AN ANALYTIC APPROACH TO KOLMOGOROV'S EQUATIONS IN INFINITE DIMENSIONS AND PROBABILITIC CONSEQUENCES

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1. Introduction

The purpose of this note and our talk at the ICMP is to present a mathematical programme designed to construct and analyze (weak) solutions to stochastic partial differential equations (SPDE) appearing in mathematical physics, e.g. in hydrodynamics or interacting infinite particle systems. Our main point is to directly solve the corresponding Kolmogorov equations to first obtain (and in fact "calculate" by finite dimensional approximations) the transition probabilities of the desired solution process. The latter is actually constructed only subsequently rendering the appropriate physical interpretation. In Section 3 we shall start with examples of typical applications giving recent references where the said programme has (at least partially) been implemented. The programme itself is described in the subsequent sections. Given the restrictions in length of this note, concerning work by other authors on these topics, we refer the reader to the discussions in our quoted papers as well as the references therein.

At this point I would like to express my sincere gratitude to all colleagues and friends who have contributed as coauthors to many of the results presented here.

2. Applications

We start with applications which are covered by our method.

2.1. *SPDE's in hydrodynamics*

(a) Stochastic generalized Burgers equation

$$dX_t(\xi) = \big[\Delta_\xi X_t(\xi) + \nabla_\xi \Psi\big(X_t(\xi)\big) + \Phi\big(X_t(\xi)\big)\big]\, dt + \sqrt{A}\, dW_t(\xi)$$

on the state space

$$E := L^2\big((0,1), ds\big)$$

(so $X_t \in E$, $\xi \in (0,1)$) with $ds :=$ Lebesgue measure on $(0,1)$, $\Psi : \mathbb{R} \to \mathbb{R}$, $\Phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. The noise term $dW_t(\xi)$ and the nature of the diffusion coefficient $\sqrt{A}$ are explained in the next section. References: [22]

(b) Stochastic Navier-Stokes equation

$$dX_t(\xi) = \big[\nu\Delta_\xi^S X_t(\xi) + \big(X_t(\xi)\cdot\nabla_\xi\big)X_t(\xi)\big]\,dt + \sqrt{A}\,dW_t(\xi)$$

on the state space

$$E := \big\{x \in L^2(D \to \mathbb{R}^2, dvol) \;\big|\; \operatorname{div} x = 0\big\}$$

(so $X_t \in E$, $\xi \in D$) with $D :=$ domain in $\mathbb{R}^2$, $dvol :=$ Lebesgue-measure on D, $\Delta_\xi^S :=$ Stokes operator, and $\nu :=$ viscosity. References: [23]

(c) Stochastic porous media equations

$$dX_t(\xi) = \Delta_\xi\big(\alpha X_t(\xi) + X_t^m(\xi)\big)\,dt + \sqrt{A}\,dW_t(\xi)$$

on the state space

$$E := H^{-1}(D) \quad (= \text{dual of } H_0^{1,2}(D))$$

(so $X_t \in E$, $\xi \in D$) with $D :=$ domain in $\mathbb{R}^d$ and $\alpha \in [0,\infty)$, $m \in \mathbb{N}$, m odd. References: [15, 16, 7]

One motivation for studying SPDE's of type (a)–(c) above is to "put back randomness" into the classical deterministic macroscopic partial differential equations by adding a function of the noise, and in a first step only taking into account the first order contribution of the Taylor expansion of this function.

2.2. *Infinite particle systems with singular interactions in $\mathbb{R}^d$ (or a Riemannian manifold)*

$$dx_t^{(k)} = \Big[\sum_{\substack{j=1\\ j\neq k}}^{\infty} \nabla V\big(x_t^{(k)} - x_t^{(j)}\big)\Big]dt + dW_t^{(k)}, \quad k \in \mathbb{N},$$

on the state space

$$E := \text{all locally finite subsets of } \mathbb{R}^d,$$

(so $X_t := \{x_t^{(k)} | k \in \mathbb{N}\} \in E$) with $(W_t^{(k)})_{t\geq 0}$, $k \in \mathbb{N}$, independent Wiener processes in $\mathbb{R}^d$, and e.g. $V(x) := \phi(|x|)$, $x \in \mathbb{R}^d$, ("two body potential") where typically $\phi : [0,\infty) \to \mathbb{R}$ has a singularity at zero, is not necessarily non-negative, and decays sufficiently fast at infinity ("Ruelle-type"). References: [4, 5, 18, 17]

With regard to the length of this paper we shall concentrate on the general framework, in which only the applications in subsection 2.1 above can be treated.

