An optimal condition for the existence of solutions to singular semilinear elliptic problems*

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Abstract

We obtain a characterization of all locally bounded functions $p \geq 0$ for which the equation (E) $\Delta u + p(x)\psi(u) = 0$ has a positive solution in Ω vanishing on the boundary, where $\Omega \subset \mathbb{R}^N$ is an arbitrary domain and $\psi > 0$ is a nonincreasing continuous function on $]0, \infty[$. For $\Omega = \mathbb{R}^N$ with $N \geq 3$, it is shown that (E) has a (unique) positive solution in $\mathbb{R}^N$ which decays to zero at infinity if and only if p is the set $\{p > 0\}$ has positive Lebesgue measure and $\lim_{|x|\to\infty} \int_{\mathbb{R}^N} p(y)|x-y|^{2-N} dy = 0$. This condition can be replaced by $\int_0^\infty rp(r)dr < \infty$ if p is radial.

1 Introduction

We study the existence of solutions to the semilinear elliptic problem

$$\begin{cases} \Delta u + p(x)\psi(u) = 0 \text{ in } \Omega, \\ u > 0 \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is a domain, p is a nonnegative Borel measurable, locally bounded function on Ω, and ψ is a nonincreasing continuous positive function on $]0, \infty[$. The boundary condition in problem (1), and in the sequel, means that $u(x) \to 0$ as $\delta(x) := \operatorname{dist}(x, \partial\Omega) \to 0$ if Ω is bounded. For unbounded domains we impose in addition that $u(x) \to 0$ as $|x| \to \infty$.

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In the special case when Ω is the whole space $\mathbb{R}^N$ with $N \geq 3$, and $\psi(u) = u^{-\gamma}$ for $\gamma > 0$, (1) is the following singular boundary value problem

$$\begin{cases} \Delta u + p(x)u^{-\gamma} = 0 \text{ in } \mathbb{R}^N, \\ u > 0 \text{ in } \mathbb{R}^N, \\ u(x) \to 0 \text{ as } |x| \to \infty, \end{cases} \tag{2}$$

which has been the subject of much study. Details about the importance of this kind of semilinear equations in scientific applications can be found in [10, 12, 15] (see also their references). In several papers (see, for instance, [3, 4, 5, 8, 10, 11, 12, 14]), we find various conditions, essentially based on the growth of the function

$$\phi : r \mapsto \sup_{|x|=r} p(x), \tag{3}$$

which are sufficient for the existence (or the nonexistence) of solutions to (2). However, as to my knowledge, no characterizations of functions p for which problem (2) admits a solution have been given. In this paper, our main goal is to establish necessary and sufficient conditions for the existence of a solution to the more general problem (1) (see Section 2 where the main results are stated).

The main tool in our approach is a connection between nonnegative solutions to the equation

$$\Delta u + p(x)\psi(u) = 0 \tag{4}$$

and nonnegative harmonic functions. More precisely, by means of an integral equation, we establish a one-to-one correspondence between nonnegative harmonic functions and nonnegative solutions to (4) in a Greenian domain Ω. This allows us to derive a characterization of all p for which problem (1) has a solution. Furthermore, we find a necessary and sufficient condition under which equation (4) has bounded positive solutions.

For $0 < \gamma < 1$, Kusano and Swanson proved in [11] that problem (2) has a solution if p is locally Hölder continuous on $\mathbb{R}^N$, $p(x) > 0$ for every $x \in \mathbb{R}^N \setminus \{0\}$, $\inf_{x\in\mathbb{R}^N} p(x)/\phi(|x|) > 0$, and

$$\int_0^\infty t^{N-1+\gamma(N-2)}\phi(t)\,dt < \infty. \tag{5}$$

This result has been extended for every $\gamma > 0$ by Delmasso in [3]. Lair and Shaker established in [13] the existence of a solution to problem (1) with $\Omega = \mathbb{R}^N$ provided p is continuous, not identically zero on $\mathbb{R}^N$, and

$$\int_0^\infty r\phi(r)\,dr < \infty. \tag{6}$$

Moreover, they pointed out that (6) is *nearly* optimal, in the sense that there is no *radial* solution to (1) if p is radial and (6) fails. In this paper, it will be shown that condition (6) is really optimal. In other words, for a nontrivial radial function p we shall prove that (1) possesses a solution if and only if (6) holds true. However, we give an example showing that (6) is, in general, not necessary for the existence of a solution to problem (2).

The paper is organized as follows: In section 2 we state the main results and we discuss some special cases. After recalling some basic tools in section 3, we study in section 4 the semilinear equation (4) in bounded regular domains. Section 5 deals with the proofs of Theorems 1 and 2. We investigate problems (8) and (1) in section 6, and we prove Theorem 3 and Corollary 2.

