The semigroup of the Glauber dynamics of a continuous system of free particles

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Abstract

We study properties of the semigroup $(e^{-tH})_{t\geq 0}$ on the space $L^2(\Gamma_X,\pi)$, where Γ_X is the configuration space over a locally compact second countable Hausdorff topological space X, π is a Poisson measure on Γ_X, and H is the generator of the Glauber dynamics. We explicitly construct the corresponding Markov semigroup of kernels $(\mathbf{P}_t)_{t\geq 0}$ and, using it, we prove the main results of the paper: the Feller property of the semigroup $(\mathbf{P}_t)_{t\geq 0}$ with respect to the vague topology on the configuration space Γ_X, and the ergodic property of $(\mathbf{P}_t)_{t\geq 0}$. Following an idea of D. Surgailis, we also give a direct construction of the Glauber dynamics of a continuous infinite system of free particles. The main point here is that this process can start in every $\gamma\in\Gamma_X$, will never leave Γ_X and has cadlag sample paths in Γ_X.

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1 Introduction

The Glauber dynamics (GD) of a continuous infinite system of particles, either free or interacting, is a special case of a spatial birth and death process on the Euclidean space $\mathbb{R}^d$, or on a more general topological space X. For a system of particles in a bounded volume in $\mathbb{R}^d$, such processes were introduced and studied by C. Preston in [16], see also [7]. In the latter case, the total number of particles is finite at any moment of time.

In the recent paper by L. Bertini, N. Cancrini, and F. Cesi, [3], the generator of the GD in a finite volume was studied. This generator corresponds to a special case of birth and death coefficients in Preston's dynamics. Under some conditions on the interaction between particles, the authors of [3] proved the existence of the spectral gap of the generator of the GD, which is uniformly positive with respect to all finite volumes Λ and boundary conditions outside Λ. An explicit estimate of the spectral gap in a finite volume was derived by L. Wu in [20].

The problem of construction of a spatial birth and death process in the infinite volume was initiated by R.A. Holley and D.W. Stroock in [7], where it was solved in a very special case of nearest neighbor birth and death processes on the real line.

In [11], the GD in infinite volume was discussed. The process now takes values in the configuration space $\Gamma_{\mathbb{R}^d}$ over $\mathbb{R}^d$, i.e., in the space of all locally finite subsets in $\mathbb{R}^d$, which is equipped with the vague topology. Using the theory of Dirichlet forms [13, 14], the authors of [11] proved the existence of a Hunt process $\mathbf{M}$ on $\Gamma_{\mathbb{R}^d}$ that is properly associated with the generator of the GD with a quite general pair potential of interaction between particles. In particular, $\mathbf{M}$ is a conservative Markov process on $\Gamma_{\mathbb{R}^d}$ with cadlag paths. An estimate of the spectral gap of the generator of the GD in infinite volume was also proved.

In the case where the interaction between particles is absent (i.e., the particles are free), the Poisson measure π on $\Gamma_{\mathbb{R}^d}$ is a stationary measure of the GD. Let us recall that the Poisson measure possesses the chaos decomposition property, and hence the space $L^2(\Gamma_{\mathbb{R}^d}, \pi)$ is unitarily isomorphic to the symmetric Fock space over $L^2(\mathbb{R}^d)$, see e.g. [18]. It can be shown that, under this isomorphism, the generator of the GD of free particles goes over into the number operator N on the Fock space. The latter operator is evidently the second quantization of the identity operator, i.e., $N = d\,\mathrm{Exp}(\mathbf{1})$.

On the other hand, a construction of a Markov process which corresponds to the Poisson space realization of the second quantization of a doubly sub-Markov generator on $\mathbb{R}^d$ (or on a more general space) was proposed by D. Surgailis in [19]. However, D. Surgailis did not discuss the following question: From which configurations is the process allowed to start so that it never leaves the configuration space? It was only proved in [19] that, for π-a.e. configuration γ, the process starting at γ will be a.s. in the configuration space at some fixed time $t > 0$.

In the case of the Brownian motion on the configuration space, i.e., in the case of the independent motion of infinite Brownian particles (cf. [1]), a solution to the above stated problem was proposed by the authors in [12]. More exactly, a subset Γ_∞ of the configuration space Γ_X over a complete, connected, oriented, and stochastically complete manifold X of dimension ≥ 2 was constructed such that the process can start at any $\gamma \in \Gamma_\infty$, will never leave Γ_∞, and has continuous sample paths in the vague topology (and even in a stronger one). In the case of a one-dimensional underlying manifold X, one cannot exclude collisions of particles, so that a modification of the construction of Γ_∞ is necessary, see [12] for details.

In this paper, we study properties of the semigroup of the GD of a continuous infinite system of free particles. So, we fix a locally compact second countable Hausdorff topological space X. We denote by π_m the Poisson measure on Γ_X with intensity m being a Radon non-atomic measure on X. In Section 2, we construct, on the space $L^2(\Gamma_X, \pi_m)$, the Dirichlet form $\mathcal{E}$, the generator H, and the semigroup $(e^{-tH})_{t\geq 0}$ for the GD of free particles in X. In particular, we derive an explicit formula of the action of the semigroup on exponential functions (Corollary 1). The results of this section are essentially preparatory.

In Section 3, we construct a Markov semigroup of kernels $(\mathbf{P}_t)_{t\geq 0}$ on $(\Gamma_X, \mathcal{B}(\Gamma_X))$ such that, for each $F \in L^2(\Gamma_X, \pi_m)$,

$$(e^{-tH}F)(\gamma) = \int_{\Gamma_X} F(\xi)\, \mathbf{P}_t(\gamma, d\xi), \qquad \pi_m\text{-a.e. } \gamma \in \Gamma_X$$

(Theorem 1 and Proposition 2). The reader is advised to compare this result with [12, Theorem 5.1].

The first main result of the paper is Theorem 2, which states that the semigroup $(\mathbf{P}_t)_{t\geq 0}$ possesses the Feller property on Γ_X (with respect to the vague topology). Notice that, though we proved, in the case of the Brownian motion on the configuration space, that the corresponding semigroup on Γ_∞ possesses a *modified* strong Feller property [12, Theorem 6.1], we were able to prove the usual Feller property only in the case $X = \mathbb{R}^d$ and only with respect to the intrinsic metric of the Dirichlet form, which induces a topology that is *much stronger* than the vague topology.

The second main result is that the semigroup $(\mathbf{P}_t)_{t\geq 0}$ is ergodic (Theorem 3). More exactly, we show that, for any probability measure μ on Γ_X, the image measure $\mathbf{P}_t\mu$ of μ under $\mathbf{P}_t$ weakly converges to π_m as $t \to \infty$. Let us recall that, in the case of a birth and death process in a finite volume in $\mathbb{R}^d$, the ergodic property of a certain class of birth and death processes was proved by C. Preston [16, Theorem 7.1]. In particular, the latter theorem holds in the case of the GD of free particles in a finite volume.

Finally, in Section 4, following the idea of D. Surgailis [19], we give a direct construction of the Glauber dynamics of an infinite system of free particles in X as a time homogeneous conservative strong Markov process on the state space Γ_X with transition probability function $(\mathbf{P}_t)_{t\geq 0}$. The main point here is that this process can start in every $\gamma \in \Gamma_X$, will never leave Γ_X and has cadlag sample paths in Γ_X (Theorem 4). In the case where $m(X) = \infty$, we also show the possibility of restricting the process to the subset $\Gamma_{X,\mathrm{inf}}$ of Γ_X consisting of all infinite configurations in Γ_X (Corollary 3).

