Finite dimensional realizations for a linear forward rate model with jumps

Eva Lütkebohmert

Institut für angewandte Mathematik, Abteilung Stochastik, Universität Bonn, Wegelerstr. 6, 53115 Bonn, Germany

Abstract

Finite dimensional realizations for a Heath, Jarrow and Morton type interest rate model with jumps are discussed in the case where the interest rate model can be described by a linear stochastic differential equation. The existence of such finite dimensional realizations can be proven under some assumptions on the Lie algebra generated by the coefficients. These finite dimensional realizations can under certain conditions be expressed in terms of invariant tangential manifolds similarly to the purely Wiener driven case studied in previous work by, e.g., T. Björk, D. Filipovic, J. Teichmann and the present author.

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1 Introduction

The problem of finite dimensional realizations for Heath, Jarrow and Morton type interest rate models is well discussed in the literature. For example the purely Wiener driven case has been studied in [Bj-S], [B-C] and in a completely general Hilbert space setting in [L1] and [Fi-T]. Also certain situations with jumps have been discussed, e.g. in [B-DM-K-R] and [B-K-R]. In the purely Wiener driven case one obtains a representation result of the form: if there exists a finite dimensional realization at a certain point then there exists also an invariant tangential manifold for the same stochastic differential equation at that point. The problem of characterizing finite dimensional realizations by invariant tangential manifolds, however, is in this situation with jumps much more complicated, since invariance might get lost whenever there is a jump, and moreover there is no obvious way of defining tangency at the jump times. However, there are certain situations where the same methodology can be applied to derive finite dimensional realizations. This is e.g. the case when the interest rate curve can be described by a stochastic differential equation with deterministic coefficients as discussed in [B-G]. In this paper we will study the more general case when the interest rate curve r can be described by a linear stochastic differential equation with stochastic coefficients. The existence of finite dimensional realizations can be proven under some assumptions on the Lie algebra generated by the coefficients and finite dimensional realizations can under certain conditions be expressed in terms of invariant tangential manifolds similarly to the purely Wiener driven case as studied, for example, in [Bj-S], [B-C], [Fi-T], [L1]. Here we require the assumption of linearity in the r-variable since then the solution of the stochastic differential equation for an initial point after the first jump is just the solution of the stochastic differential equation where we subtract the jumps with initial point before the jump plus the solution of the same stochastic differential equation with initial condition given by the value obtained just after the first jump.

Let us remark that for the general jump diffusion case, i.e. where the interest rate curve r is described by a general stochastic differential equation, an approximation of the underlying geometry is needed. In fact a treatment of the general case was given in [L2] by applying techniques of geometric measure theory. One can then represent finite dimensional realizations by approximating tangent spaces instead of invariant tangential manifolds.

2 The General Setting

2.1 The Space of Forward Rates

Consider a financial market living on a stochastic basis satisfying the usual conditions, i.e. on a filtered probability space $(\Omega, \mathcal{A}, \mathcal{F} = (\mathcal{F}_t)_{t\geq 0}, P)$.

Definition 2.1 Let $(S_t)_{t\geq 0}$ be the *family of right-shifts,* $S_t f = f(t+\cdot), t \in \mathbb{R}_+$, defined on the space $L^2(\mathbb{R}_+)$.

Proposition 2.2 *$S_t : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$ is for every $t \geq 0$ a bounded linear operator on $L^2(\mathbb{R}_+)$. Moreover, $(S_t)_{t\geq 0}$ is a strongly continuous semigroup of contractions generated by the (unbounded) operator $\tilde{A}$ defined by $\tilde{A}h = \frac{d}{dx}h$ on*

$$D(\tilde{A}) := \Big\{ h \in L^2(\mathbb{R}_+) : \tilde{A}h \in L^2(\mathbb{R}_+), \ \text{and } h(0) = 0 \Big\} = H_0^{1,2}(\mathbb{R}_+)$$

(actually h should be thought of as a Borel version of an L^2-class).

Remark 2.3 The boundary condition $h(0) = 0$ in the domain of the operator $\tilde{A}$ arises naturally since for $f \in L^2(\mathbb{R}_+)$ such that also $f' \in L^2(\mathbb{R}_+)$ we have

$$\begin{aligned} &\int_0^y f(x) f'(x) dx \\ &\qquad = \ |f(y)|^2 - |f(0)|^2 - \int_0^y f'(x) f(x) dx. \end{aligned}$$

Since the integrals converge as $x \to +\infty$ to $\int_0^\infty f(x) f'(x) dx$, respectively $\int_0^\infty f'(x) f(x) dx$, $|f(x)|^2$ tends to a limit as $x \to \infty$. Since $f \in L^2(\mathbb{R}_+)$ we have necessarily

$$\lim_{x\to\infty} |f(x)|^2 = 0.$$

Hence the boundary condition for $x \to \infty$ is satisfied automatically. Thus we only have to impose the condition $f(0) = 0$ on the functions $f \in L^2(\mathbb{R}_+)$ to achieve that $f \in D(\tilde{A})$. Hence the domain of $\tilde{A}$ corresponds to the Sobolev space $H_0^{1,2}(\mathbb{R}_+)$ of functions in $L^2(\mathbb{R}_+)$ which possess a weak derivative lying again in $L^2(\mathbb{R}_+)$ and satisfying the boundary condition $f(0) = 0$ (expressed by the index 0).

Proof of Proposition 2.2. We first show that S_t is bounded. For $t \geq 0$ we have (with $\|\cdot\|_{L^2}$ standing for $\|\cdot\|_{L^2(\mathbb{R}_+)}$)

$$\begin{aligned} \|S_t h\|_{L^2}^2 &= \int_0^\infty h^2(t+x)dx \\ &= \int_t^\infty h^2(x)dx \leq \|h\|_{L^2}^2, \end{aligned}$$

hence $\|S_t\|_{L(L^2(\mathbb{R}_+),L^2(\mathbb{R}_+))} \leq 1$, i.e. S_t is for every $t \geq 0$ a contraction.
Now we prove that $\tilde{A}$ is the generator of the C_0-semigroup $(S_t)_{t\geq 0}$. Therefore we have to show that for every $h \in D(\tilde{A})$

$$\left\| \frac{S_t h - S_0 h}{t} - \tilde{A}h \right\|_{L^2} \longrightarrow 0$$

for every $t \geq 0$. But

$$\begin{aligned} \left\| \frac{S_t h - S_0 h}{t} - \tilde{A}h \right\|_{L^2} &= \int_0^\infty \left| \frac{1}{t}(h(t+x) - h(x)) - h'(x) \right|^2 dx \\ &= \int_0^\infty \left| \frac{1}{t} \int_0^t (h'(s+x) - h'(x))ds \right|^2 dx \\ &\leq \int_0^\infty \frac{1}{t} \int_0^t \left| h'(s+x) - h'(x) \right|^2 ds \cdot dx \\ &\leq \frac{1}{t} \int_0^t \int_0^\infty |h'(s+x) - h'(x)|^2 dx ds, \end{aligned}$$

which vanishes for $t \to 0$ since $h' \in L^2(\mathbb{R}_+)$ and for every $f \in L^2(\mathbb{R}_+)$ the function

$$s \mapsto \int_0^\infty |f(s+x) - f(x)|^2 dx$$

is continuous on $[0, \infty)$ and vanishes at 0. Thus $\tilde{A}$ is the infinitesimal generator of the semigroup $(S_t)_{t\geq 0}$. □

The operator $\tilde{A}$ is skew symmetric since for any two functions $f, g \in D(\tilde{A})$ we have

$$\begin{aligned} (\tilde{A}f, g)_{L^2} &= \int_0^\infty \tilde{A}f(x)g(x)dx \\ &= \int_0^\infty f'(x)g(x)dx \qquad (1) \\ &= -\int_0^\infty f(x)g'(x)dx \quad = \quad -(f, \tilde{A}g)_{L^2}. \end{aligned}$$

Proposition 2.4 *The operator* $\hat{A}$ *defined by* $\hat{A}h = h'$ *with* $D(\hat{A}) = H^{1,2}(\mathbb{R}_+)$ *(without boundary condition) is an extension of* $\tilde{A}$ *and* $\hat{A} \neq \tilde{A}$, *hence* $i\tilde{A}$ *is not self-adjoint.*

Proof. Equation (1) holds also when $f \in D(\tilde{A})$ and $g \in H^{1,2}(\mathbb{R}_+)$ without boundary condition. Hence there exists an operator $\hat{A}$ with

$$\hat{A}h(x) = h'(x) \quad \text{and} \quad D(\hat{A}) = H^{1,2}(\mathbb{R}_+).$$

Vice versa, let $h \in D(\hat{A})$ and $\hat{A}h = h^*$. Then for all $f \in D(\tilde{A})$ the following holds

$$\begin{aligned}(\tilde{A}f, h)_{L^2} &= (f, h^*)_{L^2} \\ &= \int_0^\infty f(x)h^*(x)dx \\ &= -\int_0^\infty f(x)\frac{d}{dx}\left(-\int_0^x h^*(y)dy + C\right)dx\end{aligned}$$

where C is any arbitrary constant. Then by using partial integration it follows that

$$(A^*f, h)_{L^2} = \int_0^\infty f'(x)\left(-\int_0^x h^*(y)dy + C\right)dx \tag{2}$$

since f satisfies, as an element of $D(\tilde{A})$, the boundary condition. From (2) we obtain for arbitrary $f \in D(\tilde{A})$ the following relation

$$\begin{aligned}&\int_0^\infty f'(x)\left(h(x) - \int_0^x h^*(y)dy + C\right)dx \\ =& \int_0^\infty f'(x)h(x)dx - \int_0^\infty f'(x)\left(\int_0^x h^*(y)dy + C\right)dx \\ =& \int_0^\infty f'(x)h(x)dx + \int_0^\infty f(x)h^*(x)dx \\ =& \int_0^\infty f'(x)h(x)dx - \int_0^\infty f'(x)h(x)dx = 0.\end{aligned} \tag{3}$$

Compute C from the equation

$$\int_0^\infty \left(h(x) - \int_0^x h^*(y)dy + C\right)dx = 0$$

and take afterward for f the function

$$f_0(x) := \int_0^x \left(h(y) - \int_0^y h^*(u)du + C\right)dy,$$

which obviously belongs to $D(\tilde{A})$. Then formula (3) becomes

$$\int_0^\infty \left|h(x) - \int_0^x h^*(y)dy + C\right|^2 dx = 0.$$

Hence,

$$h(x) - \int_0^x h^*(y)dy + C = 0.$$

Thus the following equation holds Lebesgue almost everywhere

$$h'(x) = h^*(x).$$

Consequently the domain of the operator $\hat{A}$ consists of all functions $h \in H^{1,2}(\mathbb{R}_+)$ and we have

$$\hat{A}h(x) = h'(x).$$

Thus $i\tilde{A}$ is not self-adjoint, since the functions in $D(\tilde{A})$ satisfy the boundary condition and the functions in $D(\hat{A})$ do not necessarily have this property. Hence $\tilde{A} \not\subseteq \hat{A}$. □

Proposition 2.5 *There does not exist a skew symmetric strict extension of* $\tilde{A}$ *on* $L^2(\mathbb{R}_+)$.

