Lower order perturbations of Dirichlet processes

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Abstract. We consider lower order perturbations $\mathcal{M}$ of symmetric diffusions $\mathcal{M}^0$ and prove that $\mathcal{M}$ is locally absolutely continuous with respect to $\mathcal{M}^0$ up to life time. The novelty is that the absolute value of the drift b and zero order part c are merely assumed to be in $L^d(R^d)+L^\infty(R^d)$, and $L^{\frac{d}{2}}(R^d)+L^\infty(R^d)$. So, $|b|^2$ and c are not in the Kato- class (as is the case when $|b|^2, |c| \in L^p(R^d)+L^\infty(R^d)$ with $p>\frac{d}{2}$). We also consider the case where an adjoint drift is present. Finally, we use these results to prove new convergence results for diffusions.

1. Introduction and framework.

As usual, let R^d denote the Euclidean space. We assume $d \geq 3$. Let $a_{ij}(x), 1 \leq i.j \leq d$ be real-valued Borel measurable functions such that $a_{ij}(x) = a_{ji}(x)$ and the matrix-valued function $(a_{ij}(x))_{1\leq i,j\leq d}$ is uniformly elliptic,i.e., there exists a constant δ such that

$$\frac{1}{\delta}\sum_{i=1}^d y_i^2 \leq \sum_{i,j=1}^d a_{ij}(x)y_iy_j \leq \delta \sum_{i=1}^d y_i^2 \tag{1}$$

for all $y_1, y_2, ..., y_d \in R$.
It is well known that

$$\mathcal{E}^0(u,v) := \frac{1}{2}\int_{R^d}\sum_{i,j=1}^d a_{ij}(x)\frac{\partial u(x)}{\partial x_i}\frac{\partial v(x)}{\partial x_j}dx$$

$$D(\mathcal{E}^0) := H_2^1(R^d) \tag{2}$$

defines a regular Dirichlet form on $L^2(R^d)$, where $H_2^1(R^d)$ stands for the Sobolev space of order 1. Let $\mathcal{M}^0 := \{\Omega, \mathcal{F}, \mathcal{F}_t, X_t, P_x, x \in R^d\}$ denote the diffusion process associated with $(\mathcal{E}^0, D(\mathcal{E}^0))$ (see [1]). Then, Fukushima's decomposition holds :

$$X_t = x + M_t + N_t \qquad P_x - a.s., \tag{3}$$

where $M_t = (M_t^1, ..., M_t^d)$ is a $\mathcal{F}_t$-square integrable martingale additive functional with

$$< M^i, M^j >_t = \int_0^t a_{ij}(X_s)ds \tag{4}$$

N_t is a continuous additive functional of zero energy. Let γ_t be the reverse operator defined on the path space Ω by $\gamma_t(\omega)(s) = \omega(t-s)$ if $s \leq t$. One also has the Lyons-Zheng decomposition:

$$X_s - X_0 = \frac{1}{2}M_s - \frac{1}{2}(M_t \circ \gamma_t - M_{t-s} \circ \gamma_t) \quad \text{for} \quad 0 \leq s \leq t,$$

where $M_s \circ \gamma_t$ is a $\hat{\mathcal{F}}_s = \sigma(X_{t-u}, u \leq s)$-martingale with

$$< M_s^i \circ \gamma_t, M_s^j \circ \gamma_t >= \int_0^s a_{ij}(X_{t-u})du \qquad 0 \leq s \leq t.$$

The process X_t in (3) is called a Dirichlet process. Let $b = (b_1, b_2, ..., b_d)$ be a measurable vector field on R^d such that $b_i \in L^d(R^d) + L^\infty(R^d), 1 \leq i \leq d$, where $L^p(R^d), p \in (0, \infty]$ stands for the standard L^p space with respect to Lebesgue measure dx. We note that considering $L^d(R^d) + L^\infty(R^d)$ instead of each space separately widens the range of applicability essentially. For example, it includes functions like $b_i = \frac{x_i}{|x|^{\alpha+1}}, \alpha < 1$. Consider the quadratic form:

