A note on the representation of conditional expectations for non-Gaussian noise

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Abstract

For Wiener spaces conditional expectations and L^2-martingales w.r.t. the natural filtration have a natural representation in terms of chaos expansion. In this note an extension to larger classes of processes is discussed. In particular, it is pointed out that orthogonality of the chaos expansion is not required.

Recently, the martingale property and conditional expectations w.r.t. the natural filtration of Brownian motion for (generalized) processes have been studied by [Hid80], [BP96], [DPV97], and [GKS99] in the context of white noise analysis. For regular processes these characterizations are an immediate consequence of the chaos expansion w.r.t. multiple stochastic integrals. They have turned out to be useful for the study of local times, see [dFDS00] and the study of a generalized Clark-Ocone formula [AØPU00], [dFOS00], and [NS01]. This has motivated us to consider these features for a more general class of processes and more general systems of functions than multiple stochastic integrals.

We shall work throughout with the space $\mathcal{D}'(\mathbb{R})$ of generalized functions as our sample space; recall the Gelfand triple $\mathcal{D}(\mathbb{R}) \subset L^2(\mathbb{R}, dt) \subset \mathcal{D}'(\mathbb{R})$.

One equips $\mathcal{D}'(\mathbb{R})$ with the weak σ-algebra $\mathcal{F}(\mathcal{D}'(\mathbb{R}))$, i.e. the σ-algebra generated by the mappings $\omega \mapsto \langle \omega, \varphi \rangle$ for $\varphi \in \mathcal{D}(\mathbb{R})$. A probability measure P on $(\mathcal{D}'(\mathbb{R}), \mathcal{F}(\mathcal{D}'(\mathbb{R})))$ gives rise to a generalized coordinate process Φ by

$$\begin{array}{rcl} \Phi : \mathcal{D}(\mathbb{R}) \times \mathcal{D}'(\mathbb{R}) & \to & \mathbb{R} \\ (\varphi, \omega) & \mapsto & \langle \omega, \varphi \rangle . \end{array} \tag{1}$$

Example 1 *Let $(\mathcal{D}'(\mathbb{R}), \mathcal{F}(\mathcal{D}'(\mathbb{R})), P)$ be a generalized random process, with independent values at every point, more explicitly we assume that the characteristic function*

$$C_P(\varphi) := \int_{\mathcal{D}'(\mathbb{R})} e^{i\langle \omega, \varphi \rangle} P(d\omega)$$

fulfills the Lévy-Khinchin representation, i.e.

$$\ln\left(C_P(\varphi)\right) = i \int_{\mathbb{R}} \varphi(s)\nu_1(ds) - \frac{1}{2}\int_{\mathbb{R}} \varphi^2(s)\nu_2(ds)$$
$$+ \int_{\mathbb{R}} \int_{|\lambda|>0} \left[e^{i\lambda\varphi(s)} - 1 - i\lambda\varphi(s)\right] \nu_3(d\lambda, ds),$$

where ν_1 is a signed, ν_2 is a non-negative Radon-measures on $\mathbb{R}$, and ν_3, the Lévy-measure, is a non-negative Radon-measure on $(\mathbb{R} \setminus \{0\}) \times \mathbb{R}$ such that for some $\varepsilon > 0$

$$\nu(ds) := \nu_2(ds) + \int_{0<|\lambda|<1} \lambda^2 \nu_3(d\lambda, ds) + \int_{1\leq|\lambda|} e^{\varepsilon|\lambda|}\nu_3(d\lambda, ds) \tag{2}$$

is a Radon-measure. In particular, for functions $\varphi_1, \varphi_2 \in \mathcal{D}(\mathbb{R})$ with $\varphi_1 \cdot \varphi_2 = 0$ one has that $C_P(\varphi_1 \cdot \varphi_2) = C_P(\varphi_1)C_P(\varphi_2)$. Without loss of generality we consider $\nu_1 = 0$.