Proof. The domain of such an extension $\tilde{\tilde{A}}$ would have to contain a function $H \in L^2(\mathbb{R}_+)$ which does not vanish for $x = 0$. Then, however, we would have

$$\begin{aligned}(\tilde{\tilde{A}}H, H)_{L^2} &= \int_0^\infty H'(x)H(x)dx \\ &= |H(0)|^2 - \int_0^\infty H(x)H'(x)dx \\ &\neq -(H, \tilde{\tilde{A}}H)_{L^2}\end{aligned}$$

which is impossible. □

Proposition 2.6 *$\tilde{A}$ is on $L^2(\mathbb{R}_+)$ a maximal skew symmetric operator (in the sense of [AcGl]).*

Proof. $\tilde{A}$ maximal follows from the Proposition above. □

Now define $A := \hat{A}$. Then A is defined on $H^{1,2}(\mathbb{R}_+)$. Since every Hilbert space is reflexive, we can apply Lemma 10.1 and Corollary 10.6 in [Pa], pages 39 and 41, respectively, to obtain that the adjoint semigroup S_t^* of S_t is again a C_0-semigroup of contractions on $\big(L^2(\mathbb{R}_+)\big)^* = L^2(\mathbb{R}_+)$, where the adjoint S_t^* of S_t is a linear operator from $D(S_t^*) \subset L^2(\mathbb{R}_+)$ into $L^2(\mathbb{R}_+)$.

Proposition 2.7 *The operator $A = \hat{A}$ is the infinitesimal generator of the adjoint semigroup S_t^*.*

Proof. The dual operator P^* of a linear operator P from a Banach space X into another Banach space Y is defined by the following equation

$$(x, P^*y^*)_X = (Px, y^*)_{Y^*}$$

for $x \in X$ and $y^* \in Y^*$. Since $\big(L^2(\mathbb{R})\big)^* = L^2(\mathbb{R})$ we obtain for the operator S_t on $L^2(\mathbb{R})$ the following identity

$$(S_t f, g)_{L^2} = \int_{-\infty}^{\infty} f(t+x)g(x)dx = \int_{-\infty}^{\infty} f(x)g(x-t)dx = (f, S_{-t}g)_{L^2}.$$

Since the adjoint of the operator S_t on the L^2-space on the whole real line when restricted to the positive real line should coincide with the adjoint of the operator S_t on $\big(L^2(\mathbb{R}_+)\big)^* = L^2(\mathbb{R}_+)$ we obtain

$$(S_t^* f)(x) = f(x-t)$$

on $L^2(\mathbb{R}_+)$, such that $x - t \geq 0$. Then the Proposition follows similarly to the proof of Proposition 2.2, in which we did not use the boundary condition. □

Now we can define the iterated operators $A^n, n \in \mathbb{N}$, recursively by

$$D(A^n) = \big\{h \in D(A^{n-1}) | Ah \in L^2(\mathbb{R}_+)\big\} = H^{n,2}(\mathbb{R}_+).$$

Then

$$D(A^\infty) := \bigcap_{n=1}^{\infty} H^{n,2}(\mathbb{R}_+) = H^{\infty,2}(\mathbb{R}_+)$$

since $H^{n,2}(\mathbb{R}_+) \subset H^{n-1,2}(\mathbb{R}_+)$ for every $n \in \mathbb{N}$. This is a Hilbert space with inner product given by

$$(f, g)_{H^{\infty,2}} := \sum_{n \in \mathbb{N}} (f^{(n)}, g^{(n)})_{L^2}.$$

Assumption 2.8 The Hilbert space H of forward curves is assumed to satisfy the following properties:

(1) Assume that the Hilbert space H can be continuously embedded into the Hilbert space $L^2(\mathbb{R}_+)$ (that is, for every $x \in \mathbb{R}_+$ the pointwise evaluation $\mathrm{ev}_x : h \mapsto h(x)$ is a continuous linear functional on H).

(2) Assume that H is separable.

Remark 2.9 For any space H as in Assumption 2.8 one has that the right-shifts $(S_t)_{t\geq 0}$ define a strongly continuous semigroup of operators on H with infinitesimal generator A.

In fact, we only have to choose the mapping $\iota : H \to L^2(\mathbb{R}_+)$, which defines the embedding of H in $L^2(\mathbb{R}_+)$. This mapping is by assumption continuous. Since the right-shifts S_t define a continuous semigroup of operators in the space $H = L^2(\mathbb{R}_+)$ (see Proposition 2.2) the result follows by continuity of ι, when we replace in the proof the function $h \in L^2(\mathbb{R}_+)$ by $\iota(\tilde{h})$, which is again an element of $L^2(\mathbb{R}_+)$ where $\tilde{h} \in H$. For simplicity of computations we will always work with $H = L^2(\mathbb{R}_+)$ in the following.

For the following investigations the next Definition will be quite useful.

Definition 2.10 Let $B : \big(C \cap L^2(\mathbb{R}_+)\big) \times \big(C \cap L^2(\mathbb{R}_+)\big) \to \big(C \cap L^2(\mathbb{R}_+)\big)$ be defined by

$$B(f,g)(x) := f(x) \int_0^x g(y)dy$$

for $x \in \mathbb{R}_+$. We write shortly $B(f)$ instead of $B(f,f)$.

Proposition 2.11 $B : \big(C \cap L^2(\mathbb{R}_+)\big) \times \big(C \cap L^2(\mathbb{R}_+)\big) \to \big(C \cap L^2(\mathbb{R}_+)\big)$ *defines a continuous, bilinear mapping.*

Proof. Bilinearity is obvious. We first prove continuity in the first component

$$\begin{aligned}
\|B(f_1,g) - B(f_2,g)\|_{L^2}^2 &= \int_0^\infty \left| f_1(x) \int_0^x g(y)dy - f_2(x) \int_0^x g(y)dy \right|^2 dx \\
&= \int_0^\infty |f_1(x) - f_2(x)|^2 \cdot \left| \int_0^x g(y)dy \right|^2 dx \\
&\leq \|g\|_{L^2}^2 \cdot \int_0^\infty |f_1(x) - f_2(x)|^2 \, dx \\
&= \|g\|_{L^2}^2 \cdot \|f_1(x) - f_2(x)\|_{L^2}^2
\end{aligned}$$

which tends to 0 as f_1 tends to f_2 in L^2 since $g \in L^2(\mathbb{R}_+)$, hence the norm $|g|_{L^2}^2$ is bounded. Continuity in the second component follows similarly. □

Remark 2.12 One can show similarly to Remark 2.9 that for any space H in Assumption 2.8 the mapping B defined on $H \times H$ is bilinear and continuous.

Condition (2) in Assumption 2.8 is needed in our setting since the stochastic differential equation of the forward rates, which we will consider in the sequel, is a general infinite dimensional one in the sense of [D-Z], that is it is driven in particular by an infinite dimensional Q-Wiener process with values in the Hilbert space H. To define proper integration with respect to such a process, it is essential to assume that H is separable.

2.2 The Driving Processes

Let Q be a strictly positive symmetric linear operator on H with $\operatorname{Tr} Q < \infty$. Then there exists an orthonormal system $\{e_k\}_{k \geq 1}$ in H and a bounded sequence $\{\lambda_k\}_{k \geq 1}$ of positive real numbers such that

$$Qe_k = \lambda_k e_k \quad \text{for } k \in \mathbb{N}.$$

Let $\{W_t\}_{t \geq 0}$ be an H-valued Q-Wiener process in the sense of [D-Z], that is:

Definition 2.13 An H-valued stochastic process $\{W_t\}_{t \geq 0}$ is called a *Q-Wiener process* if

(i) $W_0 = 0$

(ii) W has continuous trajectories,

(iii) W has independent increments,

(iv) the law $\mathcal{L}(W_t - W_s) = \mathcal{N}(0, (t-s)Q)$ for $t \geq s \geq 0$.

If a process $\{W_t\}_{t \in [0,T]}$ satisfies (i)-(iv) for $t, s \in [0, T]$ then we say that W is a Q-Wiener process on $[0, T]$.

Proposition 2.14 *Assume that W is a Q-Wiener process, with $\operatorname{Tr} Q < +\infty$. Then the following statements hold.*

(i) W is a Gaussian process on H_0 and

$$E[W(t)] = 0 \text{ and } \operatorname{Cov}[W(t)] = tQ, \quad t \geq 0.$$

(ii) For arbitrary t, the process W has the expansion

$$W(t) = \sum_{j=1}^{\infty} \sqrt{\lambda_j} W^j(t) e_j \tag{4}$$

where

$$W^j(t) = \frac{1}{\sqrt{\lambda_j}} \langle W(t), e_j \rangle, \quad j = 1, 2, \ldots,$$

are real-valued Brownian motions mutually independent on $(\Omega, \mathcal{A}, P)$ and the series in (4) is convergent in $L^2(\Omega, \mathcal{A}, P)$.

