{(r)}), \\ x_{\vec{n}''} &= (x_{\vec{n}''}^{(1)}, \ldots, x_{\vec{n}''}^{r)}) \in \mathbf{R}^r, \vec{a} = (a^{(1)}, \ldots, a^{(r)}) \in \mathbf{R}^r, \end{aligned}$$

$$|x_{\vec{n}'} - x_{\vec{n}''} - \vec{a}|^2 = \sum_{S=1}^{r} (x_{\vec{n}'}^{(S)} - x_{\vec{n}''}^{(S)} - a^{(S)})^2 \ .$$

Here Γ is the set of pairs of r-dimensional integer vectors $\vec{n}', \vec{n}''$ such that

$$\sum_{S=1}^{r} |n_S' - n_S''| = 1 \ , \quad \sum_{S=1}^{r} (n_s' - n_S'') = 1 \ , \tag{2}$$

and the sum in the right part of the equality (1) is taken over all pairs $(\vec{n}', \vec{n}'') \in \Gamma$. We also assume that γ is a constant and $G^{(S)}(x)$ $(S = 1, \ldots, r)$ is infinitely differentiable function. If $\gamma > 0$ then from a physical point of view the system represents a net of springs that are connected at the nodes of the lattice $\mathbf{Z}_r$ and are under an external field with the potential $\sum_{S=1}^{r} G^{(S)}(x_{\vec{n}}^{(S)})$.

Further we assume that the functions $G^{(S)}(x)$, charaterizing the external forces acting along axis are represented in the form

$$G^{(S)}(x) = F^{(S)}(x) - \beta^{(S)} x \ , \tag{3}$$

where $F^{(S)}(x)$ is not identically equal to zero function having a bound derivative and $\beta^{(S)} > 0$ are parameters.

Now we consider the gradient system with infinitely number degrees of freedom:

$$\frac{dx_{\vec{n}}}{dt} = -\frac{\partial U}{\partial x_{\vec{n}}} \,, \tag{4}$$

where

$$\frac{dx_{\vec{n}}}{dt} = \left(\frac{dx^{(1)}_{\vec{n}}}{dt}, \ldots, \frac{dx^{(r)}_{\vec{n}}}{dt}\right) \in \mathbf{R}^r, \frac{\partial U}{\partial x_{\vec{n}}} = \left(\frac{\partial U}{\partial x^{(1)}_{\vec{n}}}, \ldots, \frac{\partial U}{\partial x^{(r)}_{\vec{n}}}\right) \in \mathbf{R}^r \,,$$

and for any $S = 1, \ldots, r$ the quantity $\frac{\partial U}{\partial x^{(S)}_{\vec{n}}}$ is the formal derivative of the formal series with respect to $x^{(S)}_{\vec{n}}$. Using (1) – (3) the equation (4) takes the form:

$$\frac{dx^{(S)}_{\vec{n}}}{dt} = -f^{(S)}\left(x^{(S)}_{\vec{n}}\right) + \beta^{(S)} + 2\chi\left(-2rx^{(S)}_{\vec{n}} + \sum_{\vec{n}'\in\Gamma_{\vec{n}}} x^{(S)}_{\vec{n}'}\right) \,, \tag{5}$$

where $S = 1, \ldots, r$; $\vec{n} = (n_1, \ldots, n_r) \in \mathbf{Z}^r$, $f^{(S)}(x^{(S)}_{\vec{n}}) = \frac{dF^{(S)}}{dx}(x^{(S)}_{\vec{n}})$, $\Gamma_{\vec{n}}$ is the set of all vectors $\vec{n}' = (n'_1, \ldots, n'_r) \in \mathbf{Z}^r$ such that $\sum_{S=1}^{r} |n_S - n'_S| = 1$, t is an independent variable.