2 Main results

2.1. Assumptions. In all the following, Ω is a domain of $\mathbb{R}^N$, $N \geq 1$, p is a nonnegative Borel measurable function on Ω such that $\sup_K p < \infty$ for every compact set $K \subset \Omega$, and ψ is a nonincreasing continuous positive function on $]0, \infty[$. Since equation (4) has no positive solution in a non-Greenian domain except when p is identically zero, we assume that Ω is Greenian, that is, there exists a nonconstant positive superharmonic function on Ω, which in turn means (see [6, p.27]) that the Laplace operator Δ has a Green function G_Ω in Ω ($\Delta G_\Omega(\cdot, y) = -\varepsilon_y$, where ε_y denotes the Dirac measure concentrated at y).

We note that, for dimensions $N \geq 3$, $\mathbb{R}^N$ and thereby all open subsets of $\mathbb{R}^N$ are Greenian. On the other hand, it is well known that $\mathbb{R}^1$ and $\mathbb{R}^2$ are not Greenian. However, a bounded open domain of $\mathbb{R}^N$ for $N \geq 1$ is always Greenian.

We define the function $G_\Omega p$ for every $x \in \Omega$ by $G_\Omega p(x) = \int_\Omega G_\Omega(x, y) p(y)\, dy$. The following assumptions will be often used:

(A1) $G_\Omega p$ is not identically ∞ in Ω,

(A2) $G_\Omega p$ is bounded in Ω,

(A3) $G_\Omega p$ is in the class $C_0(\Omega)$,

where $\mathcal{C}_0(\Omega)$ denotes the set of all continuous real-valued functions u on Ω such that $u = 0$ on $\partial\Omega$, which simply means (as was mentioned in the introduction) that $\lim_{x\to z} u(x) = 0$ for all $z \in \partial D$ if D is bounded, and if D is unbounded we also require that $\lim_{|x|\to\infty} u(x) = 0$.

2.2. Main theorems. The first result of this paper is the following theorem:

Theorem 1. *Assume that* (A1) *holds. Then nonnegative solutions u to equation* (4) *in Ω are in one-to-one correspondence with nonnegative harmonic functions h in Ω. This correspondence is given by the formula*

$$u - \int_{\Omega} G_{\Omega}(\cdot, y)p(y)\psi(u(y))\, dy = h. \tag{7}$$

More precisely, the following holds:

(a) *If u is a nonnegative solution to* (4) *in Ω, then h given by* (7) *is a nonnegative harmonic function in Ω.*

(b) *Conversely, for every nonnegative harmonic function h in Ω, there exists a unique nonnegative solution u to equation* (4) *satisfying* (7).

Of course, the theorem is obvious if p identically zero in Ω. If p is nontrivial, then u is the minimal solution to equation (4) satisfying $u > h$ in Ω. Conversely, for a given positive solution u to equation (4), the function h in (7) is the maximal harmonic function dominated by u in Ω. We note that a formula similar to (7) has been used by Dynkin [7] in order to construct a one-to-one correspondence between a class of nonnegative harmonic functions and a class of nonnegative solutions to $Lu + f(u) = 0$ in Ω, where f is a positive increasing locally Lipschitz function on $\mathbb{R}_+$ with $f(0) = 0$, and L belongs to a class of differential operators containing Δ.

Investigating the problem of the existence of bounded positive solutions to the singular semilinear equation (4), we obtain:

Theorem 2. *Equation* (4) *has a bounded positive solution in Ω if and only if* (A2) *is valid.*

For a given nonnegative harmonic function h in Ω, we consider the following boundary value problem

$$\begin{cases} \Delta u + p(x)\psi(u) = 0 \text{ in } \Omega, \\ u > h \text{ in } \Omega, \\ u - h = 0 \text{ in } \partial\Omega, \end{cases} \tag{8}$$

which is a more general version of problem (1). If p is nontrivial in Ω, we prove that condition (A3) is sufficient for the existence of a solution (which is unique) to problem (8). If $h = 0$ in Ω, we show that (A3) is a necessary condition as well. More precisely, the following result is obtained.

Theorem 3. *Problem* (1) *has a solution if and only if p is nontrivial in Ω and condition* (A3) *holds.*

We would like to point out that our results can easily be extended to more general second order elliptic operators than Δ. Furthermore, the hypothesis that p is locally bounded is not necessary in our approach. In fact, it is sufficient to suppose that $G_D p$ is continuous for every bounded regular open set D such that $\overline{D} \subset \Omega$. In particular, if $N \geq 3$, it is enough to assume that $p \in L^q(\Omega)$ for some $q > N/2$, or more generally, that the function $p1_\Omega$ belongs to the so-called Kato class K_N^{loc} (see, e.g., [1]).

2.3. Entire solutions. A solution to (4) in the whole space $\mathbb{R}^N$ will be called an entire solution. For $N \geq 3$, the explicit form of the Green function G of $\mathbb{R}^N$ is well known. It is defined on $\mathbb{R}^N \times \mathbb{R}^N$ by $G(x,x) = \infty$ and

$$G(x,y) = \frac{\kappa_N}{|x-y|^{N-2}} \qquad \left(\text{where } \kappa_N := \frac{\Gamma(N/2)}{2(N-2)\pi^{N/2}}\right) \tag{9}$$

for every $x, y \in \mathbb{R}^N$ such that $x \neq y$. So, taking $\Omega = \mathbb{R}^N$, Theorems 2 and 3 immediately yield the following corollary.