Proof. See [D-Z], Proposition 4.1, page 87. □

It follows from the expansion of the Q-Wiener process, that the quadratic variation of the stochastic process $S_t dW_t$ for some suitable process S_t is given by

$$\langle\langle S_t dW_t \rangle\rangle = (S_t Q^{1/2})(S_t Q^{1/2})^* dt, \tag{5}$$

since

$$\begin{aligned}
(S_t dW_t)^2 &= \langle S_t dW_t, S_t dW_t \rangle_H \\
&= \sum_{j,k=1}^{\infty} \left\langle S_t \sqrt{\lambda_j} dW_t^j e_j, S_t \sqrt{\lambda_k} dW_t^k e_k \right\rangle_H \\
&= \sum_{j,k=1}^{\infty} \left\langle S_t Q^{1/2} e_j dW_t^j, S_t Q^{1/2} e_k dW_t^k \right\rangle_H \\
&= \sum_{j=1}^{\infty} \left\langle S_t Q^{1/2} e_j, S_t Q^{1/2} e_j \right\rangle_H dt \\
&= \mathrm{Tr}\left[(S_t Q^{1/2})(S_t Q^{1/2})^* \right] dt.
\end{aligned}$$

Hence $(S_t dW_t)^2 - \langle\langle S_t dW_t \rangle\rangle$ is a local martingale (in the sense of, for example, [D-Z]).

Let μ be a marked point process on $\mathbb{R}_+ \times E$ with compensator $\nu(dt, dx)$ on a measurable Blackwell space $(E, \mathcal{E})$ (for the definition and properties of such a space see for example [De-Me] or [Ge]).

Assumption 2.15 (1) Suppose that the marked point process μ has only finitely many jumps in every finite time interval.

(2) Assume that the marked point process is càdlàg.

(3) We also assume that μ is an optional $\tilde{\mathcal{P}} - \sigma$-finite marked point process.

Then by theorem II.1.8. in [J] we know that there exists a unique compensator, denoted by ν, and we assume that $\nu(\omega; [0,t] \times E) < \infty$ P-a.s. for $\omega \in \Omega$ and for finite t. Moreover we assume that the marked point process μ has an intensity λ, i.e. the P-compensator ν is of the form

$$\nu(dt, dx) = \lambda(t, dx) dt,$$

where $\lambda(t, A)$ is a predictable process for all $A \in \mathcal{E}$. We denote by $\bar{\mu}$ the local martingale

$$\bar{\mu} := \mu - \nu.$$

It is not assumed that W and μ are independent. The filtration $\mathcal{F} = (\mathcal{F}_t)$ is the natural filtration, generated by W and μ, i.e.

$$\mathcal{F}_t = \sigma \left\{ W_s, \mu([0,s] \times A), B;\ 0 \le s \le t,\ A \in \mathcal{E}, B \in \mathcal{N} \right\},$$

where $\mathcal{N}$ is the set of all P-null sets of $\mathcal{A}$. Then $(\mathcal{F}_t)_{t \in [0,T]}$ is right continuous since the processes W and μ are right continuous. Moreover $\left(\Omega, \mathcal{A}, P, (\mathcal{F}_t)_{t \in [0,T]}\right)$ is a complete right continuous stochastic basis.

3 Interest Rate Models with Jumps

We denote by $T \in \mathbb{R}_+$ the *time-of-maturity* and by $x = T - t$ the *time-to-maturity.* Now let

$$\alpha : ([0,T] \times H \times \Omega;\ \mathcal{B}([0,T]) \otimes \mathcal{B}(H) \otimes \mathcal{A}) \longrightarrow (H, \mathcal{B}(H))$$

be a measurable, adapted, integrable, stochastic process and

$$\beta : ([0,T] \times H \times \Omega;\ \mathcal{B}([0,T]) \otimes \mathcal{B}(H) \otimes \mathcal{A}) \longrightarrow (L_2(U_0, H), \mathcal{B}(L_2(U_0, H)))$$

a measurable, adapted, square integrable stochastic process, where $L_2(U_0, H)$ is the space of Hilbert-Schmidt operators from U_0 in H, and

$$U_0 := Q^{1/2}(H).$$

We remark that the space U_0 is separable. Furthermore, let

$$\delta : ([0,T] \times H \times E \times \Omega;\ \ \mathcal{B}([0,T]) \otimes \mathcal{B}(H) \otimes \mathcal{E} \otimes \mathcal{A}) \longrightarrow (H, \mathcal{B}(H))$$

be a locally bounded, predictable stochastic process with values in the Hilbert space H. Let δ be such that

$$\int_0^T \int_E \|\delta(t, r_t, y)\|_H \, \mu(dt, dy) < +\infty.$$

Then the integral

$$\int_E \delta(\cdot, r_\cdot, y)\mu(\cdot, dy)$$

is well defined since δ is measurable with respect to $\mathcal{E}$ and $\mu(A, dy)$ is a measure on E for fixed $A \in \mathcal{B}([0,T])$. Since by assumption 2.15 the marked point process μ has càdlàg paths which are of finite variation, we obtain that μ is a π-process and the integral

$$\int_0^t \int_E \delta(s, r_s, y)\bar{\mu}(ds, dy)$$

is well defined (see [Met-P] for the definition of the integral with respect to a π-process and properties of this integral). Moreover the integral above is itself a càdlàg process. For such processes we have, in particular, a general Itô formula (see [Met-P], page 45) which we will essentially use in the following.
The coefficients α and δ take values in the Hilbert space H and β takes values in $L_2^0(U_0, H)$. Sometimes we will write for example $\alpha(t, r_t(T))$ which shall denote that we evaluate $\alpha(t, r_t)$ at the maturity times $T \in \mathbb{R}_+$, i.e. $\alpha(t, r_t(T))$ should be understood as $\alpha(t, r_t)(T)$. This is well defined since H can be continuously embedded in the space $L^2(\mathbb{R}_+, \mathbb{R})$ by definition of our Hilbert space H and hence $\alpha(t, r_t(T))$ is real valued. Similarly $\beta(t, r_t(T))$ takes values in $L_2^0(U_0, \mathbb{R})$ and hence $\beta(t, r_t(T))dW_t$ is real valued.

Now consider the following stochastic differential equation

$$\begin{aligned} df(t,T) &= \alpha(t, f(t,T))dt + \beta(t, f(t,T))dW_t \\ &\quad + \int_E \delta(t, f(t,T), y)(\mu(dt, dy) - \nu(dt, dy)). \end{aligned} \tag{6}$$

This describes the *forward rate dynamics.* The forward rate curve at time t can also depend on the past of the interest rate curve until the time t.

Let $P(t,T)$ for $t \in [0,T]$ denote the price process of the *default-free zero-coupon bond* with nominal value set to one unit, which is maturing at time T. Thus we have for time t equal to the maturity time T that $P(T,T) = 1$, and we assume that $P(t,T)$ is strictly positive. The bond price dynamics are defined via the forward rate process $f(t,T)$ such that

$$P(t,T) = \exp\left\{ -\int_t^T f(t,u)du \right\}, \quad t \leq T.$$

This is well defined since $f(t, \cdot) \in H$ is an integrable function of the maturity time T. Then $P(t,T)$ is continuously differentiable with respect to the maturity time T, and hence the forward rate can also be characterized by

$$f(t,T) = \frac{-\partial \ln P(t,T)}{\partial T}, \quad t \leq T.$$

The *short rate process* r_t is defined by $r_t = f(t,t)$. Moreover we assume that there exists an asset $B(t)$ that pays r_t, i.e. if we invest one unit of money at time 0 in this asset there is a payout at time t of

$$B(t)^{-1} := \exp\left(\int_0^t r_s ds\right).$$

B is also called the *money market account* and is usually taken as the *numéraire* so that all other derivatives are discounted by this asset, i.e. they are expressed in units of this asset. For example the *discounted bond price process* $Z(t,T)$ is given as

$$Z(t,T) := P(t,T)\cdot B(t) = \exp\left\{-\int_0^t r_s ds\right\}\cdot P(t,T).$$

Since we will work with the process $P(t,T)$ for $t \geq 0$ and not only with $0 \leq t \leq T$ we simply set the value of $P(t,T)$ for $t \geq T$ to the amount we get if we invest the money market account at time T in the money market account, i.e.

$$P(t,T) := B(t)^{-1}\cdot B(T).$$

In the following we are particularly interested in the forward rate dynamics, for which we finally would like to find finite dimensional realizations. The short rate and bond price dynamics can be handled analogously. In [B-K-R] relations for the finite dimensional case are investigated, i.e. for the case when W is an m-dimensional Wiener process. Since we want to apply methods from differential geometry and Fubini's Theorem as well as Itô's Formula, we need a few more assumptions on the coefficients α, β, δ defining the process in (6).

Assumption 3.1

1. The processes α, β and δ are smooth vector fields.
2. All processes are regular enough such that integration and differentiation can be exchanged and the order of integration can be exchanged as well.
3. The coefficients are continuous in t and for finite t and $T \geq t$ we have

   $$\int_0^T\int_t^T |\alpha(s,f(s,u))|\,duds < \infty, \text{ and } \int_0^T\int_t^T \left|\beta^k(s,f(s,u))\right|^2 duds < \infty,$$

   and

   $$\int_0^T\int_t^T\int_E |\delta(s,f(s,u),y)|^2\,du\,\nu(ds,dy) < \infty,$$

   where $\beta^k(s,f(s,u)) = \beta(s,f(s,u))e_k$ and $\{e_k\}_{k\geq 1}$ denotes an orthonormal basis of the space U_0.

As we will see below Assumption 3.1 (1) can only be satisfied when $f(t,T) \in D(A^\infty)$ since otherwise α is not even defined. We will comment on this later on. The integrability conditions in Assumption 3.1 (3.) are fulfilled if the coefficients are bounded for t,T and $f(t,T)$ from a bounded set and $\nu([0,t]\times E) < \infty$ for finite t. Assumption 3.1 (2.) (also used in [B-K-R]) is of course unprecise but it can indeed be made precise, however, at the disadvantage of becoming "rather technical" (for example involving conditions under which the Fubini Theorem holds). From the context below it should, however, become more clear what the condition actually involves.
For convenience reasons we extend the definitions of the coefficients by putting them equal to zero for $T > t$.