According to the definition of the function $F^{(S)}(x)$ in (3) for all $S = 1, \ldots, r$ there exist constants γ'_S and γ''_S satisfying the inequality $\gamma'_S < \gamma''_S$ such that

$$\gamma'_S = \min_x f^{(S)}(x) \leq f^{(S)}(x) \leq \max_x f^{(S)}(x) = \gamma''_S \,. \tag{6}$$

In this paper we obtain rigorous results about the qualitative behavior of the solutions of the system (5) for the one-dimensional case $r = 1$ and in the particular case for which in the equation (5) the constant $\chi = \frac{1}{2}$, the parameter $\beta^{(1)} = a > 0$ and the function $f^{(1)}(x) = b\sin x$, where $b > 0$ is a constant. In this case the system of equation (5) takes the form of the following infinite system of ordinary differential equations:

$$\frac{dx_n}{dt} = x_{n-1} - 2x_n + x_{n+1} - b\sin x_n + a, \quad n \in \mathbf{Z} \tag{7}$$

with respect to the infinite-dimensional vector $x = (x_n)_{n\in\mathbf{N}}$ with components x_n $(n \in \mathbf{Z})$.

In Sec. 1 we study the bifurcations of the spatially homogeneous solutions of the system (7) [i.e., solutions $x(t) = (x_n(t))$ in which the funciton $x_n(t)$ does not depend on n] for varying paramter a and given parameter b. In this section we prove Theorem 1 and the associated Corollary 1, in which we completely describe all the spatially homogeneous solutions (see the proof of Theorem 1) and find the critical value of the parameter a at which the behavior of a spatially homogeneous solution of the system (7) changes in a drastic way. In Sec. 2 we prove the existence of infinite-dimensional attractors in definite ranges of the parameters a and b, i.e., the existence in the infinite-dimensional phase space of the system (7) of invariant sets that in a well defined sense are unstable; one of them is stochastic, and two others are the sets of separatrices of the hyperbolic stationary solutions of the system (7) to which in the limit $t \to \infty$ all solutions of the system in a certain infinite-dimensional neighborhood converge in the limit $t \to \infty$. The model can be regarded as a spacetime dynamical system (the variable n determines a discrete spatial direction, and t determines the continuous time direction) in which motion is possible in both the space and time directions. For motion in the space direction, we can choose as phase space a set of two infintitely differentiable functions $x_0(t), x_1(t)$ [or a set of two infinitely differentiable functions $x_k(t), x_{k+1}(t)$ for any preassigned integer k]. Indeed, by virtue of (7) the functions $x_n(t)$ can be uniquely recovered for all n given known functions $x_0(t), x_1(t)$ [or $x_k(t), x_{k+1}(t)$]. Such a phase space was originally introduced in [3] (see also [4]), for the Frenkel'-Kontorova model [2]. We use it here to construct infinite-dimensional strange attractors and infinite-dimensional spacetime chaos. This is done as follows. From the point of view of motion in the space direction, the system (7) has a set of stationary (not depending on the time) hyperbolic solutions which the separatrices (contracting and expending) intersect, in general, at more than one point, and the separatrices of each such stationary solution, and the set of their points of intersection admits infinite-dimensional neighborhoods that converge to them as $t \to \infty$. Theorems 2 and 3 and the associated Corollaries 2 and 3, in which these attractors are constructed, are proved in Sec. 2. We note that the system we consider is one for which the existence of infinitely many degrees of freedom plays an essential role, because the solutions, for which the critical phenomena and the strange attractors are discovered, do not exist in finite-dimensional case. Nevertheless for sufficiently high dimension in finite-dimensional case the solutions which are close to solutions found here can exist.

The results of this paper are based on the paper [5].

For arbitrary natural r the results relating to the equation (5) and generalizing the results for one-dimensional case ($r = 1$) will be published in a separate paper. The proofs of these results essentially use the inequality (6) and the notion of phase space of the equation (5) which originally was introduced in [6] for the generalization of Frenkel'-Kontorova model.

1. Bifurcations of spatially homogeneous solutions

Definition. A solution $x(t) = (x_n(t))$ of the system (7) is said to be spatially homogeneous if for all t and n

$$x_n(t) = x_{n+1}(t) \ .$$

We fix the parameter $b > 0$ in the system (7) and investigate the dependence of spatially homogeneous solutions on the parameter a.

Theorem 1. Let $x(t) = (x_n(t))$ be a spatially homogeneous solution of the system (7).

