Uniqueness of Diffusion Generators for Two Types of Particle Systems with Singular Interactions

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Abstract

For two types of stochastic particle systems in $\mathbb{R}^d$ we show non-explosion in finite time by proving that their respective generators are $L^1(\mu)$-unique, where μ is their respective invariant (in these cases even symmetrizing) measure. We also prove the much harder $L^2(\mu)$-uniqueness in both models.

1 Introduction

The study of symmetric distorted Brownian motion $(X_t)_{t\geq 0}$ on $\mathbb{R}^d$ with singular drift, i.e. $(X_t)_{t\geq 0}$ is the (weak) solution to the stochastic equation

$$dX_t = \sqrt{2}\, dW_t + \frac{\nabla\rho}{\rho}(X_t)\, dt\ ,\ X_0 = x\ (\in \mathbb{R}^d), \tag{1.1}$$

with $(W_t)_{t\geq 0}$ = Brownian motion on $\mathbb{R}^d$ and ρ = Lebesgue density of the symmetrizing measure μ, started in the late of seventies (see, [2], [1]). In

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recent years, the interest in equations of type (1.1) has risen again, since generalizations of distorted Brownian motion to infinite dimensional manifolds, so called "configuration spaces", have been constructed (see e.g. [3, 4], [20]). New results for the finite dimensional case have recently been obtained in [6] where weak solutions for (1.1) starting for any given point in $\{\rho > 0\}$ have been constructed and strong Feller properties of their transition semigroups have been proved under weak assumptions on ρ which still allow the drift $\beta := \frac{\nabla \rho}{\rho}$ in (1.1) to be very singular. We shall summarize these results in Section 2 below. Uniqueness of weak solutions to (1.1) is related to the conservativity of the Dirichlet form corresponding to (1.1) (cf. Theorem 2.5 and Remark 3.3 (ii) below) or equivalently to the so - called L^1 - uniqueness of the underlying diffusion generator, i.e. $H = -\Delta - \beta \cdot \nabla$, on $L^1(\mathbb{R}^d, \rho(x)dx)$ (see [21] for the most general result on this equivalence). The main results of this paper are on L^1- and also L^2- uniqueness of H (cf. Section 3 below for the precise definitions).

We restrict ourselves to considerating two classes of models from mathematical physics where singular drifts β appear naturally (see, [5, 6]).

The first model is connected with a particle performing a random motion in Euclidean space $\mathbb{R}^d, d \geq 2$, interacting with randomly distributed impurities. This model can be formalized as follows. The impurities form a locally finite subset (i.e. configuration) $\gamma = \{x_k \,|\, k \in \mathbb{N}\} \subset \mathbb{R}^d$ and the interaction between the moving particle and particles from γ is given by a pair potential $V : \mathbb{R}^d \smallsetminus \{0\} \to \mathbb{R}$. (It is assumed that potential V is singular at zero.) The configurations γ are distributed according to a given random point process on $\mathbb{R}^d$. In mathematical physics this random point process usually corresponds to a Gibbs measure ν on the configuration space over $\mathbb{R}^d$. The stochastic dynamics of the considered particle is described by the following SDE:

$$d\xi\left(t\right) = -\sum_{k=1}^{\infty} \nabla V\left(\xi\left(t\right) - x_k\right) dt + \sqrt{2} dw\left(t\right), \tag{1.2}$$

$$\xi\left(0\right) = x \in \mathbb{R}^d \setminus \gamma,$$

where w is the standard Wiener process in $\mathbb{R}^d$. This equation describes a diffusion process with a random drift of a special type. For a review on the stochastic dynamics in random velocity fields see, e.g., [19]. Essential difficulties in the study of the solution to (1.2) originate from the singularity of the potential V induced into the drift term in (1.2) through the configuration γ.

The second model is given by a system of N particles in Euclidean space $\mathbb{R}^d, d \geq 2$, which have positions $x_k \in \mathbb{R}^d, 1 \leq k \leq N$, interacting via a singular pair potential V. In this case the stochastic motion of the particles is described by the following system of stochastic differential equations (SDE)

$$\begin{aligned} dx_k(t) &= -\sum_{j=1, j\neq k}^{N} \nabla V(x_k(t) - x_j(t))\, dt + \sqrt{2} dw_k(t) \qquad (1.3) \\ x_k(0) &= x_k \in \mathbb{R}^d, 1 \leq k \leq N, \end{aligned}$$

where $\{x_k, 1 \leq k \leq N\}$ are different points in $\mathbb{R}^d$ and $\{w_k\ , 1 \leq k \leq N\}$ are independent standard Wiener processes in $\mathbb{R}^d$.

The uniqueness problem for singular diffusion generators was extensively studied in recent years (see, e.g. [17, 10, 18, 11, 12, 21] and the references therein). In this paper we particularly use results from [18] and [21] to prove L^1-uniqueness. To prove L^2-uniqueness is more difficult in our situation. Consider for example the first case above (i.e. diffusions in random media) and the corresponding diffusion generator. In this case the density ρ has zeroes in all points of the configuration γ and, moreover, the corresponding logarithmic derivative β does not satisfy suitable global bounds. Therefore, we can not directly apply the known results of [17, 10, 18, 12]. We recall that V.Liskevich and Yu.Semenov [17] assumed that β satisfies a global integrability condition ($\beta \in L^4(\mathbb{R}^d, \rho(x)dx)$). A.Eberle [11, 12] replaced the global by a local integrability condition plus some growth condition which is not satisfied in our situation. V.I.Bogachev, N.Krylov, M.Röckner [10] do not impose any global conditions on β but they assumed that ρ is locally bounded and locally uniformly positive. V.Liskevich [18] imposed some additional local assumptions on β in the form of a weighted Hardy-type inequality outside a ball in $\mathbb{R}^d$. Unfortunately, it is not quite clear how to check this condition in the situation when ρ has zeroes. Note that it is still an open problem whether L^2-uniqueness (essential self-adjointness) holds under the assumption $\beta \in L^4_{loc}(\mathbb{R}^d, \rho(x)dx)$ only. In our special case we show L^2-uniqueness by applying the hyperbolic approximation criterium of Yu.M.Berezansky [8] together with results of [18]. More precisely, we use only a local version of [18] when β is a compactly supported function.

