Lyapunov exponents and transport in Self-Organized Criticality.

B. Cessac * Ph. Blanchard † T. Krüger ‡

We discuss the role played by the Lyapunov exponents in the dynamics of the Zhang's model of Self-Organized Criticality. We show that a wide part in the spectrum (slowest modes) is associated to energy transport in the lattice. We give in particular bounds on the first negative Lyapunov exponents in terms of the energy flux dissipated at the boundaries per unit time. We then establish an explicit formula for the transport modes, that appear as diffusion modes in a landscape where the metric is given by the density of active sites. We use a finite size scaling ansatz for the Lyapunov spectrum and relate the scaling exponent to the scaling of quantities like avalanche size, duration, density of active site, etc ...

PACS number: 02.10.Jf, 02.90+p, 05.45.+b, 05.40.+j

I. INTRODUCTION.

Within the last 10 years the notion of Self-Organized Criticality (SOC) became a new paradigm for the explanation of a huge variety of phenomena in nature and social sciences. It's origin lies in the attemp to explain the widespread appearence of power-law like statistics for characteristic events in a multitude of examples like the distribution of the size of earthquakes, 1/f-noise, amplitudes of solar flares, species extinction .... to name only a very few cases [1,2,14]. In this paradigm, the dynamics occurs as chain reactions or avalanches. Furthermore, a stationnary regime is reached, where the average incoming flux of external perturbations is compensed by the average outgoing flux that can leave the system at the boundary, or by dissipation in the bulk. Therefore, there is a constant flux through the system, leading to a non-equilibrium situation. In this stationary state, refered to as the *SOC state*, the distribution of avalanches follows a power law, namely there is scale invariance reminiscent of thermodynamic systems at the critical point. A local perturbation can induce effects at any scale and there are long-range spatial and time correlations. In other words, in this paradigm, the system reaches *spontaneously* a critical state without any fine tuning of some control parameter.

Several models have been proposed to mimic this mechanism like the sandpile model [1], the abelian sandpile [7] or the continuous energy model [26]. The results available are mainly numerical and a few rigorous results are known. The numerical simulations report the following behaviour. Fix an observable, say x, measuring some property of an avalanche (duration, size, etc ...), and compute the related probability $P_L(x)$ at stationnarity, for a system of characteristic size L. The graph of $P_L(x)$ exhibits a power law behaviour on a finite range, with a cut-off corresponding to finite size effects. As L increases the power law range increases, leading to conjecture that a critical state is indeed achieved in the thermodynamic limit, namely that $P_L(x)$ behaves like $\frac{1}{x^{\tau_x}}$ as $L \to \infty$. τ_x is called the *critical exponent* for the observable x. There is apparently no control parameter to tune to achieve the critical state, though some

*Institut Non Linéaire de Nice, 1361 Route des Lucioles, 06500 Valbonne, France
†University of Bielefeld, BiBoS, Postfach 100131, D-33501, Bielefeld, Germany
‡University of Bielefeld, BiBoS, Postfach 100131, D-33501, Bielefeld, Germany and Technische Universitaet, Str. des 17 July 135, 10623, Berlin, Germany

"hidden" parameters can be present, requiring a fine tuning to achieve criticality [23]. Despite the large number of papers written on the subject some basic problems are still opened.

Guided by the wisdom coming from the phase transition in equilibrium systems and the renormalization group analysis, it seems natural to investigate a possible classification of the models into universality classes characterized by a set of critical exponents for a family of "relevant" avalanches observables. Besides the fact that the commonly studied observables (size, duration, area, giration radius) do not necessarily consitute a *complete* set allowing to classify the models, even the computation of the critical exponents τ_x from numerical data is not easy and people do not yet agree on the way to do it. It is clear that the simple measurement of the slope of $P_L(x)$ in the linear range of a log-log plot is not reliable, due to the finite sample fluctuations and because the explicit form of the cut-off is not known in general. Therefore, refinements have to be proposed. The computation of τ_x from the behaviour of the moments is certainly a better way. However there is no agreement yet wether one can use a finite size scaling treatment [15] or more sophisticated methods (like multifractal analysis [22]). In this case the identification of a (supposed) universality class seems problematic. On a more general ground it is not known at the moment what are the key ingredients one has to put in the models to insure SOC (for example, the question wether the local conservation of energy is required remains unresolved), and what about the genericity of this behaviour. Also, an important question is the link that one can establish between the criticality of the out of equilibrium SOC models and the usual statistical mechanics of phase transition in equilibrium systems.

Recently we showed that the Zhang's model [26] can be fruitfully studied with the tools from hyperbolic dynamical system theory [3–5]. Then we were able to extract unexpected result, establishing in particular a formula relating the critical exponent of avalanche size and the spectrum of Lyapunov exponent. In this paper we developp this point of view and make a step further towards the unterstanding of the dynamical properties of this model and their link to the SOC state. We first define the model as an hyperbolic dynamical system of skew-product type. We then define in this setting the two distinguished times scale which are believed to play a key role in the SOC process: the *local time* which is the natural time for the dynamical system and the *avalanche time*, related to the avalanche duration. We introduce a natural invariant measure to characterize the statistical properties at stationnarity, and we relate the avalanche observable statistics to the ergodic local time average. We then discuss the role played by the Lyapunov exponents in the dynamics and their relation with energy transport and average avalanche observables. We show that random excitation induces a positive Lyapunov exponent, while the relaxation dynamics corresponds to negative exponents. Furthermore, we show that a wide part of the spectrum (slowest modes) is associated to energy transport in the lattice. We give in particular bounds on the first negative Lyapunov exponents in terms of the energy flux dissipated at the boundaries per unit time. We establish an explicit formula for the transport modes, that appear as diffusion modes in a landscape where the metric is given by the density of active sites. They differ dramatically from the normal difusion modes one would obtain by assuming a uniform density of active sites but for the first modes. We then use a finite size scaling ansatz for the Lyapunov spectrum and relate the scaling exponent to the critical exponent of avalanche size, duration, and anomalous diffusion.

II. DYNAMICAL SYSTEM DEFINITION AND BASIC PROPERTIES.

A. Definition.

The Zhang's model [26], widely inspired from the Bak-Tang-Wiesenfield precursor model [1], has been introduced as a possible example of a model which "self-organizes" into a critical state in the thermodynamic limit, namely without fine-tuning of a control parameter. Its beauty lies in its simplicity.

Let Λ be a d-dimensional sub-lattice in $\mathbb{Z}^d$, taken as a square of edge length L for simplicity. Call $N = \#\Lambda = L^d$, and let ${}^{\partial}\Lambda$ be the boundary of Λ, namely the set of points in $\mathbb{Z}^d \setminus \Lambda$ at distance 1 from Λ. Each site $i \in \Lambda$ is characterized by its "energy" X_i, which is a non-negative real number. Call $\mathbf{X} = \{X_i\}_{i\in\Lambda}$ a configuration of energies. Let E_c be a real, strictly positive number, called the *critical energy*, and $\mathcal{M} = [0, E_c[^N$. A configuration $\mathbf{X}$ is called "stable" iff $\mathbf{X} \in \mathcal{M}$ and "unstable" otherwise. If $\mathbf{X}$ is stable then one chooses a site i at random with some probability $\nu_L(i)$, and add to it energy δ, where δ is set to 1 in this paper (excitation). If a site i is *overcritical* or *active* ($X_i \geq E_c$), it loses a part of its energy in equal parts to its $2d$ neighbours (relaxation). Namely, we fix a parameter $\epsilon \in [0, 1[$ such that, after relaxation of the site i, the remaining energy of i is ϵX_i, while the $2d$ neighbours receive the energy $\frac{(1-\epsilon)X_i}{2d}$. Note therefore that there is *local conservation of energy*. If several nodes are simultaneously active, the local distribution rules are additively superposed, i.e. the time evolution of the system is synchronous. The succession of updating leading an unstable configuration to a stable one is called an *avalanche* (a more precise definition of an avalanche will be given below). There is dissipation at the boundaries namely the sites of ${}^{\partial}\Lambda$ have always zero energy. As a result, all avalanches are *finite*. The addition of energy is *adiabatic*. When an avalanche occurs, one waits until it stops before adding a new energy quantum. Further excitations eventually generate a new avalanche, but, because of the adiabatic rule, each new avalanche starts from *only one* active site. Note that relaxation depends on *local* conditions while excitation is conditionned by *global* constraints (all sites are quiescent). It is conjectured that a critical state is reached, independently on E_c, at least for large E_c values [1].

B. The Zhang's model as a dynamical system.

Because all avalanches are *finite* (for finite L), and since we are not interested in the transients, one can without loss of generality take all initial energy configurations $\mathbf{X} \in \mathcal{M}$. All trajectories borned from $\mathcal{M}$ belong to a compact set $\mathcal{B}$. Call $\bar{\mathcal{M}} = \mathcal{B} \setminus \mathcal{M}$. $\bar{\mathcal{M}}$ contains the set of all unstable energy configurations achievable in an avalanche borned from an energy configuration in $\mathcal{M}$.

Fix $\epsilon > 0$, and call $\alpha = \frac{1-\epsilon}{2d}$. Let h be Heaviside function. Define $H : \mathbb{R}^N \to \{0, 1\}^N$ such that $H(\mathbf{X})$ is the vector $\{h(X_i)\}_{i=1\ldots N}$. Call $\mathbf{X}_c$ the vector $\{E_c\}_{i=1\ldots N}$. Finally, let Δ be the discrete Laplacian. The dynamics is defined by the mapping $\mathbf{F} : \mathcal{B} \to \mathcal{B}$ such that:

$$\mathbf{F}(\mathbf{X}) = \mathbf{X} + \alpha.\Delta\left[H(\mathbf{X} - \mathbf{X}_c).\mathbf{X}\right] \tag{1}$$

---

[1] Strong deviations from a power law has been observed for small E_c in one dimension [3].

which redistributes the energy of the active sites in equal parts to the neighbours after one relaxation step (and is just the identity if no site is active, namely if $\mathbf{X} \in \mathcal{M}$). This is a (singular) diffusion operator where α is the diffusion coefficient. Note that $\mathbf{F}$ is *piecewise linear*, namely linear on sub-domains $\mathcal{B}_k \in \mathcal{B}$.

It is fruitful to encode the dynamics of excitation in the following way. Let Σ_Λ^+ be the set of right infinite sequence $\mathbf{a} = \{a_1, \ldots, a_k, \ldots\}, a_k \in \Lambda$, and σ be the *left shift* over Σ_Λ^+, namely $\sigma\mathbf{a} = a_2 a_3 \ldots$. The elements of Σ_Λ^+ are called *excitation sequences*. The set $\Omega = \Sigma_\Lambda^+ \times \mathcal{B}$ is the *phase space* of the Zhang's model and $\hat{\mathbf{X}} = (\mathbf{a}, \mathbf{X})$ is a point in Ω. The Zhang's model dynamics is given by a map of skew-product type $\hat{\mathbf{F}} : \Omega \to \Omega$ s.t.:

$$\mathbf{X} \in \mathcal{M} \Rightarrow \hat{\mathbf{F}}(\hat{\mathbf{X}}) = (\sigma.\mathbf{a}, \mathbf{X} + \mathbf{e}_a) \tag{2}$$

$$\mathbf{X} \in \bar{\mathcal{M}} \Rightarrow \hat{\mathbf{F}}(\hat{\mathbf{X}}) = (\mathbf{a}, \mathbf{F}(\mathbf{X})) \tag{3}$$

The knowledge of an initial energy configuration $\mathbf{X}$, and of an (infinite) sequence of excited sites $\mathbf{a}$ (resp. of an initial condition $\hat{\mathbf{X}}$) fully determines the evolution. One can give Σ_Λ^+ a probability distribution ν_L corresponding to make a random choice for the excited sites. In the original Zhang's model the excited sites are choosen at random and independently with a uniform probability. This corresponds to give Σ_Λ^+ the uniform Bernoulli measure. Throughout this paper we will often think of the left Bernoulli shift on Σ_Λ^+ as being represented by the system $z \to N.z \bmod 1, z \in [0,1]$.

