On tightness of capacities associated with sub-Markovian resolvents

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Abstract. We investigate the tightness property of the capacity induced by the reduction operator with respect to the resolvent of a right Markov process. We emphasize conditions for the tightness of capacity and quasi continuity property of the excessive functions, without assuming any sector condition. We extend and improve results of Lyons-Röckner, Ma-Röckner and Fitzsimmons, mainly obtained in the Dirichlet forms context.

1 Preliminaries and main results

Let $\mathcal{U} = (U_\alpha)_{\alpha>0}$ be a proper sub-Markovian resolvent of kernels on a Radon measurable space $(E, \mathcal{B})$ and denote by $\mathcal{E}(\mathcal{U})$ the set of all $\mathcal{U}$-excessive functions on E (recall that a positive $\mathcal{B}$-universally measurable numerical function s on E is termed **$\mathcal{U}$-excessive** if $\alpha U_\alpha s \leq s$ for all $\alpha > 0$ and $\sup_\alpha \alpha U_\alpha s = s$). We assume that the set $\mathcal{E}(\mathcal{U}) \cap p\mathcal{B}$ of $\mathcal{B}$-measurable $\mathcal{U}$-excessive functions on E is min-stable, contains the positive constant functions and generates $\mathcal{B}$; $p\mathcal{B}$ denotes the sets of all positive numerical $\mathcal{B}$-measurable functions on E. Such a resolvent will be called in the sequel **$\mathcal{B}$-sub-Markovian resolvent** on E.

Let m be a $\mathcal{U}$-excessive measure (i.e. m is a σ-finite measure on $(E, \mathcal{B})$ such that $m \circ \alpha U_\alpha \leq m$ for all $\alpha > 0$; see e.g. [7] or [12] for more details on excessive functions and measures). A σ-finite measure on $(E, \mathcal{B})$ of the form $\mu \circ U$ (where μ is a measure) is called **potential**. The set E is named m-**semisaturated** provided that every $\mathcal{U}$-excessive measure dominated by a potential $\mu \circ U$ with $\mu \circ U \ll m$ is also a potential.

We consider also a topology $\mathcal{T}$ on E, which is **natural** (with respect to $\mathcal{U}$), i.e. $(E, \mathcal{T})$ is a metrizable topological space with countable base such that every $\mathcal{T}$-open set is finely open and $\mathcal{B}$-measurable. We suppose in addition that the universally completions of $\mathcal{B}$ and $\mathcal{B}(E)$ coincide, where $\mathcal{B}(E)$ is the family of all Borel subsets of E. Notice that every Ray topology on E (i.e. the topology generated by a Ray cone, see Section 2 below) is a natural topology. We remark that if E is m-semisaturated with respect to $\mathcal{U}$ then there exists a second $\mathcal{B}$-sub-Markovian resolvent $\mathcal{U}'$ which coincides with $\mathcal{U}$ m-a.e., the topology $\mathcal{T}$ remains natural with respect to $\mathcal{U}'$ and in addition E is **semisaturated** with respect to $\mathcal{U}'$ (i.e. it is ξ-semisaturated for every $\mathcal{U}'$-excessive measure ξ).

Let $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, P^x)$ be a transient (Markov) right process with state space E, a Radon topological space, and let $\mathcal{U}$ be its associated sub-Markovian resolvent. Then (cf. [12]) there exists a σ-algebra $\mathcal{B}$ on E such that $\mathcal{B}(E) \subset \mathcal{B} \subset \mathcal{B}(E)^u$, $(E, \mathcal{B})$ is a Radon measurable space, $\mathcal{U}$ is a $\mathcal{B}$-sub-Markovian resolvent on E and the topology of E is natural (with respect to $\mathcal{U}$). Moreover the set E is semisaturated with respect to $\mathcal{U}$. Conversely, if $\mathcal{U}$ is a $\mathcal{B}$-sub-Markovian resolvent on a Lusin measurable space $(E, \mathcal{B})$, m is $\mathcal{U}$-excessive measure and E is m-semisaturated with respect to $\mathcal{U}$, then there exists a Lusin topology $\mathcal{T}$ on E such that $\mathcal{B}(E) = \mathcal{B}$ and a transient right process with state space E, such that its associated resolvent coincides with $\mathcal{U}$ m-almost everywhere (abbreviated m-a.e.).