2 Existence of strong Feller (weak) solutions

In this section we recall the main results from [6]. We start with the main conditions on the functions $\rho : \mathbb{R}^d \to \mathbb{R}_+$.

(H1) $\sqrt{\rho} \in W^{1,2}_{loc}(\mathbb{R}^d, dx),\ \rho > 0,$ dx - a.e.

(H2) $\frac{|\nabla \rho|}{\rho} = 2\frac{|\nabla \sqrt{\rho}|}{\sqrt{\rho}} \in L^{d+\epsilon}_{loc}(\mathbb{R}^d, \mu),\ \rho > 0,$ for some $\epsilon > 0$.

Here dx denotes Lebesgue measure on $\mathbb{R}^d$, $W^{s,q}_{(loc)}(\mathbb{R}^d, dx)$, $s > 0$, $q \geq 1$ the classical (local) Sobolev space of order s in $L^q_{(loc)}(\mathbb{R}^d, dx)$, and $\mu := \rho\, dx$. $L^q_{(loc)}(\mu) = L^q_{(loc)}(\mathbb{R}^d, \mu)$, $q > 0$, denote the corresponding real (local) L^p - spaces. Corresponding norms are denoted by $\|\cdot\|_{L^q(\mathbb{R}^d,\mu)}$, $\|\cdot\|_{W^{s,q}(\mathbb{R}^d,dx)}$ etc. We denote the set of bounded real Borel functions on $\mathbb{R}^d$ by $\mathcal{B}_b(\mathbb{R}^d)$.

(H1) alone already implies that the symmetric positive definite bilinear form

$$\mathcal{E}(u, v) := \int_{\mathbb{R}^d} \langle \nabla u, \nabla v \rangle \, d\mu, \qquad u, v \in C_0^\infty(\mathbb{R}^d) \tag{2.1}$$

is closable in $L^2(\mathbb{R}^d, \mu)$ and that its closure $(\mathcal{E}, D(\mathcal{E}))$ is a regular local symmetric Dirichlet form (cf. [13, 14]). We note that (H2) implies that ρ is continuous (or more precisely has a Hölder-continuous dx-version , cf. [6, Corollary 2.2]). So, the set $\{\rho > 0\}$, which we shall identify as the set of allowed starting points, is open. The main results of [6] are then the following:

Theorem 2.1. *Suppose that* (H1) *and* (H2) *with* $p := d + \epsilon$ *hold. Then there exists a diffusion process* $\mathbb{M} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in\{\rho>0\}})$ *(i.e. a strong Markov process with continuous paths) with state space* $\{\rho > 0\}$ *and cemetery* $\Delta :=$ *Alexandrov point of* $\mathbb{R}^d$*, whose transition semigroup* $(P_t)_{t>0}$ *is* $L^r(\mu)$*-strong Feller (i.e.* $P_t L^r(\mu) \subset C(\{\rho > 0\})$*),* $r \in [p, \infty)$*, and which solves* (1.1) *in the (weak) sense for all initial conditions* $x \in \{\rho > 0\}$*. If* $(\mathcal{E}, D(\mathcal{E}))$ *is, in addition, conservative, then so is* $\mathbb{M}$*. Furthermore,* $(P_t)_{t>0}$ *is strong Feller in this case (i.e.* $P_t(\mathcal{B}_b(\mathbb{R}^d)) \subset C_b(\{\rho > 0\})$ *for all* $t > 0$*).*

Remark 2.2. The notion of weak solution is equivalent to solution of the corresponding martingale problem. More precisely, for $Hu := -\Delta u - \langle \beta, \nabla u \rangle$, $u \in C_0^\infty(\mathbb{R}^d)$, and for every $x \in \{\rho > 0\}$, $\mathbb{P}_x$ from Theorem 2.1 solves the martingale problem for $(H, C_0^\infty(\{\rho > 0\}))$ with initial condition x, i.e. under

$\mathbb{P}_x$ for all $u \in C_0^\infty(\{\rho > 0\})$

$$u(X_t) - u(x) + \int_0^t Hu(X_s)ds,\ t \geq 0, \tag{2.2}$$

is an $(\mathcal{F}_t)_{t\geq 0}$-martingale starting at zero.

Also a uniqueness result was proved in [6]. For its formulation we need the following

Definition 2.3. *A diffusion process* $\mathbb{M}' = (\Omega', \mathcal{F}', (\mathcal{F}'_t)_{t\geq 0}, (X'_t)_{t\geq 0}, (\mathbb{P}'_x)_{x\in\{\rho>0\}})$ *on* $\{\rho > 0\}$ *with lifetime* ζ', *cemetry* Δ, *and semigroup* $(P'_t)_{t>0}$ *is said to satisfy the* $L^1(\{\rho > 0\}, \mu)$*-martingale problem for* $(H, C_0^\infty(\{\rho > 0\}))$, *if :*

(i) For some M', $\epsilon' \in (0, \infty)$

$$\int \mid P'_t f \mid d\mu \leq M' \int \mid f \mid d\mu,\ f \in C_b(\{\rho > 0\}),\ t \in (0, \epsilon').$$

(ii) For all $u \in C_0^\infty(\{\rho > 0\})$ *under* $\mathbb{P}'_\mu = \int \mathbb{P}'_x \mu(dx)$

$$u(X'_t) + \int_0^t Hu(X'_s)ds,\ t \geq 0,$$

is an $(\mathcal{F}_t)_{t\geq 0}$-martingale.

Proposition 2.4. *The diffusion process* $\mathbb{M}$ *from Theorem 2.1 solves the* $L_1(\{\rho > 0\}, \mu)$*-martingale problem for* $(H, C_0^\infty(\{\rho > 0\}))$.