Corollary 1. *Let $N \geq 3$. The following holds:*

(a) *Equation* (4) *has an entire bounded positive solution in $\mathbb{R}^N$ if and only if*

$$\sup_{x \in \mathbb{R}^N} \int_{\mathbb{R}^N} \frac{p(y)}{|x-y|^{N-2}}\, dy < \infty.$$

(a) *Equation* (4) *has an entire positive solution in $\mathbb{R}^N$ which decays to zero at infinity if and only if p is nontrivial and*

$$\lim_{|x| \to \infty} \int_{\mathbb{R}^N} \frac{p(y)}{|x-y|^{N-2}}\, dy = 0.$$

2.4. A special case. Consider the case when $\Omega = \mathbb{R}^N$ and p is asymptotically quasi-radial, i.e., there are $c > 0$ and a nontrivial Borel measurable function φ on $\mathbb{R}_+$ such that

$$c^{-1}\varphi(|x|) \leq p(x) \leq c\,\varphi(|x|) \quad \text{for } |x| \text{ sufficiently large.} \tag{10}$$

In this setting, if a bounded positive solution to (4) exits then it necessarily vanishes at infinity. We get the following:

Corollary 2. *Let $N \geq 3$ and assume that* (10) *holds. Then the following statements are equivalent:*

(a) *Equation* (4) *admits an entire bounded positive solution in $\mathbb{R}^N$.*

(b) *Equation* (4) *admits an entire positive solution in $\mathbb{R}^N$ vanishing at ∞.*

(c) *There exists $a \geq 0$ such that*

$$\int_a^\infty r\varphi(r)\, dr < \infty. \tag{11}$$

3 Basic notions

For every open set D of $\mathbb{R}^N$ let $\mathcal{B}(D)$ (resp. $\mathcal{C}(D)$) be the set of all numerical Borel measurable (resp. real-valued continuous) functions on D. $\mathcal{C}_c^\infty(D)$ will denote the space of infinitely differentiable functions on D with compact support. Finally, for every set $\mathcal{F}$ of numerical functions, we denote by $\mathcal{F}^+$ (resp. $\mathcal{F}_b$) the set of all functions in $\mathcal{F}$ which are nonnegative (resp. bounded).

We shall use some classical results dealing with harmonic and superharmonic functions. The reader is referred to [6] or [2] for definitions and more details about these functions. Let us recall that a bounded open set D in $\mathbb{R}^N$ is called regular if for every real continuous function f on ∂D the classical Dirichlet problem

$$\begin{cases} \Delta h = 0 \text{ in } D, \\ h = f \text{ on } \partial D \end{cases}$$

has a unique solution $H_D f$. It is easily seen that, for each point $x \in D$, the map $f \mapsto H_D f(x)$ defines a positive Radon measure on ∂D, which will be denoted by μ_x^D and is called the harmonic measure relative to x and D. The harmonic kernel on D is denoted by H_D and defined by

$$H_D f(x) = \begin{cases} \int f \, d\mu_x^D & \text{if } x \in D, \\ f(x) & \text{if } x \in \mathbb{R}^N \backslash D \end{cases}$$

for every $f \in \mathcal{B}(\mathbb{R}^N)$ for which the integral makes sense. If D is connected and $H_D|f|(x) < \infty$ for some point $x \in D$, then $H_D f$ is harmonic in D. A function $s > -\infty$ is said to be superharmonic on the open set Ω, if it is lower semicontinuous, $H_D s$ is harmonic on D and $H_D s \le s$ for every regular open set D such that $\overline{D} \subset \Omega$.

If $N \ge 3$, then $\mathbb{R}^N$ is Greenian and its Green function G is given by (9). For $N = 2$ let G be the function on $\mathbb{R}^2 \times \mathbb{R}^2$ given by $G(x, x) = \infty$ and

$$G(x, y) = \frac{1}{2\pi} \log \frac{1}{|x - y|} \quad \text{if } x \ne y.$$

We shall denote again by G_D, for every Greenian set $D \subset \mathbb{R}^N$, $N \ge 1$, the operator defined by

$$f \mapsto G_D f = \int_D G_D(\cdot, y) f(y) \, dy$$

whenever the integral has a sense. We recall (see [6, p.92]) that if D is a domain and $f \in \mathcal{B}^+(D)$, then $G_D f$ is either superharmonic or $G_D f \equiv \infty$ in D.