Two cases are possible:

I) if $b < a$, then

$$\lim_{t\to\infty} x_n(t) = \infty, \quad \dot{x}_n(t) > 0 \ ;$$

II) if $b \geq a$, then

$$\lim_{t\to\infty} x_n(t) = 2\mathrm{arctg}\left(\sqrt{\frac{b^2}{a^2} - 1} + \frac{b}{a}\right) \ .$$

Proof. Let $x_n(t) = \hat{x}(t)$. In (7) we replace $x_n(t)$ by the function $\hat{x}(t)$, and we find that the function $\hat{x}(t)$ satisfies the equation

$$\frac{d\hat{x}}{dt} = -b\sin\hat{x} + a \ .$$

Integrating this equation and setting $\delta = -\frac{a}{b}$, $\gamma = \frac{1}{\delta}$, we have to distinguish three cases:

1) if $\gamma^2 < 1$, then

$$-\frac{2}{b\delta\sqrt{1-\gamma^2}} \ \mathrm{arctg} \ \frac{\gamma + tg\frac{\hat{x}(t)}{2}}{\sqrt{1-\gamma^2}} = t + \ \mathrm{const} \ ;$$

2) if $\gamma^2 = 1$, then

$$\frac{2}{b\delta\sqrt{\gamma + tg\frac{\hat{x}(t)}{2}}} = t + \ \mathrm{const} \ ;$$

3) if $\gamma^2 > 1$, then

$$-\frac{1}{b\delta\sqrt{\gamma^2-1}} \ln \frac{\gamma + tg\frac{\hat{x}(t)}{2} - \sqrt{\gamma^2-1}}{\gamma + tg\frac{\hat{x}(t)}{2} + \sqrt{\gamma^2-1}} = t + \ \mathrm{const} \ .$$

Letting t tend to infinity in these equations, we obtain the assertion of Theorem 1.

Corollary 1. The value $a = b$ is a critical value of the parameter a for which the spatially homogeneous solution $x(t)$ of the system (7) changes its behavior fundamentally in the limit $t \to \infty$.

2. Construction of infinite-dimensional strange attractors

We introduce the space X of two-sided sequences $x = (x_n) = (..., x_1, x_0, x_1, \ldots)$, where x_n is a real number for any integer n. We introduce the two-dimensional torus

$$T = \{u, v : 0 \leq u < 2\pi, 0 \leq v < 2\pi\}$$

and consider an area-preserving transformation A of the torus T of the following form:

$$A(u,v) = (u',v') ,$$

where

$$u' = v, v' = -u + 2v + b\sin v - a \mod 2\pi .$$

Lemma 1. Suppose that there exists α such that $0 \le \alpha < 2\pi$ and that the relations $b\sin\alpha = a$, $b\cos\alpha > 0$ are satisfied. Then the point $(\alpha,\alpha) \in T$ is a fixed hyperbolic point of the mapping A.

Since in accordance with Lemma 1 the fixed point (α,α) is hyperbolic, it has a contracting separatrix Γ_- and an expanding separatrix Γ_+. We introduce the set $\Pi \subset T$, which consists of all points of intersection of Γ_- and Γ_+, $\Pi = \Gamma_- \bigcap \Gamma_+$.

Remark 1. If a and b are small, then the set Π consists of infinitely many points [7].

A solution $x(t)$ of the system (7) is said to be stationary if it does not depend on the time t. On X we introduce the set Ω, and also the sets Ω_+ and Ω_- in the following way. An element $x^0 = (x^0_n) \in \Omega$ (respectively, $x^0 \in \Omega_+$ or $x^0 \in \Omega_-$) if and only if x^0 is a stationary solution of the system (7) and for some integer k the point

$$\mathcal{T}_k = \left(2\pi\left\{\frac{x^0_k}{2\pi}\right\}, 2\pi\left\{\frac{x^0_{k+1}}{2\pi}\right\}\right) \in \Pi$$

(respectively, $\mathcal{T}_k \in \Gamma_+$ or $\mathcal{T}_k \in \Gamma_-$), where the symbol $\{a\}$ denotes the fractional part of the number a.

Remark 2. $\Omega = \Omega_+ \bigcap \Omega_+$.