$$\mathcal{E}(u,v) := \frac{1}{2}\int_{R^d}\sum_{i,j=1}^d a_{i,j}(x)\frac{\partial u(x)}{\partial x_i}\frac{\partial v(x)}{\partial x_j}dx - \int_{R^d} < b(x), \nabla u(x) > v(x)dx$$

$$D(\mathcal{E}) := H_2^1(R^d) \tag{5}$$

It is proved in [7] that for some $\alpha \in (0, \infty)$, $(\mathcal{E}_\alpha, D(\mathcal{E}))$ is a closed, local ,semi-Dirichlet form. Here as usual $\mathcal{E}_\alpha := \mathcal{E} + \alpha(,)_{L^2}$. Therefore, there is an associated diffusion, which we denote by $\mathcal{M} = \{\Omega, \mathcal{F}, \mathcal{F}_t, X_t, Q_x, x \in R^d\}$. The corresponding semigroup will be denoted by $T_t, t \geq 0$. One can regard $(\mathcal{E}, D(\mathcal{E}))$ as a drift perturbation of $(\mathcal{E}^0, D(\mathcal{E}^0))$. If $b \in L^p(R^d) + L^\infty(R^d)$ for some $p > d$, then b is in the Kato class of $\mathcal{M}^0$. In this case, it is shown in [4] that $\mathcal{M}$ is a Girsanov transform of $\mathcal{M}^0$, and that the Girsanov density is a true exponential martingale since Novikov's condition is fullfiled. In the present situation , b is merely in $L^d(R^d) + L^\infty(R^d)$ and $|b|^2$ is no longer in the Kato class of $\mathcal{M}^0$. The question is whether $\mathcal{M}$ can still be written as a Girsanov transform of $\mathcal{M}^0$. The problem seems to have been open for some time. The aim of this paper is to give a positive answer to the question.

The rest of the paper is organized as follows. In section 2, we prove a Girsanov representation for $\mathcal{M}$ and extend the representation to the case where an adjoint drift and a zero order term are added. Section 3 is devoted to the weak convergence of diffusion processes associated with quadratic forms of type (5).

2. Girsanov Representation.

Let $c(x)$ be a non-negative measurable function on R^d. Define

$$\alpha_0 = \inf\{k \geq 0; \int_{\{c(x)>k\}} c(x)^{\frac{d}{2}}dx \leq \lambda^{-d}(\frac{1}{2\delta})^{\frac{d}{2}}\} \tag{6}$$

where $\lambda := (2^{\frac{2}{3}}(d-1))/((d-2)d^{1/2})$ and δ is the constant specified in (1).

Lemma 2.1. Let $c(x)$ and α_0 be as above. Then

$$\Big(\int_{R^d}(E_x[exp(\int_0^t c(X_s)ds)f(X_t)])^2dx\Big)^{\frac{1}{2}} \le e^{\alpha_0 t}|f|_{L^2(R^d)} \quad \text{for all} \quad f\in L^2(R^d) \tag{7}$$

Proof. We can assume $f \ge 0$. Set $c_n(x) := c(x)\wedge n$. We introduce the quadratic form:

$$Q_n(u,v) = \mathcal{E}^0(u,v) - \int_{R^d} c_n(x)u(x)v(x)dx + \alpha_0\int_{R^d} u(x)v(x)dx$$

$$D(Q_n) := H_2^1(R^d) \tag{8}$$

Using the Sobolev inequality,

$$|u|_{L^{2d/(d-2)}(R^d)} \le \lambda(\int_{R^d}|\nabla u|^2dx)^{1/2} \quad u\in H_2^1(R^d),$$

we find that

$$\begin{aligned}
Q_n(u,u) &= \mathcal{E}^0(u,u) - \int_{c_n\le\alpha_0} c_n(x)u^2(x)dx + \alpha_0\int_{R^d}u^2(x)dx - \int_{c_n>\alpha_0}c_n(x)u^2(x)dx\\
&\ge \mathcal{E}^0(u,u) - (\int_{c_n>\alpha_0}c_n^{d/2}(x)dx)^{2/d}|u|^2_{L^{2d/(d-2)}(R^d)}\\
&\ge \mathcal{E}^0(u,u) - (\int_{c>\alpha_0}c^{d/2}(x)dx)^{2/d}\lambda^2\int_{R^d}|\nabla u|^2dx\\
&\ge \mathcal{E}^0(u,u) - (\int_{c>\alpha_0}c^{d/2}(x)dx)^{2/d}\lambda^2 2\delta\mathcal{E}^0(u,u) \ge 0
\end{aligned}$$