Proposition 1. *Let $D \subset \mathbb{R}^N$ be a Greenian domain, and let $f \in \mathcal{B}^+(D)$ be locally bounded such that $v = G_D f \not\equiv \infty$. Then $v \in \mathcal{C}(D)$ and $\Delta v + f = 0$ in D in the distributional sense.*

Proof. For $N = 1$ this can be shown directly using the explicit form of the Green function on a Greenian domain of $\mathbb{R}$. Let $N \geq 2$, B be an open ball such that $\overline{B} \subset D$, and define $w = G_B f$. Then $v - w$ is harmonic (and hence continuous) on B. Indeed, since

$$H_B G_D(\cdot, z)(x) = \begin{cases} G_D(x, z) - G_B(x, z) & \text{if } z \in B, \\ G_D(x, z) & \text{if } z \in D \backslash B \end{cases}$$

for every $x \in B$ (see [6, p.86]), by Fubini's theorem we get that

$$\begin{aligned} H_B v(x) = \int_{\partial B} v(y)\, d\mu_x^B(y) &= \int_D \{\int_{\partial B} G_D(y, z)\, d\mu_x^B(y)\} f(z)\, dz \\ &= \int_D H_B G_D(\cdot, z)(x) f(z)\, dz = v(x) - w(x). \end{aligned}$$

Analogously, w differs on B from $u := G(f|_B)$ by a harmonic function, where $f|_B$ denotes the restriction of f to B. On the other hand, u is continuous on B (even in the class $C^1(\mathbb{R}^N)$) by Theorem 7 in [6, p.8]. Therefore, v is continuous on every ball B with closure in D and hence v is continuous on D. Finally, the equality $\Delta v + f = 0$ can be obtained by easy computations. $\square$

Proposition 2. *If $D \subset \mathbb{R}^N$ is a bounded regular open set, then the operator G_D maps $\mathcal{B}_b(D)$ into $\mathcal{C}_0(D)$ and it is compact when $\mathcal{B}_b(D)$ is endowed with the uniform norm.*

These properties of G_D are valid for a wide class of Radon measures on D (which are called Kato measures) containing, in particular, the restriction to D of the Lebesgue measure on $\mathbb{R}^N$. See for instance Proposition 2.1 in [9].

4 Solutions to (4) in bounded regular sets

In this paper, by a solution to a partial differential equation we shall mean a continuous solution in the distributional sense. In particular, a solution to equation (4) in an open set $D \subset \Omega$ will be a function $u \in \mathcal{C}^+(D)$ such that $p\psi(u)$ is locally integrable on D and, for all $\varphi \in \mathcal{C}_c^\infty(D)$,

$$\int_D u \Delta \varphi\, dx + \int_D p\psi(u)\varphi\, dx = 0.$$

We define supersolutions and subsolution to equation (4) in the same way replacing "=" respectively by "≤", "≥" and considering nonnegative function φ in the space $\mathcal{C}_c^\infty(D)$. The following useful lemma follows easily from Proposition 1.

Lemma 1. *Let $D \subset \Omega$ be a domain and let $u \in \mathcal{B}^+(D)$ such that u, $\psi(u)$ are locally bounded, and $G_D(p\psi(u)) \not\equiv \infty$. Then u is a solution to (4) in D if and only if the function $u - G_D(p\psi(u))$ is harmonic in D.*

For an arbitrary open subset $D \subset \Omega$, which is not necessarily connected, the equivalence in the previous lemma still holds true provided $G_D(p\psi(u))$ is finite on D, which in turn means that $G_D(p\psi(u)) \in \mathcal{C}(D)$ by virtue of Proposition 1.

Lemma 2. *Let $D \subset \Omega$ be a bounded open set and let $u, v \in \mathcal{C}^+(D)$ such that*

$$\Delta u + p(x)\psi(u) \leq \Delta v + p(x)\psi(v) \quad \text{in } D.$$

If $\liminf_{x\to z}(u - v)(x) \geq 0$ for every $z \in \partial D$, then $u \geq v$ in D.

Proof. Suppose that the open set $V := \{x \in D : u(x) < v(x)\}$ is not empty. Then the function $w := u - v$ is superharmonic on V and, for every $z \in \partial V$,

$$\liminf_{x\in V, x\to z} w(x) \geq 0.$$

Consequently $w \geq 0$ on V by the classical minimum principle for superharmonic functions (see, for instance, [6, p.20]). This yields a contradiction and hence $V = \emptyset$. $\square$

By the same arguments, it can be shown that $u \geq v$ in an unbounded open subset D if $u - v \geq 0$ on ∂D, i.e., if $\liminf_{x\to z}(u - v)(x) \geq 0$ for all $z \in \partial D$ and $\liminf_{|x|\to\infty}(u - v)(x) \geq 0$.

We consider the semilinear Dirichlet problem

$$\begin{cases} \Delta u + p(x)\psi(u) = 0 \text{ in } D, \\ u = f \text{ on } \partial D \end{cases} \tag{12}$$

where D is a bounded regular open subset such that $\overline{D} \subset \Omega$ and $f \in \mathcal{C}^+(\partial D)$. In [5], del Pino investigated (12) in the case of $\psi(u) = u^{-\gamma}$ with $\gamma > 0$, $f \equiv 0$ on ∂D, and where D is smooth. He proved the existence and uniqueness of the solution provided p is nontrivial and bounded in D. In the following lemma we extend this result to the more general function ψ and where the boundary datum is any nonnegative continuous function on ∂D.

Lemma 3. *Let D be a bounded regular open set such that $\overline{D} \subset \Omega$ and let $f \in \mathcal{C}^+(\partial D)$. Then problem (12) has a unique solution $u \in \mathcal{C}^+(\overline{D})$. Furthermore, for every $x \in D$ we have*

$$u(x) = H_D f(x) + \int_D G_D(x, y)p(y)\psi(u(y))\, dy. \tag{13}$$

Proof. The uniqueness of the solution to (12) is immediate by the previous lemma. The existence of this solution and formula (13) will be proved in two steps.