2 Generator of the Glauber dynamics

Let X be a locally compact second countable Hausdorff topological space. Such a space is known to be Polish, and we fix a separable and complete metric ρ on X generating the topology. We denote by $C_0(X)$ the set of all continuous, compactly supported, real-valued functions on X. Let $\mathcal{B}(X)$ denote the Borel σ-algebra on X and let m be a Radon measure on $(X, \mathcal{B}(X))$ without atoms.

We define the configuration space Γ_X over X as the set of all locally finite subsets of X:

$$\Gamma_X := \{\gamma \subset X \mid |\gamma_\Lambda| < \infty \text{ for each compact } \Lambda \subset X\}.$$

Here, $|\cdot|$ denotes the cardinality of a set and $\gamma_\Lambda := \gamma \cap \Lambda$. One can identify any $\gamma \in \Gamma_X$ with the positive Radon measure $\sum_{x\in\gamma} \varepsilon_x \in \mathcal{M}(X)$, where $\mathcal{M}(X)$ stands for the set of all positive Radon measures on $\mathcal{B}(X)$. We endow the space Γ_X with the relative topology as a subset of the space $\mathcal{M}(X)$ with the vague topology, i.e., the weakest topology on Γ_X with respect to which all maps

$$\Gamma_X \ni \gamma \mapsto \langle \varphi, \gamma \rangle := \int_X \varphi(x)\, \gamma(dx) = \sum_{x \in \gamma} \varphi(x), \qquad \varphi \in C_0(X),$$

are continuous. We shall denote the Borel σ-algebra on Γ_X by $\mathcal{B}(\Gamma_X)$.

Let π_{zm} denote the Poisson measure on $(\Gamma_X, \mathcal{B}(\Gamma_X))$ with intensity zm, $z > 0$. This measure can be characterized by its Laplace transform

$$\int_{\Gamma_X} e^{\langle \varphi, \gamma \rangle}\, \pi_{zm}(d\gamma) = \exp\left(\int_X (e^{\varphi(x)} - 1)\, zm(dx) \right), \qquad \varphi \in C_0(X). \tag{1}$$

We refer e.g. to [10] for a detailed discussion of the construction of the Poisson measure on the configuration space. Let us recall that the Poisson measure satisfies the Mecke identity:

$$\int_{\Gamma_X} \pi_{zm}(d\gamma) \int_X \gamma(dx)\, F(\gamma, x) = \int_{\Gamma_X} \mu(d\gamma) \int_X zm(dx)\, F(\gamma \cup x, x) \tag{2}$$

for any measurable function $F : \Gamma_X \times X \to [0, +\infty]$, see [15]. Here and below, for simplicity of notation we just write x instead of $\{x\}$ for any $x \in X$.

It is easy to see that the Poisson measure π_{zm} has all local moments finite, i.e.,

$$\int_{\Gamma_X} \langle \varphi, \gamma \rangle^n\, \pi_{zm}(d\gamma) < \infty, \qquad \varphi \in C_0(X),\ \varphi \geq 0,\ n \in \mathbb{N}. \tag{3}$$

We introduce the linear space $\mathcal{F}C_{\mathrm{b}}(C_0(X), \Gamma_X)$ of all functions on Γ_X of the form

$$F(\gamma) = g_F(\langle \varphi_1, \gamma \rangle, \dots, \langle \varphi_N, \gamma \rangle), \tag{4}$$

where $N \in \mathbb{N}$, $\varphi_1, \dots, \varphi_N \in C_0(X)$, and $g_F \in C_{\mathrm{b}}(\mathbb{R}^N)$. Here, $C_{\mathrm{b}}(\mathbb{R}^N)$ denotes the set of all continuous bounded functions on $\mathbb{R}^N$. For any $\gamma \in \Gamma_X$, we consider $T_\gamma := L^2(X, \gamma)$ as a "tangent" space to Γ_X at the point γ, and for any $F \in \mathcal{F}C_{\mathrm{b}}(C_0(X), \Gamma_X)$ we define the "gradient" of F at γ as the element of T_γ given by $D^- F(\gamma, x) := D^-_x F(\gamma) := F(\gamma \setminus x) - F(\gamma)$, $x \in \gamma$. (Evidently, $D^- F(\gamma)$ indeed belongs to T_γ.)

Let now $z := 1$. We shall preserve the notation $\mathcal{F}C_{\mathrm{b}}(C_0(X), \Gamma_X)$ for the set of all π_m-classes of functions from $\mathcal{F}C_{\mathrm{b}}(C_0(X), \Gamma_X)$. The set $\mathcal{F}C_{\mathrm{b}}(C_0(X), \Gamma_X)$ is dense in $L^2(\Gamma_X, \pi_m)$. We define

$$\begin{aligned} \mathcal{E}(F, G) &:= \int_{\Gamma_X} (D^- F(\gamma), D^- G(\gamma))_{T_\gamma}\, \pi_m(d\gamma) \\ &= \int_{\Gamma_X} \pi_m(d\gamma) \int_X \gamma(dx)\, D^-_x F(\gamma)\, D^-_x G(\gamma), \qquad F, G \in \mathcal{F}C_{\mathrm{b}}(C_0(X), \Gamma_X). \end{aligned} \tag{5}$$

Notice that, for any $F \in \mathcal{F}C_{\mathrm{b}}(C_0(X), \Gamma_X)$, there exists $\varphi \in C_0(X)$ such that $|D^-_x F(\gamma)| \leq \varphi(x)$ for all $\gamma \in \Gamma_X$ and $x \in \gamma$. Hence, due to (3), the right hand side of (5) is well defined. By (2), we also get, for $F, G \in \mathcal{F}C_{\mathrm{b}}(C_0(X), \Gamma_X)$,

$$\mathcal{E}(F, G) = \int_{\Gamma_X} \pi_m(d\gamma) \int_X m(dx)\, D^+_x F(\gamma)\, D^+_x G(\gamma), \tag{6}$$

where $D_x^+ F(\gamma) := F(\gamma \cup x) - F(\gamma)$.

Using (2), (3), and (5), we see that

$$\mathcal{E}(F,G) = \int_{\Gamma_X} (HF)(\gamma) G(\gamma)\, \pi_m(d\gamma), \qquad F, G \in \mathcal{F}C_{\mathrm{b}}(C_0(X), \Gamma_X), \tag{7}$$

where

$$(HF)(\gamma) = -\int_X D_x^+ F(\gamma)\, m(dx) - \int_X D_x^- F(\gamma)\, \gamma(dx) \tag{8}$$

and $HF \in L^2(\Gamma_X, \pi_m)$. Hence, the bilinear form $(\mathcal{E}, \mathcal{F}C_{\mathrm{b}}(C_0(X), \Gamma_X))$ is well-defined and closable on $L^2(\Gamma_X, \pi_m)$ and its closure will be denoted by $(\mathcal{E}, D(\mathcal{E}))$. Furthermore, using [11, Theorem 4.1], whose proof in the Poisson case admits a direct generalization to the case of the general space X, we conclude that the operator $(H, \mathcal{F}C_{\mathrm{b}}(C_0(X), \Gamma_X))$ is essentially selfadjoint in $L^2(\Gamma_X, \pi_m)$, and we denote its closure by $(H, D(H))$. In particular, $(H, D(H))$ is the generator of the bilinear form $(\mathcal{E}, D(\mathcal{E}))$.

Let us recall that the symmetric Fock space over $L^2(X,m)$ is defined as the real Hilbert space given by

$$\mathcal{F}(L^2(X,m)) := \bigoplus_{n=0}^{\infty} \mathcal{F}_n(L^2(X,m)),$$

where $\mathcal{F}_0(L^2(X,m)) := \mathbb{R}$ and, for $n \in \mathbb{N}$, $\mathcal{F}_n(L^2(X,m)) := L^2_{\mathrm{sym}}(X^n, m^{\otimes n})$ is the subspace of $L^2(X^n, m^{\otimes n})$ consisting of all symmetric functions.