4 A Heath, Jarrow and Morton Drift Condition

Let the stochastic evolution of the forward rate be given by (6). We would like to know under what conditions for the processes α, β and δ a martingale measure exists (see e.g. [B-K-R], page 220). This is interesting since the existence of a martingale measure implies the absence of arbitrage in the market.

Now it is convenient to use Musiela notation [Mu], i.e. to describe the forward rate in the Heath, Jarrow and Morton model using a parametrization in terms of *time-to-maturity* in contrast to *time-of-maturity.*
Therefore define $r(t,\cdot) := f(t, t+\cdot)$, not to be confused with the spot rate $r_t := f(t,t)$, and assume that the forward rate dynamics for the r-process are again given by the stochastic differential equation

$$dr_s = \alpha(s, r_s)ds + \beta(s, r_s)dW_s + \int_E \delta(s, r_s, y)\bar{\mu}(ds, dy) \tag{7}$$

where the coefficients satisfy the assumptions in the former section. Assume that the initial rate r_0 is in the domain of the operator $\mathcal{A} = \frac{\partial}{\partial x}$ which, as we saw, is equal to $H^{1,2}(\mathbb{R}_+)$. This ensures that there exists a strong solution of the above equation and that r_s is again in the domain of A. Obviously the coefficients are not the same as the coefficients for the forward rate process $f(t,T)$ in the time-of-maturity parametrization.

Proposition 4.1 *For the forward rates given by equation (7) the discounted price process* $Z(t,T,r(t,T))$ *on* $[0,T]$ *satisfies the linear stochastic differential equation*

$$\begin{aligned}
&\frac{dZ(t,T,r(t,T))}{Z(t-,T,r(t-,T))} \\
= \; & \left[r(t,T-t) - r(t,0) + A(t,T-t,r(t,T)) + \frac{1}{2}\operatorname{Tr}\left[\left(S(t,T)Q^{1/2}\right)\left(S(t,T)Q^{1/2}\right)^* \right] \right] dt \\
& + S(t,T-t,r(t,T))dW_t + \int_E D(t,T-t,r(t,T),y)\bar{\mu}(dt,dy) \\
& + \int_E \left(e^{D(t,T-t,r(t,T),y)} - 1 - D(t,T-t,r(t,T),y) \right) \mu(dt,dy)
\end{aligned}$$

where

$$\begin{aligned}
A(t,T,r(t,T)) &:= -\int_0^T \alpha(t, r(t,u))du \\
S(t,T,r(t,T)) &:= -\int_0^T \beta(t, r(t,u))du \\
D(t,T,r(t,T),y) &:= -\int_0^T \delta(t, r(t,u), y)du.
\end{aligned}$$

Proof. Let $F(t,T) := Z(t, T+t)$. Applying Fubini's Theorem we get

$$\begin{aligned}
\ln F(t,T) &= -\int_0^t r_s ds - \int_0^T r(t,u)du \\
&= -\int_0^t r(s,0)ds - \int_0^T r(0,u)du - \int_0^t \int_0^T \alpha(s, r(s,u))duds \\
&\quad - \int_0^t \int_0^T \beta(s, r(s,u))dudW_s - \int_0^t \int_E \int_0^T \delta(s, r(s,u), y)du\bar{\mu}(ds,dy).
\end{aligned}$$

Now it follows from Itô's Formula (see e.g. [Met-P] for a general Itô Formula) that

$$\begin{aligned}\frac{dF(t,T)}{F(t-,T)} &= \left[-r(t,0)+A(t,T,r(t,T))+\frac{1}{2}\operatorname{Tr}\left[\left(S(t,T)Q^{1/2}\right)\left(S(t,T)Q^{1/2}\right)^*\right]\right]dt \\ &\quad +S(t,T,r(t,T))dW_t+\int_E(\exp(D(t,T,r(t,T),y))-1)\bar{\mu}(dt,dy) \\ &\quad +\int_E(\exp(D(t,T,r(t,T),y))-1-D(t,T,r(t,T),y))\nu(dt,dy)\end{aligned}$$

and thus the proposition is proved since

$$dZ(t,T)=dF(t,T-t)-\frac{\partial F(t,T-t)}{\partial x}dt=dF(t,T-t)+Z(t,T)r(t,T-t)dt.$$

□

Now we get an important result as a corollary which gives an equation for the drift coefficient which is similar to the HJM drift condition.

Corollary 4.2 *Assume that* $\nu(dt,dy)=\lambda_t(dy)dt$. *Then* P *is a martingale measure if and only if*

$$\int_0^t\int_E\left(e^{D(t,u,r(t,u),y)}-1-D(t,u,r(t,u),y)\right)\lambda_t(dy)dt<\infty$$

for finite t *and* u *, and*

(8)

$$r(t,x)=r(t,0)-A(t,x,r(t,x))-\frac{1}{2}\operatorname{Tr}\left[\left(S(t,T)Q^{1/2}\right)\left(S(t,T)Q^{1/2}\right)^*\right]-R(t,x,r(t,x))$$

$(dPdt-a.e.)$ *where*

$$R(t,x,r(t,x)):=\int_E\left(e^{D(t,x,r(t,x),y)}-1-D(t,x,r(t,x),y)\right)\lambda_t(dy)$$

and the functions $A(t,x,r(t,x)), S(t,x,r(t,x))$ *and* $D(t,x,r(t,x),y)$ *are defined as above.*

Remark 4.3 The relation (8) implies that $r(t,x)$ is an absolutely continuous function under an assumption such that the integrals given by the mappings A,S and R are all well defined and

(9)

$$\begin{aligned}\alpha(t,r(t,x)) &= -\frac{\partial}{\partial x}A(t,x,r(t,x)) \\ &= \frac{\partial}{\partial x}r(t,x)+\frac{1}{2}\frac{\partial}{\partial x}\operatorname{Tr}\left[\left(S(t,x)Q^{1/2}\right)\left(S(t,x)Q^{1/2}\right)^*\right]+\frac{\partial}{\partial x}R(t,x,r(t,x)) \\ &= \frac{\partial}{\partial x}r(t,x)+\frac{1}{2}\sum_{k=1}^{\infty}\frac{\partial}{\partial x}\left\{S(t,x)Q^{1/2}e_k\cdot S(t,x)Q^{1/2}e_k\right\}+\frac{\partial}{\partial x}R(t,x,r(t,x)) \\ &= \frac{\partial}{\partial x}r(t,x)+\sum_{k=1}^{\infty}\frac{\partial}{\partial x}S(t,x)Q^{1/2}e_k\left(S(t,x)Q^{1/2}e_k\right)+\frac{\partial}{\partial x}R(t,x,r(t,x)) \\ &= \frac{\partial}{\partial x}r(t,x)+\sum_{k=1}^{\infty}\beta^k(t,r(t,x))\int_t^x\beta^k(t,r(t,v))dv+\frac{\partial}{\partial x}R(t,x)\end{aligned}$$

where $\beta^k(t, r_t) = \beta(t, r_t)Q^{1/2}e_k \in H$ for an orthonormal basis $\{e_k\}_{k\geq 1}$ of the Hilbert space H (then the sequence $\{Q^{1/2}e_k\}_{k\geq 1}$ is an orthonormal basis of $U_0 = Q^{1/2}H$ since Q is a strictly positive nuclear operator) and

$$\beta^k(t, r(t, v)) = \beta(t, r(t, v))Q^{1/2}e_k \in \mathbb{R}.$$

Moreover,

$$\frac{\partial}{\partial x}R(t, x) = -\int_E \left(e^{D(t,x,r(t,x),y)} - 1\right) \delta(t, r(t, x), y)\lambda_t(dy).$$

One can deduce from the forward rate equation and (9) that if the model is specified under a martingale measure then the dynamics of the forward rate curve are given by the following stochastic evolution equation

$$dr(t, x) = [\mathcal{A}r(t, x) + C(t, x, r(t, x))]\, dt + \beta(t, r(t, x))dW_t + \int_E \delta(t, r(t, x), y)\bar{\mu}(dt, dy)$$

where $\mathcal{A} := \partial/\partial x$, and

$$\begin{aligned} C(t, r(t, x)) \quad := \quad & \sum_{k=1}^{\infty} \beta^k(t, r(t, x)) \int_0^x \beta^k(t, r(t, v))dv \\ & - \int_E \left(\exp(D(t, x, r(t, x), y)) - 1\right) \delta(t, r(t, x), y)\lambda_t(dy). \end{aligned} \tag{10}$$

We can indeed apply the operator $\mathcal{A}$ to the forward rate r_t since we assumed r_0 to be in the domain of the operator $\mathcal{A}$ which ensures the existence of a strong solution of the forward rate equation (7) lying again in the domain of the operator $\mathcal{A}$. Hence $\mathcal{A}r(t, x)$, understood as $\mathcal{A}r_t$ evaluated at x, is well defined.

Hence we have the following
(generalized Heath-Jarrow-Morton drift condition)

$$\alpha(t, r(t, x)) = \mathcal{A}r(t, x) + C(t, x, r(t, x)). \tag{11}$$

The relation (11) generalizes the Heath-Jarrow-Morton drift condition for forward rate models without jumps. The drift term α is uniquely determined by the diffusion volatility β , the jump volatility δ and the intensity measure λ.

5 Finite Dimensional Realizations

Now we want to discuss the existence of finite dimensional realizations as introduced in [Bj-S], [B-C], [B-G], [B-DM-K-R], [Fi], [Fi-T], i.e. one tries to find a model that is finite dimensional and represents the original infinite dimensional model. For the purely Wiener driven case this theory has been extended to our general Hilbert space setting in [L1] and [Fi-T]. The case with driving Wiener process and marked point process with deterministic coefficients has already been discussed in [B-G]. We will here derive a similar result for the case where the interest rate process is defined by a linear stochastic differential equation. Finally we will give an explicit example to illustrate our theory.