Lemma 2. If $x^0 = (x^0_n) \in \Omega$ (respectively, $x^0 \in \Omega_+$ or $x^0 \in \Omega_-$), then for all integer n

$$\mathcal{T}_n = \left(2\pi\left\{\frac{x^0_n}{2\pi}\right\}, 2\pi\left\{\frac{x^0_{n+1}}{2\pi}\right\}\right) \in \Pi$$

(respectively, $\mathcal{T}_n \in \Gamma_+$ or $\mathcal{T}_n \in \Gamma_-$) and $A\mathcal{T}_n = \mathcal{T}_{n+1}$.

Proof. Since x^0 is a stationary solution of the system (7), the following equation holds for any integer n by virtue of (7):

$$x^0_{n+1} = -x^0_{n-1} + 2x^0_n + b\sin x^0_n - a . \tag{8}$$

The assertion of Lemma 2 obviously follows from this equation and from definition of the mapping A.

Let $x^0 = (x^0_n) \in X$, m being an integer, N be a natural number, and $\epsilon > 0$. We define a neighborhood of the element x^0, denoting it by $U(x^0, m, N, \epsilon)$, as follows:

an element $x = (x_n) \in U(x^0, m, N, \epsilon)$ if and only if $x_m = x^0_m$, $x_{m+N} = x^0_{m+N}$ and for $k = 1, \ldots, N-1 \quad |x_{m+k} - x^0_{m+k}| < \epsilon$.

Lemma 3. Let $U = U(x^0, m, N, \epsilon)$ be the neighborhood of the element $x^0 \in X, y = (y_n) \in U$, t_0 be a real number. Then there exists a solution $x(t)$ of the system (7) such that $x_S(t_0) = y_S$ if $m+1 \leq S \leq m+N-1$, and for all $t \geq t_0$

$$x_m(t) = x^0_m, x_{m+N}(t) = x^0_{m+N} \; .$$

Proof. If $x(t) \in U$, then by using (7) the $(N-1)$-dimensional vector $\tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_{N-1})$ with coordinates $\tilde{x}_S = x_{m+S}$ $(S = 1, \ldots, N-1)$ satisfies the following system of equations of order $N-1$:

$$\frac{d\tilde{x}}{dt} = q + Q\tilde{x} - \tilde{f}(\tilde{x}) \; , \tag{9}$$

where $q = (q_1, \ldots, q_{N-1})$ is the vector with coordinates

$$q_S = \begin{cases} x^0_m + a \; , & if \quad S = 1 \; , \\ x^0_{m+N} + a \; , & if \quad S = N-1 \; , \\ a \; , & if \quad 1 < S < N-1 \; , \end{cases}$$

$\tilde{f}(\tilde{x}) = (\tilde{f}_1(\tilde{x}), \ldots, \tilde{f}_{N-1}(\tilde{x}))$ is the vector with coordinates $\tilde{f}_S = -b \sin \tilde{x}_S$ $(S = 1, \ldots, N-1)$,

$$Q = \begin{pmatrix} -2 & 1 & 0 & \ldots & 0 \\ 1 & -2 & 1 & 0 \ldots & 0 \\ 0 & 0 & 0 & \ldots & 1-2 \end{pmatrix} = (Q_{ij}) \tag{10}$$

is the tridiagonal Jacobi matrix of order $N-1$ with $Q_{ii} = -2$ on the diagonal and the number 1 on the two neighboring diagonals. Applying now to the system (9) the theorem about the existence and uniqueness of solutions, we find that there exists an infinitely differentiable (with respect to t) solution $\tilde{x}(t)$ of the system (9) for $t \geq t_0$ with initial condition $\tilde{x}(t_0) = (\tilde{x}_1(t_0), \ldots, \tilde{x}_{N-1}(t_0))$ having coordinates $\tilde{x}_S(t_0) = y_{m+S}$. Further it follows from the form of the system (7) that if for some integer k and $t \geq t_0$ two infinitely differentiable functions $x_k(t)$ and $x_{k+1}(t)$ are known then from them one can uniquely recover all the functions $x_n(t)$, which are the coordinates of the solution $x(t)$ of the system (7) for $t \geq t_0$. Therefore, knowing the coordinates $x_m(t) = x^0_m$, $x_{m+1}(t) = \tilde{x}_1(t)$, $x_{m+N-1}(t) = \tilde{x}_{N-1}(t)$, $x_{m+N} = x^0_{m+N}$ and using the fact that the vector $\tilde{x}(t)$ satisfies the system (9) with initial data $\tilde{x}_S(t) = y_{m+S}$ $(S = 1, \ldots, N-1)$, we determine in a unique way a solution of the system (7) satisfying Lemma 3 and for which $x_{m+S}(t) = \tilde{x}_S(t)$ if $1 \leq S \leq N-1$.