Example 2 *Let $(\Omega, \mathcal{F}(\Omega), Q)$ be an arbitrary probability space and $(M_t)_{t\in\mathbb{R}^+}$ a càdlàg L^2-martingale on this space. For Q-a.e. $\omega \in \Omega$ one can define a generalized function*

$$\begin{array}{rcl} \mathcal{D}(\mathbb{R}) & \to & \mathbb{R} \\ \varphi & \mapsto & -\displaystyle\int_0^\infty \dot{\varphi}\,(s)\, M_s(\omega)\, ds = \varphi(0)M_0(\omega) + \int_0^\infty \varphi(s)\, dM_s(\omega), \end{array}$$

where dM_s denotes the Itô integral. The image measure of Q on $(\mathcal{D}'(\mathbb{R}), \mathcal{F}(\mathcal{D}'(\mathbb{R})))$ given by this mapping we denote by P. Without loss of generality we assume $M_0 = 0$.

Condition (R) *Assume that there exists a locally convex vector space E such that $\mathcal{D}(\mathbb{R})$ is a dense subspace of E and*

$$\varphi \mapsto \int_{\mathcal{D}'(\mathbb{R})} |\langle \omega, \varphi \rangle|^2 \, P(d\omega)$$

is continuous in E. More specially, we assume that there exists a Radon measure σ such that $\cap_{p\geq 1} L^p(\mathbb{R}, \sigma)$ is a subspace of E.

E does not necessarily fit into the chain $D(\mathbb{R}) \subset L^2(\mathbb{R}, dx) \subset D'(\mathbb{R})$. Because of Condition (R) one can extend (1) in $L^2(\mathcal{D}'(\mathbb{R}), P)$-sense, i.e. for every sequence $(\varphi_n)_{n\in\mathbb{N}}$ in $D(\mathbb{R})$ which converges to $\varphi \in E$ the sequence $(\langle \cdot, \varphi_n \rangle)_{n\in\mathbb{N}}$ converges in $L^2(\mathcal{D}'(\mathbb{R}), P)$ to the same function, which we denote by $\langle \cdot, \varphi \rangle$. Denote by $E_{\mathbb{C}}$ the complexification of E and define for $\varphi_1 + i\varphi_2 \in E_{\mathbb{C}}$ the functional $\langle \cdot, \varphi_1 + i\varphi_2 \rangle := \langle \cdot, \varphi_1 \rangle + i \langle \cdot, \varphi_2 \rangle$. For $M \subset E$ define the σ-algebra $\mathcal{F}_{\mathcal{M}} \subset \mathcal{F}(D'(\mathbb{R}))$ as the σ-algebra generated by the functions $\omega \mapsto \langle \omega, \varphi \rangle$ for $\varphi \in M$. Note that for the E-closure $\overline{\mathcal{M}}$ of M it is $\mathcal{F}_{\overline{\mathcal{M}}} = \mathcal{F}_{\mathcal{M}}$. Denote for any interval $I \subset \mathbb{R}$ by $\mathcal{F}_I$ the σ-algebra generated by the subspace of all bounded measurable functions which are 0 outside of I.

Example 1 *In this case the second moment is just*

$$\begin{aligned}
&\int_{\mathcal{D}'(\mathbb{R})} |\langle \omega, \varphi \rangle|^2 \, P(d\omega) \\
&= \int_{\mathbb{R}} \varphi^2(s) \nu_2(ds) + \int_{\mathbb{R}} \varphi^2(s) \int_{|\lambda|>0} \lambda^2 \, \nu_3(d\lambda, ds) \\
&\leq c \int \varphi^2(s) \nu(ds)
\end{aligned}$$

for a constant $c > 0$. In this case we define E as the projective limit space $\cap_{p\geq 1} L^p(\mathbb{R}, \nu)$. The process $t \mapsto \langle \cdot, 1\!\!1_{[0,t]} \rangle$ has a càdlàg version which is a semi-martingale and a strong Markov process, see e.g. [Pro90].