For the following, we will denote the two projections on the first and second coordinate by $\pi^u(\hat{\mathbf{X}}) = \mathbf{a}$, and $\pi^s(\hat{\mathbf{X}}) = \mathbf{X}$. The superscript u, s means respectively *unstable* and *stable* and correspond to expansion (resp. contraction) properties of the dynamics. Let $D\hat{\mathbf{F}}_{\hat{\mathbf{X}}}$ be the tangent map of $\hat{\mathbf{F}}$ at $\hat{\mathbf{X}}$ and $D\hat{\mathbf{F}}^t_{\hat{\mathbf{X}}}$ the t-th iterate. As shown below $\pi^u(D\hat{\mathbf{F}}_{\hat{\mathbf{X}}})$ is expansive whereas $\pi^s(D\hat{\mathbf{F}}_{\hat{\mathbf{X}}})$ produces contraction. For the following we will use the notation $\hat{\mathbf{X}}(t) = \hat{\mathbf{F}}^t(\hat{\mathbf{X}})$ (resp. $\mathbf{X}(t) = \pi^s(\hat{\mathbf{F}}^t(\hat{\mathbf{X}}))$). Furthermore note that $\pi^s(D\hat{\mathbf{F}}_{\hat{\mathbf{X}}}) = D\mathbf{F}_{\mathbf{X}}$, and that $D\mathbf{F}_{\mathbf{X}} = I$, the identity matrix over $\mathbb{R}^N$, if $\mathbf{X} \in \mathcal{M}$.

Consider a point $\hat{\mathbf{X}} \in \Omega$. Its trajectory is intermittent, composed of bursts of excitation of the sites $a_1, a_2, \ldots a_n$, for those times t s.t. $\mathbf{X}(t) \in \mathcal{M}$, followed by relaxation periods when $\mathbf{X}(t) \in \bar{\mathcal{M}}$. Define the following hierarchy of *waiting times*:

$$\gamma_0(\hat{\mathbf{X}}) = 0 \tag{4}$$

$$\sigma_i(\hat{\mathbf{X}}) = \inf_{t > \gamma_{i-1}} \left\{ \mathbf{X}(t) \in \bar{\mathcal{M}} \right\}, \quad i \geq 1 \tag{5}$$

$$\gamma_i(\hat{\mathbf{X}}) = \inf_{t > \sigma_i} \left\{ \mathbf{X}(t) \in \mathcal{M} \right\}, \quad i \geq 1 \tag{6}$$

For $i \geq 1$, $\sigma_i(\hat{\mathbf{X}})$ (resp. $\gamma_i(\hat{\mathbf{X}})$) is the starting time (resp. ending time) of the i-th avalanche occuring during the evolution of $\hat{\mathbf{X}}$. We call *avalanche duration* the number of updading leading a stable configuration to the next stable one. It is given by :

$$\tau_i(\hat{\mathbf{X}}) = \gamma_i(\hat{\mathbf{X}}) - \sigma_i(\hat{\mathbf{X}}) \tag{7}$$

In the same way, one defines :

$$\omega_i(\hat{\mathbf{X}}) = \sigma_i(\hat{\mathbf{X}}) - \gamma_{i-1}(\hat{\mathbf{X}}) \tag{8}$$

which is the number of excitations between the end of the avalanche $i-1$ and the beginning of the next avalanche. In this way, one introduces naturaly two distinguished time scale : the *local time* t corresponding in one step of iteration in the dynamics, and the *avalanche time* τ_i corresponding to the duration of an avalanche (a similar description has been done in [9]). On average $\tau_i >> 1$ (adiabatic driving).

The waiting times are useful to define the usual avalanches observables. The number of relaxing sites for a given configuration is :

$$r(\hat{\mathbf{X}}) = \#\left\{i \in \Lambda,\ X_i \geq E_c\right\} \tag{9}$$

The *avalanche size* is

$$s(\hat{\mathbf{X}}) = \sum_{t=1}^{\tau(\hat{\mathbf{X}})} r(\hat{\mathbf{F}}^t(\hat{\mathbf{X}})) \tag{10}$$

where:

$$\tau(\hat{\mathbf{X}}) = \inf_{t\geq 1}\left\{\mathbf{F}^t(\mathbf{X}) \in \mathcal{M}\right\} - 1 \tag{11}$$

is the avalanche duration occuring when exciting the site a_1 in an energy configuration $\mathbf{X}$. It is zero if one drops energy witout relaxation.

The structure of an avalanche can be encoded by the sequence of active sites $A(\hat{\mathbf{X}}) = \left\{A_t(\hat{\mathbf{X}})\right\}_{1\leq t\leq \tau(\hat{\mathbf{X}})}$ where $A_t(\hat{\mathbf{X}}) = \{j \in \Lambda | X_j(t) \geq E_c\}$. (Note that $A_1(\hat{\mathbf{X}})$ is non empty and equals $\{a_1\}$ iff $\mathbf{X}+\mathbf{e}_{a_1}$ is active). Correspondingly, there exists a partition[2] of $\Sigma_\Lambda^+ \times \mathcal{M}$ into domains $\mathcal{P}_{i,k} = [i] \times \mathcal{M}_{i,k}$ where $[i]$ is the set of sequences in Σ_Λ^+ having a first digit i, such that for any energy configuration $\mathbf{X} \in \mathcal{M}_{i,k}$ the excitation of the site i leads to the same avalanche (namely the same sites relax at the same time). Under some moderates assumptions (see [5]), this allows to define a symbolic coding for the avalanche and a transition graph giving the transition rules between successives avalanches and to show that the dynamical system admits a unique, fractal, invariant set. The boundary of the domains $\mathcal{P}_{i,k}$ constitutes the *singularity set* of $\hat{\mathbf{F}}$, called $\mathcal{S}$. This is the set of points where $\hat{\mathbf{F}}$ is not continuous.

C. Stationnary state and probability of avalanche observables.

Let $\hat{\mu}_L$ be an invariant measure for the dynamical system $\left\{\Omega, \hat{\mathbf{F}}\right\}$, where L refers to the lattice size, namely $\hat{\mu}_L(\hat{\mathbf{F}}^{-1}(\mathcal{A})) = \hat{\mu}_L(\mathcal{A})$ where $\mathcal{A} \in \Omega$ is a measurable set. Since Ω has a product structure, and since the dynamical system is a skew product, $\hat{\mu}_L = \nu_L \times \mu_L$, where ν_L is the induced measure on the unstable direction or *excitation* measure and μ_L is the induced measure on $\mathcal{B}$ or measure on the *energy* configurations. For simplicity we will assume that ν_L is a Bernoulli measure, namely that the successive excited sites are chosen *independently* with fixed rates. Once we have fixed the distribution of excitation, we are interested on the possible μ_L measures. Of special physical importance are the measures obtained by iterating the Lebesgue measure μ_{Leb}[3] on $\mathcal{M}$, that is $\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \hat{\mathbf{F}}^i(\nu_L \times$

---

[2]This partition is induced by the partition of $\mathcal{B}$ into domains of continuity for $\mathbf{F}$ [5].

[3]Or any absolutely continuous measure, corresponding to select the initial energy configuration with a probability distribution having a density.

μ_{Leb}). When the excitation measure ν_L is itself the Lebesgue measure on $[0,1]$ (corresponding to choose the excited sites with a uniform probability) the measure obtained is called the Sinai-Ruelle-Bowen measure (SRB). More generally, we will call (conditional) SRB the measure $\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \hat{\mathbf{F}}^i(\nu_L \times \mu_{Leb})$, for a fixed ν_L. This is a natural invariant measure from the physicists point of view since it gives ensemble average with respect to typical initial energy configurations.

It is common in the SOC litterature to assume ergodicity. In our setting the physically relevant ergodic property is equivalent to assume that the SRB measure is unique. For the following we will assume that this property holds and that $\hat{\mu}_L$ is the unique SRB measure [6]. This implies in particular the almost-sure equality between the ensemble average and the time average, namely, if ϕ is some observable, (a function $\Omega \to \mathbb{R}$, integrable with respect to $\hat{\mu}_L$)

$$\bar{\phi}_L \stackrel{\text{def}}{=} \lim_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} \phi(\hat{\mathbf{F}}^t(\hat{\mathbf{X}})) = \int_\Omega \phi(\hat{\mathbf{X}}) d\hat{\mu}_L(\hat{\mathbf{X}}) \stackrel{\text{def}}{=} E_L[\phi] \tag{12}$$

for a typical (namely Lebesgue almost surely) initial condition $\hat{\mathbf{X}}$. Here, and for the following $\bar{}_L$ will denote the time average, while $E_L[]$ will be the ensemble average, in a lattice of size L.

From the dynamical systems point of view $\hat{\mu}_L$ is the natural object to deal with. However, in the SOC literature one is most interested in the probability distribution of some avalanche observable and its scaling properties in the thermodynamic limit. Fix an avalanche observable, say s. Call $\mathcal{P}_s$ the union of domains $\mathcal{P}_{i,k}$ such that the avalanche corresponding to each domain $\mathcal{P}_{i,k}$ has the same size s. Then the probability to have an avalanche of size s, by excitation of a *stable* configuration, is $Prob[s(\hat{\mathbf{X}}) = s | \hat{\mathbf{X}} \in \mathcal{P}] = \frac{\hat{\mu}_L(\mathcal{P}_s)}{\hat{\mu}_L(\mathcal{P})} = \frac{\hat{\mu}_L(\mathcal{P}_s)}{\mu_L(\mathcal{M})}$. In this definition we include the avalanches of size zero (excitation without relaxation). However, it is more natural from the SOC point of view to exclude this case. We define therefore $P_L(s)$ as the probability to have an avalanche of size s *strictly larger than 0*[4].

$$P_L[s] \stackrel{\text{def}}{=} \frac{\hat{\mu}_L(\mathcal{P}_s)}{p_L}, \quad s \geq 1 \tag{13}$$

where $p_L \stackrel{\text{def}}{=} Prob\left[s(\hat{\mathbf{X}}) \geq 1, \mathbf{X} \in \mathcal{M}\right]$ is the probability to *initiate* an avalanche. The average with respect to $P_L[s]$, denoted further on by $<>_L$ is :

$$< \psi(s) >_L \stackrel{\text{def}}{=} \sum_{s=1}^{\xi_L^s} P_L[s]\psi(s) \tag{14}$$

where ψ is some real function, and ξ_L^s is the maximal value that the observable s can have on a lattice of size L (note that ξ_L^s depends also on E_c, ϵ, d but is *bounded* if $L < \infty$). The same definition holds for any other avalanche observable. From the ergodic theorem:

$$< \psi(s) >_L = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \psi(s_i) \tag{15}$$

---

[4] In view of the expected critical behaviour as $L \to \infty$, one usually writes a scaling form $P_L(s) = \frac{f_L(s)}{s^\tau s}$ where $f_L(s)$ is a cut-off term accounting for finite size effects on large scales. $P_L(s)$ is not defined for $s = 0$ unless assuming very special properties for $f_L(s)$.

where s_i is the size of the i-th avalanche occuring in the trajectory of a generic point $\hat{\mathbf{X}}$.