If λ is a σ-finite measure on E we say that the right process X is **λ-standard** if it possesses left limits in E P^λ-a.e. on $[0, \zeta)$ and moreover for every increasing sequence $(T_n)_n$ of stopping times with $T_n \nearrow T$ we have $X_{T_n} \longrightarrow X_T$ P^λ-a.e. on $[T < \zeta]$, ζ being the life time of X.

We assume further that E is m-semisaturated with respect to $\mathcal{U}$.

For all $s \in \mathcal{E}(\mathcal{U})$ and every subset A of E we consider as usually the function

$$R^A s = \inf\{t \in \mathcal{E}(\mathcal{U}) / \, t \geq s \text{ on } A\},$$

called the **reduced function** of s on A. It is known that (see [1]) if $A \in \mathcal{B}$ then $R^A s$ is universally $\mathcal{B}$-measurable and if moreover A is finely open and $s \in p\mathcal{B}$ then $R^A s \in p\mathcal{B}$.

Let further λ be a finite measure on $(E, \mathcal{B})$. A property depending on the points of E is said to hold **λ-quasi everywhere** (abbreviated λ-q.e.) provided that the set of points where it does not hold is a subset of a set $A \in \mathcal{B}$ with $R^A 1 = 0$ λ-a.e. If $\mathcal{U}$ is the resolvent of right process X as above then the following fundamental result of G.A. Hunt holds for all $A \in \mathcal{B}$ and $s \in \mathcal{E}(\mathcal{U})$:

$$R^A s = E^x(s \circ X_{D_A})$$

where D_A is the entry time of A, $D_A = \inf\{t \geq 0 / \; X_t \in A\}$; see e.g. [7]. In this case the equality $R^A 1 = 0$ λ-a.e. means that the process $(1_A \circ X_t)_{t \geq 0}$ is P^λ-evanescent.

It turns out that the functional $M \longmapsto c_\lambda(M)$, $M \subset E$, defined by

$$c_\lambda(M) = \inf\{\lambda(R^G p_o) / G \in \mathcal{T}, \; M \subset G\}$$

is a Choquet capacity on $(E, \mathcal{T})$, where p_o is a fixed strictly positive, bounded $\mathcal{U}$-excessive function of the form $p_o = U f_o$, with $f_o \in p\mathcal{B}$, $0 < f_o \leq 1$.

Recall that the capacity c_λ on $(E, \mathcal{T})$ is named **tight** provided that there exists an increasing sequence $(K_n)_n$ of $\mathcal{T}$-compact sets such that

$$\inf_n c_\lambda(E \setminus K_n) = 0$$

(or equivalently $\inf_n R^{E\setminus K_n} p_o = 0$ λ-a.e.) which is also equivalent with

$$P^\lambda([\lim_n D_{E\setminus K_n} < \zeta]) = 0$$

in the probabilistic frame.

We can state now the main results of this paper. Their proofs will be presented in Section 2.

Theorem 1.1. *If $(E, \mathcal{T})$ is a co-Souslin topological space then the following assertions are equivalent for the given finite measure λ on E.*

i) For every increasing sequence $(G_n)_n$ of $\mathcal{T}$-open sets with $\bigcup_n G_n = E$ we have

$$\inf_n R^{E\setminus G_n} p_o = 0 \quad \lambda - a.e.$$

ii) For every increasing sequence $(G_n)_n$ of $\mathcal{T}$-open sets with $\overline{G_n} \subset G_{n+1}$ for all n and $\bigcup_n G_n = E$ we have

$$\inf_n R^{E\setminus G_n} p_o = 0 \quad \lambda - a.e.$$

iii) The capacity c_λ is tight.

Corollary 1.2. (Lyons-Röckner). *Assume that $\mathcal{U}$ is the resolvent of a transient right process with state space E, a co-Souslin topological space, and suppose that the process has left limits in E P^λ-a.e. on $[0, \zeta)$, where λ is the given finite measure on $(E, \mathcal{B})$. Then the capacity c_λ is tight.*

Remark. Corollary 1.2 is a result of T. Lyons and M. Röckner [10] and has important consequences in the study of the quasi-regular Dirichlet forms. In [11] are presented two proofs of it, one of them due to P.A. Meyer. Our proof is different and it is related rather to Meyer's approach.