The Poisson measure π_m possesses the chaos decomposition property, and hence the space $L^2(\Gamma_X, \pi_m)$ is unitarily isomorphic to the Fock space $\mathcal{F}(L^2(X,m))$, see e.g. [18]. More exactly, we set

$$\begin{aligned} \mathcal{F}(L^2(X,m)) \ni f &= (f^{(n)})_{n=0}^{\infty} \\ &\mapsto If := \sum_{n=0}^{\infty} (n!)^{-1/2} \int_{X^n} f^{(n)}(x_1, \dots, x_n)\, d\mathcal{X}(x_1) \cdots d\mathcal{X}(x_n) \in L^2(\Gamma_X, \pi_m). \end{aligned} \tag{9}$$

Here, $\int_{X^n} f^{(n)}(x_1, \dots, x_n)\, d\mathcal{X}(x_1) \cdots d\mathcal{X}(x_n)$ denotes the n-fold multiple stochastic integral of the function $f^{(n)}$ with respect to the Poisson random measure $\mathcal{X}(\Delta)(\gamma) := \gamma(\Delta)$, $\Delta \in \mathcal{B}(X)$, $\gamma \in \Gamma_X$, and the series on the right hand side of (9) converges in $L^2(\Gamma_X, \pi_m)$. Then, the operator I is unitary.

Denote the number operator on $\mathcal{F}(L^2(X,m))$ by N, i.e., the domain of N is given by

$$D(N) = \left\{ f = (f^{(n)})_{n=0}^{\infty} : \sum_{n=1}^{\infty} \|f^{(n)}\|^2_{\mathcal{F}_n(L^2(X,m))} n^2 < \infty \right\},$$

and $N \restriction \mathcal{F}_n(L^2(X,m)) = n\mathbf{1}$, $n \in \mathbb{Z}_+$, $\mathbf{1}$ being the identity operator. Evidently, N is a positive selfadjoint operator and, therefore, it generates a contraction semigroup $(e^{-tN})_{t \geq 0}$ in $\mathcal{F}(L^2(X,m))$. In fact, N is the second quantization of $\mathbf{1}$: $N = d\,\mathrm{Exp}(\mathbf{1})$, see e.g. [17, 2], so $e^{-tN} = \mathrm{Exp}(e^{-t\mathbf{1}})$.

Proposition 1 *Under the unitary operator I, the operator $(N, D(N))$ goes over into $(H, D(H))$.*

Proof. First, we note that, by the proof of [11, Lemma 3.2], formula (6) holds for all $F, G \in D(\mathcal{E})$. Denote by $\mathcal{FP}(C_0(X), \Gamma_X)$ the set of all functions of the form (4), where $N \in \mathbb{N}$, $\varphi_1, \dots, \varphi_N \in C_0(X)$, and g_F is a polynomial on $\mathbb{R}^N$. By the proof of [11, Theorem 4.1], $\mathcal{FP}(C_0(X), \Gamma_X) \subset D(H)$, and furthermore, H is essentially selfadjoint on $\mathcal{FP}(C_0(X), \Gamma_X)$. Analogously, we also have that $\mathcal{FP}(C_0(X), \Gamma_X) \subset D(\mathcal{E})$. Next, by [1, Theorem 5.1], $\mathcal{FP}(C_0(X), \Gamma_X)$ is a subset of the image of $D(N)$ under I, and we have

$$\begin{aligned}(INI^{-1}F, G)_{L^2(\Gamma_X, \pi_m)} &= \int_{\Gamma_X} \pi_m(d\gamma) \int_X m(dx)\, D_x^+ F(\gamma) D_x^+ G(\gamma) \\ &= \mathcal{E}(F, G) = (HF, G)_{L^2(\Gamma_X, \pi_m)}\end{aligned}$$

for all $F, G \in \mathcal{FP}(C_0(X), \Gamma_X)$. Hence, INI^{-1} coincides with H on $\mathcal{FP}(C_0(X), \Gamma_X)$. Since INI^{-1} is selfadjoint, this yields the statement. $\square$

Corollary 1 *Let*

$$\mathcal{C} := \{\varphi \in C_0(X) \mid -1 < \varphi \le 0\}.$$

Then, for each $\varphi \in \mathcal{C}$ *and* $t \ge 0$,

$$e^{-tH} \exp[\langle \log(1+\varphi), \cdot \rangle] = \exp[\langle \log(1 + e^{-t}\varphi), \cdot \rangle + (1 - e^{-t})\langle \varphi \rangle] \quad \pi_m\text{-a.e.} \tag{10}$$

Here, $\langle \varphi \rangle := \int_X \varphi(x)\, m(dx)$.

Proof. The statement follows directly from Proposition 1 by [18, formula (5.6)]. Let us shortly repeat the arguments.

For each $\varphi \in C_0(X)$, we introduce the vector

$$\operatorname{Exp} \varphi := \left(\frac{1}{n!}\, \varphi^{\otimes n} \right)_{n=0}^{\infty} \in \mathcal{F}(L^2(X, m)),$$

which is called the coherent state corresponding to the one-dimensional state φ. We have, for any $\varphi \in \mathcal{C}$,

$$I \operatorname{Exp} \varphi = \exp[\langle \log(1+\varphi), \cdot \rangle - \langle \varphi \rangle] \quad \pi_m\text{-a.e.}, \tag{11}$$

see e.g. [18, formula (5.5)]. Next, we evidently get

$$e^{-tN} \operatorname{Exp} \varphi = \operatorname{Exp}(e^{-t}\varphi). \tag{12}$$

By (11) and (12), (10) follows from Proposition 1. $\square$

3 Semigroup of the Glauber dynamics

We shall now construct probability kernels for the semigroup $(e^{-tH})_{t \ge 0}$. Let $t > 0$. Consider the two-point space $\{0, 1\}$ and define the probability measure p_t on $\{0, 1\}$ by setting $p_t(\{1\}) := e^{-t}$ and $p_t(\{0\}) := 1 - e^{-t}$. Next, let us fix an arbitrary configuration $\gamma \in \Gamma_X$, $\gamma \ne \varnothing$. On the product-space $\{0,1\}^\gamma$ equipped with the product σ-algebra, we define the product-measure $\bigotimes_{x\in\gamma} p_{t,x}$, where $p_{t,x}\equiv p_t$ for each $x\in\gamma$, . Set

$$\{0,1\}^\gamma \ni a = (a(x))_{x\in\gamma} \mapsto \mathcal{U}(a) = \sum_{x\in\gamma} a(x)\varepsilon_x \in \Gamma_X.$$

It is easy to see that the mapping $\mathcal{U}$ is measurable. We denote by $P_{t,\gamma}$ the probability measure on Γ_X that is the image of the measure $\bigotimes_{x\in\gamma} p_{t,x}$ under $\mathcal{U}$. We also set $P_{t,\varnothing} := \varepsilon_\varnothing$.