One has:

$$p_L = \hat{\mu}_L \left[\bigcup_{i=1}^{N} \{a_1 = i, X_i \in [E_c - 1, E_c[\} \right] = \sum_{i=1}^{N} p_L(i) \tag{16}$$

where:

$$p_L(i) \stackrel{\text{def}}{=} \nu_L(i).\mu_L \{X_i \in [E_c - 1, E_c[\} \tag{17}$$

is the probability that an avalanche starts at the site i. Note that the probabilities $p_L(i)$ *depend* a priori on i even if the excitation measure is uniform. In this case, however, (16) reduces to

$$p_L = \frac{1}{N}.\sum_{i=1}^{N} \mu_L \{X_i \in [E_c - 1, E_c[\} \tag{18}$$

Fix $\hat{\mathbf{X}}$ and T, then call $n(T, \hat{\mathbf{X}})$ the number of *complete* avalanches occuring until local time T for the initial condition $\hat{\mathbf{X}}$. Obviously, $n(T, \hat{\mathbf{X}}) \to \infty$ as $T \to \infty$, $\forall \hat{\mathbf{X}}$. Then from the ergodic theorem :

$$p_L = \lim_{T \to \infty} \frac{n(T, \hat{\mathbf{X}})}{T} \tag{19}$$

One can decompose T has : $T = \sum_{i=1}^{n(T,\hat{\mathbf{X}})} \tau_i + \sum_{i=1}^{n(T,\hat{\mathbf{X}})} \omega_i + K(\hat{\mathbf{X}})$ where $K(\hat{\mathbf{X}})$ is some residual time, finite, whatever T, whatever $\hat{\mathbf{X}}$ ($K(\hat{\mathbf{X}})$ is bounded by the largest avalanche duration). Then as T goes to infinity:

$$\frac{n(T, \hat{\mathbf{X}})}{T} \sim \frac{n(T, \hat{\mathbf{X}})}{\sum_{i=1}^{n(T,\hat{\mathbf{X}})} \tau_i + \sum_{i=1}^{n(T,\hat{\mathbf{X}})} \omega_i} = \frac{n(T, \hat{\mathbf{X}})}{\sum_{i=1}^{n(T,\hat{\mathbf{X}})} \tau_i} - \frac{n(T, \hat{\mathbf{X}})}{\sum_{i=1}^{n(T,\hat{\mathbf{X}})} \tau_i} \frac{\sum_{i=1}^{n(T,\hat{\mathbf{X}})} \omega_i}{T}$$

Call:

$$\bar{\omega}_L \stackrel{\text{def}}{=} \lim_{T \to \infty} \frac{1}{T}. \sum_{i=1}^{n(T,\hat{\mathbf{X}})} \omega_i = \mu_L(\mathcal{M}) \tag{20}$$

the *probability to drop energy in the system, at a given time*, (the equality holds for μ_L almost-every $\hat{\mathbf{X}}$ from the ergodic theorem). $\bar{\omega}_L(i) = Prob[a_1 = i, \mathbf{X} \in \mathcal{M}] = \nu_L(i)\bar{\omega}_L$, is the probability to drop energy on the site i, at a given time, and is called the *driving rate* in the litterature [24]. One has:

$$p_L = \frac{1 - \bar{\omega}_L}{< \tau >_L} = \frac{\mu_L(\bar{\mathcal{M}})}{< \tau >_L} \tag{21}$$

where $< \tau >_L$ is the average avalanche duration.

There exists an important relation linking the avalanches averages (average w.r.t. P_L) to the local time average (average w.r.t. $\hat{\mu}_L$). Let $\phi : \Omega \to \mathbb{R}$ be some observable *such that* $\phi(\hat{\mathbf{X}}) = 0$ *whenever* $\mathbf{X} \in \mathcal{M}$. A related avalanche observable can be defined by summing the value that ϕ takes over one avalanche. Namely, call $f_i(\hat{\mathbf{X}}) = \sum_{t=\sigma_i(\hat{\mathbf{X}})}^{\gamma_i(\hat{\mathbf{X}})} \phi(\hat{\mathbf{X}}(t))$. (An important example is when $\phi(\hat{\mathbf{X}}) = r(\hat{\mathbf{X}})$, the number of active sites in one step. Then $f_i(\hat{\mathbf{X}})$ is the size of the i th avalanche in the trajectory of $\hat{\mathbf{X}}$). One obtains:

$$\bar{\phi}_L = \lim_{T\to\infty} \frac{1}{T} \sum_{i=1}^{n(T,\hat{\mathbf{X}})} \sum_{t=\sigma_i(\hat{\mathbf{X}})}^{\gamma_i(\hat{\mathbf{X}})} \phi(\hat{\mathbf{X}}(t))$$

which yields:

$$\bar{\phi}_L = p_L . \langle f \rangle_L \tag{22}$$

In particular :

$$\bar{r}_L = p_L . \langle s \rangle_L \tag{23}$$

Finaly we define the probability that a site i is active (often called the *density of active sites* in the litterature [5]):

$$\rho_L(i) \stackrel{\text{def}}{=} \mu_L \left[X_i \geq E_c \right] \tag{24}$$

and

$$\rho_L^{av} = \frac{1}{N} \sum_{i=1}^{N} \rho_L(i) \tag{25}$$

ρ_L^{av} is believed to act as an order parameter in the Zhang's model. We investigate this point below.

III. DYNAMICAL PROPERTIES AND LYAPUNOV EXPONENTS.

A. Jacobian matrix and Lyapunov exponents.

Due to the piecewise affine structure of the map $\mathbf{F}$, the Jacobian matrix $DF_{\mathbf{X}}$ plays a central role in the Zhang's model, since it characterizes the energy transport. Indeed, the entry $DF^t_{\mathbf{X},ij}$ is the ratio of energy flowing from site j to site i in t times steps for the initial condition $\mathbf{X}$. Define $Z_k(\mathbf{X}(t)) = H(X_k(t) - E_c)$. This is a random variable, taking value 0 if $X_k(t)$ is stable, and value 1 otherwise, whose probability distribution is induced (at stationnarity) by the invariant measure $\hat{\mu}_L$. More precisely, $Prob[Z_k(\mathbf{X}(t)) = 1] = \rho_L(k)$. Let $\mathbf{Z}(\mathbf{X}) = \{Z_k(\mathbf{X})\}_{k=1}^N$, and call $S(\mathbf{X}) = \Delta\mathbf{Z}(\mathbf{X})I$ (equivalently $S(\mathbf{X})$ is the matrix of entries $S_{ij}(\mathbf{X}) = \Delta_{ij} Z_j(\mathbf{X})$). S is the "toppling" operator of the Zhang's model. The jacobian matrix writes $DF_{\mathbf{X}} = I + \alpha . S(\mathbf{X})$, while $DF^t_{\mathbf{X}}$ is given by :

$$\begin{aligned} DF^t_{\mathbf{X}} = I + \alpha \sum_{t_0=1}^{t} S(\mathbf{X}(t_0)) + \alpha^2 \sum_{t\geq t_1 > t_0 \geq 1} S(\mathbf{X}(t_1)) S(\mathbf{X}(t_0)) + \dots \\ + \alpha^r \sum_{t\geq t_{r-1} > t_{r-2} \dots > t_0 \geq 1} S(\mathbf{X}(t_{r-1})) S(\mathbf{X}(t_{r-2})) \dots S(\mathbf{X}(t_0)) + \dots + \alpha^t S(\mathbf{X}(t)) S(\mathbf{X}(t-1)) \dots S(\mathbf{X}(1)) \end{aligned} \tag{26}$$

Therefore, the generic term (say of order r) is as a "propagator" transmitting the energy in r times steps. Note that this formula is exact. It calls the following remarks:

---

[5] We will keep this terminology throughout this paper though $\rho_L(i)$ is not strictly speaking a density since $\sum_{i=1}^N \rho_L(i) \neq 1$.

- The maps $S(\mathbf{X})$ do not commute, and they depend on the state. This is a key difference from the Dhar's model since it induces a *non abelian* structure and a "toppling" operator depending *not only on the site, but also on the whole energy configuration.* In particular the propagator *is not a mere polynomial of the Laplacian.*

- The evolution depends *a priori* on the whole trajectory and therefore the strong memory effect expected in a critical phenomena, can be treated from eq. (26).

Remark also the following. If one defines the excitation times for a given trajectory by:

$$\nu_k(\hat{\mathbf{X}}) = \inf_{t>\nu_{k-1}(\hat{\mathbf{X}})} \{\mathbf{X}(t) \in \mathcal{M}\} \tag{27}$$

with $\nu_0 = \gamma_0 = 1$, the energy configuration at time T, for an initial condition $\hat{\mathbf{X}}$ is:

$$\mathbf{X}(T) = DF^T_{\mathbf{X}}.\mathbf{X} + \sum_{i=1}^{m(T,\hat{\mathbf{X}})} DF^{t-\nu_i(\hat{\mathbf{X}})}_{\mathbf{X}}.\mathbf{e}_{a_{\nu_i(\hat{\mathbf{X}})}} \tag{28}$$

where $m(T,\hat{\mathbf{X}})$ is the number of excitations on a time interval of length T for the initial condition $\hat{\mathbf{X}}$. The first term corresponds to the redistribution of the energies coming from the initial energy configuration while the second one corresponds to the redistribution of the energy quantum $\delta = 1$ dropped in the system at times $\nu_i(\hat{\mathbf{X}})$. Since the equilibrium averages are assumed to be independent of the initial condition, the first term has to decay to zero as $t \to \infty$, with a decay rate corresponding to the characteristic relaxation time to equilibrium.

It is therefore important to well understand the (spectral) properties of the $DF^t\mathbf{X}$ in the infinite time limit. Would $S(\mathbf{X})$ be the Laplace operator, would the spectrum of $DF^t\mathbf{X}$ be composed by Fourier modes, and the relaxation time to equilibrium would be the slowest mode. However, the mere presence of a singular term $Z(\mathbf{X})$ certainly makes a big difference. Since S depends on $\mathbf{X}$ one clearly has to study the decay rates averaged on a full (typical) trajectory or equivalently to compute the ensemble average. In this view, the law of the stochastic process $\{Z(\mathbf{X}(t))\}_{t=0}^{+\infty}$ (namely the density of active sites and all times correlations) certainly plays a role.

The numbers characterizing the decay (resp. expansion) rates of the norm of a small pertubation in the tangent space of the trajectory of some point $\hat{\mathbf{X}}$ under the action of the infinite product matrix $DF^t\hat{\mathbf{X}}, t \to \infty$ are the Lyapunov exponents. They are *real* numbers, well defined under some moderate assumptions on $DF\hat{\mathbf{X}}$ (see [19]) and are almost-surely independent of $\hat{\mathbf{X}}$. Furthermore they are also independent of the norm (in finite dimension).

Due to the skew product structure, the tangent map at any point $\hat{\mathbf{X}}$ admits a natural splitting $D\hat{\mathbf{F}} = (\pi^u(D\hat{\mathbf{F}}_{\hat{\mathbf{X}}}), \pi^s(D\hat{\mathbf{F}}_{\hat{\mathbf{X}}}))$ where the one dimensional map $\pi^u(D\hat{\mathbf{F}}_{\hat{\mathbf{X}}})$ is expansive. Indeed, the average expansion rate is given by :

$$\lambda_L(0) = \lim_{T\to\infty} \frac{1}{T} log(det(\pi^u(D\hat{\mathbf{F}}^T_{\hat{\mathbf{X}}}))) = \bar{\omega}_L.log(N) \tag{29}$$

since $det(\pi^u(D\hat{\mathbf{F}}^T_{\hat{\mathbf{X}}})) = N^{\sum_{i=1}^{n(T,\hat{\mathbf{X}})}\omega_i(\hat{\mathbf{X}})}$. Therefore, since $\bar{\omega}_L \neq 0$, there is a *positive Lyapunov exponent* in the dynamics. Note that this is simply due to the excitation rule, and that it reflects simply the "chaotic" properties of the Bernoulli shift.