Theorem 1.3. *Suppose that $\mathcal{U}$ is the resolvent of a transient right process with state space E, a co-Souslin topological space, and assume that the topology is generated by a Ray cone $\mathcal{R}$ such that $U_\alpha(\mathcal{R}) \subset \mathcal{R}$ for all $\alpha > 0$. If λ is a finite measure on $(E, \mathcal{B})$ then the following assertions are equivalent.*

i) The process has left limits in E P^λ-a.e. on $[0, \zeta)$.

ii) The capacity c_λ is tight.

iii) The process is λ-standard.

We shall prove in the sequel that the tightness property of the capacity c_λ always holds in two special situations. Let $\rho \circ U$ be the potential component of the $\mathcal{U}$-excessive measure m.

Theorem 1.4. *Assume that $(E, \mathcal{B})$ is a Lusin measurable space and there exists a second $\mathcal{B}$-sub-Markovian resolvent $\mathcal{U}^* = (U^*_\alpha)_{\alpha>0}$ such that the topology $\mathcal{T}$ is natural and E is m-semisaturated with respect to $\mathcal{U}^*$ too, and such that $\mathcal{U}$ and $\mathcal{U}^*$ are in* **weak duality** *with respect to m, that is*

$$\int gU_\alpha f\,dm = \int fU^*_\alpha g\,dm$$

for all $f, g \in p\mathcal{B}$ and $\alpha > 0$. Then there exists an increasing sequence $(K_n)_n$ of $\mathcal{T}$-compact sets such that

$$\inf_n R^{E\setminus K_n} p_o = 0 \quad (m+\rho)\text{-a.e.}$$

Particularly the capacity c_λ is tight for every finite measure λ on $(E, \mathcal{B})$ such that $\lambda \ll m + \rho$.

Remark. If two Borel right processes with the common (Lusin) topological space $(E, \mathcal{T})$ as state space, are in weak duality with respect to a σ-finite measure m (see e.g. [9]), then the associated resolvents satisfy the hypothesis from Theorem 1.4 and consequently the capacity c_λ is tight for every finite measure λ with $\lambda \ll m + \rho$. This assertion may be obtained alternatively, combining Corollary 1.2 and the result of J.B. Walsh [13] which ensures the existence of the left limits in the space for processes in weak duality.

Recall that a bounded function $s \in \mathcal{E}(\mathcal{U})$ is **regular** provided that for every sequence $(s_n)_n$ in $\mathcal{E}(\mathcal{U})$, $s_n \nearrow s$, we have $\inf_n R(s - s_n) = 0$, where R denotes the reduction operator. See [4] and [6] for more details on regular excessive functions. It is known that (see e.g. [7]) if $\mathcal{U}$ is the resolvent of a transient right process then a bounded function $s \in \mathcal{E}(\mathcal{U})$ is regular if and only if there exists a continuous additive functional having s as potential function. A bounded $\mathcal{U}$-excessive function s is m-**regular** if there exists a regular $\mathcal{U}$-excessive function s' such that $s = s'$ m-a.e.

Theorem 1.5. *Assume that $(E, \mathcal{T})$ is a co-Souslin topological space and the following condition is satisfied:*
$(*)$ *each $\mathcal{U}$-excessive function dominated by p_o is m-regular.*
Then the capacity c_λ is tight for every finite measure λ on $(E, \mathcal{B})$ with $\lambda \ll m+\rho$.

Corollary 1.6. *Under the assumption of Theorem 1.5, for every $\mathcal{U}$-excessive function s there exists an increasing sequence $(K_n)_n$ of $\mathcal{T}$-compact subsets of E such that $\inf_n c_\lambda(E \setminus K_n) = 0$ and $s|_{K_n}$ is $\mathcal{T}$-continuous for all n.*

Corollary 1.7. *Assume that $(E,\mathcal{T})$ is a Lusin topological space and $\mathcal{U}$ is the resolvent of a right process with state space E, such that condition $(*)$ from Theorem 1.5 holds. Then the process is λ-standard for every finite measure λ on $(E,\mathcal{B})$ with $\lambda \ll m+\rho$.*

Remark. 1. If the resolvents $\mathcal{U}$ and $\mathcal{U}^*$ are in weak duality with respect to the measure m as in Theorem 1.4, then the above property $(*)$ is precisely the axiom the m-polarity for $\mathcal{U}^*$, that is *every cosemipolar set is m-copolar*; see [5].