Now, for any $\gamma\in\Gamma_X$ and $t>0$, we define $\mathbf{P}_{t,\gamma}$ as the probability measure on Γ_X given by the convolution of the measures $P_{t,\gamma}$ and $\pi_{(1-e^{-t})m}$, i.e.,

$$\mathbf{P}_{t,\gamma}(A) := \int_{\Gamma_X} P_{t,\gamma}(d\eta_1) \int_{\Gamma_X} \pi_{(1-e^{-t})m}(d\eta_2)\, \mathbf{1}_A(\eta_1+\eta_2), \qquad A\in\mathcal{B}(\Gamma_X). \tag{13}$$

Indeed, denote by $\ddot{\Gamma}_X$ the space of all $\mathbb{N}_0\cup\{\infty\}$-valued Radon measures on X equipped with the vague topology. Then, Γ_X is a Borel-measurable subset of $\ddot{\Gamma}_X$. It is easy to see that the mapping

$$\Gamma_X\times\Gamma_X \ni (\eta_1,\eta_2)\mapsto \eta_1+\eta_2\in\ddot{\Gamma}_X$$

is measurable. Therefore, considering $\mathbf{1}_A$ as an indicator function defined on $\ddot{\Gamma}_X$, we see that (13) defines a measure on Γ_X. To see that $\mathbf{P}_{t,\gamma}$ is a probability measure, we have to show that, for $P_{t,\gamma}\otimes\pi_{(1-e^{-t})m}$-a.e. $(\eta_1,\eta_2)\in\Gamma_X\times\Gamma_X$, $\eta_1+\eta_2$ belongs to Γ_X. By construction, $P_{t,\gamma}$-a.e. $\eta_1\in\Gamma_X$ is a subset of γ. On the other hand, the set γ is of zero m measure and, therefore, $\pi_{(1-e^{-t})m}$-a.e. configuration $\eta_2\in\Gamma_X$ has empty intersection with γ, which implies the statement.

Theorem 1 *For each $F\in L^2(\Gamma_X,\pi_m)$ and $t>0$, we have*

$$(e^{-tH}F)(\gamma) = \int_{\Gamma_X} F(\eta)\,\mathbf{P}_{t,\gamma}(d\eta) \tag{14}$$

for π_m-a.e. $\gamma\in\Gamma_X$.

Proof. For $\gamma\in\Gamma_X$ and $\varphi\in\mathcal{C}$, we have by (1), (13), and the construction of $P_{t,\gamma}$:

$$\begin{aligned}
&\int_{\Gamma_X} \exp[\langle \log(1+\varphi),\eta\rangle]\,\mathbf{P}_{t,\gamma}(d\eta)\\
&\quad= \int_{\Gamma_X} P_{t,\gamma}(d\eta_1)\int_{\Gamma_X}\pi_{(1-e^{-t})m}(d\eta_2)\exp[\langle\log(1+\varphi),\eta_1+\eta_2\rangle]\\
&\quad= \int_{\Gamma_X} P_{t,\gamma}(d\eta_1)\prod_{x\in\eta_1}(1+\varphi(x))\int_{\Gamma_X}\pi_{(1-e^{-t})m}(d\eta_2)\exp[\langle\log(1+\varphi),\eta_2\rangle]\\
&\quad= \left(\prod_{x\in\gamma\cap\operatorname{supp}\varphi}((1+\varphi(x))e^{-t}+(1-e^{-t}))\right)\exp[(1-e^{-t})\langle\varphi\rangle]\\
&\quad= \exp[\langle\log(1+e^{-t}\varphi),\gamma\rangle + (1-e^{-t})\langle\varphi\rangle],
\end{aligned} \tag{15}$$

where $\prod_{x\in\varnothing} c_x := 1$.

Next, for any measurable function $F : \Gamma_X \to [0, +\infty]$, we have

$$\int_{\Gamma_X}\int_{\Gamma_X} F(\eta)\, \mathbf{P}_{t,\gamma}(d\eta)\, \pi_m(d\gamma) = \int_{\Gamma_X} F(\gamma)\, \pi_m(d\gamma). \tag{16}$$

Indeed, it is easy to check that $\{\exp[\langle \log(1+\varphi), \cdot\rangle] \mid \varphi \in \mathcal{C}\}$ is stable under multiplication and contains a countable subset separating the points of Γ_X, so it generates $\mathcal{B}(\Gamma_X)$. Therefore, we only have to check (16) for $F = \exp[\langle \log(1+\varphi), \cdot\rangle]$, $\varphi \in \mathcal{C}$. But this immediately follows from (1) and (15).

Now, by (15) and (16), the statement of the theorem follows analogously to the proof of [12, Theorem 5.1]. For the convenience of the reader, we repeat the arguments. It follows from (16) that, if $A \in \mathcal{B}(\Gamma_X)$ is of zero π_m measure, then $\mathbf{P}_{t,\gamma}(A) = 0$ for π_m-a.e. $\gamma \in \Gamma_X$. Moreover, using the Cauchy–Schwarz inequality and (16), we get

$$\begin{aligned}\int_{\Gamma_X}\left(\int_{\Gamma_X} F(\eta)\, \mathbf{P}_{t,\gamma}(d\eta)\right)^2 \pi_m(d\gamma) &\le \int_{\Gamma_X}\int_{\Gamma_X} |F(\eta)|^2\, \mathbf{P}_{t,\gamma}(d\eta)\, \pi_m(d\gamma)\\ &= \int_{\Gamma_X} |F(\gamma)|^2\, \pi_m(d\gamma).\end{aligned}$$

Thus, for each $t > 0$, we can define a linear continuous operator

$$\mathbf{P}_t : L^2(\Gamma_X, \pi_m) \to L^2(\Gamma_X, \pi_m)$$

by setting

$$(\mathbf{P}_t F)(\gamma) := \int_{\Gamma_X} F(\eta)\, \mathbf{P}_{t,\gamma}(d\eta).$$

By Corollary 1 and (15), the action of the operator $\mathbf{P}_t$ coincides with the action of e^{-tH} on the set $\{\exp[\langle \log(1+\varphi), \cdot\rangle] \mid \varphi \in \mathcal{C}\}$, which is total in $L^2(\Gamma_X, \pi_m)$ (i.e., its linear hull is a dense set in $L^2(\Gamma_X, \pi_m)$). Hence, we get the equality $\mathbf{P}_t = e^{-tH}$, which proves the theorem. $\square$

In what follows, for a measurable function F on Γ_X, we set

$$(\mathbf{P}_t F)(\gamma) := \int_{\Gamma_X} F(\eta)\, \mathbf{P}_{t,\gamma}(d\eta), \qquad t > 0,\ \gamma \in \Gamma_X,$$

provided the integral on the right hand side exists. Thus, by Theorem 1, $\mathbf{P}_t F$ is a π_m-version of e^{-tH} for each $F \in L^2(\Gamma_X, \pi_m)$.

We also define a family of probability kernels $(\mathbf{P}_t)_{t\ge 0}$ on the space $(\Gamma_X, \mathcal{B}(\Gamma_X))$ setting

$$\mathbf{P}_t(\gamma, A) := \mathbf{P}_{t,\gamma}(A), \qquad \gamma \in \Gamma_X,\ A \in \mathcal{B}(\Gamma_X),\ t \ge 0.$$

Since $\gamma \mapsto \mathbf{P}_t F(\gamma)$ is measurable for F in the linear span of $\{\exp[\langle \log(1+\varphi), \cdot\rangle] \mid \varphi \in \mathcal{C}\}$ by (15), a monotone class argument shows that, indeed, $\gamma \mapsto \mathbf{P}_t(\gamma, A)$ is $\mathcal{B}(\Gamma_X)$-measurable for all $A \in \mathcal{B}(\Gamma_X)$.