Theorem 2.5. *Assume (in addition to (H1),(H2)) that* $(\mathcal{E}, D(\mathcal{E}))$ *is conservative. Let* $\mathbb{M}' = (\Omega', \mathcal{F}', (\mathcal{F}'_t)_{t\geq 0}, (X'_t)_{t\geq 0}, (\mathbb{P}'_x)_{x\in\{\rho>0\}})$ *on* $\{\rho > 0\}$ *be a diffusion process on* $\{\rho > 0\}$ *with transition semigroup* $(P'_t)_{t>0}$ *such that* $\mathbb{M}'$ *satisfies the* $L_1(\{\rho > 0\}, \mu)$*-martingale problem for* $(H, C_0^\infty(\{\rho > 0\}))$. *Then* $\mathbb{P}'_x = \mathbb{P}_x$ *for* μ*-a.e.* $x \in \{\rho > 0\}$, *where* $\mathbb{P}_x$, $x \in \{\rho > 0\}$, *are the probability measures of* $\mathbb{M}$ *in Theorem 2.1. If, in addition,* $P'_t(C_0^\infty(\{\rho > 0\})) \subset C(\{\rho > 0\})$ *for all* $t > 0$, *then* $\mathbb{P}'_x = \mathbb{P}_x$ *for every* $x \in \{\rho > 0\}$.

The above results apply to the two models described in the introduction and analyzed in the subsequent sections (cf. also [6], Section 6). The essential part is to show the conservativity of $(\mathcal{E}, D(\mathcal{E}))$ or equivalently the L^1-uniqueness of $(H, C_0^\infty(\mathbb{R}^d))$ on $L^1(\mu)$. In Sections 3 and 4 below we shall prove both L^1 and the much harder L^2-uniqueness for both models.

3 Diffusions in a random media

In this section we suppose that the density $\rho : \mathbb{R}^d \to \mathbb{R}$ has the form

$$\rho(x) := \exp(-E(x)),$$

where E is the potential energy of the particle in the configuration $\gamma \subset \mathbb{R}^d$

$$E(x) := E_\gamma(x) = \sum_{y \in \gamma} V(x-y),\ x \in \mathbb{R}^d.$$

We assume that the function $V : \mathbb{R}^d \to \mathbb{R}$, called potential, satisfies the following conditions:

$$V \in C^1(\mathbb{R}^d \setminus \{0\}),\ V \geq -a,\ \text{for some } a > 0, \tag{3.1}$$

and

$$\exp(-\frac{1}{2}V) \in W^{1,2}_{loc}(\mathbb{R}^d, dx). \tag{3.2}$$

Note that then $\nabla V \in L^2_{loc}(\mathbb{R}^d, \exp(-V)\,dx)$ (but not vice versa in general). We also assume a decay condition at infinity: there exist constants $c, k_0 > 0$ and $\alpha > d$ such that

$$|V(x)| + |\nabla V(x)| \leq c(1+|x|)^{-\alpha} \quad \text{if} \quad |x| \geq k_0. \tag{3.3}$$

To be able to control the drift in (1.2) we will restrict the class of admissible configurations. By $B(x,r) := \{y \in \mathbb{R}^d \mid |y-x| < r\}$ we denote the open ball of radius $r > 0$ with center at point x. Define the set Γ_{ad} of admissible configurations in $\mathbb{R}^d$ as

$$\Gamma_{ad} := \{\gamma \mid \forall r > 0\ \exists c(\gamma, r) > 0 : |\gamma \cap B(x,r)| \leq c(\gamma,r)\log(1+|x|)\}. \tag{3.4}$$

Here $|A|$ denotes the cardinality of a set A. Note that for many classes of probability measures ν on configuration spaces we have $\nu(\Gamma_{ad}) = 1$, see [16]. In particular, this is true for the well-known Ruelle measures corresponding to superstable pair potentials [15].

Below as before we set $\mu := \rho dx$, $\beta := \frac{\nabla \rho}{\rho}$. For $A \subset \mathbb{R}^d$ set $A^c := \mathbb{R}^d \setminus A$.

Lemma 3.1. *Assume that (3.1) holds and fix $\gamma \in \Gamma_{ad}$. Consider the decomposition $E = E^{(1)} + E^{(2)}$, where for $x \in \mathbb{R}^d$*

$$E^{(1)}(x) := \sum_{y \in \gamma \cap B(x,k_0)} V(x-y), \quad E^{(2)}(x) := \sum_{y \in \gamma \cap B(x,k_0)^c} V(x-y).$$

Then:

(i) Assume V satisfies (3.3). Then $E^{(2)} \in C^1(\mathbb{R}^d)$ and there exist $b_1, b_2 \in (0,\infty)$ (only depending on γ, α and d) such that for all $x \in B(0,r)$, $r > 0$,

$$E(x) \geq -b_1 - (b_2 + ac(\gamma, k_0)) \log(1+r).$$

(ii) Assume V satisfies (3.2) and (3.3). Then $\sqrt{\rho} = \exp(-\frac{1}{2}E) \in W^{1,2}_{loc}(\mathbb{R}^d, dx)$ (in particular $|\beta| \in L^2_{loc}(\mathbb{R}^d, \mu)$), so (H1) from Section 2 is satisfied. Furthermore, $\beta = -\nabla E$ on $\mathbb{R}^d \setminus \gamma$, in particular, $|\beta| \in L^p_{loc}(\mathbb{R}^d, \mu)$, $p \in [2,\infty)$, if $\nabla V \in L^p_{loc}(\mathbb{R}^d, \exp(-V)\,dx)$. So, in case $p > d$, (H2) from Section 2 holds.

Proof. (i): Note that for $k \in \mathbb{N}$, $x \in \mathbb{R}^d$, and $D(x,k) := B(x,k+1) \setminus B(x,k)$

$$|\gamma \cap D(x,k)| \leq c_d k^{d-1} c(\gamma,1) \log(1+|x|+k)$$

for some constant c_d only depending on the dimension d. Hence by (3.3) for all $x \in B(0,r)$, $r > 0$,

$$\sum_{k=k_0}^{\infty} \sum_{y \in \gamma \cap D(x,k)} (|V(x-y)| + |\nabla V(x-y)|) \leq$$
$$cc_d c(\gamma,1) \sum_{k=k_0}^{\infty} \frac{k^{d-1}\log(1+|x|+k)}{(1+k)^{\alpha}} < \infty.$$