2. Suppose that E is a co-Souslin topological space and $\mathcal{U}$ is the resolvent of a right process with state space E, which is associated with a semi-Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(E,\nu)$, where ν is a given σ-finite measure. P.J. Fitzsimmons [8] proved that such a semi-Dirichlet form is quasi-regular. In fact, in this case if we consider a strictly positive bounded function $f_o \in p\mathcal{B}\cap L^1(E,\nu)$ with Uf_o bounded, then $p_o = Uf_o$ lies in the extended domain $\widetilde{D(\mathcal{E})}$ of the form and the measure $m = U^*f_o\cdot\nu$ is $\mathcal{U}$-excessive (where $(U^*_\alpha)_{\alpha>0}$ is the dual of the family $\mathcal{U}$ as operators on $L^2(E,\nu)$). We claim that condition $(*)$ holds in this case. Indeed, if $s\in\mathcal{E}(\mathcal{U})$, $s\le p_o$, and $(s_n)_n \subset \mathcal{E}(\mathcal{U})$ with $s_n \nearrow s$, then s, $s_n \in \widetilde{D(\mathcal{E})}$ and $(s_n)_n$ converges to s in $\widetilde{D(\mathcal{E})}$. The reduction operator in $\widetilde{D(\mathcal{E})}$ being continuous, we conclude that $R(s-s_n)\searrow 0$, hence s is m-regular. Consequently we can derive the above result of Fitzsimmons applying Theorem 1.5 and Corollary 1.6. Notice that Theorem 1.5 offers sufficient conditions for tightness of the capacity and quasi continuity property for excessive functions, without assuming any sector condition.

2 Proofs of main results

Proof of Theorem 1.1. The implications $iii) \Longrightarrow i) \Longrightarrow ii)$ are clear.

$ii) \Longrightarrow iii)$. Let $(\widetilde{E},\widetilde{\mathcal{T}})$ be a metrizable compact space such that $E\subset\widetilde{E}$ and $\widetilde{\mathcal{T}}|_E = \mathcal{T}$. We denote by $\widetilde{c}_\lambda$ the Choquet capacity on $(\widetilde{E},\widetilde{\mathcal{T}})$ such that $\widetilde{c}_\lambda(G) = \lambda(R^{G\cap E}p_o) = c_\lambda(G\cap E)$ for all $G\in\widetilde{\mathcal{T}}$. The tightness of the capacity c_λ is equivalent with the fact that $\widetilde{c}_\lambda(\widetilde{E}\setminus E)=0$. Since E is a co-Souslin set in $(\widetilde{E},\widetilde{\mathcal{T}})$, the last equality is equivalent with: $\widetilde{c}_\lambda(K)=0$ for all compact subsets K of $\widetilde{E}\setminus E$. Let further fix a compact subset K of $\widetilde{E}\setminus E$ and $(\Gamma_n)_n$ be a decreasing sequence of $\widetilde{\mathcal{T}}$-open subsets of $\widetilde{E}$ such that $\overline{\Gamma}_{n+1}\subset\Gamma_n$ for all n and $\bigcap_n \Gamma_n = K$. If we put $G_n = E\setminus\overline{\Gamma}_n$ then $\overline{G}_n\subset G_{n+1}$ for all n, $\bigcup_n G_n = E$ and by hypothesis we have $\inf_n R^{E\setminus G_n}p_o = 0$ λ-a.e. or equivalently $\inf_n \lambda(R^{E\setminus G_n}p_o)=0$. We conclude that $\widetilde{c}_\lambda(K)\le \inf_n \widetilde{c}_\lambda(\Gamma_n) = \inf_n \lambda(R^{E\cap\Gamma_n}p_o) = \inf_n\lambda(R^{E\setminus G_n}p_o)=0$, completing the proof.

Proof of Corollary 1.2. Let $(G_n)_n$ be an increasing sequence of $\mathcal{T}$-open sets with $\bigcup_n G_n = E$. By Theorem 1.1 we have to show that $\inf_n R^{E\setminus G_n} p_o = 0$ λ-a.e. If X is the right process having $\mathcal{U}$ as associated resolvent and we consider the entry time $D_{E\setminus G_n}$ of the set $E \setminus G_n$, then we have $\sup_n D_{E\setminus G_n} \geq \zeta$ P^λ-a.e. on Ω. Indeed, in the contrary case we have $P^\lambda([\sup_n D_{E\setminus G_n} < \zeta]) > 0$, contradicting the fact that $X_{D_{E\setminus G_n}}(\omega) \in E \setminus G_n$ for all n and the limit $\lim_n X_{D_{E\setminus G_n}}(\omega)$ exists in $(E, \mathcal{T})$ P^λ-a.e. on $[0, \zeta)$. From $\lambda(R^{E\setminus G_n} p_o) = E^\lambda(\int_{D_{E\setminus G_n}}^{\zeta} f_o \circ X_t dt)$ we deduce that $\inf_n R^{E\setminus G_n} p_o = 0$ λ-a.e. and the proof is completed.