5.1 The Continuous Case

We consider forward rate dynamics of the form

$$(12)\qquad \left\{ \begin{array}{rcl} dr_t(x) & = & \alpha(t,r_t(x))dt + \beta(t,r_t(x)) \circ dW_t + \int_E \delta(t,r_t(x),y)\bar{\mu}(dt,dy) \\ r_s(x) & = & r^0(x), \end{array} \right\}$$

where $\{r^0(x); x \geq 0\}$ is a given initial process for a fixed initial time $s \in \mathbb{R}_+$ and the coefficients α, β and δ are defined as in section 3. Here $\bar{\mu} = \mu - \nu$ is the local martingale associated to a general continuous marked process μ on a Blackwell mark space $(E, \mathcal{E})$ with compensator ν and with intensity measure λ.

Remark 5.1 We use the notation $\int_E \delta(t,r_t(x),y)\bar{\mu}(dt,dy)$ which is analogous to the usual notation for a marked point process although in this section $\bar{\mu}$ shall denote a continuous process instead of a jump process. The reason for this notation is that in the next section where we discuss the jump diffusion case with linear coefficients we will use the results for the continuous case and in this context the mentioned notation will turn out to be useful.

We start with a Stratonovich stochastic differential equation for simplicity reasons, since we then can apply the usual differentiation rules which are more convenient especially since we will describe finite dimensional realizations by invariant manifolds. Of course, one could also start with an Itô stochastic differential equation and transform it into a Stratonovich stochastic differential equation (since r_t is a semi martingale) which would just add a term to the drift part. For such a forward rate process we have the following relation between the drift and the volatilities which is in analogy to equation (11) for continuous $\bar{\mu}$.

$$(13)\qquad \begin{array}{rcl} \alpha(t,r_t(x)) & = & \frac{\partial}{\partial x} r_t(x) + \sum_{k=1}^{\infty} \beta^k(t,r_t(x)) \int_0^x \beta^k(t,r_t(v))dv \\ & & - \int_E \left(e^{D(t,x,r_t(x),y)} - 1 \right) \delta(t,r_t(x),y)\lambda_t(dy), \end{array}$$

where $\beta^k = \beta \cdot e_k$ for some orthonormal basis $\{e_k\}_{k \geq 1}$ of U_0 and D is defined by

$$D(t,x,r_t(x),y) = -\int_0^x \delta(t,r_t(u),y)du.$$

We now define finite dimensional realizations in a fashion similar to the one used in the theory for purely Wiener driven forward rates (cf. [L1]).

Definition 5.2 The stochastic differential equation (12) has a local d-*dimensional realization* at the initial curve (r^0, s) if there exist a point $z_0 \in V$, smooth vector fields $a : V \to H$, $b : V \to L_2(U_0, H)$ and $c : V \times E \to H$ on an open d-dimensional subset V of H and a smooth mapping $g : V \to H$ with $g(z_0) = r^0(x)$, such that r has a local representation

$$(14)\qquad r_t(x) = g(Z_t)(x),$$

where Z is the solution of the V-valued Stratonovich stochastic differential equation:

$$(15)\qquad \left\{ \begin{array}{rcl} dZ_t & = & a(Z_t)dt + b(Z_t) \circ dW_t + \int_E c(Z_t,y)\bar{\mu}(dt,dy) \\ Z_s & = & z_0. \end{array} \right\}$$

Here "local" means that the representation holds for all times t with $s \leq t < \tau(r^0, s)$ P-a.s., where $\tau(r^0, s)$ is a strictly positive stopping time for every $(r^0, s) \in D(A) \times \mathbb{R}_+$ with $\tau(r^0, s) > s$.

In analogy to [L1] we define the forward curve manifold G via the finitely parameterized family of forward rate curves g.

Definition 5.3 Given a mapping $g : V \to D(A^\infty)$, the *forward curve manifold* $G \subseteq D(A^\infty)$ is defined as $G = \mathrm{Im}[g] = \{g(z) : z \in V\}$, where V is an open d-dimensional connected subset of H.

Throughout this section we will suppose that the following assumptions holds.

Assumption 5.4 We assume that for every initial point $r^0 \in G$, there exists a unique strong solution in H of the forward rate equation

$$dr_t(x) = \alpha(t, r_t(x))dt + \beta(t, r_t(x)) \circ dW_t + \int_E \delta(t, r_t(x), y)\bar{\mu}(dt, dy).$$

This is only possible if the initial point r^0 is in the domain $D(A)$ of the operator A. Since we will later on impose a condition on the Lie algebra generated by drift and volatilities, which is sufficient for the existence of finite dimensional realizations, and since this Lie algebra is only defined for an initial point $r^0 \in D(A^\infty)$ we will here already assume the same. Hence we suppose as above that the forward curve manifold G is a subset of the space $D(A^\infty)$ and we choose the initial point r^0 in G. The reason why we define the forward curve manifold G as a subset of $D(A^\infty)$ instead of just taking an initial point in $D(A^\infty)$ and defining $G \subset H$ is that we will later on give a condition for the existence of finite dimensional realizations in terms of invariant tangential manifolds. G is invariant if $r_t \in G$ for all times t near the initial time s (see Definition 5.8). The solution r_t however exists and is well defined only if r_t is in the domain of the operator A.
Moreover we will have to impose some assumption on the mapping g in equation (14) as well as for the mapping g of the forward rate manifold $G = \mathrm{Im}[g]$.

Assumption 5.5 Let $\dot{g}$ and g' denote the Fréchet derivatives of $g : V \to D(A^\infty)$ with respect to the x and z variables respectively. We assume the following:

(1) The mapping $z \mapsto g(z)$ is injective and the Fréchet derivative $g'(z)$ with respect to the z-variable is injective for all $z \in V$.

(2) The mapping $z \mapsto \dot{g}(z)$ is a continuous map from V to $D(A^\infty)$.

Definition 5.6 Consider an interest rate model M, that is a specification of the volatility functions $\beta(t, r_t)$ and $\delta(t, r_t, y)$, as well as a given forward curve manifold G. The drift term α is then uniquely determined via the generalized Heath-Jarrow-Morton drift condition (13). We say that G is locally r*-invariant* under the action of the forward rate process r_t if, for each initial curve $r^0 \in G \subset D(A^\infty)$, there exists a strictly positive (Q-a.s.) stopping time $\tau(r^0, s)$, and a stochastic process Z with state space V and possessing a Stratonovich differential of the form

$$dZ_t = a(Z_t)dt + b(Z_t) \circ dW_t + \int_E c(Z_t, y)\bar{\mu}(dt, dy),$$

such that, for all t with $0 \le t \le \tau(r^0)$, Q-a.s., we have the representation

$$r_t(x) = g(Z_t)(x),$$

for all x, Q-a.s.. If $\tau(r^0) = +\infty$ for all $r^0 \in G \subset D(A^\infty)$, Q-a.s., we say that G is *globally* r*-invariant.*

Proposition 5.7 *Let* r^0 *be an initial curve lying in* $D(A^\infty)$. *Under Assumption 5.4 the following statements are equivalent:*

(i) There exists a local finite dimensional realization at the initial point (r^0, s) *for the stochastic differential equation (12) given via a mapping* g *and a finite dimensional process* Z *so that* $r_t = g(Z_t)$, *and the mapping* g *satisfies Assumptions (1) and (2) in 5.5.*

(ii) There exists an invariant (w.r.t. the stochastic differential equation (12)) finite dimensional submanifold $G = \mathrm{Im}[g]$ *with* $r^0 \in G \subset D(A^\infty)$ *such that* g *satisfies assumptions (1) and (2) in 5.5, where invariance is meant in the sense of Definition (5.8).*

Definition 5.8 A submanifold G in $D(A^\infty)$ is said to be *invariant* under the action of the stochastic differential equation (12), if $r_t \in G$ for all $t \ge s$ and all $(r^0, s) \in G \times \mathbb{R}_+$. G is said to be *locally invariant* under the action of the stochastic differential equation (12), if $r_t \in G$ for all $s \le t < \tau(r^0, s)$ for every choice of $(r^0, s) \in G \times \mathbb{R}_+$ where $\tau(r^0, s)$ is a positive stopping time greater than s.

Proof. This proof is analogous to the proof of proposition 4.2 in [B-C].
If we know that there exists a locally finite dimensional realization of r near the initial point $r^0 \in D(A^\infty)$ given via a mapping g satisfying assumptions 5.5 (1) and (2) then the solution r_t exists for t near the initial time s and is well defined, that is in particular $r_t \in D(A^\infty)$. Hence $\mathrm{Im}[g]$ is a submanifold on $D(A^\infty)$ and since g defines a finite dimensional realization this obviously implies invariance by definition.
Conversely if we know that there exists an invariant finite dimensional submanifold G with $r^0 \in G \subset D(A^\infty)$ we can write the initial point as $r^0 = g(z_0)$ for a unique $z_0 \in V$, where

$$G = \mathrm{Im}[g] = \{g(z) : z \in V\}$$

for some open finite dimensional subset V of H and g satisfies assumption 5.5. Thus g is differentiable and $g'(z_0)$ is injective and we conclude that $g'(z_0)$ has a bounded left inverse L , that is

$$Lg'(z_0) = z_0$$

is the identity on V. Define that mapping $h : V \to V$ by $h(z) = Lg(z)$. Applying the standard inverse function theorem (see e.g. [Sp]) to h we obtain (locally) a mapping $f_0 : V \to V$ such that

$$f_0(h(z)) = z, \quad \text{for all } z \in U,$$

where U is a neighborhood of z_0. The mapping $f : H \to V$ defined by $f(r) = f_0(Lr)$ is then a local left inverse of g, i.e. there exist neighborhoods U and W of z_0 and $g(z_0)$, respectively, such that $f(g(z)) = z$, for all $z \in U$. We now define the process Z by $Z_t = f(r_t)$. From the Stratonovich dynamics of r, the Z-dynamics are

$$dZ_t = f'(r_t)\alpha(t, r_t(x))dt + f'(r_t)\beta(t, r_t(x)) \circ dW_t + \int_E f'(r_t)\delta(t, r_t(x), y)\bar{\mu}(dt, dy).$$

Thus, Z is the solution to a finite dimensional stochastic differential equation (15) with

$$\begin{aligned} a(z) &= f'(g(z))\alpha(g(z)), \\ b(z) &= f'(g(z))\beta(g(z)), \\ c(z,y) &= \int_E f'(g(z))\delta(g(z),y)\bar{\mu}(\cdot,dy). \end{aligned}$$

By construction $Z_t = f(r_t)$ and since g is inverse to f we have

$$g(Z_t) = g(f(r_t)) = r_t$$

locally to r^0, so we have proved that there exists a locally finite dimensional realization near r^0. □

The proposition shows that under Assumptions 5.4 and 5.5 r-invariance is equivalent to invariance in the sense of Definition 5.8.