We conclude that $E^{(2)} \in C^1(\mathbb{R}^d)$ and because $\log(1+|x|+k) \leq \log(1+|r|) + \log(1+k)$ for $x \in B(0,r)$, there exist $b_1, b_2 \in (0,\infty)$ (only depending on γ, α and d) such that for all $x \in B(0,r)$, $r > 0$,

$$|E^{(2)}(x)| \leq b_1 + b_2 \log(1+r)$$

Furthermore, by (3.1) for all $x \in B(0,r)$, $r > 0$,

$$E^{(1)}(x) = \sum_{y \in \gamma, |y-x| < k_0} V(x-y) \geq -ac(\gamma,k_0)\log(1+|x|) \geq -ac(\gamma,k_0)\log(1+r),$$

and (i) is proved.
(ii): Obviously,

$$\sqrt{\rho}(x) = \exp(-\frac{1}{2}E^{(2)}(x)) \prod_{y \in \gamma \cap B(x,k_0)} \exp(-\frac{1}{2}V(x-y)) \tag{3.5}$$

with all factors (as functions of x) in $W^{1,2}_{loc}(\mathbb{R}^d, dx) \cap L^\infty_{loc}(\mathbb{R}^d, dx)$. Now all parts of the assertion are obvious. □

If V and γ satisfies (3.1) - (3.4), then (H1) holds, so

$$Hu := -\Delta u - \langle \beta \,, \nabla u \rangle,\ u \in C_0^\infty(\mathbb{R}^d), \tag{3.6}$$

defines an operator $(H, C_0^\infty(\mathbb{R}^d))$ on $L^p(\mu)$, $p \in [1,2]$. If, in addition, $\nabla V \in L^p_{loc}(\mathbb{R}^d, \exp(-V)\,dx)$, also for $p > 2$, this is true for all $p \in [1, \infty)$.

We recall the following notion.

Definition 3.2. *Let $p \in [1,\infty)$. $(H, C_0^\infty(\mathbb{R}^d))$ is called $L^p(\mu)$-unique, if its closure $(\tilde{H}, D(\tilde{H}))$ on $L^p(\mu)$ generates a C_0-semigroup on $L^p(\mu)$.*

Remark 3.3. (i) Due to a result of W.Arendt [7, A-II, Theorem 1.33] $(H, C_0^\infty(\mathbb{R}^d))$ is $L^p(\mu)$-unique if and only if it has exactly one closed extension on $L^p(\mu)$ generating a C_0-semigroup.
(ii) By [21, Corollary 2.2 and Remark 2.4] $(H, C_0^\infty(\mathbb{R}^d))$ is $L^1(\mu)$-unique if and only if $(\mathcal{E}, D(\mathcal{E}))$ (and hence the diffusion process in Theorem 2.1) is conservative.

We start with $L^1(\mu)$-uniqueness.

Theorem 3.4. *Let $\gamma \in \Gamma_{ad}$ and suppose that the potential V satisfies conditions (3.1)–(3.3). Then $(H, C_0^\infty(\mathbb{R}^d))$ is $L^1(\mu)$-unique and both Theorems 2.1 and 2.5 apply, provided $\nabla V \in L^{d+\epsilon}_{loc}(\mathbb{R}^d, \exp(-V)\,dx)$ for some $\epsilon > 0$.*

Proof. Since we already know by Lemma 3.1 (ii) that $|\beta| \in L^2_{loc}(\mathbb{R}^d, d\mu)$, by [18, Theorem 4 and Remark 2] we have to find constants $A, B > 0$ such that

$$\mu(B(0,r) \leq A \exp(Br^2) \quad \text{for all} \quad \text{r} > 0.$$

But by Lemma 3.1 (i) we know that up to constants depending only on γ, α and d $\mu(B(0,r)$ is dominated by $e^{b_1}(1+r)^{b_2+ac(\gamma,k_0)+d}$. The last part of the assertion follows by Lemma 3.1 (ii). □

Next we consider $L^2(\mu)$-uniqueness. In this case we need stronger assumptions on V, namely, we suppose that (instead of (3.1)-(3.3)):

$$V \in C^2(\mathbb{R}^d \setminus \{0\}),\ V \geq -a,\ \text{for some } a > 0, \tag{3.7}$$

$$\exp(-\frac{1}{2}V) \in W^{2,2}_{loc}(\mathbb{R}^d, dx), \quad \nabla V \in L^4_{loc}(\mathbb{R}^d, \exp(-V)\, dx). \tag{3.8}$$

and there exist constants $c, k_0 > 0$ and $\alpha > d$ such that

$$|V(x)| + |\nabla V(x)| + |\Delta V(x)| \leq c(1 + |x|)^{-\alpha} \quad \text{if} \quad |x| \geq k_0 > 0. \tag{3.9}$$

We start with a simple technical result.

Lemma 3.5. *Let $\gamma \in \Gamma_{ad}$ and suppose that the potential V satisfies conditions (3.7)–(3.9). Then $\rho^{1/2} = \exp(-\frac{1}{2}E) \in W^{2,2}_{loc}(\mathbb{R}^d, dx)$ (in particular $\operatorname{div} \beta \in L^2_{loc}(\mathbb{R}^d, \mu)$) and $|\beta| \in L^4_{loc}(\mathbb{R}^d, \mu)$.*

Proof. The same arguments as in the proof of Lemma 3.1 (i) show that $E^{(2)} \in C^2(\mathbb{R}^d)$. Furthermore, by (3.5) $\rho^{1/2}$ is a (finite) product of functions from $W^{2,2}_{loc}(\mathbb{R}^d, dx) \cap L^\infty_{loc}(\mathbb{R}^d, dx)$. The fact that $|\beta| \in L^4_{loc}(\mathbb{R}^d, \mu)$ follows from Lemma 3.1 (ii). □

Remark 3.6. It directly follows from the proof that for compactly supported V the assertion of Lemma 3.5 is true for configurations γ which are locally finite, i.e. $|\gamma \cap B_r| < \infty$ for any $r > 0$.