by the choice of α_0. By the boundeness of c_n, it is easy to see that $(Q_n, D(Q_n))$ is also a closed form on $L^2(R^d)$. Thus there exists a strongly continuous contraction semigroup on $L^2(R^d)$, denoted by $P_t^n, t \ge 0$, associated with $(Q_n, D(Q_n))$. Moreover, the Feynman-Kac representation holds:

$$P_t^n f(x) = E_x[exp(\int_0^t c_n(X_s)ds - \alpha_0 t)f(X_t)],$$

where E_x denotes expectation with respect to P_x. Hence,

$$\int_{R^d}(E_x[exp(\int_0^t c_n(X_s)ds - \alpha_0 t)f(X_t)])^2dx \le |f|^2_{L^2(R^d)} \tag{9}$$

Letting $n\to\infty$ in (9), the assertion follows by Fatou's lemma.

Since $(a^{-1}b)^T \in L^2_{loc}(R^d, dx)$, by [1] $\int_0^t (a^{-1}b)^T(X_s)dM_s$ defines a continuous local-martingale additive functional. Hence

$$Z_t := exp\{\int_0^t (a^{-1}b)^T(X_s)dM_s - \frac{1}{2}\int_0^t ba^{-1}b^T(X_s)ds\} \tag{10}$$

is a supermartingale multiplicative functional. For $x \in R^d$, define

$$d\hat{Q}_x|_{\mathcal{F}_t \cap \{t<\xi\}} = Z_t dP_x \tag{11}$$

where ξ stands for the life time. Then by a result of Kunita [2], $\hat{\mathcal{M}} := \{\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \hat{Q}_x, x \in R^d_\Delta\}$ gives rise to a diffusion process, where Δ denotes the cemetery of the process. We want to show that $\mathcal{M}$ and $\hat{\mathcal{M}}$ are equivalent, i.e. $Q_x = \hat{Q}_x$ for $dx - a.e.\ x \in R^d$. For this end, it suffices to prove that the semigroup $T_t, t \geq 0$ of $\mathcal{M}$ is given by

$$T_t f(x) = E_x[Z_t f(X_t)]$$

This will be a consequence of Theorem 2.3 below. For $f \in \mathcal{B}(R^d)$ with $f = 0$ on $\{\Delta\}$, define

$$\hat{T}_t f(x) := \hat{E}_x[f(X_t)], \tag{12}$$

where $\hat{E}_x$ denotes expectation with respect to $\hat{Q}_x$. First we show that each $\hat{T}_t, t > 0$ extends to a bounded linear operator on $L^2(R^d)$. Let $b \in L^d(R^d \to R^d) + L^\infty(R^d \to R^d)$. Choose a sequence $b_n \in L^\infty(R^d \to R^d), n \geq 1$, of functions such that $\lim_{n\to\infty}(b_n - b) = 0$ in $L^d(R^d \to R^d)$. For $n \geq 1$, define

$$\mathcal{E}^n(u,v) := \frac{1}{2}\int_{R^d} \sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial u(x)}{\partial x_i} \frac{\partial v(x)}{\partial x_j} dx - \int_{R^d} < b_n(x), \nabla u(x) > v(x) dx$$

$$D(\mathcal{E}^n) := H^1_2(R^d) \tag{13}$$

Let $T^n_t, t \geq 0$ be the semigroup associated with $(\mathcal{E}^n, D(\mathcal{E}^n))$. It is known (for example , see [5]) that the Girsanov's formula holds:

$$T^n_t f(x) = E_x[Z^n_t f(X_t)] \tag{14}$$

where

$$Z^n_t := exp\{\int_0^t (a^{-1} b_n)^T (X_s) dM_s - \frac{1}{2}\int_0^t b_n a^{-1} b^T_n (X_s) ds\}$$

Lemma 2.2. For any $t \geq 0$, $\hat{T}_t$ extends to a bounded linear operator on $L^2(R^d)$.