A more important issue concerns $\pi^s(D\hat{\mathbf{F}}_{\hat{\mathbf{X}}})) = D\mathbf{F}_{\mathbf{X}}$. The Oseledec theorem [19] asserts that under mild conditions on $D\mathbf{F}_{\mathbf{X}}$ there exists a hierarchy of Lyapunov exponents $\lambda_L(1) > \ldots \lambda_L(N)$, Lebesgue almost-surely constant if $\hat{\mu}_L$ is the SRB measure, and a hierarchy of nested subspaces (Oseledec splitting):

$$\mathbb{R}^N = \mathcal{V}_1(\hat{\mathbf{X}}) \supset \mathcal{V}_2(\hat{\mathbf{X}}) \supset \mathcal{V}_N(\hat{\mathbf{X}})$$

depending on $\hat{\mathbf{X}}$, such that the norm of a perturbation $\mathbf{v} \in \mathcal{V}_i(\hat{\mathbf{X}}) \setminus \mathcal{V}_{i+1}(\hat{\mathbf{X}})$ is given by :

$$\|D\mathbf{F}^t_{\mathbf{X}}.\mathbf{v}\| = C(\hat{\mathbf{X}}, t)e^{\lambda_L(i).t}.\|\mathbf{v}\| \tag{30}$$

where $\lim_{t\to\infty} \frac{1}{t} log C(\hat{\mathbf{X}}, t) = 0$ a.s., namely $\lambda_L(i)$ is the exponential rate of variation of $\|\mathbf{v}\|$. Define $M(\mathbf{X}, t) = \tilde{D\mathbf{F}}^t_{\mathbf{X}}.D\mathbf{F}^t_{\mathbf{X}}$ and $\Lambda = \lim_{t\to\infty} M(\mathbf{X}, t)^{\frac{1}{2t}}$,(the Oseledec multiplicative ergodic theorem insures that this limit exists almost-surely and is a constant). Then the Lyapunov exponents are the logarithm of the eigenvalues of Λ. $M(\mathbf{X}, t)$ being symmetric it admits an orthogonal basis $\{\mathbf{v}_i(\mathbf{X}, t)\}_{i=1}^N$ and eigenvalues $\mu_i(\mathbf{X}, t)$ such that $\lambda_L(i) = \lim_{t\to\infty} \frac{1}{2t} log(\mu_i(\mathbf{X}, t))$. Furthermore, each $\mathbf{v}_i(\mathbf{X}, t)$ converges exponentially to a vector $\mathbf{v}_i(\hat{\mathbf{X}})$ in $\mathbb{R}^N$, depending on $\hat{\mathbf{X}}$ [21]. The $\mathbf{v}_i(\hat{\mathbf{X}})$'s constitutes therefore a basis for the Oseledec splitting. We call them the *Oseledec modes* for the trajectory of $\hat{\mathbf{X}}$. They can be numerically obtained from the QR decomposition used in the computation of the Lyapunov spectrum (see [11]). It has been shown in [4] that the $\lambda_L(i)$ are all negative for finite L, namely all vectors in $\mathbb{R}^N$ are asymptotically contracted. Finally, the Lyapunov spectrum is closely related to the (local) fractal properties of the invariant set through the Kaplan-Yorke and the Ledrappier-Young formula [11,16].

From this discussion, one expects a close connection between the Lyapunov spectrum and the energy transport in the Zhang's model. In particular, the following formula can be proved [5]

$$\sum_{i=1}^{N} \lambda_L(i) = log(\epsilon).(1 - \bar{\omega}_L)\frac{< s >_L}{< \tau >_L} = p_L log(\epsilon). < s >_L \tag{31}$$

It relates the Lyapunov spectrum which characterizes *local* properties of the *microscopic* dynamics to the avalanche *statistical* properties of the *macroscopic* system. Note that the exponent $\lambda_L(i)$ give the contraction rate in the direction $\mathbf{v}_i(\hat{\mathbf{X}})$ versus the *local time*. One can also define the average contraction *per avalanche*, $\chi_L(i)$ given from formula (22) by:

$$\chi_L(i) = \frac{< \tau >_L}{1 - \bar{\omega}_L}\lambda_L(i) = \frac{\lambda_L(i)}{p_L} \tag{32}$$

Then, the sum of χ_L's, giving the average volume contraction *per avalanche* is related to the average avalanche size by :

$$\sum_{i=1}^{N} \chi_L(i) = log(\epsilon). < s >_L \tag{33}$$

Note that $< s >_L$ corresponds to the total energy transport within one avalanche and is believed to be related to the total response function [24]. Our formula shows that it is also equal to the volume contraction in the phase space, produced on average by one avalanche.

B. Oseledec modes.

To each negative Lyapunov exponent $\lambda_L(i),\ i = 1 \dots N$ is associated a characteristic time $t_L(i) = |\lambda_L(i)|^{-1}$, the time of vanishing of a perturbation in the Oseledec direction i. This defines therefore a hierarchy:

$$t_L(1) > t_L(2) > \dots t_L(N)$$

Note that there is no contradiction with the expected critical behaviour in the thermodynamic limit, since as $L \to \infty$ there are an infinite number of characteristic time scales.

From the physical point of view a perturbation can be viewed as a small modification of the initial energy landscape $\mathbf{X}$. It can be localized (for example one site perturbed) or spread. The attenuation is due to two distincts effects :

- Propagation through the lattice.
- Dissipation at the boundaries.

We show in the following that these distinct effects (essentially) correspond to distinct regions in the spectrum. Note that according to the Oseledec mode under consideration the contraction can be due (on average) to the effect of one avalanche (if $t_L(i)$ is small compared to the average avalanche size), or to the cumulative effect of many avalanches (if $t_L(i)$ is large). The coefficient $\chi_L(i)$ (eq. (32)) gives the average contraction per avalanche for the i-th Oseledec mode. Therefore the number $\frac{1}{\chi_L(i)}$ gives an estimate of the number of avalanches needed to have a reduction of the initial perturbation of a ratio $\frac{1}{e}$ for this mode. Therefore a crossover point can be estimated by :

$$\chi_L(i_c) \sim 1 \tag{34}$$

We will call slow (resp. fast) modes the Oseledec modes corresponding to $\lambda_L(i) << \lambda_L(i_c)$ (resp. $\lambda_L(i) >> \lambda_L(i_c)$).

1. Bounds on the first negative Lyapunov exponent.

We give a bound on the first Lyapunov exponent related to the energy dissipation at the boundaries. Call $\Phi_j^{out}(t, \hat{\mathbf{X}}) \stackrel{\text{def}}{=} 1 - \sum_{i=1}^{N} DF^t_{\mathbf{X},ij}$. The energy being locally conserved, $\Phi_j^{out}(t, \hat{\mathbf{X}})$ is the ratio of the initial energy X_j given by the site j to the boundary $^{\partial}\Lambda$ in t time steps. In other words, the energy coming from X_j and lost at time t is $\Phi_j^{out}(t, \hat{\mathbf{X}}).X_j$. The following holds:

Proposition 1 *The largest negative Lyapunov exponent, $\lambda_L(1)$ admits the following bounds:*

$$0 > \lim_{t\to\infty} \frac{1}{t} log(1 - \min_j(\Phi_j^{out}(t, \hat{\mathbf{X}}))) \geq \lambda_L(1) \geq \lim_{t\to\infty} \frac{1}{t} log(1 - \max_j \Phi_j^{out}(t, \hat{\mathbf{X}})) \tag{35}$$

This interprets as follows. As $t \to \infty$, $\Phi_j^{out}(t, \hat{\mathbf{X}}) \to 1,\ \forall j,\ \forall \hat{\mathbf{X}}$, since, finally, all the initial energy coming from $\mathbf{X}$ has been lost at the boundaries. The limit $\lim_{t\to\infty} \frac{1}{t} log(1 - \Phi_j^{out}(t, \hat{\mathbf{X}}))$ gives the exponential rate of convergence of $\Phi_j^{out}(t, \hat{\mathbf{X}})$ to 1. In other words, it gives the exponential decrease for the initial energy still in the lattice at given time. The maximal negative Lyapunov exponent is bounded by the extremal dissipation rates. One sees therefore that the

contraction in the principal Oseledec mode is mainly due to the dissipation at the boundaries. We shall see later on that $\lambda_L(1)$ is essentially related to the so-called dissipation rate.

Proof It is easy to show that there exists a time t_s depending on E_c, ϵ, d such that, whatever $\hat{\mathbf{X}}$ each site in the lattice has relaxed at least once after this time and therefore all sites of the boundary have dissipated energy. At time t the energy coming from a site j with initial energy X_j and redistributed into the lattice is $\sum_{i=1}^{N} DF^t_{\mathbf{X},ij} X_j$. For $t \geq t_s$, $\Phi_j^{out}(t, \hat{\mathbf{X}}) > 0$ and $\sum_{i=1}^{N} DF^t_{\mathbf{X},ij}$ is bounded away from 1. $D\mathbf{F}^t_{\hat{\mathbf{X}}}$ being a matrix with positive entries:

$$\min_j(\sum_{i=1}^{N} D\mathbf{F}^t_{\mathbf{X},ij}) = 1 - \max_j \Phi_j^{out}(t, \hat{\mathbf{X}}) \leq \rho(D\mathbf{F}^t_{\mathbf{X}}) \leq \max_j(\sum_{i=1}^{N} DF^t_{\mathbf{X},ij}) = 1 - \min_j \Phi_j^{out}(t, \hat{\mathbf{X}}) < 1 \tag{36}$$

where $\rho(D\mathbf{F}^t_{\mathbf{X}})$ is the spectral radius of $D\mathbf{F}^t_{\mathbf{X}}$.

The largest negative Lyapunov exponent is given by :

$$\lambda_L(1) = \lim_{t\to\infty} \frac{1}{t} log(\|DF^t_{\mathbf{X}}\|_2) \tag{37}$$

where $\| \ \|_2$ is the L_2 norm. In (37) the limit does not depends on $\hat{\mathbf{X}}$, provided $\hat{\mathbf{X}}$ belongs to the support of $\hat{\mu}_L$. One has $\rho(D\mathbf{F}^t_{\mathbf{X}}) \leq \|DF^t_{\mathbf{X}}\|_2$ and therefore:

$$\lambda_L(1) \geq \lim_{t\to\infty} \frac{1}{t} log(1 - \max_j \Phi_j^{out}(t, \hat{\mathbf{X}}))$$

Furthermore, all norms being equivalent in finite dimension eq. (37) holds also for the L_1 norm where $\|DF^t_{\mathbf{X}}\|_1 = \sup_{\mathbf{X}} \frac{\sum_{i,j=1}^{N} D\mathbf{F}^t_{\mathbf{X},ij}|X_j|}{\sum_{j=1}^{N}|X_j|}$. The $D\mathbf{F}^t_{\mathbf{X},ij}$'s being positive the supremum is certainly achieved for positive X_i values. Therefore,

$$\|DF^t_{\mathbf{X}}\|_1 = \sup_{\mathbf{X}} \frac{\sum_{j=1}^{N}(1 - \Phi_j^{out}(t, \hat{\mathbf{X}}))X_j}{\sum_{j=1}^{N} X_j} = 1 - \inf_{\mathbf{X}} \frac{1}{\sum_{j=1}^{N} X_j} \sum_{j=1}^{N} \Phi_j^{out}(t, \hat{\mathbf{X}}).X_j \leq 1 - \inf_{\mathbf{X}} \min_j(\Phi_j^{out}(t, \hat{\mathbf{X}}))$$

When taking the limit, one has to take initial conditions $\hat{\mathbf{X}}$ in the support of $\hat{\mu}_L$. For these points the limit $\lim_{T\to\infty} log(1 - \min_j(\Phi_j^{out}(t, \hat{\mathbf{X}})))$ does not depend on $\hat{\mathbf{X}}$, hence:

$$\lambda_L(1) \leq \lim_{t\to\infty} \frac{1}{t} log(1 - \min_j(\Phi_j^{out}(t, \hat{\mathbf{X}})))$$

□

2. Stabilizing modes

The contraction in the principal Oseledec mode (first negative Lyapunov exponent) is mainly due to the dissipation at the boundaries. On the other hand it is possible to have a large contraction in one local time step without reaching the boundaries, because the tangent matrix $DF_{\mathbf{X}}$ has the following property, which can be checked by direct computation.

Proposition 2 *Let* $\Lambda = \Lambda_c(\mathbf{X}) \oplus \Lambda_n(\mathbf{X})$ *where* $\Lambda_c(\mathbf{X}) = \{i \in \Lambda \mid X_i \geq X_c\}$, $\Lambda_n(\mathbf{X}) = \{i \in \Lambda \mid X_i < X_c\}$ *and* $n_c(\mathbf{X}) = \#\Lambda_c(\mathbf{X})$, *then* $DF_{\mathbf{X}}$ *has* $n_c(\mathbf{X})$ *eigenvalues* ϵ *corresponding to the eigenvectors*

$$k_i(\mathbf{X}) = 2.d.\mathbf{e}_i - \sum_{j \in \mathcal{U}_i} \mathbf{e}_j \; ; \quad i \in \Lambda_c(\mathbf{X}) \tag{38}$$

where $\mathcal{U}_i$ *denotes the set of sites in* Λ *at distance* 1 *from* i. *There are also* $N - n_c(\mathbf{X})$ *neutral eigenvalues associated to the eigenvectors* $\mathbf{e}_i$, $i \in \Lambda_n(X)$.