Theorem 3.7. *Let $\gamma \in \Gamma_{ad}$ and suppose that the potential V satisfies conditions (3.7)–(3.9). Then $(H, C_0^\infty(\mathbb{R}^d))$ is $L^2(\mu)$-unique (i.e. H is essentially self-adjoint on $C_0^\infty(\mathbb{R}^d)$ in the space $L^2(\mu)$).*

Proof. Define the "renormalized" potential

$$\widehat{V}(x) := \frac{1}{4}|\nabla E(x)|^2 - \frac{1}{2}\Delta E(x) \equiv \frac{1}{4}|\beta(x)|^2 + \frac{1}{2}\operatorname{div} \beta(x). \tag{3.10}$$

By Lemma 3.5 $\widehat{V} \in L^2_{loc}(\mathbb{R}^d, d\mu)$ and the "renormalized" Hamiltonian

$$\widehat{H} := -\Delta + \widehat{V} \tag{3.11}$$

defined in $L^2(\mathbb{R}^d, dx)$ on the domain $\mathcal{D} := \rho^{1/2} C_0^\infty(\mathbb{R}^d)$. The operators $(H, C_0^\infty(\mathbb{R}^d))$ and $(\widehat{H}, \mathcal{D})$ are unitary equivalent under the linear map

$$\mathcal{H} \ni u \ \rightarrow \ \rho^{1/2} u \in L^2(\mathbb{R}^d, dx). \tag{3.12}$$

Below by $\widehat{H}$ we denote the closure of the operator $(\widehat{H}, \mathcal{D})$. Clearly, $\widehat{H}$ is a non-negative definite symmetric operator in $L^2(\mathbb{R}^d, dx)$. We shall show that the operator $\widehat{H}$ is self-adjoint in $L^2(\mathbb{R}^d, dx)$ and therefore H is self-adjoint in $L^2(\mu)$. We use the hyperbolic approximation criterium developed by Yu.M. Berezansky (see, e.g. [8, 9]). As the configuration γ is a locally finite set we can choose sequences r_n, $d_n > 0$, $r_n \uparrow \infty$ such that

$$\overline{B(0, r_n + d_n) \setminus B(0, r_n)} \cap \gamma = \emptyset. \tag{3.13}$$

Here $\bar{A}$ denotes the closure of a set A. Let χ_n be a cut-off function such that $\chi_n \in C_0^\infty(\mathbb{R}^d))$, $\chi_n(x) = 1$ if $|x| \leq r_n$ and $\chi_n(x) = 0$ if $|x| \geq r_n + d_n$. Define the cut-off energy $E_n(x) = E(x)\chi_n(x)$ and the cut-off density $\rho_n(x) := \exp(-E_n(x))$. Set $\mu_n := \rho_n \, dx$, $\beta_n := \frac{\nabla \rho_n}{\rho_n}$. Let H_n be the operator associated with the cut-off Dirichlet form

$$(H_n u, v) := \int_{\mathbb{R}^d} \langle \nabla u, \nabla v \rangle \, d\mu_n. \tag{3.14}$$

in $\mathcal{H}_n = L^2(\mathbb{R}^d, \, \mu_n)$. By (3.13), (3.9) (see also the proof of Lemma 3.1) $E \nabla \chi_n \in C_0^2(\mathbb{R}^d)$ and $\beta_n = -(\chi_n \nabla E + E \nabla \chi_n) \in L^4(\mathbb{R}^d, \, \mu_n)$. By [18] H_n is essentially self-adjoint on $C_0^\infty(\mathbb{R}^d)$. Therefore, its unitary image (under the linear map $\mathcal{H}_n \ni u \ \rightarrow \ \rho_n^{1/2} u \in L^2(\mathbb{R}^d, dx)$)

$$\widehat{H}_n := -\Delta + \widehat{V}_n \tag{3.15}$$

is an essentially self-adjoint non-negative operator in $L^2(\mathbb{R}^d, dx)$ on the domain $\mathcal{D}_n := \rho_n^{1/2} C_0^\infty(\mathbb{R}^d)$. Here

$$\widehat{V}_n := \frac{1}{4} |\beta_n|^2 + \frac{1}{2} \operatorname{div} \beta_n \in L^2(\mathbb{R}^d, \, \mu_n). \tag{3.16}$$

(Note that $\operatorname{div} \beta_n = -(\chi_n \Delta E + 2 \langle \nabla \chi_n, \nabla E \rangle + E \Delta \chi_n) \in L^2(\mathbb{R}^d, \, \mu_n)$.) Consider the Cauchy problem

$$\frac{d^2 u_n}{dt^2}(t) + (\widehat{H}_n u_n)(t) = 0, \ u_n(0) = \varphi_0, \ u_n'(0) = \varphi_1. \tag{3.17}$$

Note that the operator $\widehat{H}_n$ can be approximated in the strong resolvent sense by Schrödinger operators $H_{nk} := -\Delta + V_{nk}$ with smooth potentials V_{nk} (e.g., V_{nk} must be chosen such that $(V_{nk} - \widehat{V_n})\varphi \to 0$ as $k \to \infty$ for any $\varphi \in \mathcal{D}_n$). It follows that the strong solution of (3.17) has a finite rate of propagation, i.e. $\mathrm{supp}\, \mathrm{u_n(t)} \subset \mathrm{B(0, r + t)}$, under the condition $\mathrm{supp}\, \varphi_\mathrm{i} \subset \mathrm{B(0, r)}$, $i = 0, 1$ (by the support of a function from $L^2(\mathbb{R}^d, dx)$ we understand the support of the corresponding distribution). To prove the essential self-adjointness of $(\widehat{H}, \mathcal{D})$ it is sufficient to show (see [8, 9]) that for $\varphi_i \in \mathcal{D}$ $(i = 0, 1)$ and any $T > 0$ the strong solutions of (3.17) satisfy the relations

$$u_n(t) \in D(\widehat{H}),\ t \in [0, T],\ n > n_0 = n_0(T,\ \varphi_0,\ \varphi_1) \tag{3.18}$$

and

$$\int_0^T (u(t), (\widehat{H} - \widehat{H}_n)u_n(t))_{L^2(\mathbb{R}^d, dx)} dt \to 0,\ as\ n \to \infty, \tag{3.19}$$

for any strong solution u of the Cauchy problem

$$\frac{d^2u}{dt^2}(t) + (\widehat{H}^* u)(t) = 0,\ u(T) = 0,\ u'(T) = 0. \tag{3.20}$$