Proof. Define for $n \geq 1$,

$$\alpha_n = \inf\{k \geq 0; \int_{\{|b_n|(x) > \sqrt{k}\}} |b_n|^d(x) dx \leq (2\delta\lambda)^{-d}\}$$

and set $\alpha = \sup_n \alpha_n$. Then $\alpha < \infty$ and

$$|\int_{R^d} < b_n(x), \nabla u(x) > u(x) dx| \leq \frac{1}{4\delta}\int_{R^d} |\nabla u|^2(x) dx + \delta \int_{R^d} |b_n|^2(x) u^2(x) dx$$

$$\leq \frac{1}{4\delta}\int_{R^d} |\nabla u|^2(x) dx + \delta\alpha \int_{R^d} u^2(x) dx$$

$$+\delta\int_{|b_n|>\sqrt{\alpha}}|b_n|^2(x)u^2(x)dx$$

$$\leq \frac{1}{4\delta}\int_{R^d}|\nabla u|^2(x)dx+\delta\alpha\int_{R^d}u^2(x)dx$$

$$+\delta\Big(\int_{|b_n|>\sqrt{\alpha}}|b_n|^d(x)dx\Big)^{\frac{2}{d}}\lambda^2\int_{R^d}|\nabla u|^2(x)dx$$

$$\leq \frac{1}{2\delta}\int_{R^d}|\nabla u|^2(x)dx+\delta\alpha\int_{R^d}u^2(x)dx$$

$$\leq \mathcal{E}^0(u,u)+\delta\alpha\int_{R^d}u^2(x)dx \tag{15}$$

Hence, $\mathcal{E}^n_{\delta\alpha}(u,u)=\mathcal{E}^n(u,u)+\delta\alpha(u,u)\geq 0$. Thus it follows that

$$\int_{R^d}(T^n_t f(x))^2dx\leq exp(2\delta\alpha t)\int_{R^d}f^2(x)dx \tag{16}$$

Since $b_n-b\to 0$ in $L^d(R^d\to R^d)$, it follows from (16) and Fatou's Lemma that

$$\int_{R^d}(\hat{T}_t f(x))^2dx\leq exp(2\delta\alpha t)\int_{R^d}f^2(x)dx \tag{17}$$

which completes the proof.

Next we prove a general Girsanov representation result for the semigroup associated with a quadratic form, which also contains an adjoint drift and a zero order term. Vector field b is the same as before. Let $d(x)$ be an another vector field such that $d\in L^d(R^d\to R^d)+L^\infty(R^d\to R^d)$ and $c(x)$ be a measurable real-valued function on R^d with $c\in L^{\frac{d}{2}}(R^d)+L^\infty(R^d)$. Consider the bilinear form

$$\mathcal{Q}(u,v):=\frac{1}{2}\int_{R^d}\sum_{i,j=1}^d a_{ij}(x)\frac{\partial u(x)}{\partial x_i}\frac{\partial v(x)}{\partial x_j}dx-\int_{R^d}<b(x),\nabla u(x)>v(x)dx$$

$$-\int_{R^d}<d(x),\nabla v(x)>u(x)dx-\int_{R^d}c(x)u(x)v(x)dx$$

$$D(\mathcal{Q}):=H^1_2(R^d) \tag{18}$$

It was shown in [7] that $(\mathcal{Q},D(\mathcal{Q}))$ is a closed ,lower bounded sectorial bilinear form on $L^2(R^d)$. Let us denote by $Q_t, t\geq 0$, the associated strongly continuous semigroup .

Theorem 2.3. Let $f\in L^2(R^d)$. Then for $dx-a.e.x\in R^d$,

$$Q_tf(x)$$

$$=E_x[f(X_t)exp\{\int_0^t(a^{-1}b)^T(X_s)dM_s-\int_0^t(a^{-1}d)^T(X_{t-s})d(M_s\circ\gamma_t)$$

$$-\frac{1}{2}\int_0^t (b-d)a^{-1}(b-d)^T(X_s)ds + \int_0^t c(X_s)ds\}] \tag{19}$$

Remark. Here $\int_0^t (a^{-1}d)^T(X_{t-s})d(M_s \circ \gamma_t)$ denotes the stochastic integral with respect to the backward martingale under $P_{dx} := \int_{R^d} P_x dx$, which appears in the well known Lyons-Zheng decomposition (see [4].