Step 1. Assume first that $f \geq \varepsilon$ on ∂D for some $\varepsilon > 0$, and consider the sequences (u_n), (v_n) defined by $u_0 = H_D f$ and for $n \geq 0$

$$\begin{aligned} u_{n+1} &= H_D f + v_n, \\ v_n &= \int_D G_D(\cdot, y) p(y) \psi(u_n(y))\, dy. \end{aligned} \tag{14}$$

Since ψ is nonincreasing and $u_n \geq H_D f \geq \varepsilon$ for all $n \geq 0$ we get

$$0 \leq p(y)\psi(u_n(y)) \leq \psi(\varepsilon) \sup_D p < \infty$$

for every $y \in D$. Therefore, by Proposition 2, (v_n) possesses a subsequence (v_{n_k}) which is uniformly convergent on D. Then (u_{n_k}) converges uniformly on D to a function $u \in \mathcal{C}_b(D)$ which satisfies the integral equation (13) in view of (14) and the dominated convergence theorem. Hence u is a solution to (4) by Lemma 1. On the other hand, the fact that D is regular and $p\psi(u)$ is bounded on D yields that $G_D(p\psi(u)) \in \mathcal{C}_0(D)$ (see Proposition 2). Consequently, for every $z \in \partial D$ we have

$$\lim_{x \to z} u(x) = \lim_{x \to z} H_D f(x) = f(z).$$

So the theorem is proved for every $f \in \mathcal{C}(\partial D)$ such that $f > 0$ in ∂D.

Step 2. We now turn to the general case of a nonnegative continuous function f on ∂D. For every $k \geq 1$ let $f_k = k^{-1} + f$ and define u_k to be the solution to problem (12) for $f = f_k$ which is already determined in the first step. Applying Lemma 2, we see that (u_k) is a nonincreasing sequence in $\mathcal{C}^+(D)$. On the other hand,

$$u_k = H_D f_k + \int_D G_D(\cdot, y) p(y) \psi(u_k(y))\, dy$$

for every $k \geq 1$. Define $u := \lim_{k \to \infty} u_k$ (i.e., $u = \inf_{k \geq 1} u_k$). Letting k tend to ∞ in the above formula, it follows that $G_D(p\psi(u)) < \infty$ and that u satisfies (13). Therefore u is a solution to equation (4) by Lemma 1. Furthermore, the inequality $H_D f \leq u$ yields that

$$f(z) = \liminf_{x \to z} H_D f(x) \leq \liminf_{x \to z} u(x)$$

for every $z \in \partial D$. On the other hand, since $u \leq u_k$ for all $k \geq 1$ we get that

$$\limsup_{x \to z} u(x) \leq \limsup_{x \to z} u_k(x) = f(z) + k^{-1} \to f(z) \text{ as } k \to \infty.$$

Whence $u = f$ on ∂D and the proof is complete. $\square$

It should be noticed that in the previous proof, it is of vital importance that the operator $v \mapsto G_D(pv)$ is compact on the space $\mathcal{B}_b(D)$ endowed with the uniform norm. This is not necessarily true for the Greenian domain Ω.

5 Proofs of Theorems 1 and 2

The purpose of this section is to prove Theorems 1 and 2. First we need two further lemmas.

Lemma 4. *Let $u_i \in \mathcal{B}^+(\Omega)$ be real-valued such that $G_\Omega(p\psi(u_i)) < \infty$ on Ω and let $h_i := u_i - G_\Omega(p\psi(u_i))$ for $i = 1, 2$. Suppose that $h_1 - h_2$ is a nonnegative superharmonic function on Ω. Then $u_1 \geq u_2$ on Ω.*

Proof. Obviously

$$h_1 - h_2 + G_\Omega(v^+) = u_1 - u_2 + G_\Omega(v^-)$$

where $v := p(\psi(u_1) - \psi(u_2))$, and $t^\pm := \max(\pm t, 0)$. Since $u_1 - u_2 \geq 0$ on the subset $\{v^- > 0\}$ of $\{\psi(u_1) < \psi(u_2)\}$, we see that the inequality

$$h_1 - h_2 + G_\Omega(v^+) \geq G_\Omega(v^-) \tag{15}$$

is valid in $\{v^- > 0\}$. On the other hand $h_1 - h_2 + G_\Omega(v^+)$ is nonnegative and superharmonic on Ω. Therefore, the domination principle [6, p.67] implies that (15) holds true everywhere in Ω. This yields that $u_1 \geq u_2$ on Ω. □

Lemma 5. *Let (u_n) be a nondecreasing sequence of nonnegative solutions to equation (4) in a domain $\Omega' \subset \Omega$. Then $u := \sup_{n \geq 1} u_n$ is either ∞ or a solution to (4) in Ω'.*