Here $\widehat{H}^*$ is the adjoint of the operator $\widehat{H}$. Suppose that $\mathrm{supp}\, \varphi_\mathrm{i} \subset \mathrm{B(0, r)}$ for some $r > 0$. Then $\mathrm{supp}\, \mathrm{u_n(t)} \subset \mathrm{B(0, r + T)}$ for $t \in [0, T]$. Choose n_0 in such a way that $r_{n_0} > r + T$. Note (see (3.10)) that $\widehat{V}_n(x) = \widehat{V}(x)$ for $|x| \le r_n$. Then by Lemma 3.8 below (3.18) is valid and $(\widehat{H}_n u_n)(t) = (\widehat{H} u_n)(t)$. In particular the relation (3.19) is fulfilled. Therefore to finish the proof of Theorem 3.7 we only need to prove the following lemma. □

Lemma 3.8. *Suppose that* $f \in D(\widehat{H}_n)$ *and* $\mathrm{supp\, f} \subset \mathrm{B(0, r)}$ *for some* $r \in (0, r_n)$. *Then* $f \in D(\widehat{H})$ *and* $\widehat{H} f = \widehat{H_n} f$.

Proof. Let $\{f_m\}$ be a sequence in $\mathcal{D}_n$ converging to f in the graph norm of the operator $\widehat{H_n}$. As γ is a locally finite set we can choose r', r'' such that $r < r' < r'' < r_n$ and

$$\overline{B(0, r'') \setminus B(0, r')} \cap \gamma = \emptyset.$$

Let α be a cut-off function such that $\alpha \in C_0^\infty(\mathbb{R}^d))$, $\alpha(x) = 1$ if $|x| \le r'$ and $\alpha(x) = 0$ if $|x| \ge r''$. Set $g_m := \alpha f_m$. As $\rho_n(x) = \rho(x)$ for $x \in B(0, r_n)$, we have that $g_m \in \mathcal{D} \cap \mathcal{D}_n$ and $\widehat{H} g_m = \widehat{H}_n g_m$. Clearly, $g_m \to f$ in $L^2(\mathbb{R}^d, dx)$ and we only need to prove that $\{\widehat{H}_n g_m\}$ converges in $L^2(\mathbb{R}^d, dx)$. We have

$$\widehat{H}_n g_m = (-\Delta + \widehat{V}_n) g_m = \alpha \widehat{H}_n f_m + (-\Delta\alpha) f_m - 2\langle \nabla\alpha, \nabla f_m \rangle. \tag{3.21}$$

The first two terms of (3.21) converge in $L^2(\mathbb{R}^d, dx)$. Consider the last term. By the definition of $\mathcal{D}_n$ we have that $f_m = \rho_n^{1/2} \phi_m$ with $\phi_m \in C_0^\infty(\mathbb{R}^d)$). Furthermore,

$$\langle \nabla\alpha, \nabla f_m \rangle = \rho_n^{1/2} \langle \nabla\alpha, \nabla\phi_m \rangle + \frac{1}{2} \langle \beta, \nabla\alpha \rangle f_m. \tag{3.22}$$

(Clearly, $\langle \beta, \nabla\alpha \rangle = \langle \beta_n, \nabla\alpha \rangle$ with $\beta_n := \frac{\nabla\rho_n}{\rho_n}$.) By the construction $\operatorname{supp} \nabla\alpha \cap \gamma = \emptyset$. It follows that β is bounded on $\operatorname{supp} \nabla\alpha$ and $\langle \beta, \nabla\alpha \rangle f_m$ converges in $L^2(\mathbb{R}^d, dx)$. Since the operators H_n and $\widehat{H}_n$ are unitary equivalent, we see that ϕ_m converges in the graph norm of the operator H_n. In particular $\nabla\phi_m$ converges in the Hilbert space $\mathcal{H}_n = L^2(\mu_n)$. It follows that $\langle \nabla\alpha, \nabla\phi_m \rangle$ converges in $L^2(\mu_n)$ and by (3.22) $\langle \nabla\alpha, \nabla f_m \rangle$ converges in $L^2(\mathbb{R}^d, dx)$. This completes the proofs of Lemma 3.8 and Theorem 3.7. □

Note that Lemma 3.8 shows that the domains of the operators $\widehat{H}$ and $\widehat{H}_n$ locally coincide. If, instead of assumption (3.3) we suppose that V is compactly supported, we can prove that the assertion of Theorem 3.7 is valid for all locally finite configurations γ.

Theorem 3.9. *Suppose that the potential V is compactly supported and satisfies conditions (3.7)–(3.8). Then the operator $(H, C_0^\infty(\mathbb{R}^d))$ is essentially self-adjoint on $L^2(\mathbb{R}^d, d\mu)$ for all locally finite configurations γ.*

Proof. The proof is done by the same arguments as used for proving Theorem 3.7. We only need to take into account Remark 3.6. □

4 N-particle systems with gradient dynamics

In this section we consider a model of N interacting particles in the Euclidean space $\mathbb{R}^d$ (see (1.3)). We introduce the potential energy of the system

$$E(x) := \sum_{1 \le k < j \le N} V(x_k - x_j), \; x = (x_1, ... x_N) \in \mathbb{R}^{Nd}$$

and the density

$$\rho(x_1,...x_N) := \exp(-E(x_1,...x_N)).$$

We shall suppose that conditions (3.1) and (3.2) are satisfied. Then $\rho^{1/2} = \exp(-\frac{1}{2}E) \in W^{1,2}_{loc}(\mathbb{R}^{Nd}, dx)$ and E can have singularities only on the set

$$S = \bigcup_{1\le k<j\le N} S_{kj}, \quad S_{kj} = \{x = (x_1,...x_N) \in \mathbb{R}^{Nd} \,|\, x_k = x_j\}.$$

As before we set $\mu := \rho dx$. We start with the problem of L^1-uniqueness for the operator H given by (3.6) with $\mathbb{R}^d$ replaced by $\mathbb{R}^{Nd}$ and

$$\beta(x) := \frac{\nabla\rho}{\rho}(x) = (-\sum_{j\neq k} \nabla V(x_k - x_j))_{k=1}^N \in \mathbb{R}^{Nd},\ x \notin S.$$

Theorem 4.1. *Suppose that the potential V satisfies conditions (3.1) and (3.2). Then $(H, C_0^\infty(\mathbb{R}^{Nd}))$ is $L^1(\mu)$- unique.*