Example 2 *In this case one may choose a measure σ closely related to the Föllmer-Doleans measure, the quadratic variation $[\cdot, \cdot]$, and the compensator $\langle \cdot, \cdot \rangle$ respectively, see e.g. [WW90] and [Pro90],*

$$\int_{\mathcal{D}'(\mathbb{R})} |\langle \omega, \varphi \rangle|^2 \, P(d\omega) = \mathbb{E}_P \left[\int_0^\infty \varphi^2(s) \, d\,[M]_s \right]$$

$$
\begin{aligned}
&= \mathbb{E}_P\left[\int_0^\infty \varphi^2(s)\, d\langle M\rangle_s\right] \\
&=: \int_{\mathbb{R}} \varphi^2(s)\sigma(ds).
\end{aligned}
$$

For E we consider $L^2(\mathbb{R},\sigma)$. Note that for $0 \le T$, $\left\langle \cdot, 1\!\!1_{(-\infty,T]}\right\rangle$ and M_T have the same distribution and for all $\varphi \in E$ it holds P-a.s. that

$$
\langle \cdot, \varphi\rangle = \left\langle \cdot, \varphi 1\!\!1_{[0,\infty)}\right\rangle .
$$

Different σ-algebras generated by the martingale itself can be expressed in the following way: let $T_2 < T_2$

$$
\mathcal{F}_{\mathcal{D}((T_1,T_2))} = \sigma\left(M_t - M_s \middle|\, t,s \in (T_1,T_2)\right) = \sigma\left(M_t - M_s \middle|\, t,s \in [T_1,T_2)\right).
$$

Furthermore, $\mathcal{F}_{\mathcal{D}((-\infty,T_2))} = \sigma\left(M_t \middle|\, t < T_2\right)$.

Denote by $E_{\mathbb{C}}^{\hat{\otimes} n}$ the n-th symmetric algebraic tensor product of E and by $\mathrm{Exp}_{\mathrm{alg}}(E_{\mathbb{C}})$ the space of all sequences $\varphi := (\varphi_n)_{n\in\mathbb{N}_0}$ with $\varphi_n \in E_{\mathbb{C}}^{\hat{\otimes} n}$ for which only finite many φ_n are unequal to 0.

Condition (C) *There exist linear mappings $P_n : E_{\mathbb{C}}^{\hat{\otimes} n} \to L^2(\mathcal{D}'(\mathbb{R}),P)$ such that the set*

$$
\left\{ P_n(\varphi_n) \middle| n \in \mathbb{N},\, \varphi_n \in E_{\mathbb{C}}^{\hat{\otimes} n} \right\}
$$

is a total subset of $L^2(\mathcal{D}'(\mathbb{R}),P)$.

One can define a linear mapping

$$
\begin{aligned}
I : \mathrm{Exp}_{\mathrm{alg}}(E_{\mathbb{C}}) &\to L^2(\mathcal{D}'(\mathbb{R}),P) \\
(\varphi_n)_{n=1}^\infty &\mapsto \sum_{n=0}^\infty P_n(\varphi_n).
\end{aligned}
$$

using the fact that the sum is actually finite. The image $I\left(\mathrm{Exp}_{\mathrm{alg}}(E_{\mathbb{C}})\right)$ we denote by $\mathcal{P}(\mathcal{D}'(\mathbb{R}))$ and hence we obtain a triple $\mathcal{P}(\mathcal{D}'(\mathbb{R})) \subset L^2(\mathcal{D}'(\mathbb{R}),P) \subset \mathcal{P}'_P(\mathcal{D}'(\mathbb{R}))$. By $E_{\mathbb{C}}^{\hat{\otimes} n\prime}$ we denote the space of all linear forms on $E_{\mathbb{C}}^{\hat{\otimes} n}$, rather than we try to interpret the elements as distributions. This is due to the fact that $E_{\mathbb{C}}$ is not necessarily a subspace of $L^2(\mathbb{R},dx)$. For $\phi_n \in E_{\mathbb{C}}^{\hat{\otimes} n\prime}$ we can construct a distribution $Q_n(\phi_n) \in \mathcal{P}'_P(\mathcal{D}'(\mathbb{R}))$ by the following definition: for all $m \in \mathbb{N}$ and all $\varphi_m \in E_{\mathbb{C}}^{\hat{\otimes} m}$