3. General framework and strategy

All examples in Subsection 2.1 belong to the following general class of stochastic differential equations on a seperable Hilbert space $(E, \langle\, , \,\rangle)$,

$$dX_t = \big[HX_t + F(X_t)\big]\,dt + \sqrt{A}\,dW_t \tag{1}$$
$$X_0 = x \in E \quad \text{(typically a function space)}$$

Here H is a (in general unbounded) linear operator on E generating a strongly continuous semigroup on E, A is a positive definite symmetric bounded linear operator on H (often assumed to be of trace class), $F : \mathrm{Dom}(F) \subset E \to F$ is a non-linear map, and $(W_t)_{t\geq 0}$ is a (cylindrical) Brownian motion on E.

The associated infinitesimal generator, also called "Kolmogorov operator", (obtained by heuristically applying Itô's formula to (1)) is given as follows:

$$\begin{aligned} L\varphi(x) &= \mathrm{Tr}(AD^2\varphi)(x) + \big\langle Hx + F(x), D\varphi(x)\big\rangle \\ &= \frac{1}{2}\sum_{i,j=1}^{N} \langle Ae_i, e_j\rangle \frac{\partial^2\varphi}{\partial e_i \partial e_j}(x) + \sum_{i=1}^{N} \langle Hx + F(x), e_i\rangle \frac{\partial \varphi}{\partial e_i}(x), \end{aligned}$$

$x \in E$, $\varphi = g(\langle e_1, \cdot\rangle, \ldots, \langle e_N, \cdot\rangle)$ with $N \in \mathbb{N}$, $g \in C_b^2(\mathbb{R}^N)$, and $\{e_i | i \in \mathbb{N}\}$ is a (in the respective application properly chosen) orthonormal basis of E. Such functions $\varphi : E \to \mathbb{R}$ are called finitely based and the set they form is denoted by $\mathcal{F}C_b^2$. D, D^2 denote first and second Frêchet derivatives and $\frac{\partial}{\partial e_i}$ stands for partial derivative in direction e_i.

The corresponding Kolmogorov equation in infinitely many variables (= generalized heat equation on the Hilbert space E) is then given by

$$\frac{\partial}{\partial t} u(t,x) = \bar{L}u(t,x), \quad u(0, \cdot) = f, \tag{2}$$

where $f : E \to \mathbb{R}$ is the initial condition of this infinite dimensional deterministic PDE. We emphasize that in (2) we have to consider an appropriate extension $\bar{L}$ of L with domain $\mathcal{F}C_b^2$, since due to the highly non-diagonal nature of (1) caused by the non-linearity F, even if f is finitely based this will never be the case for the solution $u(t, \cdot)$, apart from trivial uninteresting cases. As in the classical finite dimensional case the connection between (1) and (2) is that for $f := I_A$ (= indicator function of the set) $A \subset E$, we have

$$\begin{aligned} u(t,x) &= \text{probability that the solution of (1) with initial condition } x \text{ is in } A \text{ at time } t \\ &= \mathrm{Prob}(X_0 = x, X_t \in A), \end{aligned} \tag{3}$$

i.e. the solution to (2) is the "transition semigroup" for the stochastic process solving (1).

For many years the approach to solve the Kolmogorov equation (2) in infinite dimensions was to first solve (1) and use (3) as a definition for $u(t,x)$ and check under what additional conditions it really solves (2). The reason for this was that in infinite dimensions there were many techniques known to solve (1), albeit under quite strong assumptions on the coefficients, but almost none was known to solve (2) directly.

Our main point is

FIRST solve (2) and then (1).

This turns out to be possible under much weaker regularity conditions on the coefficients. In addition, in contrast to previous approaches this opens up ways to "calculate" the transition probabilities in (3) by finite dimensional approximations.

Our strategy to solve (2) is as follows:

Solve (2) via

$$u(t,x) = \big(e^{t\bar{L}} f\big)(x) \quad \text{in } L^p(E,\mu) \ (p \in [1,\infty])$$

for a suitably chosen probability measure μ on E. Construct from this strongly continuous semigroup on $L^p(E,\mu)$ a semigroup of probability kernels on E, and then using Kolmogorov's scheme a stochastic (Markov) process having these kernels as transition semigroup. Finally, show that this process solves (1) in the sense of a martingale problem uniquely.

4. Programme to implement the strategy

Now we summarize six single steps to implement this strategy to solve (2). For each step we give at least one published reference, which is basic for the respective main underlying ideas. More recent references (≥ 2003) have been mentioned already in Section 2 and more will be given in the next section.

Step 1: Reference measures.
Find an appropriate reference measure on the Hilbert space E. It has turned out, that they can be obtained as solutions to the elliptic equation

$$L^*\mu = 0 \tag{4}$$

that is, μ is a probability measure on E such that

$$\int L\varphi \, d\mu = 0 \quad \forall\, \varphi \in \mathcal{F}C_b^2.$$

Such measures are called "infinitesimally invariant". Often it is enough to find a probability measure μ on E such that for some $\lambda \in [0,\infty)$

$$\int L\varphi \, d\mu \leq \lambda \int \varphi \, d\mu \quad \forall\, \varphi \in \mathcal{F}C_b^2,\ \varphi \geq 0.$$

We stress that such a measure μ essentially always exist if SDE (1) at all has a solution for sufficiently many starting points $x \in E$.