Proof. The proof directly follows from Liskevich's result [18]. We already mentioned that (3.2) implies that $\rho^{1/2} = \exp(-\frac{1}{2}E) \in W^{1,2}_{loc}(\mathbb{R}^{Nd}, dx)$ (in particular $\beta \in L^2_{loc}(\mathbb{R}^{Nd}, \mu)$). Moreover, in this case ρ is bounded and therefore $\mu(B(0,r) \le Cr^{Nd}$ for some $C > 0$. □

Next we turn to the $L^2(\mu)$-uniqueness of the operator $(H, C_0^\infty(\mathbb{R}^{Nd}))$. For simplicity we restrict ourselves to the case of radially-symmetric potentials V. More precisely, we assume that

$$V(x) = v(|x|),\ v \in C^2(0,\infty),\ x \in \mathbb{R}^{Nd}, \tag{4.1}$$

and that there exist constants c_2, $\varepsilon > 0$ such that

$$v^{(2)}(r) \sim c_2/r^{2+\varepsilon},\ r \to 0, \tag{4.2}$$

where $a(r) \sim b(r),\ r \to 0$ means that $a(r)/b(r) \to 1,\ r \to 0$. It is easy to see that (4.2) implies that

$$v^{(i)}(r) \sim (-1)^i c_i/r^{i+\varepsilon},\ c_i, \varepsilon > 0,\ i = 0,1,\ r \to 0. \tag{4.3}$$

Note that these assumptions yield (3.8) and hence $\rho^{1/2} = \exp(-\frac{1}{2}E) \in W^{2,2}_{loc}(\mathbb{R}^{Nd}, dx)$ and $\beta \in L^4_{loc}(\mathbb{R}^{Nd}, \mu)$.

Theorem 4.2. *Suppose that a real-valued bounded from below potential V satisfies conditions (4.1)-(4.2). Then the operator H is essentially self-adjoint on $C_0^\infty(\mathbb{R}^{Nd})$ in $L^2(\mathbb{R}^{Nd}, \mu)$.*

Proof. We will follow the line of the proof of Theorem 3.7. However, the asymptotic condition (4.2) will simplify our arguments. Define the renormalized potential $\widehat{V}$ and the renormalized Hamiltonian $\widehat{H}$ by (3.10), (3.11). It is easy to see that conditions (4.2), (4.3) imply that $\widehat{V} \in L^2_{loc}(\mathbb{R}^{Nd}, \rho(x)dx)$ and, moreover, $\widehat{V}$ is locally semibounded from below on $\mathbb{R}^{Nd}$ (the singularities of $\widehat{V}$ appear only on S where one can apply the asumptotics (4.2), (4.3)). In the following we again denote by $\widehat{H}$ the closure of the operator $(\widehat{H}, \mathcal{D})$ where $\mathcal{D} := \rho^{1/2} C_0^\infty(\mathbb{R}^{Nd})$. Analogously to the proof of Theorem 3.7 we take a cut-off function $\chi_n \in C_0^\infty(\mathbb{R}^{Nd})$ such that $\chi_n(x) = 1$ if $|x| \le n$ and $\chi_n(x) = 0$ if $|x| \ge n+1$. Set $E_n(x) := E(x)\chi_n(x)$, $\rho_n(x) := \exp(-E_n(x))$ and $\mu_n = \rho_n(x)dx$. Let H_n be the operator associated with the cut-off Dirichlet form

$$(H_n u, v) := \int\limits_{\mathbb{R}^{Nd}} \langle \nabla u, \nabla v \rangle \, d\mu_n \tag{4.4}$$

on $\mathcal{H}_n := L^2(\mathbb{R}^{Nd}, \mu_n)$. Clearly, $\beta_n = -(\chi_n \nabla E + E \nabla \chi_n) \in L^4(\mathbb{R}^{Nd}, \mu_n)$ and by [18] H_n is essentially self-adjoint on $C_0^\infty(\mathbb{R}^{Nd})$. Therefore, its unitary image (under the linear map $\mathcal{H}_n \ni u \;\to\; \rho_n^{1/2} u \in L^2(\mathbb{R}^{Nd}, dx)$)

$$\widehat{H}_n := -\Delta + \widehat{V}_n \tag{4.5}$$

is an essentially self-adjoint non-negative definite operator in $L^2(\mathbb{R}^{Nd}, dx)$ on the domain $\mathcal{D}_n := \rho_n^{1/2} C_0^\infty(\mathbb{R}^{Nd})$ (the potential $\widehat{V}_n \in L^2(\mathbb{R}^{Nd}, \mu_n)$ is defined by (3.16)). Clearly, $\widehat{V}_n \in C^2(\mathbb{R}^{Nd} \setminus S)$ and $\operatorname{supp} \widehat{\mathrm{V}}_\mathrm{n} \subset \overline{\mathrm{B}(0, \mathrm{n}+1)}$. We will show that for any $x_0 \in S$ $\widehat{V}_n(x) \to \infty$ and therefore the potential $\widehat{V}_n$ is semibounded from below. First suppose that $x^0 \in S_{jk}$ and $x^0 \notin S_{lm}$ for $|j-l| + |k-m| > 0$. Then by (4.2), (4.3)

$$\begin{aligned} \widehat{V}_n(x) &= \frac{1}{4} |\nabla (V(x_j - x_k)\chi_n(x))|^2 - \frac{1}{2} \Delta (V(x_j - x_k)\chi_n(x)) + O(|x - x^0|) \\ &\sim \frac{c_1^2 \chi_n^2(x^0)}{4} r_{jk}^{-2-2\varepsilon} - \frac{c_2 \chi_n(x^0)}{2} r_{jk}^{-2-\varepsilon} \to +\infty, \; x \to x^0. \end{aligned}$$

Here $r_{jk} := |x_j - x_k|$. Consider the more difficult case $x^0 \in S_{jk} \cap S_{lm}$ and $x^0 \notin S_{j'k'}$ for the other indices j', k'. For simplicity suppose that $j = 1,\ k = l = 2,\ m = 3$. Then

$$\widehat{V}_n(x) = \frac{1}{4}|\nabla((V(x_1 - x_2) + V(x_2 - x_3) + V(x_3 - x_1))\chi_n(x))|^2$$
$$-\frac{1}{2}\Delta((V(x_1 - x_2) + V(x_2 - x_3) + V(x_3 - x_1))\chi_n(x)) + O(|x - x^0|). \quad (4.6)$$