Proposition 2 *$(\mathbf{P}_t)_{t\ge 0}$ is a Markov semigroup of kernels on $(\Gamma_X, \mathcal{B}(\Gamma_X))$.*

Proof. We only have to prove the semigroup property: $\mathbf{P}_t\mathbf{P}_s = \mathbf{P}_{t+s}$, $t, s \ge 0$. Let $\varphi \in \mathcal{C}$. By (1) and (15), we get, for any $\gamma \in \Gamma_X$,

$$\begin{aligned}
&\int_{\Gamma_X}\int_{\Gamma_X} \exp[\langle \log(1+\varphi), \eta_1\rangle]\, \mathbf{P}_{s,\eta}(d\eta_1)\, \mathbf{P}_{t,\gamma}(d\eta)\\
&\quad = \int_{\Gamma_X} \exp[\langle \log(1+e^{-s}\varphi), \eta\rangle + (1-e^{-s})\langle\varphi\rangle]\, \mathbf{P}_{t,\gamma}(d\eta)\\
&\quad = \exp[\langle \log(1+e^{-(t+s)}\varphi), \gamma\rangle + (1+e^{-t})e^{-s}\langle\varphi\rangle + (1-e^{-s})\langle\varphi\rangle]\\
&\quad = \exp[\langle \log(1+e^{-(t+s)}\varphi), \gamma\rangle + (1-e^{-(t+s)})\langle\varphi\rangle]\\
&\quad = \int_{\Gamma_X} \exp[\langle \log(1+\varphi), \eta\rangle]\, \mathbf{P}_{t+s,\gamma}(d\eta). \qquad (17)
\end{aligned}$$

Analogously to the proof of (16), we conclude from (17) that

$$\int_{\Gamma_X}\int_{\Gamma_X} F(\eta_1)\, \mathbf{P}_{s,\eta}(d\eta_1)\, \mathbf{P}_{t,\gamma}(d\eta) = \int_{\Gamma_X} F(\eta)\, \mathbf{P}_{t+s,\gamma}(d\eta)$$

holds for any measurable function $F : \Gamma_X \to [0,\infty]$, in particular, for $F = \mathbf{1}_A$, $A \in \mathcal{B}(\Gamma_X)$. $\square$

Theorem 2 *The semigroup $(\mathbf{P}_t)_{t\ge 0}$ possesses the Feller property, i.e., $\mathbf{P}_t : C_{\mathrm{b}}(\Gamma_X) \to C_{\mathrm{b}}(\Gamma_X)$, $t \ge 0$, where $C_{\mathrm{b}}(\Gamma_X)$ denotes the set of all continuous bounded functions on Γ_X.*

Proof. First, we note that the space Γ_X is metrizable, see e.g. [9], and so the continuity of a function $G : \Gamma_X \to \mathbb{R}$ follows if we can show the convergence $G(\gamma_n) \to G(\gamma)$ as $\gamma_n \to \gamma$ vaguely in Γ_X .

Let us fix any $x_0 \in X$ and denote by $B(r)$ an open ball in X centered at x_0 and of radius $r > 0$ with respect to the metric ρ. We recall that $\gamma_n \to \gamma$ vaguely in Γ_X if and only if, for any $r > 0$, there exists $N \in \mathbb{N}$ such that $|\gamma_n \cap B(r)| = |\gamma \cap B(r)| =: l$ for all $n \ge N$ and there exists a numeration of the points of the configurations $\gamma_n \cap B(r)$, $n \ge N$, and $\gamma \cap B(r)$, denoted by $\{x_k^{(n)}\}_{k=1}^l$, $\{x_k\}_{k=1}^l$, respectively, such that $x_k^{(n)} \to x_k$ in X as $n \to \infty$, $k = 1, \dots, l$. From here, by an easy modification of the proof of (6.9) in [12], we can conclude the following

Claim. Let $\gamma_n \to \gamma$ as $n \to \infty$ vaguely in Γ_X. Then, there exists a numeration of the points of the configurations γ_n, $n \in \mathbb{N}$, and γ, denoted by $\{x_k^{(n)}\}_{k\ge 1}$, $\{x_k\}_{k\ge 1}$, respectively, such that: 1) either for each $k \in \mathbb{N}$ if $|\gamma| = \infty$, or for each $k \in \{1, \dots, |\gamma|\}$ if $|\gamma| < \infty$, there exists $N \in \mathbb{N}$ such that $|\gamma_n| \ge k$ for all $n \ge N$ and $x_k^{(n)} \to x_k$ in X as $n \to \infty$.

So, we fix any $t > 0$, $F \in C_{\mathrm{b}}(\Gamma_X)$ and γ_n, $n \in \mathbb{N}$, γ from Γ_X such that $\gamma_n \to \gamma$ vaguely, and we fix a numeration of the points of γ_n, $n \in \mathbb{N}$, and γ as described in the claim.

Analogously to the above, on the product-space $\{0,1\}^{\mathbb{N}}$ we consider the product measure $\bigotimes_{k\in\mathbb{N}} p_{t,k}$, where $p_{t,k} \equiv p_t$. Then, as easily seen from (13),

$$(\mathbf{P}_t F)(\gamma_n) = \int_{\{0,1\}^{\mathbb{N}}} \bigotimes_{k\in\mathbb{N}} p_{t,k}(d(a_1, a_2, \dots)) \int_{\Gamma_X} \pi_{(1-e^{-t})m}(d\eta)\, F\left(\sum_{k=1}^{|\gamma_n|} a_k \varepsilon_{x_k^{(n)}} + \eta\right). \qquad (18)$$

Set $D := (\bigcup_{n=1}^{\infty} \gamma_n) \cup \gamma$. Since D has zero m measure, $\pi_{(1-e^{-t})m}$-a.e. $\eta \in \Gamma_X$ has empty intersection with D. Furthermore, for any $\eta \in \Gamma_X$ with empty intersection with D and for any $(a_1, a_2, \dots) \in \{0,1\}^{\mathbb{N}}$, we easily get, using the claim, that

$$\sum_{k=1}^{|\gamma_n|} a_k \varepsilon_{x_k^{(n)}} + \eta \to \sum_{k=1}^{|\gamma|} a_k \varepsilon_{x_k} + \eta \quad \text{vaguely in } \Gamma_X \text{ as } n \to \infty. \tag{19}$$

By the monotone convergence theorem, we now have from (18) and (19):

$$\begin{aligned}(\mathbf{P}_t F)(\gamma_n) \to & \int_{\{0,1\}^{\mathbb{N}}} \bigotimes_{k\in\mathbb{N}} p_{t,k}(d(a_1, a_2, \dots)) \int_{\Gamma_X} \pi_{(1-e^{-t})m}(d\eta)\, F\left(\sum_{k=1}^{|\gamma|} a_k \varepsilon_{x_k} + \eta\right) \\ & = (\mathbf{P}_t F)(\gamma) \quad \text{as } n \to \infty,\end{aligned}$$

which yields the theorem. $\square$

Before formulating the next theorem, let us recall that we use the notation $\ddot{\Gamma}_X$ for the space of all $\mathbb{N}_0 \cup \{\infty\}$-valued Radon measures on X endowed with the vague topology.