Theorem 2. For any $x^0 = (x_n^0) \in \Omega$ there exist numbers M_- and M_+ and a sequence of positive numbers $\epsilon_1, \epsilon_2, \ldots$ such that if for an integer m and a natural number N at least one of the inequalities $m \geq M_+$ or $m+N \leq M_-$ is satisfied and the solution $x(t) = (x_n(t))$ of the system (7) at the initial time t_0 satisfies the condition $x(t_0) \in U(x^0, m, N, \epsilon_N)$ and for $t \geq t_0$ $x_m(t) = x_m^0$, $x_{m+N}(t) = x_{m+N}^0$, then for any integer n $\quad x_n(t) \to x_n^0$ as $t \to \infty$.

Proof. Since $x^0 = (x_n^0) \in \Omega$, by Lemmas 1 and 3

$$\lim_{n \to +\infty} \left(2\pi \left\{ \frac{x_n^0}{2\pi} \right\} \right) = \lim_{n \to -\infty} \left(2\pi \left\{ \frac{x_n^0}{2\pi} \right\} \right) = \alpha \ ,$$

where α is the number introduced in Lemma 1.

Therefore, by virtue of Lemma 1 there exist integer M_+ and M_- such that for $n \geq M_+$ and $n < M_-$

$$\cos x_n^0 > 0 \ . \tag{11}$$

Now suppose that for the integer m and natural N at least one of the inequalities

$$m \geq M_+, m + N \leq M_-$$

is satisfied and consider the system of equation (9) discussed in Lemma 3 in the neighborhood of the equilibrium position $\tilde{x}^0 = (\tilde{x}_1^0, \ldots, \tilde{x}_{N-1}^0)$, where for $S = 1, \ldots, N-1$ $\tilde{x}_S^0 = x_{m+S}^0$. Linearization of the system (9) at the point $\tilde{x}^0$ leads to the linear system

$$\frac{d\tilde{x}}{dt} = \mathcal{D}\tilde{x} \ ,$$

where $\mathcal{D}$ is the matrix of partial derivatives of the right-hand side of the system (9) at the point $\tilde{x}^0$, which has the form

$$\mathcal{D} = Q + \Lambda \ ,$$

where Q is the matrix introduced in (10), and

$$\Lambda = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_{N-1} \end{pmatrix}$$

is a diagonal matrix with diagonal elements

$$\lambda_S = -b \cos \tilde{x}^0 \quad (S = 1, \ldots, N-1) \ . \tag{12}$$

Since $b > 0$, it follows from the relations (12) and (11) and from the form of the matrix Q in (10) that the matrix $-\mathcal{D} = -Q - \Lambda$ is an oscillatory matrix [8], and therefore all its eigenvalues are real, positive, and pairwise distinct, while the eigenvalues of the matrix $\mathcal{D}$ are negative and pairwise distinct. Thus, the equilibrium position $\tilde{x}^0$ of the system (9) is asymptotically stable in the sense of Lyapunov. The assertion of Theorem 2 now follows directly from Lemma 3 and the form of the system (7).

Theorem 3. For any $x^0 = (x_n^0) \in \Omega_+$ (respectively, $x^0 \in \Omega_-$) there exists a number M_- (respectively M_+) and a sequence of positive numbers $\epsilon_1, \epsilon_2, \ldots$ such that if for integer m and a natural number N the inequality $m + N \leq M_-$ (respectively, $m \geq M_+$) holds and the solution $x(t) = (x_n(t))$ of the system (7) at the initial time t_0 satisfies the condition $x(t_0) \in U(x^0, m, N, \epsilon_N)$ and for $t \geq t_0$ $\quad x_m(t) = x_m^0$, $x_{m+N}(t) = x_{m+N}^0$, then for any integer n

$$\lim_{t\to\infty} x_n(t) = x_n^0 \ .$$

Theorem 3 is proven in the same way as Theorem 2.