Theorem 5.9 *Let g be a mapping such that assumption (5.5) holds. The forward curve manifold $G = \mathrm{Im}[g]$ given by Definition 5.3 is locally invariant for the forward rate process $r_t(x)$ defined by the stochastic differential equation (12) with initial curve $r^0 \in G \subset D(A^\infty)$ if and only if the following conditions hold for all $t \in [0,T]$*

1. (consistent drift condition)

$$\dot{g}(z) + \sum_{k=1}^{\infty} B\beta^k(t,r) - \int_E \delta(t,r,y)\left(e^{D(t,x,r,y)} - 1\right)\lambda_t(dy) \in \mathrm{Im}[g'(z)]$$

2. (consistent volatility conditions)

$$\begin{aligned} \beta^k(t,r) &\in \mathrm{Im}[g'(z)] \\ \delta(t,r,y) &\in \mathrm{Im}[g'(z)], \quad \forall y \in E, \end{aligned}$$

for all $k \in \mathbb{N}$ where $r \in H$ such that $r = g(z)$ for some $z \in V$ and B is defined by

$$(Bh)(\cdot) = h(\cdot)\int_0^{\cdot} h(s)ds$$

and β^k is defined as $\beta^k(t,r) := \beta(t,r)e_k$ for all $k \in \mathbb{N}$ where $\{e_k\}_{k\in\mathbb{N}}$ is a countable basis for the separable Hilbert space $U_0 = Q^{1/2}H$.

We remark that the sum

$$\sum_{k=1}^{\infty} B\beta^k(t,r_t) = \sum_{k=1}^{\infty} \beta^k(t,r_t(\cdot))\int_0^{\cdot} \beta^k(t,r_t(s))ds$$

is well defined since β is a square integrable stochastic process.

As proved in Section 4, there exists a generalized Heath-Jarrow-Morton drift condition between drift α and the other coefficients which can be described in our continuous case as:

$$\begin{aligned} \alpha(t,r_t(x)) &= \frac{\partial}{\partial x} r_t(x) + \sum_{k=1}^{\infty} \beta^k(t,r_t(x))\int_s^x \beta^k(t,r_t(s))ds \\ &\quad - \int_E \left(e^{D(t,x,r_t(x),y)} - 1\right)\delta(t,r_t(x),y)\lambda_t(dy) \end{aligned}$$

and the forward rate equation can be written as

$$\begin{aligned} dr_t(x) &= \Big(\frac{\partial}{\partial x}r_t(x) + \sum_{k=1}^{\infty}\beta^k(t,r_t(x))\int_s^x \beta^k(t,r_t(x))ds \\ &\quad - \int_E \left(e^{D(t,x,r_t(x),y)}-1\right)\delta(t,r_t(x),y)\lambda_t(dy)\Big)dt \\ &\quad + \beta(t,r_t(x))\circ dW(t) + \int_E \delta(t,r_t(x),y)\bar{\mu}(dt,dy). \end{aligned}$$

Proof of Theorem 5.9. $\Rightarrow$ Assume the forward curve manifold G is locally invariant for the forward rate process r_t defined by the stochastic differential equation (12). Thus Proposition 5.7 states that there exists a locally finite dimensional realization at the initial point $r^0 \in D(A^\infty)$ for (12). Taking the differential in the equation $r_t(x) = g(Z_t)(x)$, where Z_t is a process as in definition 5.2 one obtains
(16)

$$dr_t(x) = g'(Z_t)(x)a(Z_t)dt + g'(Z_t)(x)b(Z_t)\circ dW_t + \int_E g'(Z_t)(x)c(Z_t,y)\bar{\mu}(dt,dy)$$

Moreover we have

$$\begin{aligned} dr_t(x) &= \Big(\frac{\partial}{\partial x}r_t(x) + \sum_{k=1}^{\infty}B\beta^k(t,r_t(x)) \\ &\quad - \int_E \delta(t,r_t(x),y)\left(e^{D(t,x,r_t(x),y)}-1\right)\lambda_t(dy)\Big)dt \\ &\quad + \beta(t,r_t(x))\circ dW_t + \int_E \delta(t,r_t(x),y)\bar{\mu}(dt,dy). \end{aligned}$$

Comparing this with (16) and equating coefficients produces the three conditions in the theorem. Since the initial point r^0, and thus $Z_s = z^0$, can be chosen arbitrarily in $D(A^\infty)$ the conditions hold in full generality.
$\Leftarrow$ Assume that the conditions in the theorem hold. The linear map $g'(z)$ is injective for each z by Assumption 5.5. Thus there exist unique $a(z), b(z)$ and $c(z,y)$ such that

$$(17)\qquad \left\{\begin{aligned} g'(z)(x)a(z)(x) &= \dot{g}(z)(x) + \sum_{k=1}^{\infty}B\beta^k(t,r_t(x)) \\ &\qquad - \int_E \delta(t,r_t(x),y)\left(e^{D(t,x,r_t(x),y)}-1\right)\lambda_t(dy), \\ g'(z)(x)b^k(z)(x) &= \beta^k(t,r_t(x)), \\ g'(z)(x)c(z,y)(x) &= \delta(t,r_t(x),y), \end{aligned}\right\}$$

where $b^k(z)(x) = (b(z)e_k)(x)$ with $\{e_k\}_{k\geq 1}$ an orthonormal basis of $U_0 = Q^{1/2}H$. a, b and c are defined on the finite dimensional subspace V of H and $a(z)(x) \in \mathbb{R}$ is understood as $a(z) \in H$ evaluated at the point x (similarly for b and c). Since V is finite dimensional, $g'(z)$ has closed range. $g'(z)^*g'(z)$ is invertible since $g'(z)$ is injective and has closed range (here $g'(z)^*$ denotes the adjoint of $g'(z)$). If $u = g'(z)x$, then $H(z)u = x$ where

$$H(z) := (g'(z)^*g'(z))^{-1}g'(z)^*.$$

$H(z)$ is called left inverse of $g'(z)$. Then

$$\begin{aligned} a(z) &= H(z)\alpha(g(z)), \\ b(z) &= H(z)\beta(g(z)), \\ c(z,y) &= H(z)\delta(g(z),y), \end{aligned}$$

with

$$\begin{aligned}\alpha(t,r_t(x)) &= \frac{\partial}{\partial x}r_t(x)+\sum_{k=1}^{\infty}B\beta^k(t,r_t(x))\\ &\quad -\int_E \delta(t,r_t(x),y)\left(e^{D(t,x,r_t(x),y)}-1\right)\lambda_t(dy).\end{aligned}$$

The mapping $g'(z)\mapsto H(z)$ is smooth in the operator norm since the mappings

$$g'(z)\mapsto g'(z)^* \text{ and } L\mapsto L^{-1}$$

for invertible L are smooth. Hence a,b and c are smooth functions and thus locally Lipschitz continuous. Define Z as the unique strong solution to equation (15), and define the infinite dimensional process u^n by $u(t,x)=g(Z(t))(x)$. Then we obtain

$$du_t = g'(Z_t)a(Z_t)dt+g'(Z_t)b(Z_t)\circ dW(t)+\int_E g'(Z_t)c(Z_t,y)\bar{\mu}(dt,dy).$$

Considering equation (17) we see that u is a strong solution to the infinite dimensional stochastic differential equation

$$dr_t(x) = \alpha(t,r_t(x))dt+\beta(t,r_t(x))\circ dW_t+\int_E \delta(t,r_t(x),y)\bar{\mu}(dt,dy).$$

with initial condition $u_0=g(z_0)\in D(A^\infty)$. For arbitrary $r^0\in G\subset D(A^\infty)$ we may uniquely select $z_0\in V$ such that $r_s(x)=u_0(x)=g(z_0)(x)$. Thus r and u both solve equation (12) with the same initial condition strongly. By uniqueness of the strong solution to this forward rate equation, we obtain that $r_t=u_t=g(Z_t)$. Thus we have shown r-invariance, and r-invariance obviously implies invariance w.r.t. the stochastic differential equation (12) in the sense of Definition 5.8. □

As a consequence we obtain the following proposition.

Proposition 5.10 *Let r^0 be a given initial curve in $D(A^\infty)$ at a given initial time $s\geq 0$. The stochastic differential equation (12) has a locally finite dimensional realization at (r^0,s) so that g in (14) satisfies assumption 5.5 if and only if there exists a finite dimensional tangential manifold $G=\mathrm{Im}[g]\subset D(A^\infty)$ for*

$$\{\alpha(t,r),\beta(t,r)(u),\delta(t,r,y):\ u\in U_0,y\in E\}$$

for all t near the initial time $s\in[0,T]$ and all $r\in H$ near the initial curve r^0, containing the initial curve r^0 and so that g satisfies Assumption 5.5. The dimension of a minimal realization coincides with the dimension of the minimal tangential manifolds.