Theorem 3 *The semigoup $(\mathbf{P}_t)_{t\ge 0}$ is ergodic in the following sense: For an arbitrary probability measure μ on $(\Gamma_X, \mathcal{B}(\Gamma_X))$, the image measure $\mathbf{P}_t\mu$ of μ under $\mathbf{P}_t$ converges to π_m weakly on $\ddot{\Gamma}_X$ as $t \to \infty$.*

Proof. It is evidently enough to prove that, for each $\gamma \in \Gamma_X$, $\mathbf{P}_{t,\gamma}$ converges to π_m weakly on $\ddot{\Gamma}_X$ as $t \to \infty$. To this end, it suffices to show that, for each $\varphi \in C_0(X)$, $\varphi \le 0$,

$$\int_{\Gamma_X} e^{\langle \varphi, \eta\rangle}\, \mathbf{P}_{t,\gamma}(d\eta) \to \exp\left(\int_X (e^{\varphi(x)} - 1)\, m(dx)\right) \quad \text{as } t \to \infty, \text{ for each } \gamma \in \Gamma_X, \tag{20}$$

see (1) and [8], Chapter 4, in particular, Theorem 4.2. But analogously to (15), we get:

$$\int_{\Gamma_X} e^{\langle \varphi, \eta\rangle}\, \mathbf{P}_{t,\gamma}(d\eta) = \prod_{x\in\gamma\cap \operatorname{supp}\varphi} (e^{-t}(e^{\varphi(x)} - 1) + 1) \exp\left((1 - e^{-t}) \int_X (e^{\varphi(x)} - 1)\, m(dx)\right). \tag{21}$$

Since the support of φ is compact, we have $|\gamma \cap \operatorname{supp}\varphi| < \infty$, which implies that the right hand side of (21) converges to the right hand side of (20) as $t \to \infty$. $\square$

4 Glauber dynamics

A Markov process on Γ_X with the generator H we shall call a Glauber dynamics of a continuous system of free particles.

Theorem 4 *The Glauber dynamics of a continuous system of free particles may be realized as the unique, time homogeneous Markov process*

$$\mathbf{M} = (\boldsymbol{\Omega}, \mathbf{F}, (\mathbf{F}_t)_{t\ge 0}, (\boldsymbol{\theta}_t)_{t\ge 0}, (\mathbf{P}_\gamma)_{\gamma\in\Gamma_X}, (\mathbf{X}_t)_{t\ge 0})$$

on the state space $(\Gamma_X, \mathcal{B}(\Gamma_X))$ with transition probability function $(\mathbf{P}_t)_{t\ge 0}$ and with cadlag paths, i.e., right continuous on $[0, \infty)$ and having left limits on $(0, \infty)$ (cf. e.g. [5]).

Proof. In what follows, for a metric space E, we shall denote by $D([0,\infty),E)$ the space of all cadlag functions from $[0,\infty)$ into E equipped with the corresponding cylinder σ-algebra.

Now, we consider on the two-point space $\{0,1\}$ the time homogeneous Markov process with cadlag paths, whose transition probabilities are given by $p_{t,1}(\{1\}) = e^{-t}$, $p_{t,1}(\{0\}) = 1-e^{-t}$, and $p_{t,0}(\{0\}) = 1$, $t \ge 0$. Let $\Omega := D([0,\infty),\{0,1\})$ and let $\bar p$ denote the probability measure on Ω defined as the law of the above process starting at 1.

For an arbitrary configuration $\gamma \in \Gamma_X$, $\gamma \neq \varnothing$, on the product space Ω^γ we consider the product measure $\bigotimes_{x\in\gamma} \bar p_x$, where $\bar p_x := \bar p$, $x \in \gamma$. We define

$$\Omega^\gamma \ni \omega = (\omega_x)_{x\in\gamma} \mapsto Y(\omega) = Y(\omega)(t) := \sum_{x\in\gamma} \omega_x(t)\varepsilon_x, \quad t \ge 0. \tag{22}$$

As easily seen, for each $\omega \in \Omega^\gamma$, $Y(\omega)$ is an element of $\mathbf{\Omega} := D([0,\infty),\Gamma_X)$ and, furthermore, the mapping Y is measurable. Let P^Y_γ denote the image of $\bigotimes_{x\in\gamma} \bar p_x$ under Y. We also set $P^Y_\varnothing$ to be the delta measure concentrated at $Y(t) \equiv \varnothing$, $t \ge 0$.

Next, we consider the configuration space $\Gamma_{X\times\mathbb{R}_+}$ over $X \times \mathbb{R}_+$, where $\mathbb{R}_+ := (0,+\infty)$. Let $\pi_{m\otimes d\tau}$ denote the Poisson measure on $\mathcal{B}(\Gamma_{X\times\mathbb{R}_+})$ with intensity measure $m \otimes d\tau$, where $d\tau$ denotes the Lebesgue measure on $\mathbb{R}_+$. By $\widetilde\Gamma_{X\times\mathbb{R}_+}$ we denote the subset of $\Gamma_{X\times\mathbb{R}_+}$ which consists of those configurations ξ for which $(x,t),(x',t') \in \xi$, $(x,t) \neq (x',t')$ implies $x \neq x'$. It is not hard to see that $\widetilde\Gamma_{X\times\mathbb{R}_+} \in \mathcal{B}(\Gamma_{X\times\mathbb{R}_+})$. Furthermore, since the measure m is non-atomic, we have that $\widetilde\Gamma_{X\times\mathbb{R}_+}$ is of full $\pi_{m\otimes d\tau}$ measure.

Now, for any $\xi \in \widetilde\Gamma_{X\times\mathbb{R}_+}$, $\xi \neq \varnothing$, we consider on the product space Ω^ξ the product measure $\bigotimes_{(y,\tau)\in\xi} \bar p_{(y,\tau)}$, where $\bar p_{(y,\tau)} \equiv \bar p$, and define

$$\Omega^\xi \ni \omega = (\omega_{(y,\tau)})_{(y,\tau)\in\xi} \mapsto Z(\omega) = Z(\omega)(t) := \sum_{(y,\tau)\in\xi} \mathbf{1}_{[\tau,\infty)}(t)\omega_{(y,\tau)}(t-\tau)\varepsilon_y. \tag{23}$$

Again, for each $\omega \in \Omega^\xi$, $Z(\omega) \in \mathbf{\Omega}$, Z is measurable, and by P^Z_ξ we denote the probability measure on $\mathbf{\Omega}$ that is the law of the process Z. Furthermore, a monotone class argument shows that, for each measurable set A in $\mathbf{\Omega}$, the mapping $\widetilde\Gamma_{X\times\mathbb{R}_+} \ni \xi \mapsto P^Z_\xi(A) \in \mathbb{R}$ is measurable. Therefore, we can define the probability measure

$$P^Z := \int_{\widetilde\Gamma_{X\times\mathbb{R}_+}} \pi_{m\otimes d\tau}(d\xi) P^Z_\xi \tag{24}$$

on $(\mathbf{\Omega},\mathbf{F})$, where $\mathbf{F}$ denotes the cylinder σ-algebra on $\mathbf{\Omega}$.

Analogously to (13), we define, for each $\gamma \in \Gamma_X$, $\mathbf{P}_\gamma$ as the probability measure on $(\mathbf{\Omega},\mathbf{F})$ given by

$$\mathbf{P}_\gamma(A) := \int_{\mathbf{\Omega}} P^Y_\gamma(d\omega_1) \int_{\mathbf{\Omega}} P^Z(d\omega_2)\,\mathbf{1}_A(\omega_1+\omega_2), \qquad A \in \mathbf{F}. \tag{25}$$

(Notice that $\omega_1+\omega_2 \in \mathbf{\Omega}$ for $P^Y_\gamma \otimes P^Z$-a.e. $(\omega_1,\omega_2) \in \mathbf{\Omega}$.) We evidently have that $\omega(0) = \gamma$ for $\mathbf{P}_\gamma$-a.e. $\omega \in \mathbf{\Omega}$.