Corollary 2. Consider the set

$$S = \bigcup_{x^0\in\Omega} \ \bigcup_{\substack{m\geq M_+(x_0)\\ m+N\leq M_-(x_0)}} \ \bigcup_{N=1}^{\infty} U(x^0, m, N, \epsilon) \ ,$$

where the numbers $M_+ = M_+(x^0), M_- = M_-(x^0)$ and ϵ_N are defined as in Theorem 2. Then the set of initial data of the solutions $x(t)$ of the system (7) are such that $x(t_0) \in S$ and $\lim_{t\to\infty} x(t) \in \Omega$ is infinite dimensional.

Corollary 3. Consider the sets

$$\begin{aligned} S_+ &= \bigcup_{x^0\in\Omega_+} \ \bigcup_{m+N\leq M_-(x_0)} \ \bigcup_{N=1}^{\infty} U(x^0, m, N, \epsilon_n) \ , \\ S_- &= \bigcup_{x^0\in\Omega_-} \ \bigcup_{m\geq M_+(x_0)} \ \bigcup_{N=1}^{\infty} U(x^0, m, N, \epsilon_n) \ , \end{aligned}$$

where the numbers $M_+ = M_+(x^0)$, $M_- = M_-(x_0)$ and ϵ_N are defined as in Theorem 3. Then the set of initial data of the solutions $x(t)$ of the system (7) such that $x(t_0) \in S_+$ and $\lim_{t\to\infty} x(t) \in \Omega_+$ is infinite dimensional and the set of initial data of the solution $x(t)$ of the system (7) such that $x(t) \in S_-$ and $\lim_{t\to\infty} x(t) \in \Omega_-$ is infinite dimensional.

Corollaries 2 and 3 follow directly from Theorem 2 and 3, respectively, and from the fact that the phase space of solutions of the system (7) in the region $U(x^0, m, N, \epsilon_N)$ has dimension $N - 1$ nd N ranges over all natural numbers.

3. Summary and conclusions

This article describes a model consisting of infinitely many numbers of particles which interact only with the nearest-neighbor particles and are moreover under the action of external fields. It discusses critical phenomena and chaos in some physical systems. The one dimensional case is investigated in more details. In this case the bifurcations of spatially homogeneous solutions are obtained and strange attractors are constructed. Bifurcations give a rigorous justification for the possibility of phase transitions in nonconservative physical systems having very high dimension, and the existence of strange attractors is the signature of chaotic behavior.

4. References

[1] V.I. Arnold, V.S. Afrajmovich, Yu.S. Il'yashenko, L.P. Shil'nikov. Bifurcation Theory and Catastrophe Theory. Springer (1999).

[2] Ya.I. Frenkel and T.A. Kontorova. On the theory of plastic deformation and doubling, Zh. Eksper. Teoret. Fiz. 8 (1938), 89–97.

[3] L.D. Pustyl'nikov. A dynamical system with an infinite number of degrees of freedom and a solution of the Frenkel-Kontorova problem, Theoret. Mat. Fiz. 68 (1986), 58–68; English transl. in Theoret. and Math. Phys. 68 (1986).

[4] L.D. Pustyl'nikov. Infinite-dimensional non-linear ordinary differential equation and the KAM theory, Uspekhi Mat. Nauk. 52:3 (1997), 105–160; English transl. in Russian Math. Surveys 52:3 (1997), 551–604.

[5] L.D. Pustyl'nikov. Infinite-dimensional strange attractors and bifurcation in a dynamical system with an infinite number of degrees of freedom, Teoret. Mat. Fiz. 92 (1992), 85-91; English transl. in Theoret. and Math. Phys. 92 (1993).

[6] T.V. Lokot' and L.D. Pustyl'nikov. On some solutions of infinite-dimensional systems of ordinary differential equations originating in statistical mechanics, Sinai's Moscow Seminar on Dynamical Systems. Adv. Math. Sci. AMS Transl. (2) 171 (1996), 173–183.

[7] V.G. Gelfreich, V.F. Lazutkin, N.V. Svanidze. Refined Formula to Separatrix Splitting for the Standard Map, Warwick Preprints, 59/1992, Nov. 1992.

[8] F.R. Gantmacher. The Theory of Matrices, Transl. of first Russ. ed., Chelsea, New York (1959).