Remark 5.11 Recall that $U_0=Q^{1/2}H$. In the Proposition β has to depend on $u\in U_0$ since β takes values in the space $L_2^0(U_0,H)$. Hence we have to evaluate β at u for all $u\in U_0$. The existence of a finite dimensional tangential manifold G for

$$\{\alpha(t,r),\beta(t,r)(u),\delta(t,r,y):\ u\in U_0,y\in E\}$$

is then understood in the sense that for all t near the initial time s and for all r near the initial curve r^0 G should be tangential for

$$\{\alpha(t,r),\beta(t,r)(u),\delta(t,r,y):\ u\in U_0,y\in E\}.$$

Proof. $\Rightarrow$ Suppose there exists a locally finite dimensional realization at (r^0, s) for the stochastic differential equation (12) with initial point $r^0 \in D(A^\infty)$. Hence there exist a point z_0 in a finite dimensional subset V of H, smooth vector fields $a : V \to H$ and $b : V \to L_2^0(H)$ and $c : V \times E \to H$ and a smooth mapping $g : V \to D(A^\infty)$ with $g(z_0) = r^0(x)$ such that r has the local representation

$$r_t(x) = g(Z_t)(x)$$

where Z is the solution of the finite dimensional Stratonovich stochastic differential equation:

$$\left\{ \begin{array}{rcl} dZ_t &=& a(Z_t)dt + b(Z_t) \circ dW_t + \int_E c(Z_t, y)\bar{\mu}(dt, dy). \\ Z_s &=& z^0. \end{array} \right\}$$

Thus we define the forward curve manifold G as the image of the mapping g

$$G = \mathrm{Im}[g] = \{g(z) : z \in V\} \subset D(A^\infty).$$

Obviously the initial point r^0 is contained in G. This manifold is invariant under the action of the stochastic differential equation (12) since $r_t \in G$ by definition. By applying Theorem 5.9 we know that

$$\begin{array}{rcl} \alpha(t, r_t(x)) &\in& T_G(r_t(x)) \\ \beta(t, r_t(x))(u) &\in& T_G(r_t(x)) \\ \delta(t, r_t(x), y) &\in& T_G(r_t(x)) \end{array}$$

for all r_t near $r^0 \in D(A^\infty)$ where α is given by the generalized Heath-Jarrow-Morton drift condition (13) for the stochastic differential equation (12) as in Theorem (5.9). Hence G is also tangential for

$$\{\alpha(t, r_t), \beta(t, r_t)(u), \delta(t, r_t, y) : \ u \in U_0, \ y \in E\}.$$

$\Leftarrow$ Now assume that there exists a finite dimensional tangential manifold $G \subset D(A^\infty)$ for

$$\{\alpha(t, r_t), \beta(t, r_t)(u), \delta(t, r_t, y) : \ u \in U_0, \ y \in E\}.$$

By Theorem 5.9 this is equivalent to the fact that G is invariant under the forward rate process r_t with initial point $r^0 \in D(A^\infty)$. This, however, means that $r_t \in G$ and thus there exists a finite dimensional realization for the stochastic differential equation (12) near the initial point $r^0 \in D(A^\infty)$. $\square$

Definition 5.12 Let $f, g : U \to X$ be smooth vector fields on an open subset U of a real Banach space X. Their *Lie bracket* is the vector field

$$[f, g](x) = f'(x)g(x) - f(x)g'(x),$$

where $f'(x)[g(x)]$ denotes the Fréchet derivative of f and similarly for g. Let F be the smooth distribution on U generated by f and g, that is for every $x \in U$

$$\mathrm{span}\{f(x), g(x)\} = F(x).$$

F is called *involutive* if for f and g their Lie bracket also lies in F, that is if for all $x \in U$, $[f, g](x) \in F(x)$. The *Lie algebra* generated by F, denoted by $\{F\}_{LA}$ or $\{f, g\}_{LA}$, is defined as the minimal involutive distribution containing F. Hence the Lie algebra generated by f and g is the minimal distribution containing f, g, their bracket and their brackets of brackets, and so on.

In the following we denote by $\{\alpha, \beta, \delta\}_{LA}$ the Lie algebra generated by the smooth vector fields $\alpha(t, \cdot), \beta(t, \cdot)(u)$ and $\delta(t, \cdot, y)$ on H for $t \in [0, T]$, $u \in U_0$ and $y \in E$.

Theorem 5.13 *Assume that the vector fields α, β, δ and an initial curve $\hat{r} \in D(A^\infty)$ are given. Then the following statements are equivalent:*

(i) For each choice of initial point r^0 near $\hat{r} \in D(A^\infty)$, there exists a local d-dimensional realization at (r^0, s) of the infinite dimensional stochastic differential equation (12).

(ii) The Lie algebra $\{\alpha, \beta, \delta\}_{LA}$ has dimension d near $(\hat{r}, s)$.

Remark 5.14 The Lie algebra $\{\alpha, \beta, \delta\}_{LA}$ is well defined in a neighborhood of $\hat{r} \in D(A^\infty)$ since the drift term α which is given via the Heath, Jarrow and Morton drift condition (11)

$$\begin{aligned} \alpha(t, r_t(x)) &= \frac{\partial}{\partial x} r_t(x) + \sum_{k=1}^{\infty} \beta^k(t, r_t(x)) \int_s^x \beta^k(t, r_t(s)) ds \\ &\quad - \int_E \left(e^{D(t,x,r_t(x),y)} - 1 \right) \delta(t, r_t(x), y) \lambda_t(dy) \end{aligned}$$

is infinitely often weak differentiable for $r_t(x) \in D(A^\infty)$ by definition of the Hilbert space $D(A^\infty) = H^{\infty,2}(\mathbb{R}_+)$ and since all coefficients are assumed to be smooth vector fields.

Proof. From the above observations it follows that G is a tangential manifold for the distribution generated by

$$\{\alpha, \beta(u), \delta(y) : u \in U_0, y \in E\}$$

if and only if it is tangential for

$$\{\alpha, \beta, \delta\}_{LA}.$$

The Lie algebra is only defined for $r_t \in D(A^\infty)$. Then the rest follows from Proposition 5.10, which states that there exists a finite dimensional realization at (r^0, s) for $r^0 \in D(A^\infty)$ if and only if there exists a finite dimensional tangential manifold $G \subset D(A^\infty)$ for α, β, δ containing the initial point. In this situation the condition that G is tangential for the Lie algebra is well defined since $G \subset D(A^\infty)$. □

Now we formulate the main theorem for the case of a continuous forward rate model.

Theorem 5.15 (Main Theorem) *Let β and δ be as above and consider an initial forward rate curve $r^0 \in D(A^\infty)$. Then the forward rate model (12) generated by β and δ admits a locally finite dimensional realization at (r^0, s) if and only if*

$$\dim \{\alpha, \beta, \delta\}_{LA} < \infty$$

in a neighborhood of $r^0 \in D(A^\infty)$ and the drift α is given as

$$\alpha(t, r_t(x)) = \frac{\partial}{\partial x} r_t(x) + \sum_{k=1}^{\infty} B\beta^k(t, r_t(x)) - \int_E \delta(t, r_t(x), y) \left(e^{D(t,x,r_t(x),y)} - 1 \right) \lambda_t(dy).$$

where B is defined as above.

Proof. From Theorem 5.13 it follows that the condition

$$\dim \{\alpha, \beta, \delta\}_{LA} < \infty$$

at (r^0, s) is equivalent to the fact that for every initial value $r^0 \in D(A^\infty)$ there exists a d-dimensional realization of the infinite dimensional stochastic differential equation (12). The proof of Theorem 5.9 gives the representation of α. □

5.2 The Linear Case

Now we consider an interest rate model as in Section 3 driven by a Wiener process as well as a marked point process which will be denoted by μ and is defined as in general setting 2 and $\bar{\mu} = \mu - \nu$ where $\nu = \lambda_t(dy)dt$ is the P-compensator of μ. Thus we have the following r-dynamics

$$\left\{ \begin{array}{rcl} dr_t(x) & = & \alpha(t, r_t(x))dt + \beta(t, r_t(x)) \circ dW_t + \int_E \delta(t, r_t(x), y)\bar{\mu}(dt, dy) \\ r_s(x) & = & r^0(x), \end{array} \right\} \tag{18}$$

where r^0 is a given initial curve in an arbitrary separable Hilbert space H satisfying Assumption 2.8 for a fixed initial time $s \in \mathbb{R}_+$ and the coefficients α, β and δ are defined as in Section 3, however, on an arbitrary separable Hilbert space H.

We now define finite dimensional realizations in a similar fashion to the theory for purely Wiener driven forward rates.

Definition 5.16 The stochastic differential equation (18) has a local d*-dimensional realization* at the initial curve r_0 if there exist a point $z_0 \in V$, smooth vector fields $a : V \to H$, $b : V \to L_2(H)$ and $c : V \times E \to H$ on an open d-dimensional subset V of H and a smooth mapping $g : V \to H$ with $g(z^0) = r^0$, such that r has a local representation

$$r_t(x) = g(Z_t)(x), \tag{19}$$

where Z is the solution of the V-valued Stratonovich stochastic differential equation:

$$\left\{ \begin{array}{rcl} dZ(t) & = & a(Z(t))dt + b(Z(t)) \circ dW_t + \int_E c(Z(t), y)\bar{\mu}(dt, dy) \\ Z_s & = & z^0. \end{array} \right\} \tag{20}$$

Here "local" means that the representation holds for all times t with $s \leq t < \tau(r^0(x), s)$ P-a.s., where $\tau(r^0, s)$ is a strictly positive stopping time for every $(r^0, s) \in D(A) \times \mathbb{R}_+$ with $\tau(r^0, s) > s$.

However there are some difficulties since we now have jumps at certain times and hence we will no longer be able to describe finite dimensional realizations in terms of invariant tangential submanifolds since whenever there is a jump invariance will get lost and moreover tangency can no longer be defined in the usual way. Furthermore, we will have to change the initial curve r^0 after any jump. Thus, if we assume that there exists a finite dimensional realization to equation (18) near r^0 for an initial time s before the first jump occurs, we have to add the jump to the initial curve at the jump time to receive the new initial curve for the next time interval, i.e. up to the next jump. However this will change the whole solution of the stochastic differential equation (18). This is why it is not possible to describe finite dimensional realization for the stochastic differential equation (18) when the coefficients are arbitrary functions of t, x and the forward rate r_t itself. However, if the coefficients are linear in r the solution r of the stochastic differential equation (21) as below for the new initial point after the first jump would just be the solution of the stochastic differential equation before the jump without any jumps plus the solution of the same stochastic differential equation but with initial condition given by the value obtained just after the first jump. We will explain this in more detail later on. Thus in such a setting it is possible to describe finite dimensional realizations via invariant tangential submanifolds.