Proof of Theorem 2.3. Choose $b_n, d_n \in L^\infty(R^d \to R^d)$ and $c_n \in L^\infty(R^d \to R)$ such that both $b_n - b \to 0, d_n - d \to 0$ in $L^d(R^d)$ and $c_n - c \to 0$ in $L^{\frac{d}{2}}(R^d)$. Let a quadratic form $(\mathcal{Q}^n, D(\mathcal{Q}^n))$ be defined as in (18) with b_n, d_n, c_n in place of b, d, c. The corresponding semigroup is denoted by $Q_t^n, t \geq 0$. It was shown in [8] that $Q_t^n f(x) \to Q_t f(x)$ in $L^2(R^d)$ for any $f \in L^2(R^d)$. Let now $f, g \in L^2(R^d) \cap L^\infty(R^d)$. Then, it is shown in [5] that,

$$\int_{R^d} Q_t^n f(x) g(x) dx$$

$$= \int_{R^d} dx E_x[f(X_t)g(X_0)exp\{\int_0^t (a^{-1}b_n)^T(X_s)dM_s - \int_0^t (a^{-1}d_n)^T(X_{t-s})d(M_s \circ \gamma_t)$$

$$-\frac{1}{2}\int_0^t (b_n-d_n)a^{-1}(b_n-d_n)^T(X_s)ds + \int_0^t c_n(X_s)ds\}] \tag{20}$$

Thus it reduces to show

$$\int_{R^d} dx E_x[f(X_t)g(X_0)exp\{\int_0^t (a^{-1}b_n)^T(X_s)dM_s - \int_0^t (a^{-1}d_n)^T(X_{t-s})d(M_s \circ \gamma_t)$$

$$-\frac{1}{2}\int_0^t (b_n-d_n)a^{-1}(b_n-d_n)^T(X_s)ds + \int_0^t c_n(X_s)ds\}]$$

converges to

$$\int_{R^d} dx E_x[f(X_t)g(X_0)exp\{\int_0^t (a^{-1}b)^T(X_s)dM_s - \int_0^t (a^{-1}d)^T(X_{t-s})d(M_s \circ \gamma_t)$$

$$-\frac{1}{2}\int_0^t (b-d)a^{-1}(b-d)^T(X_s)ds + \int_0^t c(X_s)ds\}] \tag{21}$$

for $f, g \in C_0^+(R^d)$. Define a measure m on $(\Omega, \mathcal{F})$ by

$$m(A) := \int_{R^d} dx E_x[f(X_t)g(X_0)\chi_A], A \in \mathcal{F}.$$

Denote by E_m the integral with respect to m. By the choice of b_n, d_n, c_n, we see that

$$\hat{Z}_t^n := exp\{\int_0^t (a^{-1}b_n)^T(X_s)dM_s - \int_0^t (a^{-1}d_n)^T(X_{t-s})d(M_s \circ \gamma_t)$$

$$-\frac{1}{2}\int_0^t (b_n-d_n)a^{-1}(b_n-d_n)^T(X_s)ds + \int_0^t c_n(X_s)ds\}$$

converges in measure m to

$$exp\{\int_0^t (a^{-1}b)^T(X_s)dM_s - \int_0^t (a^{-1}d)^T(X_{t-s})d(M_s \circ \gamma_t)$$

$$-\frac{1}{2}\int_0^t (b-d)a^{-1}(b-d)^T(X_s)ds + \int_0^t c(X_s)ds\}.$$

So, all we need to show is that

$$\sup_n E_m[(\hat{Z}_t^n)^2] < \infty. \tag{22}$$

By Hölder's inequality,

$$E_m[(\hat{Z}_t^n)^2] \leq (A_t^n \times B_t^n \times C_t^n)^{\frac{1}{3}}$$

where

$$A_t^n := E_m[exp\{6\int_0^t (a^{-1}b_n)^T(X_s)dM_s - 18\int_0^t b_n a^{-1} b_n^T(X_s)ds\}]$$

$$B_t^n := E_m[exp\{-6\int_0^t (a^{-1}d_n)^T(X_{t-s})d(M_s\circ\gamma_t) - 18\int_0^t d_n a^{-1} d_n^T(X_s)ds\}]$$

$$C_t^n := E_m[exp\{18\int_0^t d_n a^{-1} d_n^T(X_s)ds + 18\int_0^t b_n a^{-1} b_n^T(X_s)ds -$$
$$3\int_0^t (b_n-d_n)a^{-1}(b_n-d_n)^T(X_s)ds + 6\int_0^t c_n(X_s)ds\}].$$