Set

$$A(x) := |\nabla((V(x_1 - x_2) + V(x_2 - x_3) + V(x_3 - x_1))\chi_n(x))|^2,$$
$$B(x) := \Delta((V(x_1 - x_2) + V(x_2 - x_3) + V(x_3 - x_1))\chi_n(x)).$$

Clearly,

$$B(x) \sim c_2 \chi_n(x^0) \sum_{1 \le k < j \le 3} r_{kj}^{-2-\varepsilon},\ x \to x^0.$$

Set $\omega_{ij} := \frac{x_i - x_j}{r_{ij}},\ i \ne j$. By (4.3)

$$A(x) \sim c_1^2 \chi_n^2(x^0)(|v'(r_{12})\omega_{12} + v'(r_{13})\omega_{13}|^2 + |v'(r_{21})\omega_{21} + v'(r_{23})\omega_{23}|^2 +$$
$$|v'(r_{32})\omega_{32} + v'(r_{31})\omega_{31}|^2),\ x \to x^0.$$

Consider the following cases:
I. One of the r_{ij} (e.g., r_{12}) tends to zero faster than the other two, i.e. $\liminf \frac{r_{23}}{r_{12}} > 1$ and $\liminf \frac{r_{31}}{r_{12}} > 1$ Then

$$A(x) \sim 2c_1^2 \chi_n^2(x^0) r_{12}^{-2-2\varepsilon},\ x \to x^0.$$

2. One of the r_{ij} (e.g., r_{12}) tends to zero slowlier than the other two, and, moreover, r_{23} and r_{31} tend to zero with the same rate, i.e. $\limsup \frac{r_{23}}{r_{12}} < 1$ and $\limsup \frac{r_{31}}{r_{12}} < 1$ and $\lim \frac{r_{23}}{r_{31}} = 1$. In this case

$$A(x) \ge c_1^2 \chi_n^2(x^0)(r_{23}^{-2-2\varepsilon} + r_{31}^{-2-2\varepsilon}),\ x \to x^0.$$

3. All r_{ij} tend to zero with the same rate. Then

$$A(x) \sim c_1^2 \chi_n^2(x^0) r_{ij}^{-2-2\varepsilon}(|\omega_{12} + \omega_{13}|^2 + |\omega_{21} + \omega_{23}|^2 + |\omega_{32} + \omega_{31}|^2),\ x \to x^0.$$

Note that

$$|\omega_{12}+\omega_{13}|^2+|\omega_{21}+\omega_{23}|^2+|\omega_{32}+\omega_{31}|^2 =$$
$$2(3+\langle\omega_{12},\omega_{13}\rangle+\langle\omega_{21},\omega_{23}\rangle+\langle\omega_{32},\omega_{31}\rangle)\geq 3.$$

Here we have used the following well-known estimate: for any triangle with angles $\alpha_1,\alpha_2,\alpha_3$

$$|\cos(\alpha_1)+\cos(\alpha_2)+\cos(\alpha_3)|\leq 3/2.$$

It follows that for some $c>0$ $A(x)\geq c r_{ij}^{-2-2\varepsilon},\ x\to x^0$.

Therefore $A(x)$ tends to infinity faster then $B(x)$ in all three cases and (see (4.6)) $\widehat{V}_n(x)\to\infty$ as $x\to x_0$. A similar analysis shows that this is true for all $x^0\in S$. It follows that $\widehat{V}_n$ is semibounded from below. Now one repeats the arguments from the proof of Theorem 3.7. It should only be noted that due to the semiboundedness of $\widehat{V}_n$ the proof of the analogue of Lemma 3.8 is even simpler. We give the proof for the convenience of the reader. □

Lemma 4.3. *Suppose that the potential V satisfies the conditions of Theorem 4.2. Let $f\in D(\widehat{H}_n)$ such that* $\operatorname{supp} f\subset \mathrm{B}(0,r)$ *for some* $r\in(0,n)$. *Then* $f\in D(\widehat{H})$ *and* $\widehat{H}f=\widehat{H}_n f$.

Proof. Let $\{f_m\}$ be a sequence in $\mathcal{D}_n$ converging to f in the graph norm of the operator $\widehat{H_n}$. Choose a cut-off function α such that $\alpha\in C_0^\infty(\mathbb{R}^{Nd})$, $\alpha(x)=1$ if $x\in B(0,r)$ and $\operatorname{supp}\alpha\subset \mathrm{B}(0,\mathrm{n})$. Set $g_m=\alpha f_m$. As $\rho_n(x)=\rho(x)$ for $x\in B(0,n)$, we have that $g_m\in\mathcal{D}\cap\mathcal{D}_n$ and $\widehat{H}g_m=\widehat{H}_n g_m$. Clearly, $g_m\to f$ in $L^2(\mathbb{R}^{Nd},dx)$ and we only need to prove that $\{\widehat{H}_n g_m\}$ converges in $L^2(\mathbb{R}^{Nd},dx)$. We have

$$\widehat{H}_n g_m=(-\Delta+\widehat{V}_n)g_m=\alpha\widehat{H}_n f_m+(-\Delta\alpha)f_m-2\langle\nabla\alpha,\nabla f_m\rangle. \tag{4.7}$$

The first two terms of (4.7) clearly converge in $L^2(\mathbb{R}^{Nd},dx)$. Consider the last term. We have already mentioned that the potential $\widehat{V_n}$ is semibounded from below. Therefore, for some $\alpha>0$

$$(\widehat{H}_n f_m,f_m)_{L^2(\mathbb{R}^{Nd})}+\alpha(f_m,f_m)_{L^2(\mathbb{R}^{Nd})}\geq\int\limits_{\mathbb{R}^{Nd}}\langle\nabla f_m,\nabla f_m\rangle\,dx.$$

It follows that ∇f_m converges in $L^2(\mathbb{R}^{Nd},dx)$. □

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