Proof. Without loss of generality we consider only the case when p is nontrivial. Suppose that u is not identically infinite on Ω'. Then u is superharmonic on Ω' by Theorem 3.1.4 in [2] and thereby $H_D u$ is harmonic in every regular open set D such that $\overline{D} \subset \Omega'$ (see [6, p.35]). Choose a regular open set D with closure in Ω' and let $\alpha = \inf_D u_1$. Then $\alpha > 0$ and for every $n \geq 1$ we have $\psi(u) \leq \psi(u_n) \leq \psi(\alpha)$ in D. Consequently, in view of (13) we get that

$$\begin{aligned} u_n(x) &= H_D u_n(x) + \int_D G_D(x,y) p(y) \psi(u_n(y))\, dy \\ &\leq H_D u(x) + \psi(\alpha) \int_D G_D(x,y) p(y)\, dy \end{aligned} \tag{16}$$

for every $x \in D$. This proves that u is locally bounded on D. Passing to the limit in (16) and using Lemma 1, it follows that u is a solution to (4) in D. Since this is true for every regular open subset D with closure in Ω', the function u is a solution to equation (4) in Ω'. □

In the remainder of this paper, we fix a sequence (D_n) of bounded regular open sets such that $\overline{D_n} \subset D_{n+1}$ for all $n \geq 1$ and $\bigcup_{n\geq 1} D_n = \Omega$. We note that a such sequence exists by Corollary 6.6.13 in [2]. If $\Omega = \mathbb{R}^N$, a simple choice of (D_n) is the following:

$$D_n = B(0,n) := \left\{x \in \mathbb{R}^N : |x| < n\right\}.$$

From Proposition 1, we recall that (A1) is equivalent to the condition:

$$\int_\Omega G_\Omega(\cdot, y)p(y)\, dy \ \in \ \mathcal{C}(\Omega). \tag{17}$$

Proof of Theorem 1. The statements will be trivial if p vanishes almost everywhere in Ω. So we assume that the set $\{p > 0\}$ has positive Lebesgue measure, which implies that every nonnegative solution to (4) in Ω is (strictly) positive.

(a) Let u be a positive solution to (4) in Ω. By Lemma 3, for every $n \geq 1$

$$u = H_{D_n}u + \int_{D_n} G_{D_n}(\cdot, y)p(y)\psi(u(y))\, dy \quad \text{ in } D_n. \tag{18}$$

Since u is superharmonic in Ω, the sequence $(H_{D_n}u)$ is nonincreasing and thereby the limit function $h := \inf_{n\geq 1} H_{D_n}u$ is well defined and harmonic in Ω. On the other hand, it is well known that (G_{D_n}) is nondecreasing and $\sup_{n\geq 1} G_{D_n} = G_\Omega$ (see, e.g., [6, p.94]). So, letting n tend to ∞ in (18) we obtain formula (7) which completes the proof of (a). It should be clear that assertion (a) always holds true even if (A1) does not hold.

(b) In virtue of Lemma 4 there exists at most one solution u to (4) in Ω which satisfies (7). As in the proof of Lemma 3, in order to show the existence of u we first consider a harmonic function h in Ω such that $\inf_\Omega h \geq \varepsilon > 0$. Let u_n denote the solution to equation (4) in D_n with the boundary condition $u_n = h$ on ∂D_n. Again by Lemma 3 we have

$$u_n = h + \int_{D_n} G_{D_n}(\cdot, y)p(y)\psi(u_n(y))\, dy \quad \text{ in } D_n. \tag{19}$$

In particular, $u_{n+1}(z) \geq h(z) = u_n(z)$ for every $z \in \partial D_n$ and consequently $u_{n+1} \geq u_n$ in D_n for every $n \geq 1$ by Lemma 2. Since $u_n \geq h \geq \varepsilon$ in D_n it follows from (19) that

$$u_n \leq h + \psi(\varepsilon) \int_{D_n} G_{D_n}(\cdot, y)p(y)\, dy \leq h + \psi(\varepsilon) \int_\Omega G_\Omega(\cdot, y)p(y)\, dy.$$

Therefore, $u := \sup_{n\geq 1} u_n$ is a solution to (4) in Ω by (A1) and Lemma 5. Furthermore, applying the dominated convergence theorem and using (A1) (or equivalently (17)) we obtain that

$$\lim_{n\to\infty} \int_{D_n} G_{D_n}(\cdot, y)p(y)\psi(u_n(y))\, dy = \int_\Omega G_\Omega(\cdot, y)p(y)\psi(u(y))\, dy < \infty.$$

Hence, passing to the limit in (19) we get (7). This finishes the proof of (b) in the case when $\inf_\Omega h > 0$.