Recall now some facts about the Ray cones and Ray topologies. Let $\mathcal{V} = (V_\alpha)_{\alpha>0}$ be a bounded sub-Markovian resolvent on $(E, \mathcal{B})$ such that the $\mathcal{V}$-excessive functions coincide with the $\mathcal{U}$-excessive ones.

A **Ray cone** (associated with $\mathcal{V}$) is a convex cone $\mathcal{R}$ of bounded $\mathcal{B}$-measurable, $\mathcal{U}$-excessive functions such that:

– The cone $\mathcal{R}$ contains the positive constant functions and is min-stable.

– $V_0((\mathcal{R} - \mathcal{R})_+) \subset \mathcal{R}$ and $V_\alpha(\mathcal{R}) \subset \mathcal{R}$ for all $\alpha > 0$.

– The cone $\mathcal{R}$ is separable with respect to the uniform norm.

– The σ-algebra on E generated by $\mathcal{R}$ coincides with $\mathcal{B}$.

One can show that for every countable set $\mathcal{A}$ of bounded $\mathcal{B}$-measurable, $\mathcal{U}$-excessive functions there exists a Ray cone including $\mathcal{A}$.

The topology $\mathcal{T}_\mathcal{R}$ on E generated by a Ray cone $\mathcal{R}$ (i.e. the coarsest topology on E for which every function from $\mathcal{R}$ is continuous) is called the **Ray topology** induced by $\mathcal{R}$.

Proof of Theorem 1.3. The implication $i) \Longrightarrow ii)$ follows from Corollary 1.2 and $iii) \Longrightarrow i)$ is clear.

$ii) \Longrightarrow iii)$. Let $(K_n)_n$ be an increasing sequence of $\mathcal{T}$-compact subsets of E such that $\inf_n R^{E\setminus K_n} p_o = 0$ λ-a.e. We denote by Y the Ray compactification of E with respect to $\mathcal{R}$. Since for every $s \in \mathcal{R}$ the process $(s \circ X_t)_{t\geq 0}$ is a bounded right continuous supermartingale with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$ it follows that (cf. [7]) this process has left limits P^λ-a.e. The Ray cone $\mathcal{R}$ being separable with respect to the uniform norm it results that the process $(X_t)_{t\geq 0}$ has left limits in Y P^λ-a.e. From $\lim_n R^{E\setminus K_n} p_o = 0$ λ-a.e. and $\lambda(R^{E\setminus K_n} p_o) = E^\lambda(\int_{T_{E\setminus K_n}}^{\zeta} f_o \circ X_t dt)$ we deduce that $\sup_n T_{E\setminus K_n} \geq \zeta$ P^λ-a.e. Hence for every $\omega \in \Omega$ with $T_{E\setminus K_n}(\omega) < \zeta(\omega)$ we have $X_t(\omega) \in K_n$ provided that $t < T_{E\setminus K_n}(\omega)$ and so $X_{t-}(\omega) \in K_n$. Consequently the process $(X_t)_{t\geq 0}$ has left limits in E P^λ-a.e. on $[0, \zeta)$.

Let now $(T_n)_n$ be an increasing sequence of stopping times and $T = \lim_n T_n$. We show that $\lim_n X_{T_n} = X_T$ P^λ-a.e. on $[T < \zeta]$. From the above considerations there exists in Y the limit $Z = \lim_n X_{T_n}$ P^λ-a.e. and $Z(\omega) \in E$ if $T(\omega) < \zeta(\omega)$. It remains to prove that $Z = X_T$ P^λ-a.e. on $[T < \zeta]$. For all $f, g \in \mathcal{R}$, $\alpha > 0$ and $n < m$ we have $E^\lambda(g(X_{T_n}) \cdot \widetilde{U_\alpha f}(X_{T_m})) = E^\lambda(g(X_{T_n}) \cdot E^{X_{T_m}}(\int_0^\zeta e^{-\alpha t} f \circ X_t dt)) = E^\lambda(g(X_{T_n}) \cdot e^{-\alpha T_m} \int_{T_m}^\zeta e^{-\alpha t} f \circ X_t dt)$, where $\widetilde{g}$ denotes the Ray continuous extension of g to Y. Letting $m \to \infty$ we obtain

$$E^\lambda(g(X_{T_n}) \cdot \widetilde{U_\alpha f}(Z)) = E^\lambda(g(X_{T_n}) \cdot e^{-\alpha T} \int_T^\zeta e^{-\alpha t} f \circ X_t dt) = E^\lambda(g(X_{T_n}) \cdot U_\alpha f(X_T)).$$