$$\langle Q_n(\phi_n), P_m(\varphi_m)\rangle_{L^2(\mathcal{D}'(\mathbb{R}),P)} := \delta_{n,m}\, \phi_n(\varphi_n).$$

Therefore, for any distribution $\Phi \in \mathcal{P}'_P(\mathcal{D}'(\mathbb{R}))$ there exists a unique sequence $(\phi_n)_{n\in\mathbb{N}_0}$ with $\phi_n \in E_{\mathbb{C}}^{\hat{\otimes} n\prime}$ such that

$$\Phi = \sum_{n=0}^{\infty} Q_n(\phi_n).$$

Explicitly, ϕ_n is determined by the equation

$$\phi_n(\varphi_n) := \langle \Phi, P_n(\varphi_n)\rangle_{L^2(\mathcal{D}'(\mathbb{R}),P)}\,, \quad \varphi_n \in E_{\mathbb{C}}^{\hat{\otimes} n}.$$

So we can write any $F \in L^2(\mathcal{D}'(\mathbb{R}), P)$ as $F = \sum_{n=0}^{\infty} Q_n(\phi_n)$ for unique $\phi_n \in E^{\hat{\otimes} n\prime}, n \in \mathbb{N}_0$.

Example 1 *A natural class of polynomials for the Lévy-Khinchin processes are given by the so-called generalized Appell-polynomials of non-Gaussian analysis, see [ADKS96], [KSWY98], [KSS97]. According to assumption (2) the Fourier transform of P is holomorphic in a neighborhood of 0. Then one can construct the polynomials via the following generating functional*

$$e_{\alpha,P}(\varphi,\omega) := \frac{e^{i\langle\omega,\alpha(\varphi)\rangle}}{\mathbb{E}_P\left[e^{i\langle\cdot,\alpha(\varphi)\rangle}\right]}, \quad \varphi \in \mathcal{D}(\mathbb{R}), \omega \in \mathcal{D}'(\mathbb{R}),$$

where $\alpha : \mathbb{C} \to \mathbb{C}$ is a function which is holomorphic and invertible around zero with $\alpha(0) = 0$. As $e_{\alpha,P}(\cdot,\omega)$ is also a holomorphic function near 0 the polynomials are defined by the Taylor expansion

$$e_{\alpha,P}(z\varphi,\omega) = \sum_{n=0}^{\infty} \frac{z^n}{n!} P_n(\varphi^{\otimes n})(\omega).$$

Condition (C) holds, because by a direct calculation one sees that $\|P_n(\varphi^{\otimes n})\|_{L^2(\mathcal{D}'(\mathbb{R}),P)}$ can be expressed as a polynomial of the terms: $n, m \in \mathbb{N}$

$$\int_{\mathbb{R}} |\varphi(s)|^2 \nu_2(ds) \quad \text{and} \quad \int_{\mathbb{R}} \overline{\varphi(s)^m} \varphi(s)^n \int_{|\lambda|>0} \lambda^{n+m} \nu_3(d\lambda, ds).$$

Hence this norm is continuous in $E = \cap_{p\geq 1} L^p(\mathbb{R},\nu)$. In certain cases one can choose α in such a way that these polynomials are orthogonal, see for example [KSS97]. In [NS00] the authors construct a complete system of orthogonal polynomials for Lévy-processes using powers of the jump parts. Using the Lévy decomposition (a Lévy process can be written as a mixture of a Brownian motion and Poisson processes) one can construct another chaotic orthogonal decomposition of $L^2(\mathcal{D}'(\mathbb{R}), P)$, for details see [Itô56] and [Der90].