Now, let h be a nonnegative harmonic function in Ω and define $h_k := h + k^{-1}$. By the first part of the present proof there exists a sequence (u_k) of solutions to (4) in Ω such that

$$u_k = h_k + \int_\Omega G_\Omega(\cdot, y)p(y)\psi(u_k(y))\, dy.$$

Since (u_k) is nonincreasing by Lemma 4, a simple application of the monotone convergence theorem proves that h and the limit function $u := \lim_{n\to\infty} u_n$ satisfy equality (7). Therefore, u is a solution to (4) in Ω by virtue of Lemma 1. So the proof of Theorem 1 is complete. $\square$

Proof of Theorem 2. Let u be a bounded positive solution to (4) in Ω. Then, as the proof of statement (a) in Theorem 1 shows,

$$u = h + \int_\Omega G_\Omega(\cdot, y)p(y)\psi(u(y))\, dy \quad \text{in } \Omega$$

where $h = \inf_{n\geq 1} H_{D_n} u$ is the greatest harmonic minorant of u. Hence for every $x \in \Omega$ we have

$$\int_\Omega G_\Omega(x, y)p(y)\, dy \leq \frac{1}{\psi(M)} \int_\Omega G_\Omega(x, y)p(y)\psi(u(y))\, dy \leq \frac{M}{\psi(M)}$$

where $M = \sup_\Omega u$. Thus (A2) holds and the proof of the necessity is finished.

Suppose now that p satisfies (A2) and let $c > 0$ be a real constant. By assertion (b) in Theorem 1 there exists a positive solution u to (4) satisfying (7) for $h \equiv c$. Therefore, using the fact that ψ is nonincreasing it follows that

$$c \leq u \leq c + \psi(c) \sup_{x\in\Omega} \int_\Omega G_\Omega(x, y)p(y)\, dy.$$

Thus, u is bounded in Ω and the proof is complete. $\square$

6 Boundary value problems

Before dealing with the proof of Theorem 3 we first establish an existence and uniqueness result for the solution to the more general boundary value problem (8).

Theorem 4. *Assume that p is nontrivial in Ω and* (A3) *holds. Then for every nonnegative harmonic function h in Ω, there exists one and only one solution u to problem* (8)*. This solution is given by* (7).

Proof. Let h be a nonnegative harmonic function in Ω and let u be the positive solution to (4) in Ω associated to h by formula (7). We claim that $u - h \in \mathcal{C}_0(\Omega)$. Indeed, consider $h_k = k^{-1} + h$ and denote by u_k the solution to (4) given by Theorem 1. Then, it is easy to see that for every $x \in \Omega$ we have

$$\begin{aligned} 0 \le u_k(x) - h_k(x) &= \int_\Omega G_\Omega(x,y)p(y)\psi(u_k(y))\,dy \\ &\le \psi(k^{-1}) \int_\Omega G_\Omega(x,y)p(y)\,dy. \end{aligned}$$

Whence, by assumption (A3), $u_k - h_k \in \mathcal{C}_0(\Omega)$ for every $k \ge 1$. On the other hand, we know (see the proof of Theorem 1.b) that the sequence (u_k) is nonincreasing and $\lim_{n\to\infty} u_k = u$. So, for all $x \in \Omega$ and all $k \ge 1$

$$0 \le u(x) - h(x) \le u_k(x) - h_k(x) + k^{-1}.$$

Hence, if $z \in \partial\Omega$ then for every $k \ge 1$

$$0 \le \limsup_{x\to z}(u(x) - h(x)) \le k^{-1},$$

which yields that $\limsup_{x\to z}(u(x) - h(x)) = 0$. Analogously, we show that $\lim_{|x|\to\infty}(u(x) - h(x)) = 0$ if Ω is unbounded. So $u - h \in \mathcal{C}_0(\Omega)$ and the claim is proved.

Assume that v is a positive solution to (8). Then v and the harmonic function $g := \inf_n H_{D_n} v$ satisfy formula (7) and we have $v \ge g \ge h$ in Ω. Hence

$$0 \le g - h \le v - h$$

which yields that $g - h \in \mathcal{C}_0(\Omega)$ and consequently $g = h$ in Ω. Therefore $v = u$ by Lemma 4. Thus u is the unique solution to problem (8). □

Proof of Theorem 3. If $p = 0$ in Ω then obviously problem (1) has no solution. By the previous theorem, if p is nontrivial then condition (A3) is sufficient for (1) to admit a solution. So, it only remains to show that (A3) holds whenever (1) is solvable. To do this, let u denote the solution to problem (1) and define

$M := \sup_\Omega u$. Clearly $0 < M < \infty$ because $u \in \mathcal{C}_0(\Omega)$. By Propositions 1 and 2 we see that for every $n \geq 1$, the function

$$v_n := \int_{D_n} G_{D_n}(\cdot, y)p(y)\,dy$$

is the unique solution to the problem

$$\begin{cases} \Delta v_n + p = 0 \text{ in } D_n, \\ v_n = 0 \text{ on } \partial D_n. \end{cases}$$

It then follows that

$$\begin{cases} \Delta(u - \psi(M)v_n) = p(\psi(M) - \psi(u)) \leq 0 \text{ in } D_n, \\ u - \psi(M)v_n \geq 0 \text{ on } \partial D_n \end{cases}$$

Thereby $u \geq \psi(M)v_n$ in D_n by the classical minimum principle for superharmonic functions. Consequently, we obtain that

$$\int_\Omega G_\Omega(x, y)p(y)\,dy \leq \frac{u(x)}{\Psi(M)}$$

for every $x \in \Omega$. Since $u \in \mathcal{C}_0(\Omega)$ we conclude that (A3) is fulfilled. □

In [12, 13] Lair and Shaker investigated problems (2) and (1) and proved that for a nontrivial continuous function p in $\mathbb{R}^N$, condition (6) is sufficient for equation (4) to have a positive solution vanishing at infinity. Since $Gp \leq G\phi$ (ϕ is given by (3)), this follows immediately from the if-part of Theorem 3. In the following remark, we give an example showing that (A3) is weaker than (6), which yields that (6) is not necessary for the existence of a solution to problem (1). However, by Corollary 2 we observe that conditions (A3) and (6) are equivalent if p is radial.