Define now $\mathbf{X}_t(\omega) := \omega(t)$, $\omega \in \mathbf{\Omega}$, $t \ge 0$, and $\mathbf{F}_t := \sigma\{\mathbf{X}_s,\ 0 \le s \le t\}$. Define the translations $(\boldsymbol{\theta}\omega)(s) := \omega(s+t)$, $\omega \in \mathbf{\Omega}$, $s,t \ge 0$, and

$$\mathbf{M} := (\mathbf{\Omega}, \mathbf{F}, (\mathbf{F}_t)_{t\ge0}, (\boldsymbol{\theta}_t)_{t\ge0}, (\mathbf{P}_\gamma)_{\gamma\in\Gamma_X}, (\mathbf{X}_t)_{t\ge0}).$$

To show that $\mathbf{M}$ is a time homogeneous Markov process on $(\Gamma_X, \mathcal{B}(\Gamma_X))$ with transition probability function $(\mathbf{P}_t)_{t\ge 0}$, it suffices to show that the finite-dimensional distributions of $\mathbf{X}_t$ are determined by the Markov semigroup of kernels $(\mathbf{P}_t)_{t\ge0}$ (see e.g. [4, Ch. 1, Sect. 3]).

By (24) and (25), we get, for any $\gamma\in\Gamma_X$, $0<t_1<t_2<\dots<t_n$, $\varphi_1,\dots,\varphi_n\in\mathcal{C}$, $n\in\mathbb{N}$,

$$\begin{aligned}&\int_{\mathbf{\Omega}}\mathbf{P}_\gamma(d\omega)\prod_{k=1}^n\exp[\langle\log(1+\varphi_k),\omega(t_k)\rangle]\\&\quad=\int_{\mathbf{\Omega}}P_\gamma^Y(d\omega_1)\prod_{k=1}^n\exp[\langle\log(1+\varphi_k),\omega_1(t_k)\rangle]\\&\qquad\times\int_{\widetilde{\Gamma}_{X\times\mathbb{R}_+}}\pi_{m\otimes d\tau}(d\xi)\int_{\mathbf{\Omega}}P_\xi^Z(d\omega_2)\prod_{k=1}^n\exp[\langle\log(1+\varphi_k),\omega_2(t_k)\rangle].\end{aligned}\tag{26}$$

By (22) and the construction of the measure $P_{t,\gamma}$, we evidently get:

$$\begin{aligned}&\int_{\mathbf{\Omega}}P_\gamma^Y(d\omega_1)\prod_{k=1}^n\exp[\langle\log(1+\varphi_k),\omega_1(t_k)\rangle]\\&\quad=\int_{\Gamma_X}P_{t_1,\gamma}(d\eta_1)\exp[\langle\log(1+\varphi_1),\eta_1\rangle]\int_{\Gamma_X}P_{t_2-t_1,\eta_1}(d\eta_2)\exp[\langle\log(1+\varphi_2),\eta_2\rangle]\\&\quad\times\dots\times\int_{\Gamma_X}P_{t_n-t_{n-1},\eta_{n-1}}(d\eta_n)\exp[\langle\log(1+\varphi_n),\eta_n\rangle].\end{aligned}\tag{27}$$

For any $\Delta_1,\dots,\Delta_k\in\mathcal{B}(X\times\mathbb{R}_+)$, $k\in\mathbb{N}$, which are of finite $m\otimes d\tau$ measure and disjoint, the random variables $\langle\mathbf{1}_{\Delta_1},\cdot\rangle,\dots,\langle\mathbf{1}_{\Delta_k},\cdot\rangle$ are independent under $\pi_{m\otimes d\tau}$. Therefore, by (23),

$$\begin{aligned}&\int_{\widetilde{\Gamma}_{X\times\mathbb{R}_+}}\pi_{m\otimes d\tau}(d\xi)\int_{\mathbf{\Omega}}P_\xi^Z(d\omega_2)\prod_{k=1}^n\exp[\langle\log(1+\varphi_k),\omega_2(t_k)\rangle]\\&\quad=\int_{\widetilde{\Gamma}_{X\times\mathbb{R}_+}}\pi_{m\otimes d\tau}(d\xi)\prod_{(y,\tau)\in\xi:\,y\in\bigcup_{k=1}^n\operatorname{supp}\varphi_k}\int_\Omega\bar p(d\omega_{(y,\tau)})\\&\qquad\times\prod_{k=1}^n\exp[\langle\log(1+\varphi_k),\mathbf{1}_{[\tau,\infty)}(t_k)\omega_{(y,\tau)}(t_k-\tau)\varepsilon_y\rangle]\\&\quad=\prod_{k=1}^n\int_{\widetilde{\Gamma}_{X\times(t_{k-1},t_k)}}\pi_{m\otimes d\tau}(d\xi)\prod_{(y,\tau)\in\xi}\int_\Omega\bar p(d\omega_{(y,\tau)})\\&\qquad\times\prod_{l=k}^n\exp[\langle\log(1+\varphi_l),\omega_{(y,\tau)}(t_l-\tau)\varepsilon_y\rangle],\end{aligned}\tag{28}$$

where $t_0:=0$. Using the Markov property, we get, for any fixed $k\in\{1,\dots,n\}$ and $(y,\tau)\in X\times(t_{k-1},t_k)$:

$$\int_\Omega\bar p(d\omega)\prod_{l=k}^n\exp[\langle\log(1+\varphi_l),\omega(t_l-\tau)\varepsilon_y\rangle]$$

$$= e^{-(t_k-\tau)}(1+\varphi_k(y))\int_{\{0,1\}} p_{t_{k+1}-t_k,1}(da_{k+1})\exp[\langle\log(1+\varphi_{k+1}), a_{k+1}\varepsilon_y\rangle]$$

$$\times\cdots\times\int_{\{0,1\}} p_{t_n-t_{n-1},a_{n-1}}(da_n)\exp[\langle\log(1+\varphi_n), a_n\varepsilon_y\rangle] + (1-e^{-(t_k-\tau)}). \tag{29}$$

Hence, by (1), (28), and (29), we get:

$$\int_{\widetilde{\Gamma}_{X\times\mathbb{R}_+}} \pi_{m\otimes d\tau}(d\xi)\int_{\mathbf{\Omega}} P_\xi^Z(d\omega_2)\prod_{k=1}^n \exp[\langle\log(1+\varphi_k), \omega_2(t_k)\rangle]$$

$$= \exp\Bigg[\sum_{k=1}^n (1-e^{-(t_k-t_{k-1})})\int_X m(dy)\Bigg(-1+(1+\varphi_k(y))$$

$$\times\int_{\{0,1\}} p_{t_{k+1}-t_k,1}(da_{k+1})\exp[\langle\log(1+\varphi_{k+1}), a_{k+1}\varepsilon_y\rangle]$$

$$\times\cdots\times\int_{\{0,1\}} p_{t_n-t_{n-1},a_{n-1}}(da_n)\exp[\langle\log(1+\varphi_n), a_n\varepsilon_y\rangle]\Bigg)\Bigg]. \tag{30}$$