By the supermartingale property of

$$exp\{6\int_0^t (a^{-1}b_n)^T(X_s)dM_s - 18\int_0^t b_n a^{-1} b_n^T(X_s)ds\}$$

it is seen that A_t^n is bounded by $||f||_{L^\infty(R^d)}||g||_{L^\infty(R^d)}$. Interchanging the role of f and g, and using the reversibility of the process X_t with respect to $\int_{R^d} P_x dx$, we see that the same is also true for B_t^n. By Lemma 2.1,

$$C_t^n \leq ||g||_{L^\infty(R^d)}(\int_{R^d}(E_x[|f|(X_t)exp\{18\int_0^t d_n a^{-1} d_n^T(X_s)ds + 18\int_0^t b_n a^{-1} b_n^T(X_s)ds -$$
$$3\int_0^t (b_n-d_n)a^{-1}(b_n-d_n)^T(X_s)ds + 6\int_0^t c_n(X_s)ds\}])^2 dx)^{\frac{1}{2}}$$
$$\leq ||g||_{L^\infty(R^d)} e^{\beta t}|f|_{L^2(R^d)} \tag{23}$$

where β can be chosen independently of n. Thus (22) is proven, hence the Theorem.

Corollary 2.4. $T_t f(x) = E_x[Z_t f(X_t)]$ and hence $\mathcal{M}$ is equivalent to $\hat{\mathcal{M}}$.

3. Convergence of Diffusions.

As before, we consider the diffusion process $\mathcal{M} = \{\Omega, \mathcal{F}, \mathcal{F}_t, X_t, Q_x, x \in R^d\}$ associated with the quadratic form.

$$\mathcal{E}(u,v) = \frac{1}{2}\int_{R^d}\sum_{i,j=1}^d a_{i,j}(x)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}dx - \int_{R^d} < b(x), \nabla u(x) > v(x)dx$$

$$D(\mathcal{E}) = H^1_2(R^d) \tag{24}$$

In addition to the assumption $b \in L^d(R^d \to R^d) + L^\infty(R^d \to R^d)$, we also impose the following condition

(3.1). $\int_{R^d} < b(x), \nabla u(x) > dx \leq 0$ for all non-negative $u \in C_0^\infty(R^d)$.

Remark. It is easy to see that (3.1) always implies that we even have $\int_{R^d} < b(x), \nabla u(x) > dx = 0$ for $u \in H^1_2(R^d)$.

In this section, we will study the convergence of the diffusion processes when the corresponding coefficients converge.

Proposition 3.1. Assume (3.1). Then the associated diffusion $\mathcal{M} = \{\Omega, \mathcal{F}, \mathcal{F}_t, X_t, Q_x, x \in R^d\}$ is conservative, i.e., $Q_x(\xi = \infty) = 1$ for $dx - a.e. x \in R^d$, where ξ stands for the life time of $\mathcal{M}$.

Proof. Recall that $T_t, t \geq 0$, stands for the semigroup of $\mathcal{M}$. Under assumption (3.1), the adjoint semigroup T_t^* of T_t is also Markovian, which implies that T_t is a contraction semigroup on $L^1(R^d)$. Hence,

$$\int_{R^d} dx Q_x[(\int_0^t |b|^2(X_s)ds)^{\frac{d}{2}}] \leq C_t \int_0^t ds \int_{R^d} dx T_s |b|^d(x) dx$$

$$\leq C_t t \int_{R^d} dx |b|^d(x) < \infty.$$

In particular, $\int_0^t |b|^2(X_s)ds < \infty$ $Q_x - a.e$ for dx almost all $x \in R^d$. Define $\tau_n = \inf\{t > 0, \int_0^t |b|^2(X_s)ds > n\}$. Then $\lim_{n\to\infty} \tau_n = \infty$, Q_x-a.e., for almost all $x \in R^d$. For $t > 0$, we have

$$Q_x(\tau_n \wedge t < \xi) = E_x[Z_{\tau_n \wedge t}] = 1.$$

Letting $n, t \to \infty$, we obtain $Q_x(\xi = \infty) = 1$ which completes the proof.