Remark 1. *There exists a nonnegative continuous function p on $\mathbb{R}^N$, $N \geq 3$, such that $Gp \in \mathcal{C}_0(\mathbb{R}^N)$ and (6) does not hold.*

Proof. Let $I = \{(x_1, 0, \cdots, 0) \in \mathbb{R}^N : 0 \leq x_1 \leq 1\}$. For every $\eta > 0$, define $A_\eta = \cup_{x \in I} B(x, \eta)$ and $v_\eta = G(1_{A_\eta})$ where $1_{A_\eta}(x)$ takes the value 1 if $x \in A_\eta$ and zero otherwise. It is easy to see that $v_\eta \in \mathcal{C}_0(\mathbb{R}^N)$ and for every $x \in \mathbb{R}^N$

$$v_\eta(x) \downarrow 0 \text{ as } \eta \downarrow 0.$$

Therefore, in view of Dini's theorem, (v_η) converges uniformly on $\mathbb{R}^N$ to zero as η tends to zero. So, for every $n \geq 0$ we may find $\eta_n > 0$ such that $v_{\eta_n} \leq 2^{-n}$ in $\mathbb{R}^N$. Choose $q_n \in \mathcal{C}(\mathbb{R}^N)$ such that

$$1_I \leq q_n \leq 1_{A_{\eta_n}}.$$

For every $x \in \mathbb{R}^N$, define $p_n(x) = q_n(x - a_n)$ and

$$p(x) = \sum_{n=0}^{\infty} p_n(x)$$

where $a_n = (n, 0, \cdots, 0)$ for every $n \geq 0$. Since the previous sum is locally finite, the function p is continuous on $\mathbb{R}^N$. Moreover, $Gp_n \in \mathcal{C}_0(\mathbb{R}^N)$ and

$$\sup_{x \in \mathbb{R}^N} Gp_n(x) \leq \sup_{x \in \mathbb{R}^N} v_{\eta_n} \leq 2^{-n}$$

for every $n \geq 0$, whence $Gp \in \mathcal{C}_0(\mathbb{R}^N)$. On the other hand, for all $n \geq 0$ and all $x_1 \in \mathbb{R}$ such that $n \leq x_1 \leq n + 1$ we have

$$p(x_1, 0, \cdots, 0) \geq q_n(x_1 - n, 0, \cdots, 0) \geq 1_I(x_1 - n, 0, \cdots, 0) = 1.$$

Whence $\phi(r) \geq 1$ for all $r \geq 0$ and thereby condition (6) does not holds. $\square$

We conclude this paper by the proof of Corollary 2 which gives a characterization of all asymptotically quasi-radial function p for which problem (1) has a solution.

Proof of Corollary 2. Hypothesis (10) means that for some sufficiently large real R we have $c^{-1}\varphi(|x|) \leq p(x) \leq c\varphi(|x|)$ for all $x \in \mathbb{R}^N$ such that $|x| \geq R$. Notice that if (11) is valid for some $a > 0$ then it holds for $a = 0$. This follows from the fact that φ is locally bounded on $\mathbb{R}_+$.

(c)$\Rightarrow$(b): Let $v := G\varphi$. Using spherical coordinates we obtain that

$$v(0) = \frac{1}{N-2} \int_0^{\infty} r\varphi(r)\, dr.$$

Then (11) implies that $v(0) < \infty$ which in turn implies that v is a continuous superharmonic function on $\mathbb{R}^N$ (see Proposition 1). Furthermore $\inf_{\mathbb{R}^N} v = 0$ (see Theorem in [6, p.48]). From the explicit form of G we clearly see that v is radial on $\mathbb{R}^N$. Consequently, the classical minimum principle yields that $v(x) \geq v(y)$ for all $x, y \in \mathbb{R}^N$ such that $|x| \leq |y|$. Hence

$$\lim_{|x| \to \infty} v(x) = \inf_{\mathbb{R}^N} v = 0$$

which means that $v \in \mathcal{C}_0(\mathbb{R}^N)$. To see that Gp is also in $\mathcal{C}_0(\mathbb{R}^N)$, it is enough to observe that

$$Gp \leq G(p 1_{B(0,R)}) + cG\varphi.$$

So statement (b) follows from (b) in Corollary 1.

(b)⇒(a): Trivial.

(a)⇒(c): The function Gp is bounded on $\mathbb{R}^N$ by Theorem 2. Seeing that

$$v \leq G(\varphi 1_{B(0,R)}) + cGp,$$

we deduce that v is bounded on $\mathbb{R}^N$ as well. Hence $v(0) < \infty$ and (c) holds true. □

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