The eigenvectors k_i produces arbitrary large contraction as $\epsilon \to 0$. In particular, in the original Zhang's model where $\epsilon = 0$ they correspond to *kernel* modes, which have eigenvalues 0. Note that, in this case, the dimension of the kernel of the product tangent map $DF^t_{\mathbf{X}}$ increases with t. However, it is strictly lower than N as $t \to \infty$ [5]. It is easy to check that these modes have zero energy, except if some of the $\mathbf{e}_i$'s correspond to sites neighbours to the boundary. This occurs with small (but non zero) probability. These modes act as directions where a one local time step is enough to reduce the initial perturbation by a factor ϵ, with on average, small variation of the total energy. They dynamically correspond to directions transverse to the attractor, and their contraction corresponds to a fast convergence onto the attractor. For this reason we call them *stabilizing modes*. In the Lyapunov spectrum they can be identified because the corresponding Lyapunov exponents go to $-\infty$ as $\epsilon \to 0$ while the other part of the spectrum weakly depend on ϵ (see Fig 1).

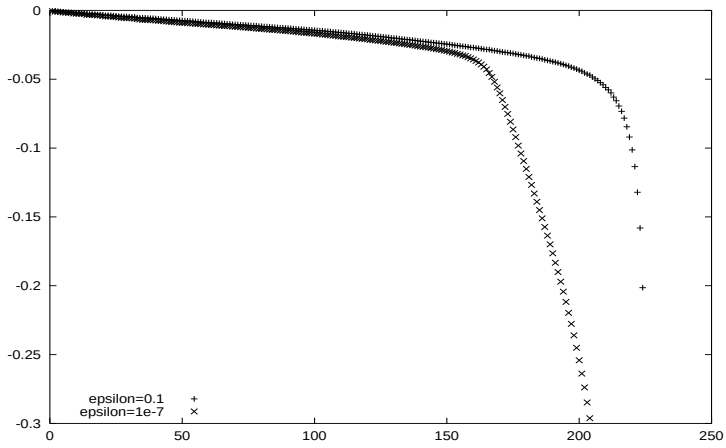

FIG. 1. Lyapunov spectrum for $\epsilon = 0.1$ and $\epsilon = 10^{-7}$, $L = 15$, $E_c = 2.2$.

3. The transport operator.

It is usually not possible to give an explicit formula for the whole Lyapunov spectrum, except in some specific cases [18]. We propose here a mean-field ansatz which gives good results for the slowest modes, and has a nice interpretation in terms of random walk. It is based on the following observations.

The Lyapunov exponents are the eigenvalues of the matrix $\Lambda = E_L[\Lambda] = E_L\left[\lim_{t\to\infty}\left(\tilde{D}\mathbf{F}^t_{\mathbf{X}}.D\mathbf{F}^t_{\mathbf{X}}\right)^{\frac{1}{2t}}\right]$. Since the matrix $\tilde{D}\mathbf{F}^t_{\mathbf{X}}.D\mathbf{F}^t_{\mathbf{X}}$ is bounded in L_2 norm, $\forall \mathbf{X}, \forall t$, one has, from the Lebesgue theorem : $\Lambda = \lim_{t\to\infty} E_L\left[\left(\tilde{D}\mathbf{F}^t_{\mathbf{X}}.D\mathbf{F}^t_{\mathbf{X}}\right)^{\frac{1}{2t}}\right]$. On the other hand, the matrix

$$\mathcal{L}(t) = E_L\left[D\mathbf{F}_{\mathbf{X}}^t\right] \tag{39}$$

caracterizes the (ensemble) average energy transport in t time steps. However, the connexion between $\mathcal{L}(t)$ and Λ (if any) is loose.

Would the transport be normal, namely would be $D\mathbf{F}_{\mathbf{X}}$ be independent of $\mathbf{X}$ and of the form $D\mathbf{F}_{\mathbf{X}} = I + \gamma.\Delta$, where γ is some constant, would $\mathcal{L}(t)$ write $\mathcal{L}(t) = (I + \gamma.\Delta)^t$. In this case, $\mathcal{L}(t)$ would be constant and symmetric. Then $E_L\left[\left(\tilde{D}\mathbf{F}_{\mathbf{X}}^t.D\mathbf{F}_{\mathbf{X}}^t\right)^{\frac{1}{2t}}\right] = I + \gamma.\Delta$ and $\Lambda = I + \gamma.\Delta$. In this case the Lyapunov exponents would be the eigenvalues of a one step transport operator $\mathcal{L} = I + \gamma.\Delta$ (Fourier modes).

More generally, the (naive) hope would be to find an effective transport operator $\mathcal{L}$ such that $\mathcal{L}(t) = \mathcal{L}^t$ and whose singular values (or eigenvalues if self-adjoined) would give the Lyapunov spectrum. There is however a priori no hope to find such an operator in general. Note in particular that the assumption of independence of the matrices $D\mathbf{F}_{\mathbf{X}(t)}$, first step towards a mean-field approach, is not a sufficient condition. Since, in this case, $E_L\left[D\mathbf{F}_{\mathbf{X}}^t\right] = E_L\left[D\mathbf{F}_{\mathbf{X}}\right]^t$, it suggests to propose $\mathcal{L} = E_L\left[D\mathbf{F}_{\mathbf{X}}\right]$ as a one step operator. However, one needs further conditions to insure that the singular values of $\mathcal{L}$ give the Lyapunov exponents (see for example [18,10]). It appears nevertheless that in the Zhang's model an effective transport operator can be found from a mean-field treatment well approximating the *slowest modes*.

The first obstacle towards a mean-field approach lies in the independence assumption. The matrix $\mathcal{L}(t)$ is a sum of time correlations terms of the form $E_L\left[S(\mathbf{X}(t_{r-1}))S(\mathbf{X}(t_{r-2}))\dots S(\mathbf{X}(t_0))\right]$ whose entry (i,j) writes $\sum_{i_1,\dots,i_{r-1}} \Delta_{i,i_{r-1}}\dots\Delta_{i_2,i_1}\Delta_{i_1,j} Prob\left[Z_{i_{r-1}}(\mathbf{X}(t_{r-1})) = 1,\dots,Z_{i_1}(\mathbf{X}(t_1)) = 1, Z_j(\mathbf{X}(t_0)) = 1\right]$ Clearly, the non vanishing terms in this sum are those corresponding to a path from j to i where each intermediate site has been active at least once. A simple glance to this formula shows that *a priori all time correlations function of the joint probability of active sites*, $Prob\left[Z_{i_{r-1}}(\mathbf{X}(t_{r-1})) = 1,\dots,Z_{i_1}(\mathbf{X}(t_1)) = 1, Z_j(\mathbf{X}(t_0)) = 1\right]$ have to be considered.

However, the Zhang's model, as an hyperbolic dynamical system, has exponential correlation decay (for finite L). Our numerical investigations show that the correlation decay rate is quite larger than the caracteristic times related to the slow modes (for example the correlation decay rate of a site with itself is about -0.025 for $E_c = 2.2, \epsilon = 0.1, L = 20$, corresponding roughly to the 320-th exponent in the spectrum, while the slowest lyapunov exponent value is -0.000209871). On the other hand, for these modes, a small perturbation has essentially no variation during one step of an avalanche. In other words it does not feel the fast changes (one local time step) of the individual matrices $D\mathbf{F}_{\mathbf{X}}^t$ (resp. the fluctuations of the $Z_j(\mathbf{X}(t))$'s) but is rather sensitive to the average variations on the caracteristic time scale $t_L(i) = \frac{1}{\lambda_L(i)}$, which is quite longer than a local time step. This suggests to consider the projection of the matrices $D\mathbf{F}_{\mathbf{X}(t)}$'s on the slow Oseledec space as independent, in a first approximation, and to propose $E_L\left[D\mathbf{F}_{\mathbf{X}}\right] = I + \alpha.\Delta.\rho_L.I$ as a one step transport operator. Note that we obtain the same by assuming that the $Z_i(\mathbf{X}(t))$'s are *independent*. Indeed, in this case, $Prob\left[Z_{i_{r-1}}(\mathbf{X}(t_{r-1})) = 1,\dots,Z_{i_1}(\mathbf{X}(t_1)) = 1, Z_j(\mathbf{X}(t_0)) = 1\right] = \rho_L(i_{r-1})\dots\rho_L(i_1)\rho_L(j)$. Then $\mathcal{L}(t) = \sum_{k=1}^t C_t^k((\Delta.\rho_L.I)^k = (I + \alpha\Delta.\rho_L.I)^t$.

This approximation gives correct results .... provided one multiplies the density of active sites by 2 !!! It neglects indeed an important effect. Provided $E_c > \frac{\epsilon}{1-\epsilon}$ a site *cannot relax two successive time steps* [5], and therefore it relaxes at most only half of the time during one avalanche. This means in particular that the random variables

$Z_i(\mathbf{X}(t)), Z_i(\mathbf{X}(t+1))$ are *not independent* and that the probability the one site relaxes at time $t+1$ depends on what happened at time t. Besides, two neighbouring sites cannot be simultaneously active. In certain sense, the lattice is "blinking" : during one avalanche all active sites are at pairwise distance [5]. This introduces therefore strong correlations between $\mathbf{Z}(\mathbf{X}(t))$ and $\mathbf{Z}(\mathbf{X}(t+1))$.

One can however circumvent the problem by reparametrizing the time and considering the evolution of the process any *two times steps*. Equivalently, one replace the stochastic process $\{\mathbf{Z}(\mathbf{X}(t))\}_{t=1}^{+\infty}$ by a new process $\{\mathbf{Y}(t')\}_{t'=1}^{+\infty} = \{\mathbf{Z}(\mathbf{X}(t)), \mathbf{Z}(\mathbf{X}(t+1))\}_{t=1}^{+\infty}$ whose components $Y_k(t)$ take values in $\{0,1\}^2$, where the event $(1,1)$ has zero probability and where $t' = \frac{t}{2}$. One can then encode the $Y_k(t')$ values by $0 \to (0,0)$ (no relaxation at times $t, t+1$) and $1 \to (0,1),(1,0)$ (relaxation at time t or at time $t+1$). This leads to define a new "density of active sites" $\rho'_L(i) = Prob\,[Y_i(t') = 1] = Prob\,[Z_i(\mathbf{X}(t)) = 1 \; or \; Z_i(\mathbf{X}(t+1)) = 1]$. Since the events $\{Z_i(\mathbf{X}(t) = 1\}$ and $\{Z_i(\mathbf{X}(t+1) = 1\}$ are disjoint we have: $\rho'_L(i) = Prob\,[\mathbf{Z}(\mathbf{X}(t)) = 1] + Prob\,[\mathbf{Z}(\mathbf{X}(t+1)) = 1] = 2.\rho_L(i)$. Assuming now that the $Y_k(t')$'s are independent and considering $\rho'_L(i)$ as the effective density of active sites one obtains an effective transport operator:

$$\mathcal{L} = I + 2\alpha\Delta.\rho_L.I \tag{40}$$

Calling γ_i the singular values of $\mathcal{L}$, our mean-field ansatz suggests that the slowest Lyapunov modes are given by :

$$\lambda_L(i) = log(\gamma_i) \tag{41}$$

Note that this operator is self-adjoined for the metric $\rho_L.I$ and that the corresponding matrix can be made symmetric by the variable change $\rho_L^{-\frac{1}{2}}.I$.

To check the validity of this ansatz, we first computed the density of active sites on a 20×20 lattice and found numerically the γ_i's from these data [6]. In the same time we computed the Lyapunov spectrum. A plot of the two curves is drawn fig. 2. One finds a very good agreement for a wide part of the spectrum, and the discrepancy increases when going to small times scale, as expected.

---

[6] We weren't able to go beyond $L = 20$ in the Lyapunov spectrum computation. We used a version of the Eckmann-Ruelle algorithm [11] revisited from Von Bremmen et al. [25]. Nevertheless, we needed two weeks of computation on a Pentium II 300 for the case $L = 20$, with a relative accuracy of 10^{-3}.