Letting now $n \to \infty$ we get $E^\lambda(\tilde{g}(Z) \cdot \widetilde{U_\alpha f}(Z)) = E^\lambda(\tilde{g}(Z) \cdot \widetilde{U_\alpha f}(X_T))$ and therefore, by monotone class argument, we have

$$E^\lambda((h \cdot 1_E)(Z) \alpha \widetilde{U_\alpha f}(Z)) = E^\lambda((h \cdot 1_E)(Z) \alpha \widetilde{U_\alpha f}(X_T))$$

for all $h \in p\mathcal{B}(Y)$. Letting $\alpha \to \infty$ we derive that

$$E^\lambda((h \cdot 1_E)(Z) \cdot \tilde{f}(Z)) = E^\lambda((h \cdot 1_E)(Z) \cdot f(X_T)).$$

Again by monotone class argument we deduce that

$$E^\lambda((h \cdot 1_E)(Z) \cdot (k \cdot 1_E)(Z)) = E^\lambda((h \cdot 1_E)(Z) \cdot k \mid_E (X_T))$$

for all $k \in p\mathcal{B}(Y)$ and as a consequence

$$E^\lambda(G \cdot 1_{E \times E}(Z, Z)) = E^\lambda(G \cdot 1_{E \times E}(Z, X_T))$$

for all $G \in p\mathcal{B}(Y \times Y)$. Taking G the characteristic function of the diagonal of $Y \times Y$ we conclude that

$$P^\lambda(\{\omega \in \Omega / Z(\omega), X_T(\omega) \in E,\ Z(\omega) \neq X_T(\omega)\}) = 0,$$

completing the proof.

Proof of Theorem 1.4. Let μ be a finite measure on $(E, \mathcal{B})$ such that the negligible sets are the same for μ and $m + \rho$. Let $(G_n)_n$ be an increasing sequence of $\mathcal{T}$-open sets with $\overline{G_n} \subset G_{n+1}$ for all n and $\bigcup_n G_n = E$. We show that the function $u = \inf_n R^{E \setminus G_n} p_o$ equals zero μ-a.e. Clearly the function u is $\mathcal{U}$-excessive and $u \cdot m \leq U f_o \cdot m = (f_o \cdot m) \circ U^*$, where $f_o \in bp\mathcal{B}$ is the strictly positive function such that $p_o = U f_o$ and we assume that $m(f_o) < \infty$ and $U^* f_o \leq 1$. Since E is m-semisaturated with respect to $\mathcal{U}^*$, there exists a measure ν on $(E, \mathcal{B})$ with $u \cdot m = \nu \circ U^*$. Obviously $\nu(1) \leq (f_o \cdot m)(1) < \infty$. The set E being m-semisaturated with respect to $\mathcal{U}$ too, for every n there exists a measure θ_n on

$(E,\mathcal{B})$ such that $\theta_n(s)=\mu(R^{E\setminus\overline{G}_n}s)$ for all $s\in\mathcal{E}(\mathcal{U})$. Hence $\mu(R^{E\setminus\overline{G}_n}u)=\theta_n(u)=\inf_k\theta_n(R^{E\setminus\overline{G}_k}p_o)=\inf_k\mu(R^{E\setminus\overline{G}_n}(R^{E\setminus\overline{G}_k}p_o))=\inf_k\mu(R^{E\setminus\overline{G}_k}p_o)=\mu(u)$. It results that $R^{E\setminus\overline{G}_n}u=u$ μ-a.e. If $g\in p\mathcal{B}$ then by Theorem 1.2 in [5] we have $\nu(U^*g)=\int gudm=\int gR^{E\setminus\overline{G}_n}udm=L^*(R^{E\setminus\overline{G}_n}u\cdot m,U^*g)=L^*(u\cdot m,{}^*R^{E\setminus\overline{G}_n}U^*g)=\nu({}^*B^{E\setminus\overline{G}_n}U^*g)$, where L^* denotes the energy functional and ${}^*R$ the reduction operator with respect to the resolvent $\mathcal{U}^*$. Because ${}^*R^{E\setminus\overline{G}_n}U^*f_o<U^*f_o$ on G we conclude that ν is carried by $E\setminus G_n$ for all n and therefore $\nu=0$ which implies $u\cdot m=0$, $\inf_n R^{E\setminus G_n}p_o=0$ μ-a.e. By Theorem 1.1 there exists an increasing sequence $(K_n)_n$ of $\mathcal{T}$-compact sets such that $\inf_n R^{E\setminus K_n}p_o=0$ μ-a.e. or equivalently $\inf_n R^{E\setminus K_n}p_o=0$ $(m+\rho)$-a.e.