Example 2 *For $\varphi \in \mathcal{D}(\mathbb{R})$ define iteratively the multiple Itô-integrals*

$$\begin{aligned} I_1(\varphi,t) &:= \int_0^t \varphi(s)\, dM_s \\ I_n(\varphi,t) &:= \int_0^t \varphi(s)\, I_{n-1}^-(\varphi,s)\, dM_s, \end{aligned}$$

where $I_{n-1}^-(\varphi,t) := \lim_{s\uparrow t} I_{n-1}(\varphi,s)$. If the compensator $\langle M\rangle$ is deterministic then also I_n is a L^2-martingale and

$$\mathbb{E}_P\left[I_n(\varphi,t) I_m(\psi,t)\right] = \delta_{n,m}\frac{1}{n!}\left(\int_0^t \varphi(s)\psi(s)\sigma(ds)\right)^n, \qquad (3)$$

see [Mey76]. We define $P_n(\varphi^{\otimes n}) := I_n(\varphi,\infty)$. Due to (3) the mapping P_n can also be extended in L^2-sense to $E_{\mathbb{C}}^{\hat{\otimes} n}$. Injectivity of this mapping follows from orthogonality. Condition (C) is called chaos representation property in this context and does not hold automatically, i.e. Lévy processes have a deterministic compensator, however the only Lévy processes which have the chaos representation property are the trivial mixtures of a pure Gaussian and a pure Poissonian process, cf. [Der90].

If for a closed subspace $\mathcal{M} \subset E_{\mathbb{C}}$ the conditional expectation w.r.t. $\mathcal{F}_{\mathcal{M}}$ preserves the polynomials, i.e. for every n there exists a mapping $\pi_{\mathcal{M},n} : E_{\mathbb{C}}^{\hat{\otimes} n} \to \mathcal{M}^{\hat{\otimes} n}$ such that for all $\varphi_n \in E_{\mathbb{C}}^{\hat{\otimes} n}$

$$\mathbb{E}_P\left[P_n(\varphi_n) | \mathcal{F}_{\mathcal{M}}\right] = P_n(\pi_{\mathcal{M},n}(\varphi_n)).$$

then for any function $F \in L^2(\mathcal{D}'(\mathbb{R}), P)$ with $F = \sum_{n=0}^\infty Q_n(\phi_n)$ it is

$$\mathbb{E}_P\left[F | \mathcal{F}_M\right] = \sum_{n=0}^\infty Q_n(\pi^*_{\mathcal{M},n}(\phi_n)).$$

Indeed, because for any $\varphi_n \in E_{\mathbb{C}}^{\hat{\otimes} n}$ it is

$$\begin{aligned}
& \int_{\mathcal{D}'(\mathbb{R})} \mathbb{E}_P \left[F \middle| \mathcal{F}_M \right](\omega)\ P_n(\varphi_n)(\omega)\, P(d\omega) \\
= & \int_{\mathcal{D}'(\mathbb{R})} F(\omega)\ \mathbb{E}_P \left[P_n(\varphi_n) \middle| \mathcal{F}_M \right](\omega)\, P(d\omega) \\
= & \int_{\mathcal{D}'(\mathbb{R})} F(\omega)\ P_n(\pi_{\mathcal{M},n}(\varphi_n))(\omega)\, P(d\omega) \\
= & \ \phi_n\left(\pi_{\mathcal{M},n}(\varphi_n)\right) = \pi^*_{\mathcal{M},n}(\phi_n)\left(\varphi_n\right) \\
= & \left\langle \sum_{n=0}^{\infty} Q_n(\pi^*_{\mathcal{M},n}(\phi_n)),\ P_n(\varphi_n) \right\rangle_{L^2(\mathcal{D}'(\mathbb{R}),P)} .
\end{aligned}$$

Obviously, also $\left(\pi_{\mathcal{M},n}\right)^2 = \pi_{\mathcal{M},n}$.