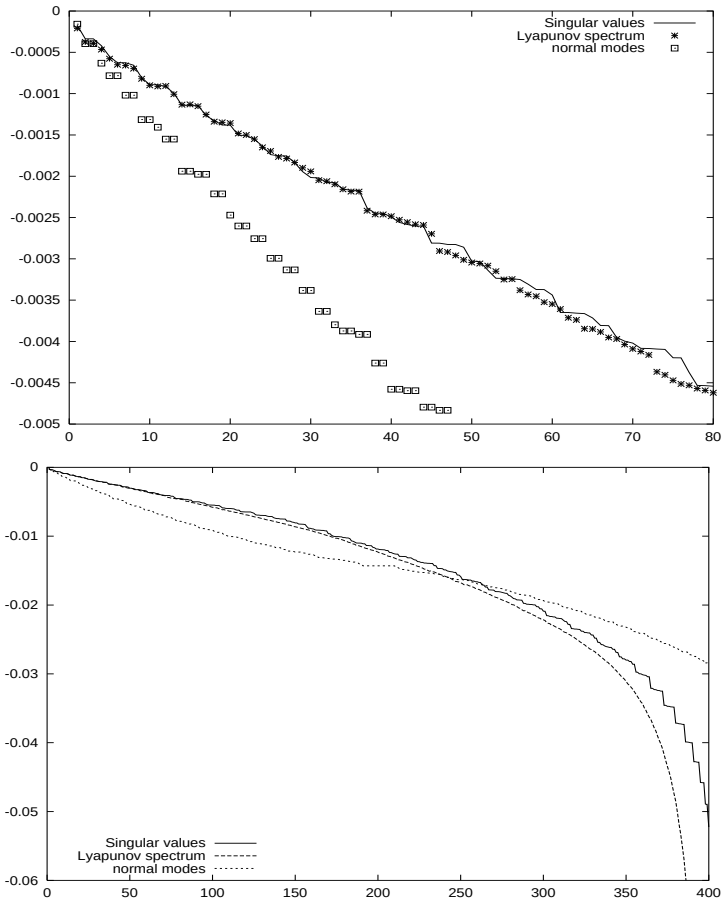

FIG. 2. Lyapunov spectrum, logarithm of the $\mathcal{L}$ singular values, and normal diffusion modes for $E_c = 2.2, \epsilon = 0.1, L = 20$. Fig 2a : 80 first modes; Fig 2b Full spectra drawn with lines in order to better see the shape.

4. The role of the spatial variations of the density of active sites.

It is usually assumed in the SOC litterature, when dealing with the spatial properties of the model, that only the density of active site, and more precisely, its lattice average, ρ_L^{av}, acting then as an order parameter, has to be taken into account. One neglects therefore the spatial dependence of ρ_L. In our approach this would lead to an effective transport operator $I + 2.\alpha\rho_L^{av}\Delta$, corresponding to normal transport (see below). In this case the slowest Oseledec modes would simply be the eigenmodes of the Laplacian with zero boundaries conditions on ${}^{\partial}\Lambda$, and the Lyapunov exponents would the correspond to the normal diffusion modes, namely:

$$\lambda_L(i) = log(|1 + 2.\alpha.\rho_L^{av}.\sum_{k=1}^{d}(cos(\frac{\pi.n_k}{L+1}) - 1)|) \tag{42}$$

where the Laplacian modes are parametrized by the quantum numbers $\mathbf{n} = (n_k), k = 1 \ldots d$, sorted such that the corresponding eigenvalues are decreasing (and i refers to the place of the exponent in this sequence). We plot also in fig. (2) the diffusion modes of eq. (42). The computed Lyapunov exponents are different from these values but for the largest ones. This approximation is therefore too crude and gives a wrong spectrum. Note in particular that the *shape* of the spectrum differs, namely the discrepancy cannot be corrected by a mere multiplication of ρ_L^{av} by some

factor. As far as the Lyapunov exponents contains all the relevant informations about the dynamics at stationnarity, our conclusion is that the non homogeneity of $\rho_L(i)$ plays a key role in computing dynamical quantities, and implies unfortunately that the zero-th order "mean-field" approaches which would approximate the density of active sites as a constant would lead to incorrect estimates for finite size when dealing with intermediate time scales. On the other hand, this should lead to correct results when dealing with the longest time scales, since the first modes are well approximated by the transport operator where $\rho_L(i)$ is considered as uniform.

As a corrollary note that the transport over the longest times scales (namely when averaging over a large number of avalanches) is therefore *normal*. Though this is an often used assumption in the litterature, it is somehow confusing since in the same time it is claimed that SOC is characterized by anomalous diffusion. There is however no contradiction since the anomalous diffusion exponent z characterizes the average transport within *one* avalanche, namely on time scales corresponding roughly to the crossover point $\lambda_L(i_c)$ where the normal diffusion operator gives a wrong exponent.

5. The random walk picture.

The operator $\mathcal{L}$ is the Laplace-Beltrami operator associated to a random diffusion in a "medium" or a landcaspe *that is not flat*, corresponding to the metric $g = \rho_L.I$ where I is the identity matrix on $\mathbb{R}^N$. It has a nice interpretation in the so-called random walk picture [7]. Assume for a while that the energy of a site is composed by (undivisible) energy quanta η that can be made arbitrary small (this is way to "map" the Zhang's model to a sandpile). Assume that we are in the stationnary regim, and that at initial time we drop a grain in some place and study its motion. At each time step where it is involved in a relaxation process this grain makes a jump at random in one of the $2.d$ directions in the lattice. From this point of view, the stochastic dynamics of the grain is driven by the underlying dynamics of eq.(1). *If we assume that the evolution is Markovian*, the probability to jump from i to some nearest neighbour j depends only on the state of i at time t and is given by a transition rate $W_{ij} = \alpha.\rho_L(i)$ while the probability to stay at the same place is $1 - \rho_L(i)(1-\epsilon)$ (remember that only an amount $\frac{1-\epsilon}{2d}$ of the energy is transfered to the neighbours when a site relaxes). From this one gets the equation for the probability $P(i,t)$ that a grain is at place i at time t, before it leaves the lattice $P(.,t+1) = [I + \alpha.\Delta\rho_L].P(.,t) = [I + \alpha.\Delta\rho_L]^t.P(.,1)$ and recovers the operator obtained above when assuming that the $Z_i(t)$'s were independent. Indeed, the independence assumption of the $Z_i(t)$'s is *equivalent to the Markovian assumption in the random walk picture.* The probability of jump for a grain a time t depends *a priori* on its whole past via a Chapman-Kolmogorov equation whose transfert matrix is a sum of terms containing conditionnal probabilities

$$Prob\left[Z_{i_{r-1}}(\mathbf{X}(t_{r-1})) = 1 | Z_{i_{r-2}}(\mathbf{X}(t_{r-2})) = 1, \ldots, Z_{i_1}(\mathbf{X}(t_1)) = 1, Z_j(\mathbf{X}(t_0)) = 1\right] =$$

$$\frac{Prob\left[Z_{i_{r-1}}(\mathbf{X}(t_{r-1})) = 1, Z_{i_{r-2}}(\mathbf{X}(t_{r-2})) = 1, \ldots, Z_{i_1}(\mathbf{X}(t_1)) = 1, Z_j(\mathbf{X}(t_0)) = 1\right]}{Prob\left[Z_{i_{r-2}}(\mathbf{X}(t_{r-2})) = 1, \ldots, Z_{i_1}(\mathbf{X}(t_1)) = 1, Z_j(\mathbf{X}(t_0)) = 1\right]} =$$

[7] B.C. is very grateful to P. Grassberger and D. Dhar for illuminating discussion on this point in Trieste.

$$Prob\left[Z_{i_{r-1}}(\mathbf{X}(t_{r-1}))=1\right]$$

where the last equality holds when the $Z_k(t)$'s are independent. In this case $W_{ij}=\alpha.Prob\left[Z_i(\mathbf{X}(t))=1\right]=\alpha\rho_L(j)$ for the j nearest neighbours of i.

However, we saw above that the process is not strictly Markovian since a jump from a given site cannot arise two successive time steps. In other word the probability of jump $i\to j$ depends on the state of i at time t and time $t-1$. The system has some memory (at least two time steps). However, defining the random variables $Y_k(t)$'s as above and assuming them independent amounts to make the random walk Markovian by a suitable reparametrization of the process, and gives a transfert equation

$$P(.,t+1)=[I+2.\alpha.\Delta\rho_L].P(.,t)=\mathcal{L}^t.P(.,1) \tag{43}$$

Therefore, the operator $\mathcal{L}$ characterizing the decay of a small perturbation can also be interpreted as the transfert matrix of a random walk in a medium where the diffusion rate depends on the location.

6. Density of active site and average energy flowing to the boundaries.

Eq. (43) characterizes the energy transport into the lattice [8], but does not take into account the source term (addition of grain) required to reach stationnarity. For, each times a grain goes out of the lattice, one must add another grain at random place i, with probability $\bar{\omega}_L(i)$ which is a source term. Call $\mathcal{P}_L$ the equilibrium state and $\mathcal{V}(\partial\lambda)$ the set of sites at distance one from the boundary. The probability for a grain to go out is $2\alpha\sum_{j\in\mathcal{V}(\partial\lambda)}\rho_L(i).\mathcal{P}_L(i)$. It is obviously proportional to the outgoing energy flux, which is, at stationnarity, equal to incoming flux (resp. the probability to add a grain in the lattice), namely [9]:

$$\bar{\omega}_L=2\alpha\sum_{j\in\mathcal{V}(\partial\lambda)}\rho_L(j).\bar{X}_L(j) \tag{44}$$

The local source term in the energy equation is $\bar{\omega}_L(i)$. Therefore, the complete equation for the energy at stationnarity is :

$$2\alpha.\Delta[\rho_L.\bar{\mathbf{X}}_L]+\bar{\omega}_L(i)=0 \tag{45}$$

with zero boundaries conditions and with the constraint (44).

In this equation one distinguishes a *local* transport term, and a source term which depends on a *global* constraint. When the excitation is uniform (45) reduces to $\Delta[\rho_L.\bar{\mathbf{X}}_L]+\frac{\bar{\omega}_L}{2\alpha.L^d}=0$.

---

[8] Note that the average energy at site i, $\bar{X}_L(i)$ being proportional to the probability $P_L(i)$ the equation for $\bar{X}_L(i)$ is the same, up to some constant

[9] The cautious reader has noticed that this equation is not dimensionaly correct since no energy term appears on the l.h.s.. One should indeed multiply by δ, the input energy quantum, which is set to one throughout this paper.

The difficulty in solving it is that it deals with the product $\rho_L.\bar{\mathbf{X}}_L$. On the other hand, it is known in the litterature that $\bar{\mathbf{X}}_L$ converges to a uniform energy distribution over the lattice as $L \to \infty$ [13]. Assume, now that we can write $\bar{\mathbf{X}}_L$ as:

$$\bar{\mathbf{X}}_L = \bar{\mathbf{X}}_0 + f(L) \tag{46}$$

where $\|f(L)\|$ goes to zero as $L \to \infty$ and where $\bar{\mathbf{X}}_0$ is spatially uniform, i.e. $\bar{\mathbf{X}}_0(i) = cste = \bar{X}_L$. At the zero-th order, corresponding to a flat energy landscape, one gets for ρ_L the following equation.

$$\Delta\rho_L + \frac{\bar{\omega}_L}{2.\alpha L^d \bar{X}_L} = 0 \tag{47}$$

where $L^d \bar{X}_0 = E_{tot}$, the total energy in the lattice. The solution of this equation can be easily found by decomposition on the eigenmodes of the Laplacian. The general solution is:

$$\rho_L(\mathbf{x}) = \sum_{\mathbf{n}} A_{\mathbf{n}} \prod_{i=1}^{d} sin(k_i.x_i) \tag{48}$$

where $\mathbf{n} = (n_1, \ldots, n_d)$ is the set of quantum numbers parametrizing the eigenmodes of the discrete Laplace operator, $s_{\mathbf{n}} = 2(\sum_{i=1}^{d} cos(k_i) - d)$ is the corresponding eigenvalue with $k_i = \frac{n_i \pi}{L+1}$,

$$A_{\mathbf{n}} = -\frac{2^{d-1}.\bar{\omega}_L}{\alpha E_{tot}(L+1)^d} \frac{\prod_{i=1}^{d} C_{n_i}}{s_{\mathbf{n}}}$$

and

$$C_{n_i} = \sum_{x=1}^{L} sin(k_i.x) = (-1)^{m_i} \frac{sin(\frac{n_i \pi L}{2(L+1)})}{sin(\frac{n_i \pi}{2(L+1)})}$$

where $n_i = 2m_i + 1$. Surprisingly, this already gives a quite good approximation for ρ_L which is better and better as L increases (see fig. 3 and fig. 4 in the next section.).