Proof of Theorem 1.5. Let μ be a finite measure on $(E,\mathcal{B})$ such that the negligible sets with respect to μ and $m+\rho$ are the same. By Theorem 1.1 it suffices to show that for every increasing sequence $(G_n)_n$ of $\mathcal{T}$-open sets with $\overline{G}_n\subset G_{n+1}$ for all n and $\bigcup G_n=E$, the function $u=\inf_n R^{E\setminus G_n}p_o$ equals zero μ-a.e. Clearly u is $\mathcal{U}$-excessive and $u\leq p_o$ and by hypothesis there exists a regular $\mathcal{U}$-excessive function u_0 such that $u_0\preccurlyeq u$ and $u=u_0$ m-a.e. (we have denoted by $\preccurlyeq$ the specific order in the cone of potentials of all finite $\mathcal{U}$-excessive functions). Let further $(E_1,\mathcal{B}_1)$ be the saturation of E with respect to $\mathcal{U}$ (cf. [2]) and V the kernel on $(E_1,\mathcal{B}_1)$ such that $V1=\widetilde{u}_0$, where $\widetilde{u}_0$ denotes the extension by fine continuity of u_0 to E_1. Since E is m-semisaturated it follows that the set $E_1\setminus E$ is m-semipolar and consequently we get $V(1_{E_1\setminus E})=0$ m-a.e. on E and so $V(1_{E_1\setminus E})|_E=0$ μ-a.e. Using again the m-semisaturation of E we deduce that for all n there exists a finite positive measure μ_n on $(E,\mathcal{B})$ such that $\mu(R^{E\setminus\overline{G}_n}s)=\mu_n(s)$ for all $s\in\mathcal{E}(\mathcal{U})$. Hence $\mu_n(u)=\inf_k\mu_n(R^{E\setminus\overline{G}_k}p_o)=\inf_k\mu(R^{E\setminus\overline{G}_k}p_o)=\mu(u)$, $\mu(R^{E\setminus\overline{G}_n}u)=\mu(u)$. As a consequence $R^{E\setminus\overline{G}_n}u=u$ m-a.e. If Γ_n is a finely open set in $\mathcal{B}_1$ with $\Gamma_n\cap\overline{E}=G_n$ then $V(1_{E_1\setminus\Gamma_n})=V(1_{E\setminus G_n})$ μ-a.e. and $V(1_{E_1\setminus\Gamma_n})=V1\curlywedge R^{E_1\setminus\Gamma_n}V1$, where $\curlywedge$ denotes the infimum with respect to the specific order $\preccurlyeq$. Let M_n be the fine closure in E_1 of the set $[R^{E_1\setminus\Gamma_n}V1<V1]$. The set $M_n\cap E$ being μ-polar and μ-negligible and from $V(1_{(E_1\setminus\Gamma_n)\cup M_n})=R^{(E_1\setminus\Gamma_n)\cup M_n}V1=V1$ μ-a.e. it results that $V1\preccurlyeq V(1_{E_1\setminus\Gamma_n})+V(1_{M_n})=V(1_{E_1\setminus\Gamma_n})$ μ-a.e. and we conclude that $u=V1=0$ μ-a.e., completing the proof.