It is easy to see that, for any $t>0$ and any $\gamma_1,\gamma_2\in\Gamma_X$, $\gamma_1\cap\gamma_2=\varnothing$, the measure $P_{t,\gamma_1+\gamma_2}$ on Γ_X is the convolution of the measures P_{t,γ_1} and P_{t,γ_2}. Therefore, by (1), (13), (26), (27), and (30),

$$\int_{\Gamma_X}\mathbf{P}_{t_1,\gamma}(d\gamma_1)\exp[\langle\log(1+\varphi_1),\gamma_1\rangle]\int_{\Gamma_X}\mathbf{P}_{t_2-t_1,\gamma_1}(d\gamma_2)\exp[\langle\log(1+\varphi_2),\gamma_2\rangle] \tag{31}$$

$$\times\cdots\times\int_{\Gamma_X}\mathbf{P}_{t_n-t_1,\gamma_{n-1}}(d\gamma_n)\exp[\langle\log(1+\varphi_n),\gamma_n\rangle]$$

$$= \int_{\Gamma_X} P_{t_1,\gamma}(d\eta_1)\int_{\Gamma_X}\pi_{(1-e^t)m}(d\theta_1)\exp[\langle\log(1+\varphi_1),\eta_1+\theta_1\rangle]$$

$$\times\int_{\Gamma_X} P_{t_2-t_1,\eta_1+\theta_1}(d\eta_2)\int_{\Gamma_X}\pi_{(1-e^{-(t_2-t_1)})m}(d\theta_2)\exp[\langle\log(1+\varphi_2),\eta_2+\theta_2\rangle]$$

$$\times\cdots\times\int_{\Gamma_X} P_{t_n-t_{n-1},\eta_{n-1}+\theta_{n-1}}(d\eta_n)$$

$$\times\int_{\Gamma_X}\pi_{(1-e^{-(t_n-t_{n-1})})m}(d\theta_n)\exp[\langle\log(1+\varphi_n),\eta_n+\theta_n\rangle]$$

$$= \int_{\Gamma_X} P_{t_1,\gamma}(d\eta_1)\exp[\langle\log(1+\varphi_1),\eta_1\rangle]\int_{\Gamma_X} P_{t_2-t_1,\eta_1}(d\eta_2)\exp[\langle\log(1+\varphi_2),\eta_2\rangle]$$

$$\times\cdots\times\int_{\Gamma_X} P_{t_n-t_{n-1},\eta_{n-1}}(d\eta_n)\exp[\langle\log(1+\varphi_n),\eta_n\rangle]$$

$$\times\prod_{k=1}^n\int_{\Gamma_X}\pi_{(1-e^{-(t_k-t_{k-1})})m}(d\theta_k)\exp[\langle\log(1+\varphi_k),\theta_k\rangle]$$

$$\times\int_{\Gamma_X} P_{t_{k+1}-t_k,\theta_k}(d\eta_{k+1})\exp[\langle\log(1+\varphi_{k+1}),\eta_{k+1}\rangle]$$

$$\times\cdots\times\int_{\Gamma_X} P_{t_n-t_{n-1},\eta_{n-1}}(d\eta_n)\exp[\langle\log(1+\varphi_n),\eta_n\rangle]$$

$$
\begin{aligned}
&= \int_{\mathbf{\Omega}} P^Y_\gamma(d\omega_1) \prod_{k=1}^n \exp[\langle \log(1+\varphi_k), \omega_1(t_k)\rangle] \\
&\quad \times \prod_{k=1}^n \int_{\Gamma_X} \pi_{(1-e^{-(t_k-t_{k-1})})m}(d\theta_k) \exp[\langle \log(1+\varphi_k), \theta_k\rangle] \\
&\quad \times \prod_{x\in\theta_k} (1+\varphi_k(x)) \int_{\{0,1\}} p_{t_{k+1}-t_k,1}(da_{k+1}) \exp[\langle \log(1+\varphi_{k+1}), a_{k+1}\varepsilon_x\rangle] \\
&\quad \times \cdots \times \int_{\{0,1\}} p_{t_n-t_{n-1},a_{n-1}}(da_n) \exp[\langle \log(1+\varphi_n), a_n\varepsilon_x\rangle] \\
&= \int_{\mathbf{\Omega}} \mathbf{P}_\gamma(d\omega) \prod_{k=1}^n \exp[\langle \log(1+\varphi_k), \omega(t_k)\rangle]. \qquad (32)
\end{aligned}
$$

Hence, analogously to the proof of (16), we conclude from (32) that, for any $A_1, \dots, A_n \in \mathcal{B}(\Gamma_X)$,

$$
\mathbf{P}_\gamma(\mathbf{X}_{t_1} \in A_1, \dots, \mathbf{X}_n \in A_n) = \int_{A_1} \mathbf{P}_{t_1,\gamma}(d\gamma_1) \int_{A_2} \mathbf{P}_{t,\gamma_1}(d\gamma_2) \cdots \int_{A_n} \mathbf{P}_{t_n,\gamma_{n-1}}(d\gamma_n).
$$

Finally, we note that any measure on the space $(\mathbf{\Omega}, \mathbf{F})$ is uniquely determined by its finite-dimensional distributions, and therefore the constructed Markov process with cadlag paths is unique. □

Corollary 2 *The Markov process* $\mathbf{M}$ *from Theorem* 4 *is a strong Markov process.*

Proof. The statement follows directly from Theorems 2, 4 and [6, Theorem 5.10]. □

Remark 1 It is easy to see that the process $(\mathbf{X}_t)_{t\ge0}$ constructed in the course of the proof of Theorem 4 is even Markov with respect to the filtration $(\mathbf{F}_{t+})_{t\ge0}$, where $\mathbf{F}_{t+} := \bigcap_{s>t} \mathbf{F}_t$.

Suppose that the space X is non-compact and by $\Gamma_{X,\,\mathrm{inf}}$ denote the subset of Γ_X consisting of all infinite configurations $\gamma \in \Gamma_X$. It is not hard to show that $\Gamma_{X,\,\mathrm{inf}} \in \mathcal{B}(\Gamma_X)$. We endow $\Gamma_{X,\,\mathrm{inf}}$ with the vague topology and denote by $\mathcal{B}(\Gamma_{X,\,\mathrm{inf}})$ the corresponding Borel σ-algebra.

Corollary 3 *Suppose that* $m(X) = \infty$. *Then, in the formulation of Proposition* 2, *Theorems* 1, 2, 4, *and Corollary* 2, *we can replace* Γ_X *by* $\Gamma_{X,\,\mathrm{inf}}$.

Proof. It is well known that the condition

$$
m(X) = \infty
$$

implies that $\pi_{zm}(\Gamma_{X,\,\mathrm{inf}}) = 1$, $z > 0$. Furthermore, by (13),

$$
\mathbf{P}_{t,\gamma}(\Gamma_{X,\,\mathrm{inf}}) = 1, \qquad t > 0,\ \gamma \in \Gamma_{X,\,\mathrm{inf}}.
$$

Therefore, by Proposition 2, $(\mathbf{P}_t)_{t\ge0}$ is a Markov semigroup of kernels on $(\Gamma_{X,\,\mathrm{inf}}, \mathcal{B}(\Gamma_{X,\,\mathrm{inf}}))$, and we can replace Γ_X by $\Gamma_{X,\,\mathrm{inf}}$ in Theorems 1, 2.

Next, by using the Borel-Cantelli lemma, we easily get:

$$P_{t,\gamma}(\Gamma_{X,\mathrm{inf}}) = 1, \qquad t > 0,\ \gamma \in \Gamma_{X,\mathrm{inf}}. \tag{33}$$

By the construction of P_γ^Y, we also have:

$$P_\gamma^Y(\omega \in \mathbf{\Omega} : \omega(\tau) \supset \omega(t)\, \forall \tau \in [0,t]) = 1, \qquad t \in \mathbb{N},\ \gamma \in \Gamma_{X,\mathrm{inf}}. \tag{34}$$

Taking (33) by (34) into account, we can now replace Γ_X by $\Gamma_{X,\mathrm{inf}}$ in the proof of Theorem 4, to get the respective modification of the latter theorem. $\square$

Remark 2 Notice that $D([0,\infty),\Gamma_{X,\mathrm{inf}})$ does not belong to the cylinder σ-algebra on $D([0,\infty),\Gamma_X)$, so that in the proof of Corollary 3 we could not simply state that $D([0,\infty),\Gamma_{X,\mathrm{inf}})$ is a set of full $\mathbf{P}_\gamma$ measure for each $\gamma \in \Gamma_{X,\mathrm{inf}}$.

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