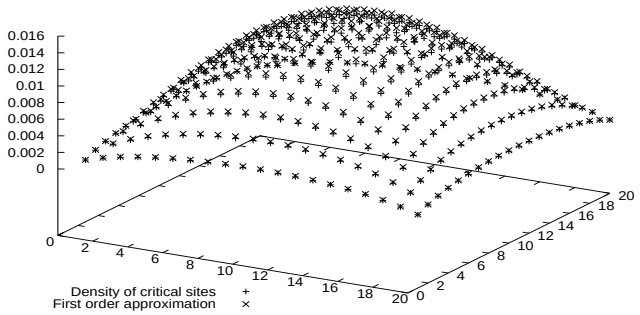

FIG. 3. Plot of the density of active sites and solution of eq. (47) for $E_c = 2.2, \epsilon = 0.1, L = 20$.

Away from the boundaries, one expects rotational invariance for $\rho_L(\mathbf{x})$. This can be checked by expanding the sin near to $x_i = \frac{L}{2}, i = 1 \ldots d$ up to the first order. One obtains the well known paraboloid form [12] $\rho_L(\mathbf{x}) \sim K_0 - K_1 \sum_{i=1}^{d} x_i^2$, where the constants K_0, K_1 can be easily deduced from eq. (47).

One also obtains the average density of active sites, $\rho_L^{av} = \frac{1}{L^d} \sum_{i=1}^{N} \rho_L(i)$.

$$\rho_L^{av} = -\frac{2^{d-1}.\bar{\omega}_L}{L^d(L+1)^d \alpha E_{tot}} \sum_{\mathbf{n}} \frac{\prod_{i=1}^{d} C_{n_i}^2}{s_{\mathbf{n}}} \tag{49}$$

which is expected to hold for sufficiently large L. We give an example fig. 4 where one notices that this formula gives already a quite good estimate for $L = 15$.

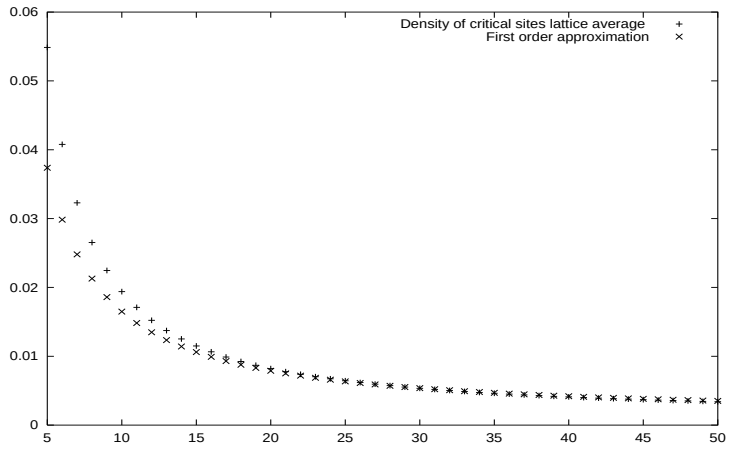

FIG. 4. Plot of ρ_L^{av} and solution of eq. (49) versus L, for $E_c = 2.2, \epsilon = 0.1$.

IV. SCALING PROPERTIES OF THE LYAPUNOV SPECTRUM.

The Zhang's model, as an hyperbolic dynamical system does not exhibit a critical behaviour for finite size. However, since a critical behaviour is expected in the thermodynamic limit, it is of crucial importance to know the behaviour of the Lyapunov exponents as $L \to \infty$. In this section, we use a Finite Size Scaling ansatz to study the scaling of the Lyapunov spectrum with L and discuss its relation with the scaling of other characteristic observables like the average size or duration.

A. Finite Size-Scaling of the Lyapunov spectrum.

An approximate expression for the modes related to the transport in the lattice is obtained from eq. (43), whereas an approximate equation for ρ_L is given by eq. (49). However, at the moment we don't have an analytic expression for the solution of eq. (43) . In this paper we use a standard statistical mechanics ansatz, the Finite Size Scaling. This is a standard tool in the SOC approach with respect to the scaling of the probability distribution of avalanches observables, though its validity has been asked in this case [22]. Our point of view is different since we apply the Finite Size Scaling on the Lyapunov spectrum itself. Namely, we assume that, for any L, there exists a change of coordinates $i \to \phi_L(i), \lambda_L \to \psi_L(\lambda_L)$, depending on L such that the points of the spectrum $\{i, \lambda_L(i)\}$ are mapped

onto the same "universal" curve [10] $\{x, \lambda(x)\}$, where $\lambda(x) = \psi_L \circ \lambda_L \circ \phi_L^{-1}(x)$. Furthermore we assume (as in usual Finite Size Scaling) that the coordinate changes are simple dilatations where $\phi_L(x) = L^{\beta_\lambda}.x,\ \psi_L(x) = L^{\beta_\lambda.\tau_\lambda}(x)$. Then:

$$\lambda(x) = L^{\beta_\lambda.\tau_\lambda}.\lambda_L(x.L^{-\beta_\lambda}) \tag{50}$$

Equivalently, knowing the curve $\{x, \lambda(x)\}$ the spectrum for a given size is

$$\lambda_L(i) = L^{-\beta_\lambda.\tau_\lambda}\lambda(iL^{-\beta_\lambda}),\ i = 1 \ldots L^d \tag{51}$$

We have no rigourous justification for this scaling form. Nevertheless we remark that FSS holds for the slow modes of the transport operator $1 + 2.\alpha.\rho_L^{av}\Delta$ obtained when ρ_L is considered as uniform. This operator being self-adjoined, the Lyapunov exponents are $log(|1 + 2\alpha.\rho_L^{av}s_L(i)|)$ where $s_L(i) = 2(\sum_{k=1}^{d} cos(\frac{n_k\pi}{L+1}) - d)$ and where we associate to the set of quantum numbers $\mathbf{n} = (n_k)$ an index i corresponding to the position of the eigenvalue in the ordered sequence. For the slow modes $log(|1 + 2\alpha.\rho_L^{av}s_L(i)|) \sim \rho_L^{av}s_L(i)$. Since ρ_L^{av} decay polynomialy, say like $\rho_L^{av} \sim L^{-\gamma}$, the points of the spectrum fall on the same curve $2(\sum_{k=1}^{d} cos(\pi.x) - d)$, by using a FSS transformation where $\beta_\lambda = d, \beta_\lambda.\tau_\lambda = \gamma$. At the moment we do not know how to extend this analysis to the general case. Note anyway that FSS may not hold for the full spectrum and in particular for the kernel modes (tail of the spectrum). This is however not central since the kernel modes are irrelevant w.r.t. to the energy transport. Finally, we find a good data collapse (see fig. 5).

Since the set of indices $i \in \{1 \ldots L^d\}$ it is evident that :

$$\beta_\lambda = d \tag{52}$$

The exponent τ_λ can be numerically computed by several means. A first one is to minimize the euclidean distance between the spectra obtained for different lattice size, with respect to τ_λ. Another way is to compute the sum of Lyapunov exponents. Indeed:

$$S_L \stackrel{\text{def}}{=} \sum_{i=1}^{N} \lambda_L(i) = L^{-d.\tau_\lambda}.\sum_{y=L^{-d}}^{1} \lambda(y) \sim L^{d.(1-\tau_\lambda)} \int_{L^{-d}}^{1} \lambda(y).dy$$

Assuming that $\lambda(y)$ is bounded as $y \to 0$ and that $0 < K = \int_0^1 \lambda(y)dy < \infty$ one gets:

$$S_L \sim K.L^{d.(1-\tau_\lambda)} \tag{53}$$

which allows to compute τ_λ. The value of τ_λ for $d = 2, \epsilon = 0.1$ and different E_c values are given Table 1. These value were obtained for a sample a spectra from $L = 10$ to $L = 20$. We note in particular that τ_λ depends on E_c. At the moment we have no way to know wether this effects remains in the thermodynamic limit. Note that this value are given as indications but the correct estimation of τ_λ requires certainly further investigations for consequently larger system size, which are beyond our computer performances.

---

[10] Note that this curve depends on the parameters E_c, ϵ, d

The data collapse of spectra is drawn Fig. 5

Ec	τ_λ
0.6	0.632
1.1	0.622
1.5	0.621
2.2	0.560
4.1	0.524

Table 1:Computed values of τ_λ versus E_c, obtained from formula (53), from samples of size L=10 to L=20.

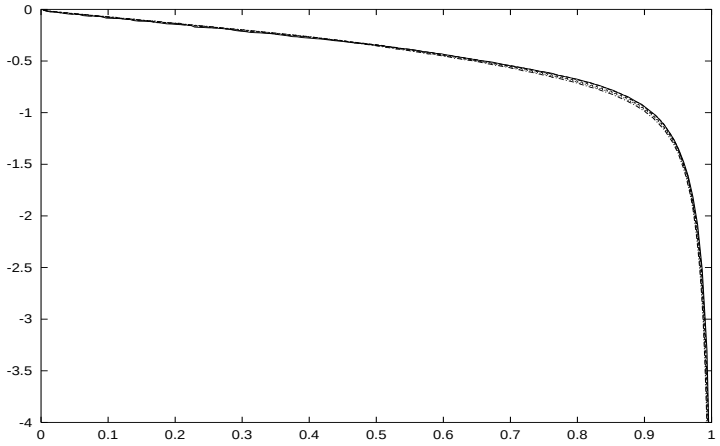

FIG. 5. Data collapse of the Lyapunov spectrum for Ec=2.2,eps=0.1,L=12,14,16,18.

B. Relation between τ_λ and other characteristic exponents.

The average value of observables like size, duration, etc ... is known to diverge with a power law scaling $\langle x \rangle_L \sim L^{\gamma_x}$. From simple scaling arguments we conjecture that γ_s, γ_τ (and other characteristic exponents like z, the anomalous diffusion exponents) might be related to τ_λ. Note that γ_x is related to the critical exponent τ_x [11] and therefore our discussion suggests that there is a link between τ_λ and the critical exponents τ_s, τ_τ.

The equation (49) can be written:

$$\rho_L^{av} = -\frac{2^{d-1}.\bar{\omega}_L}{L^d \alpha E_{tot}}.\gamma_L \tag{54}$$

where $(L+1)^d.\gamma_L = \sum_{\mathbf{n}} \frac{\prod_{i=1}^d C_{n_i}^2}{s_{\mathbf{n}}}$. Let us estimate the scaling of this sum as $L \to \infty$. Set first $d=1$ and fix $\alpha > 0$ arbitrary small. The sum over $n = n_1$ can be split into a part such that $n < (L+1)\alpha$ and another part where $n \geq (L+1)\alpha$. In the first sum, $C_n \sim \frac{2(L+1)}{n\pi}$, $s_{\mathbf{n}} \sim -\frac{(\pi.n)^2}{(L+1)^2}$, while the second sum is lower than $(L+1)C(\alpha)$ where

[11] Under the finite-size scaling assumption of $P_L(x)$ one finds that $\gamma_x = \beta_x.(2-\tau_x)$ where L^{β_x} is the scaling for the maximal value of x in a lattice of size L.

$C(\alpha)$ is bounded for $\alpha > 0$. Therefore $\gamma_L \sim (L+1)^3.S$ where $S \sim \sum_{n<(L+1)\alpha} \frac{1}{n^4}$ stays bounded as $L \to \infty$. Then $\gamma_L \sim (L+1)^3$. This argument can be generalized for any d by splitting the sum over $\mathbf{n} = (n_1, n_2, \ldots n_d)$ into sums where k indexes are smaller that $\alpha(L+1)$, k going from 0 to d. It is easy to see that the major contribution is due to the terms such that the d indexes are $< \alpha(L+1)$ giving a leading contribution $\sum (L+1)^{2d+2}$ and $\gamma_L \sim L^{d+2}$. We conclude therefore that ρ_L^{av} scales like :

$$\rho_L^{av} \sim \frac{\bar{\omega}_L}{E_{tot}} \frac{(L+1)^{d+2}}{L^d} \sim \frac{\bar{\omega}_L.L^2}{E_{tot}} = \frac{\bar{\omega}_L.L^2}{L^d \bar{X}_0} \tag{55}$$

Set $h = \frac{\bar{\omega}_L}{L^d}$ for the driving rate and call $e = \frac{h}{\rho_L^{av}} = \bar{X}_L.L^{-2}$. One gets the energy conservation equation : $h = \rho_L^{av} e$ and therefore e is the energy dissipated per active site and per unit time. It corresponds to the *dissipation rate* introduced by Vespignani et al. [24]. Since $0 < \bar{X}_L < E_c, \forall L$, $\bar{X}_L$ plays no role in the asymptotic scalings in L and therefore $e \sim L^{-2}$ as already anticipated by a mean-field approach in [24].