In general, the diffusion $\mathcal{M} = \{\Omega, \mathcal{F}, \mathcal{F}_t, X_t, Q_x, x \in R^d\}$ is not a semimartingale since $a_{ij}(x)$ are merely measurable. One consequence of our next result is that $\mathcal{M} = \{\Omega, \mathcal{F}, \mathcal{F}_t, X_t, Q_x, x \in R^d\}$ can be approximated weakly by semimartingales. Let $(a^n_{ij}(x))_{1\leq i,j\leq d}$ be a sequence of matrix-valued functions such that $a^n_{ij}(x) = a^n_{ji}(x)$ and

$$\frac{1}{\delta}\sum_{i=1}^d y_i^2 \leq \sum_{i,j=1}^d a^n_{ij}(x) y_i y_j \leq \delta \sum_{i=1}^d y_i^2 \tag{25}$$

for all $y_1, y_2, ..., y_d \in R$, where δ is independent of n.
Let b_n be a sequence of vector fields that satisfy (3.1). For example, $b_n = \phi_n * b$ for some $\phi_n \in C_0^\infty(R^d)$. Define quadratic forms $(\mathcal{E}^{0,n}, D(\mathcal{E}^{0,n}))$ and $(\mathcal{E}^n, D(\mathcal{E}^n))$ as in (2) and (5) with $a_{ij}(x), b(x)$ replaced by $a^n_{ij}(x)$ and b_n. Let $\mathcal{M}^{0,n} := \{\Omega, \mathcal{F}, \mathcal{F}_t, X_t, P^n_x, x \in R^d\}$ and $\mathcal{M}^n := \{\Omega, \mathcal{F}, \mathcal{F}_t, X_t, Q^n_x, x \in R^d\}$

denote respectively the diffusion processes associated with $(\mathcal{E}^{0,n}, D(\mathcal{E}^{0,n}))$ and $(\mathcal{E}^n, D(\mathcal{E}^n))$. These processes are semimartingales if the coefficients are smooth. Denote by M_t^n the martingale part of $\mathcal{M}^{0,n}$ in (3). Take $h \in L^1(R^d) \cap L^\infty(R^d)$ with $h(x) > 0$ dx-a.e. and $\int_{R^d} h(x)dx = 1$. Define probability measures on the path space $C([0,1] \to R^d)$ by

$$P_h^n(\cdot) = \int_{R^d} h(x)P_x^n(\cdot)dx \qquad Q_h^n(\cdot) = \int_{R^d} h(x)Q_x^n(\cdot)dx \tag{26}$$

Theorem 3.2. Assume $a_{ij}^n(x) \to a_{ij}(x)$ dx-almost everywhere and $b_n \to b$ weakly* in $L^d(R^d)$. Then Q_h^n converges weakly to Q_h on the path space $C([0,1] \to R^d)$ equipped with the topology of uniform convergence.

Proof. Let $0 < t_1 < t_2 < t_3 < \cdots < t_m \le 1$ and $f_1, f_2, \cdots f_m \in C_0^\infty(R^d)$. Since $T_t^n \to T_t$ strongly in $L^2(R^d)$, we have that

$$Q_h^n[f_1(X_{t_1})f_2(X_{t_2})\cdots f_m(X_{t_m})] = \int_{R^d} h(x)dxT_{t_1}^n[f_1T_{t_2-t_1}^n[f_2\cdots T_{t_m-t_{m-1}}^n[f_m]\cdots]](x)dx$$

converges to

$$\int_{R^d} h(x)dxT_{t_1}[f_1T_{t_2-t_1}[f_2\cdots T_{t_m-t_{m-1}}[f_m]\cdots]](x)dx = Q_h[f_1(X_{t_1})f_2(X_{t_2})\cdots f_m(X_{t_m})].$$