Proposition 2.1. *There exists a Ray topology on E which is finer than the natural topology $\mathcal{T}$.*

Proof. Let $\mathcal{G}$ be a countable base for the topology $\mathcal{T}$. We remark that for every $G\in\mathcal{G}$ and $x\in G$ there exists $G'\in\mathcal{G}$ with $x\in G'$ and $\overline{G'}\subset G$. We consider now a Ray cone $\mathcal{R}$ such that $p_o\in\mathcal{R}$ and $R^{E\setminus\overline{G}}p_o\in\mathcal{R}$ for all $G\in\mathcal{G}$. Notice that the function $R^{E\setminus\overline{G}}p_o$ belongs to $p\mathcal{B}$. If $G,G'\in\mathcal{G}$ and $\overline{G'}\subset G$ then $G'\subset[R^{E\setminus\overline{G'}}p_o<p_o]\subset\overline{G'}\subset G$ and we conclude that every $\mathcal{T}$-open set is Ray open. □

Proof of Corollary 1.6. Let λ be a finite measure, $\lambda \ll m+\rho$, and $s \in \mathcal{E}(\mathcal{U}) \cap p\mathcal{B}$. By Proposition 2.1 there exists a Ray cone $\mathcal{R}$ such that $\inf(s,k) \in \mathcal{R}$ for every natural number k and $\mathcal{T} \subset \mathcal{T}_{\mathcal{R}}$, where recall that $\mathcal{T}_{\mathcal{R}}$ denotes the topology generated by $\mathcal{R}$. By Theorem 1.5 we deduce that there exists an increasing sequence $(K_n)_n$ of $\mathcal{T}_{\mathcal{R}}$-compact subsets of E such that $\inf_n R^{E\setminus K_n} p_o = 0$ λ-a.e. Since $\mathcal{T} \subset \mathcal{T}_{\mathcal{R}}$ we get $\mathcal{T}|_{K_n} = \mathcal{T}_{\mathcal{R}}|_{K_n}$ for all n and $\inf(s,k)|_{K_n}$ is $\mathcal{T}$-continuous for all k and n. If $s \in \mathcal{E}(\mathcal{U})$ then there exists $s' \in \mathcal{E}(\mathcal{U}) \cap p\mathcal{B}$, $s' \le s$, such that $s' = s$ λ-q.e.

Proof of Corollary 1.7. By Proposition 2.1 we may consider a Ray cone $\mathcal{R}$ with respect to $\mathcal{U}$ such that $U_\alpha(\mathcal{R}) \subset \mathcal{R}$ for all $\alpha > 0$ and such that the Ray topology $\mathcal{T}_{\mathcal{R}}$ on E is finer than $\mathcal{T}$. From Theorem 1.5 it results that the capacity c_λ on $(E, \mathcal{T}_{\mathcal{R}})$ is tight and let $(K_n)_n$ be an increasing sequence of $\mathcal{T}_{\mathcal{R}}$-compact subsets of E such that $\inf_n R^{E\setminus K_n} p_o = 0$ λ-a.e. By Theorem 1.3 the process X having $\mathcal{U}$ as associated resolvent is λ-standard with respect to the topology $\mathcal{T}_{\mathcal{R}}$. From $\inf_n R^{E\setminus K_n} p_o = 0$ λ-a.e. it follows that $\sup_n T_{E\setminus K_n} \ge \zeta$ P^λ-a.e. Because K_n is $\mathcal{T}_{\mathcal{R}}$-compact and $\mathcal{T}_{\mathcal{R}}$ is finer than $\mathcal{T}$ we deduce that $\mathcal{T}|_{K_n} = \mathcal{T}_{\mathcal{R}}|_{K_n}$ for all n. Since $X_t(\omega) \in K_n$ for all $t < T_{E\setminus K_n}(\omega)$ we deduce that P^λ-a.e. there exists in E $\mathcal{T} - \lim_{s \nearrow t} X_s(\omega)$ for all $t < T_{E\setminus K_n}(\omega)$ and so X has left limits in E P^λ-a.e. on $[0, \zeta)$.

Let now $(T_n)_n$ be an increasing sequence of stopping times and $T = \lim_n T_n$. We have $[T < \zeta] = \bigcup_k [T < T_{E\setminus K_k}]$ P^λ-a.e. If $\omega \in [T < T_{E\setminus K_k}]$ then we have $T(\omega) < T_{E\setminus K_k}(\omega) \le \zeta(\omega)$, $X_{T_n(\omega)}(\omega) \in K_k$ for all n and consequently

$$\mathcal{T} - \lim_n X_{T_n} = \mathcal{T}_{\mathcal{R}} - \lim_n X_{T_n} = X_T \quad P^\lambda\text{-a.e. on } [T < T_{E\setminus K_k}].$$

We conclude that

$$\lim_n X_{T_n} = X_T \quad P^\lambda\text{-a.e. on } [T < \zeta].$$

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