The scaling exponent of ρ_L^{av} can be obtained from τ_λ. Indeed the first Lyapunov exponent is given with a good accuracy by the normal diffusion operator $1 + 2.\rho_L^{av}\Delta$ (see fig.2), which implies that $\lambda_L(1) \sim \rho_L^{av} L^{-2}$. Another way to argue is to note that from theorem 1, $\lambda_L(1)$ scales like the average ratio of energy dissipated from one site. In other words, $\lambda_L(1).\bar{X}_L \sim h$ from which we get $\lambda_L(1) \sim \rho_L^{av} L^{-2}$.

The scaling of $\lambda_L(1)$ can be obtained from the FSS form since $\lambda_L(1) = L^{-d.\tau_\lambda}.\lambda(L^{-d})$. However the analyticity properties of λ near to zero is not known. Assume that $\lambda(x) \sim x^\alpha, x \sim 0$ where α may depend on d (seemingly $\alpha = 1$ for $d = 2$). Then $\lambda_L(1) \sim L^{-d.\tau_\lambda - d\alpha}$ gives:

$$\rho_L^{av} \sim L^{-d.\tau_\lambda + 2 - d\alpha} \tag{56}$$

Furthermore, since $0 \leq \bar{\omega}_L \leq 1$ eq. (21) implies $p_L = f(\frac{1}{\langle\tau\rangle_L}) = \frac{a_1}{\langle\tau\rangle_L} - \frac{a_2}{\langle\tau\rangle_L^2} + O(\frac{1}{\langle\tau\rangle_L^3})$. This is in particular clear for $E_c < 1$ since $p_L = \bar{\omega}_L$ which implies $p_L = \frac{1}{1+\langle\tau\rangle_L}$ and therefore $a_1 = 1, a_2 = 1$. For general E_c using this form gives, from eq. (21) $a_1 = 1$ and :

$$p_L \sim \frac{1}{\langle\tau\rangle_L} - \frac{a_2}{\langle\tau\rangle_L^2} \tag{57}$$

and

$$\bar{\omega}_L \sim \frac{a_2}{\langle\tau\rangle_L} \tag{58}$$

as $L \to \infty$. Therefore $\rho_L^{av} \sim L^{-\gamma_\tau + 2 - d}$ which implies $\gamma_\tau + d = d(\tau_\lambda + \alpha)$.

Finally from eq. (31) one gets:

$$d\tau_\lambda = d - \gamma_s + \gamma_\tau \tag{59}$$

which gives:

$$\gamma_s = 2 \tag{60}$$

$$\gamma_\tau = d\tau_\lambda + 2 - d \tag{61}$$

and

$$\alpha = \frac{2}{d} \tag{62}$$

The equation for γ_s has been already anticipated from many authors on the basis of numerical simulations [13] mean-field approach [24] and proved in the Dhar's model for $d = 2$ by Dhar himself [8]. The equation for α is well verified in $d = 1$ and $d = 2$. This relation deserves however further investigations in larger dimensions. It suggests in particular that the curve $\lambda(x)$ is not C^1 at zero for $d > 2$, namely the largest exponents do not go to zero in a smooth way as $L \to \infty$.

Finally, the anomalous diffusion exponent z, characterizing the average transport within one avalanche, is equal to γ_τ if one assumes that the average avalanche radius scales like L in any dimension [14]. Equivalently one can notice that the crossover point for the $\chi_L(i)$'s spectrum (eq. (32)) is $\sim \frac{L^z}{<\tau>_L}$ and does not depend on L. From eq. (61) it follows that the transport on time scales of order $< \tau >_L$ is anomalous ($z < 2$) iff $\tau_\lambda < 1$. Note however that this argument assumes that the FSS is still valid at the crossover point.

This result suggests therefore that the exponents γ_x usually computed in the litterature can be obtained from τ_λ. Note that in this case, the E_c dependance appearing in table 1 would have to be elucidated since it would suggest that the critical exponents depend on E_c. This was already argued in [3–5] and revealed from numerical simulations (though not discussed) in [17]. Note however that the dependence on the dynamical quantities in the control parameter in a dynamical system is more a rule that an exception. One certainly needs very special properties to insure that the critical exponents are constant in the limit $L \to \infty$, whatever E_c. If it happened to be true this would mean that the Zhang's model is somehow non generic at least from the dynamical systems point of view.

V. CONCLUSION

In this paper, we investigated the dynamics of the Zhang's model in terms of the Lyapunov exponents and Oseledec modes. Due to the piecewise affine structure of the model, the Lyapunov exponents, usually related to the local properties of the dynamics (expansion rates, fractal dimensions, entropy), also appears as characteristic rates of energy transport in the system. We showed that the spectrum is roughly divided into two parts, the slow modes corresponding to transport and dissipation and the fast ones essentially associated to the attractor stability. Even if the Oseledec modes are the analogous of Fourier modes in normal diffusion, they are not normal modes, because the density of active sites is not spatially homogeneous. The slow Oseledec modes correspond rather to a diffusion in a metric which is not flat, given by the density of active sites. Only for the slowest mode is the Lyapunov exponent the same as the largest rate in normal diffusion. This is nevertheless important since the slowest mode characterize the equilibrium properties of the model. This means that the usual mean-field approaches, replacing the density of active sites by its lattice average, are correct if one considers properties related to the longest time scales.

We also investigated the scaling properties of the spectrum with respect to the lattice size and found that it obeys Finite Size Scaling with a very good accuracy. In particular we extracted a critical exponent τ_λ which might be related to the usual critical exponents computed in the litterature. The scaling form show also that in the thermodynamic limit a part of the spectrum goes to zero, corresponding to translation invariance, and zero dissipation. In this way

the Zhang's model is not hyperbolic in the thermodynamic limit. This is expected. Indeed, an hyperbolic system exhibits exponential correlation decay and cannot therefore be a critical system with polynomial correlation decay. Only if the largest correlation decay rates go to zero in the thermodynamic limit can the system achieve a critical state. On the other hand, it means that our approach is valid only for finite size.

Nevertheless, our approach gives a description of the dynamics equivalent to the Fourier decomposition in normal diffusion. We believe that it can be extended to other models, like the Dhar's one with continuous energy values, though it may not give new insight in this case. We also believe that this approach could shed some lights on some remaining open problems in SOC. In particular, the problem of the right scaling form to compute the critical exponents (Finite Size Scaling, multifractal analysis, explicit form of the cut-off) could be adressed. Also, the dependence of the critical exponents in the control parameters (E_c in the Zhang's model), which might be related to the variation of the Haussdorf dimension, could be investigated [5]. Finally, the vanishing of correlations in the thermodynamic limit suggests that some connexion with usual critical phenomena could be made.

Acknowledgement. B.C. is very grateful to D. Dhar, P. Grassberger, and H. Jensen for illuminating discussions in Trieste. He also greatly acknowledge J.L. Meunier for the many exciting debates they have, and ideas they exchange together. This work was partially supported by the Procope grant : 98008

---

[1] Bak P., Tang C., Wiesenfeld K., "Self organised criticality: An explanation of $1/f$ noise",Phys. Rev. Lett., Vol 59, Num 4, (1987), 381- 384Bak P., Tang C., Wiesenfeld K., "Self organised criticality", Phys. Rev. A., Vol 38, Num 1, (1988), 364-374.
[2] Bak P. "How nature works.", Springer-Verlag (1996).
[3] Blanchard Ph., Cessac B., Krüger T., "A dynamical systems approach for SOC models of Zhang's type", Jour. of Stat. Phys., 88, (1997), 307-318 .
[4] Cessac B., Blanchard Ph., Krueger T., "A dynamical system approach to Self-Organized Criticality" Mathematical results in Statistical Mechanics", Marseille 1998, Word Scientific Singapore.
[5] Blanchard Ph., Cessac B., Krüger T., "What can one learn about Self-Organized Criticality from Dynamical System theory ?", *Jour. of Stat. Phys., 98* (2000), 375-404.
[6] Blanchard Ph., Cessac B., Krüger T., "Ergodicity in the Zhang's model of Self-Organized Criticality", in preparation.
[7] D. Dhar, Phys. Rev. Lett. **64**, 1613 (1990); D. Dhar and S.N. Majumdar, J. Phys. A **23**, 4333 (1990); S.N. Majumdar and D. Dhar, Physica A **185**, 129 (1992). D. Dhar and R. Ramaswamy, Phys. Rev. Lett. **63**, 1659 (1989).
[8] D. Dhar, Phys. Rev. Lett. **64**, 1613 (1990).
[9] Giacometti A., Diaz-Guilera A., "Dynamic renormalization group approach to self-organized critical phenomena", Phys. Rev. E 58, 247 (1998).
[10] Eckmann J.P., Gat O., " Hydrodynamic Lyapunov Modes in Translation Invariant Systems", preprint chao-dyn/9908018.
[11] Eckmann J.P., Ruelle D., "Ergodic Theory of Strange attractors" Rev. of Mod. Physics, 57, 617 (1985). Eckmann J.P., Kamphorst O., Ruelle D., Cilliberto S. Phys. Rev. A, 34, 4971 (1986).
[12] Grassberger P., private communication.
[13] Grassberger P., Manna S., J. Phys France, 51:1077, (1990)
[14] Jensen H.J., "Self-Organized Criticality : Emergent Complex Behavior in Physical and Biological systems", Cambridge Lecture Notes in Physics 10, Cambridge University Press (1998).
[15] Kadanoff L.P., Nagel S.R.,Wu L., Zhou S.,"Scaling and universality in avalanches", Phys. Rev. A, Vol 39, Num 12, 6524-6527, (1989)
[16] Ledrappier F., Young L. S., "The metric entropy for diffeomorphisms", Annals of Math. 122, (1985) 509-574
[17] Luebeck S. "Large-scale simulations of the Zhang sandpile model", Phys. Rev. E., Vol. 56, Num. 2, 1590-1594, (1997)
[18] Newmann C. M., "The distribution of Lyapunov exponents: Exact results for Random Matrices", Com. Math. Phys., 103, 121-126 (1986).

[19] Oseledec V.I., "A multiplicative ergodic theorem. Characteristic Lyapunov exponents of dynamical systems.", Transaction of the Moscow Mathematical Society, 19 (1968), 179-210.

[20] Pietronero L., Tartaglia P., Zhang Y.C., "Local Rigidity and Self-Organized criticality for avalanches.", Europ. Let., 129 (2), 111-116 (1995). ,"Theoretical studies of Self-Organized Criticality", Physica A 173, (1991), 22-44; Cafiero R., Loreto V., Pietronero L., Vespignani A. , Zapperi S., ; L. Pietronero, A. Vespignani, and S. Zapperi, Phys. Rev. Lett. **72**, 1690 (1994); A. Vespignani, S. Zapperi, and L. Pietronero, Phys. Rev. E **51**, 1711 (1995).

[21] Pollicott M. "Lectures on ergodic theory and Pesin Theory on compact manifolds", London Math. Soc., Lect. Notes Series 180, Cambridge University Press, (1993).

[22] Tebaldi C. , De Menech M., Stella A., "Multifractal scaling in the Bak-Tang-Wiesenfeld Sandpile and edge events", preprint cond-mat/9903270.

[23] Dickman R., Vespignani A., Zapperi S., "Self-organized criticality as an absorbing-state phase transition" *Physical Review E57, 5095* (1998)

[24] Vespignani A., Zapperi S., "How self-organized criticality works : A unified mean-field picture." *Physical Review E57, 6345* (1998)

[25] Von Bremen H.F.,Udwadia F.E., Proskuroswki W., "An efficient QR based method for the computation of Lyapunov exponents", Physica D, 101 (1997) 1-6.

[26] Zhang H.Y. "Scaling theory of Self-Organized Criticality", Phys. Rev. Let., Vol 63, Num 5 (1988), 470-473.