Example 1 *Let $I \subset \mathbb{R}$ be an interval. For $\mathcal{M} := \cap_{p\geq 1} L^p(I,\nu)$ we want to compute the conditional expectation w.r.t. $\mathcal{F}_I = \mathcal{F}_{\mathcal{M}}$. First, we observe that we can write any $\varphi \in E$ in the form $\varphi = \varphi_1 + \varphi_2$ with $\varphi_1 := \varphi 1\!\!1_I \in \mathcal{M}$. According to the infinite divisibility of P*

$$\mathbb{E}_P\left[e^{i\langle \cdot, \alpha(\varphi)\rangle}\right] = \mathbb{E}_P\left[e^{i\langle \cdot, \alpha(\varphi_1)\rangle}\right] \mathbb{E}_P\left[e^{i\langle \cdot, \alpha(\varphi_2)\rangle}\right]$$

and therefore, $e_{\alpha,P}(\varphi,\omega) = e_{\alpha,P}(\varphi_1,\omega) e_{\alpha,P}(\varphi_2,\omega)$. Since $\left\{ e^{i\langle \cdot, \psi\rangle} \middle| \psi \in \mathcal{M} \right\}$ generates the σ-algebra $\mathcal{F}_I$ one obtains for any $F = e^{i\langle \cdot, \psi\rangle}$

$$\begin{aligned}
& \sum_{n=0}^{\infty} \frac{z^n}{n!} \mathbb{E}\left[F\ \mathbb{E}_P \left[\left\langle P_n(\cdot), \varphi^{\otimes n} \right\rangle \middle| \mathcal{F}_I \right]\right] \\
= & \ \mathbb{E}\left[e^{i\langle \cdot, \psi\rangle}\ e_{\alpha,P}(\varphi,\omega) \right] = \mathbb{E}\left[e^{i\langle \cdot, \psi\rangle}\ e_{\alpha,P}(\varphi_1,\omega) e_{\alpha,P}(\varphi_2,\omega) \right] \\
= & \ \mathbb{E}\left[F\ e_{\alpha,P}(\varphi_1,\omega) \right] \mathbb{E}\left[e_{\alpha,P}(\varphi_2,\omega) \right] = \mathbb{E}\left[F\ e_{\alpha,P}(\varphi_1,\omega) \right].
\end{aligned}$$

Thus

$$\mathbb{E}_P \left[\left\langle P_n(\cdot), \varphi^{\otimes n} \right\rangle \middle| \mathcal{F}_I \right] = \left\langle P_n(\cdot), (\varphi 1\!\!1_I)^{\otimes n} \right\rangle .$$

Due to linearity of P_n we obtain for all $\varphi_n \in E_{\mathbb{C}}^{\hat{\otimes} n}$ that

$$\mathbb{E}_P \left[\left\langle P_n(\cdot), \varphi_n \right\rangle \middle| \mathcal{F}_I \right] = \left\langle P_n(\cdot), \varphi_n 1\!\!1_I^{\otimes n} \right\rangle$$

and for any $F \in L^2(\mathcal{D}'(\mathbb{R}), P)$ of the form $F = \sum_{n=0}^{\infty} Q_n(\phi_n)$ one can write

$$\mathbb{E}_P \left[F \middle| \mathcal{F}_I \right] = \sum_{n=0}^{\infty} Q_n\left(\phi_n(1\!\!1_I^{\otimes n} \cdot) \right).$$

Example 2 *Let $T > 0$ and denote by $\mathcal{F}_{[0,T]} := \mathcal{F}_{L^2([0,T],\sigma)}$. For any $\varphi \in L^2(\mathbb{R},\sigma)$ it is*

$$\begin{aligned}
\mathbb{E}_P\left[P_n(\varphi^{\otimes n})\middle|\,\mathcal{F}_{[0,T]}\right] &= \mathbb{E}_P\left[\int_0^\infty \varphi(s)\, I_{n-1}^-(\varphi,s)\,dM_s\middle|\,\mathcal{F}_{[0,T]}\right] \\
&= \int_0^T \varphi(s)\, I_{n-1}^-(\varphi,s)\,dM_s \\
&= P_n\left((\varphi \mathbb{1}_{[0,T]})^{\otimes n}\right).
\end{aligned}$$