For techniques how to solve (4) we refer to [6] for the symmetric case and to [9], [10] for the general case.

Step 2: L^p-solutions for PDE (2).
Construct $e^{t\bar{L}}$, $t \geq 0$, on $L^p(E,\mu)$, where $\bar{L}$ denotes the closure of L. Then

$$(L^p(E,\mu)-)\quad \frac{d}{dt}\big(\underbrace{e^{t\bar{L}}f}_{u(t,\,\cdot\,)}\big) = \bar{L}\big(\underbrace{e^{t\bar{L}}f}_{u(t,\,\cdot\,)}\big), \quad t > 0. \tag{5}$$

This is the solution of the Kolmogorov equation (2) in $L^p(E,\mu)$. A possibly quite useful reference here is [14] (see also [21]).

Step 3: Regularity of L^p-solutions of PDE (2).
To find probability kernels p_t, $t \geq 0$, on E such that for $t \geq 0$

$$\text{“}\ e^{t\bar{L}}f(x) = \text{”} \int f(y)\, p_t(x,dy) =: p_t f(x)$$

and $\forall\, \varphi \in \mathcal{F}C_b^\infty$

$$p_t\varphi(x) - \varphi(x) = \int_0^t p_s(L\varphi)(x)\, ds \quad \forall\, x \in E_\mu \subset E \tag{6}$$

with $\mu(E_\mu) = 1$. (6) means that we habe a pointwise solution of the Kolmogorov equation on E_μ. Furthermore, try to prove

$$p_t\big(C_b^{(w)}(E)\big) \subset C_b^{(w)}(E) \quad \text{"Feller-property"}$$

or even

$$p_t\big(\mathcal{B}_b(E)\big) \subset C_b(E) \quad \text{"strong Feller-property"}$$

where $C_b(E)$, $C_b^{(w)}(E)$ denote the set of real-valued bounded continuous, weakly sequentially continuous functions on E respectively. $\mathcal{B}_b(E)$ denotes the set of bounded Borel measurable functions on E. Reference: e.g. [14].

Step 4: Weak solutions of SDE (1).
Construct a (strong Markov) process with continuous sample paths on E_μ with transition semigroup $(p_t)_{t\geq 0}$, which uniquely solves SDE (1) for all starting points $x \in E_\mu$. Reference: [27].

Step 5: Characterization of E_μ.
Typically, we have that E_μ is the topological support of μ, which often can be identified as all of E or a well-described subset thereof, as e.g. a closed ball. In this case everything is independent of the chosen measure μ. We refer to [14] for examples.

Step 6: Finite dimensional approximations of solutions.
Construct a Galerkin-type approximation of PDE (2), to calculate p_t on E as a limit for $N \to \infty$ of solutions p_t^N for corresponding finite dimensional Kolmogorov equations (with Lipschitz coefficients) on $E_N := \text{span}\{e_1, \ldots, e_N\}$.

5. Some results and consequences

The above programme has been

(a) completely implemented e.g. for

- stochastic generalized Burgers equations (see [22])
- 2D-stochastic Navier-Stokes equation (see [23])

(b) partially implemented e.g. for

- stochastic porous media equations (see [15], [16], [7])
- infinite particle systems in $\mathbb{R}^d$ (or in a Riemannian manifold) with singular interactions (see [5])
- 3D-stochastic Navier-Stokes equation (work in progress, e.g. existence of infinitesimally invariant measures has already been proved in [9])

Then: Analysis of solutions of (1) and (2)
(We only give very recent references, i.e. ≥ 2003) For example:

- large time asymptotics and invariant measures e.g. existence and uniqueness of invariant measures for non-linearities F of gradient type (i.e. the Gibbsian case),

see [3], [1], [2], also [11] with respect to the classical problem whether invariance $\Rightarrow$ Gibbsian
· small time asymptotics and large deviations, [18], [26]
· small noise limits and large deviations, [12] (for paths), [13] (for the invariant measures)
· scaling limits [17]
· explicitly time dependent coefficients [8]
· spectral properties of the Kolmogorov operator L, [24], [25]
· Yamada-Watanabe principle and existence of strong solutions, [19]
· non-local Kolmogorov operators, [20]

etc.

Acknowledgement

Financial support of the BiBoS-Research Centre and the German Science Foundation (DFG) through the DFG-Research Group "Spectral Analysis, Asymptotic Distributions and Stochastic Dynamics" is gratefully acknowledged. We also thank the Faculty of Mathematics at Bielefeld University for the substantial contribution to the travel expenses for participating in the ICMP.

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