Thus, it only remains to show that the family $\{Q_h^n, n \ge 1\}$ is tight. By the Girsanov's transform in section 2,

$$dQ_h^n = Z^n dP_h^n \tag{27}$$

where

$$Z^n = exp\{\int_0^1 ((a^n)^{-1}b_n)^T(X_s)dM_s^n - \frac{1}{2}\int_0^1 b_n(a^n)^{-1}b_n^T(X_s)ds\}$$

Observe that

$$P_h^n[Z^n log Z^n]$$

$$= P_h^n[Z^n(\int_0^1 ((a^n)^{-1}b_n)^T(X_s)dM_s^n - \frac{1}{2}\int_0^1 b_n(a^n)^{-1}b_n^T(X_s)ds)]$$
$$= \frac{1}{2}P_h^n[\int_0^1 Z_s^n b_n(a^n)^{-1}b_n^T(X_s)ds] \le \frac{1}{2}cP_h^n[\int_0^1 Z_s^n|b_n|^2(X_s)ds]$$
$$\le \frac{1}{2}\frac{1}{\delta}\int_0^1 ds(Q_h^n[|b_n|^d(X_s)])^{\frac{2}{d}} \le \frac{1}{2}\frac{1}{\delta}\int_0^1 ds(\int_{R^d} T_s^n(|b_n|^d)(x)dx)^{\frac{2}{d}}$$
$$\le \sup_n\{\frac{1}{2}\frac{1}{\delta}(\int_{R^d}|b_n|^d(x)dx)^{\frac{2}{d}}\} = K < \infty$$

where we have used the fact that $\{T_t^n\}$ is a contraction semigroup on $L^1(R^d)$, due to (3.1). Now we are ready to pove $\{Q_h^n, n \ge 1\}$ is tight. Given $\varepsilon > 0$, choose first $L > 0$ so that $\frac{K}{logL} < \frac{\varepsilon}{2}$. By [3], the family $\{P_h^n, n \geq 1\}$ is tight. Therefore, there exists a compact subset $F \subset C([0,1] \to R^d)$ so that $P_h^n(F^c) < \frac{\varepsilon}{2L}$ for all n. Thus, we have for all $n \geq 1$

$$Q_h^n(F^c) = P_h^n[Z^n \chi_{F^c}] = P_h^n[Z^n \chi_{F^c}, Z^n > L] + P_h^n[Z^n \chi_{F^c}, Z^n \leq L]$$

$$\leq \frac{\sup_n P_h^n[Z^n logZ^n]}{logL} + LP_h^n(\chi_{F^c}) \leq \frac{\varepsilon}{2} + L\frac{\varepsilon}{2L} = \varepsilon$$

which proves the tightness.

References

[1]. M.Fukushima,Oshima and Takeda,Dirichlet Forms and Symmetric Markov Processes, de Gruyter, 1994.

[2]. H.Kunita, Absolute continuity of Markov processes. Seminaire de Probabilites X., Lecture Notes in Mathematics 511 (1976) 44-77.

[3]. T.J.Lyons and T.S.Zhang, Note on convergence of Dirichlet processes, Bull.London Math.Soc. 25 (1993) 353-356.

[4]. T.J.Lyons and W.A.Zheng, Acrossing estimate for the canonical process on a Dirichlet space and tighteness result, Colloque Paul Levy Sur Les processus stochastiques, Asterisque 157-158 (1989) 248-272.

[5]. J.Lunt,T.J.Lyons and T.S.Zhang, Integrability of Functionals of Dirichlet Processes, Probabilistic Representations of Semigroups and Estimates of Heat Kernels, Journal of Functional Analysis 153:2 (1998) 320-342.

[6]. Z.M.Ma and M.Röckner,Introduction to the Theory of (non-symmetric)Dirichlet Forms, Springer, Berlin-New York, 1992.

[7]. M.Röckner and B.Smuland,Quasi-regular Dirichlet forms: examples and counterexamples, Can.J.Math. 47 (1995), 165-200.

[8]. M.Röckner and T.S.Zhang, Convergence of operators semigroups generated by elliptic operators, Osaka J. Math. 34(1997) 923-932.

[9]. W.Stannat, The Theory of Generalized Dirichlet Forms and Its Applications in Analysis and Stochastics, Memoirs of the American Mathematical Society No:678, (1999)