Hence $\pi_{[0,T],n}(\varphi_n) = \varphi_n \mathbb{1}_{[0,T]}^{\otimes n}$. Thus for any $F \in L^2(\mathcal{D}'(\mathbb{R}),P)$ of the form $F = \sum_{n=0}^\infty Q_n(\phi_n)$ one can write

$$\mathbb{E}_P\left[F\middle|\,\mathcal{F}_{[0,T]}\right] = \sum_{n=0}^\infty Q_n\left(\phi_n(\mathbb{1}_{[0,T]}^{\otimes n}\cdot)\right).$$

Thus in both examples $\pi_{[0,T],n}$ is the multiplication by $\mathbb{1}_{[0,T]}^{\otimes n}$. This allows us to characterize martingales:

Proposition 1 *Let $(\mathcal{D}'(\mathbb{R}),\mathcal{F}(\mathcal{D}'(\mathbb{R})),P)$ be a probability space fulfilling condition (R) and (C). Consider the filtration $\mathcal{F}_{[0,T]}$, $T > 0$. Assume that for every $T > 0$ and $\varphi_n \in E_{\mathbb{C}}^{\hat{\otimes} n}$ it is $\mathbb{1}_{[0,T]}E \subset E$ and*

$$\mathbb{E}_P\left[P_n(\varphi_n)\middle|\,\mathcal{F}_{[0,T]}\right] = P_n(\varphi_n \mathbb{1}_{[0,T]}^{\otimes n}).$$

Let $F : \mathcal{D}'(\mathbb{R}) \times [0,\infty) \to \mathbb{R}$ be a $(\mathcal{F}_t)_{t\geq 0}$-adapted L^2-process. Denote by $f_n(t,\cdot) \in \left(E_{\mathbb{C}}^{\otimes n}\right)'$ the kernels of $F(t,\cdot)$, i.e., $F(t,\cdot) = \sum_{n=0}^\infty Q_n(f_n(t,\cdot))$. Then $(F(t,\cdot))_{t\geq 0}$ is a martingale iff for all $s \leq t$ one has $f_n(s,\cdot) = f_n(t,\mathbb{1}_{[0,s]}^{\otimes n}\cdot)$. If F is closed by a L^2-random variable $F(\infty,\cdot) = \sum_{n=0}^\infty Q_n(f_n(\infty,\cdot))$ then $f_n(s,\cdot) = f_n(\infty,\mathbb{1}_{[0,s]}^{\otimes n}\cdot)$.

Remark 2 *Example 1 and Example 2 provide probability spaces fulfilling the assumption of Proposition 1.*

Proof: If $F(t,\cdot)$ is a martingale then by definition for any $s \leq t$ is holds

$$\begin{aligned}
\sum_{n=0}^\infty Q_n(f_n(s,\cdot)) &= F(s,\cdot) = \mathbb{E}_P\left[F(t,\cdot)\,|\mathcal{F}_s\right] \\
&= \sum_{n=0}^\infty Q_n(f_n(t,\mathbb{1}_{[0,s]}^{\otimes n}\cdot)).
\end{aligned}$$

Due to Condition (C), $f_n(s,\cdot) = f_n(t, 1\!\!1_{[0,s]}^{\otimes n}\cdot)$. The converse follows by the same calculation. ■

In order to have a richer analytical structure on the space of distributions, larger spaces of test functions, equipped with weaker topologies, have to be considered, see [BP96] and [GKS99]. The authors used that the multiplication w.r.t. $1\!\!1_{[0,s]}^{\otimes n}$ is a projection also for the scalar-products generating these topologies.

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