Now we suppose that the coefficients in the forward rate model (18) are linear in the r-variable and independent of the time t, i.e. we assume an interest rate model of the form

$$(21) \qquad \left\{ \begin{array}{rcl} dr_t(x) & = & \alpha(x) \cdot r_t(x)dt + \beta(x) \cdot r_t(x) \circ dW_t + \int_E \delta(x,y) r_t(x) \bar{\mu}(dt,dy) \\ r_s(x) & = & r^0(x). \end{array} \right\}$$

r^0 is again a given initial curve and $s \in \mathbb{R}_+$ an initial time. In this sequel we use the Musiela parameterization, where x denotes the time-to-maturity rather than the standard HJM parameterization where x would denote the time-of-maturity. This is supposed to be a general interest rate model and not a particular forward rate model. In the case of a forward rate model we would have to take into account the generalized Heath-Jarrow-Morton drift condition (11) which determines the drift term uniquely by the volatility terms. Within that framework it would not be possible to have a linear drift part when the volatilities are linear in r. This, however will be of great importance in the following as already mentioned above. In contrast to the former continuous setting where we consider an Heath, Jarrow and Morton forward rate model, we can choose here the initial curve arbitrary in the Hilbert space H. The reason therefore is that we here consider a general linear interest rate model and not an HJM model any longer. Hence we do not have an HJM drift condition as before. Thus the drift term $\alpha(x)r_t(x)$ above does not, in general, depend on the operator $\mathcal{A}$ as before. The condition we will need later on is that the Lie algebra generated by drift and volatilities is well defined, i.e. all coefficients have to be smooth vector fields.

To overcome the problems connected with the jumps we define stopping times $T_n(\omega)$ for a fixed $\omega \in \Omega$, namely the jump times, by

$$\begin{array}{rcl} T_0(\omega) & := & s \in \mathbb{R}_+ \\ T_n(\omega) & := & \min\{t : \bar{\mu}_{t+}(\omega) - \bar{\mu}_{t-}(\omega) > 0 \text{ and } t > T_{n-1}(\omega)\}, \quad n \geq 1. \end{array}$$

Then we consider the stochastic differential equation (21) for the times between the jumps $0 \leq t < T_1(\omega)$, $T_1(\omega) \leq t < T_2(\omega), \ldots$. In these time intervals r is only driven by a continuous process and thus we can apply the standard theory developed in section 5.1. Here we assume that $\alpha, \beta(u)$, $u \in U$, and δ are smooth vector fields on $H \times \mathbb{R}_+$ and on $H \times \mathbb{R}_+ \times E$ respectively.
Now we consider the stochastic differential equation (21) without jumps, that is at every time $T_n(\omega)$ when there would be a jump we subtract the jump size so that we receive a continuous interest rate r as solution. Hence we consider the stochastic differential equation

$$(22) \qquad \left\{ \begin{array}{rcl} d\tilde{r}_t(x) & = & \alpha(x)\tilde{r}_t(x)dt + \beta(x)\tilde{r}_t(x) \circ dW_t + \int_E \delta(x,y)\tilde{r}_t(x)\bar{\mu}(dt,dy) \\ & & -\sum_{n \in \mathbb{N}} \sum_{(T_n(\omega); T_n(\omega) \leq t)} \int_E \delta(x,y)\tilde{r}_{T_n(\omega)}(x)\bar{\mu}(dt,dy) \\ \tilde{r}_s(x) & = & r^0(x). \end{array} \right\}$$

All the statements in the former subsection still hold in this linear case since we only used the generalized Heath-Jarrow-Morton drift condition in the theorem (16) which can be easily reformulated as follows.

Theorem 5.17 *Let g be a mapping such that assumption (5.5) holds. The forward curve manifold G given by definition (5.3) is locally invariant for the interest rate process $\tilde{r}_t(x)$ without jumps defined by the stochastic differential equation (22) with initial curve r^0 if and only if the following conditions hold*

1. (consistent drift condition)
$$\alpha(x)\tilde{r}_t \ \in \ \mathrm{Im}[g'(z)]$$
2. (consistent diffusion volatility condition)
$$\beta^k(x)\tilde{r}_t \ \in \ \mathrm{Im}[g'(z)]$$
3. (consistent jump volatility condition)
$$\delta(x,y)\tilde{r}_t - \sum_{n\in\mathbb{N}} \sum_{(T_n(\omega);T_n(\omega)\leq t)} \delta(x,y)\tilde{r}_{T_n(\omega)} \in \mathrm{Im}[g'(z)],$$

for all $k \in \mathbb{N}$ *where* $\tilde{r} = g(z)$ *and* B *is defined by*
$$(Bh)(x) = h(x)\int_0^x h(s)ds$$
and β^k *is defined as* $\beta^k(x)\tilde{r} := \beta(x)\tilde{r}\cdot e_k$ *for all* $k \in \mathbb{N}$ *where* $\{e_k\}_{k\in\mathbb{N}}$ *is a countable basis for the separable Hilbert space* $U_0 = Q^{1/2}H$.

The notation *forward curve manifold* is of course not really suitable in this situation since we consider a general interest rate model rather than a special forward rate model. However we will keep this notation since only the name should be different and the rest is just the same in both models. Hence a new notation would probably only lead to confusion. The proof of Theorem (5.17) is similar to the proof of Theorem 5.9 if we replace the forward rate drift by our linear drift $\alpha(x)r_t(x)$.

Proof. $\Rightarrow$ Assume the forward curve manifold G is locally invariant for the interest rate process $r_t(x)$ defined by the stochastic differential equation (22) with initial point $r^0 \in H$. Thus Proposition 5.7 states that there exists a locally finite dimensional realization near r^0 for (22). Taking the differential in the equation $\tilde{r}_t(x) = g(Z_t)(x)$, where Z_t is a process as in Definition 5.16 one obtains
(23)
$$\begin{aligned} d\tilde{r}_t(x) &= g'(Z_t)(x)a(Z_t)dt + g'(Z_t)(x)b(Z_t)\circ dW_t + \int_E g'(Z_t)(x)c(Z_t,y)\bar{\mu}(dt,dy) \\ &\quad -\sum_{n\in\mathbb{N}} \sum_{(T_n(\omega);T_n(\omega)\leq t)} \int_E g'(Z_{T_n(\omega)})(x)c(Z_{T_n(\omega)},y)\bar{\mu}(T_n(\omega),dy) \end{aligned}$$
Moreover we have
$$\begin{aligned} d\tilde{r}_t(x) &= \alpha(x)\cdot\tilde{r}_t(x)dt + \beta(x)\cdot\tilde{r}_t(x)\circ dW_t + \int_E \delta(x,y)\cdot\tilde{r}_t(x)\bar{\mu}(dt,dy) \\ &\quad -\sum_{n\in\mathbb{N}} \sum_{(T_n(\omega);T_n(\omega)\leq t)} \int_E \delta(x,y)\cdot\tilde{r}_{T_n(\omega)}(x)\bar{\mu}(T_n(\omega),dy). \end{aligned}$$
Comparing this with (16) and equating coefficients produces the two conditions in the theorem. Since the initial point $r_0(x)$, and thus Z_s, can be chosen arbitrarily the conditions hold in full generality.
$\Leftarrow$ Assume that the conditions in the theorem hold. The linear map $g'(z)$ is injective for each z by Assumption 5.5. Thus there exist unique $a(z), b(z)$ and $c(z,y)$ such that
$$\begin{aligned} g'(z)a(z) &= \alpha(x)\cdot\tilde{r}_t(x) \\ g'(z)\langle b(z),e_k\rangle_H &= \beta^k(x)\cdot\tilde{r}_t(x), \\ g'(z)c(z,y) &= \delta(x,y)\cdot\tilde{r}_t(x) - \sum_{n\in\mathbb{N}} \sum_{(T_n(\omega);T_n(\omega)\leq t)} \delta(x,y)\cdot\tilde{r}_{T_n(\omega)}(x). \end{aligned} \tag{24}$$

Since V is finite dimensional, $g'(z)$ has closed range. $g'(z)^* g'(z)$ is invertible since $g'(z)$ is injective and has closed range. If $u = g'(z)x$, then $H(z)u = x$ where

$$H(z) := (g'(z)^* g'(z))^{-1} g'(z)^*.$$

$H(z)$ is called left inverse of $g'(z)$. Then

$$\begin{aligned} a(z) &= H(z)\alpha(g(z)), \\ b(z) &= H(z)\beta(g(z)), \\ c(z,y) &= H(z)\delta(g(z),y). \end{aligned}$$

The mapping $g'(z) \mapsto H(z)$ is smooth in the operator norm since the mappings

$$g'(z) \mapsto g'(z)^*$$

and $A \mapsto A^{-1}$ for invertible A are smooth. Hence a, b and c are smooth functions and thus locally Lipschitz continuous. Define Z as the unique strong solution to equation (22), and define the infinite dimensional process u by $u(t,x) = g(x, Z(t))$. Then we obtain

$$\begin{aligned} du_t &= g'(Z_t)a(Z_t)dt + g'(Z_t)b(Z_t) \circ dW(t) + \int_E g'(Z_t)c(Z_t,y)\bar{\mu}(dt,dy) \\ &\quad - \sum_{n\in\mathbb{N}} \sum_{(T_n(\omega);T_n(\omega)\le t)} \int_E g'(Z_{T_n(\omega)})c(Z_{T_n(\omega)},y)\bar{\mu}(T_n(\omega),dy). \end{aligned}$$

Considering equation (24) we see that u solves the infinite dimensional stochastic differential equation

$$\begin{aligned} d\tilde{r}_t(x) &= \alpha(x)\cdot\tilde{r}_t(x)dt + \beta(x)\cdot\tilde{r}_t(x) \circ dW_t + \int_E \delta(x,y)\cdot\tilde{r}_t(x)\bar{\mu}(dt,dy) \\ &\quad - \sum_{n\in\mathbb{N}} \sum_{(T_n(\omega);T_n(\omega)\le t)} \int_E \delta(x,y)\cdot\tilde{r}_{T_n(\omega)}(x)\bar{\mu}(T_n(\omega),dy). \end{aligned}$$

with initial condition $u_0 = g(z^0)$. For arbitrary $r^0 \in G$ we may uniquely select $z^0 \in V$ such that $r^0 = \tilde{r}_s = u_0 = g(z^0)$. Thus $\tilde{r}$ and u both solve equation (22) with the same initial condition. By uniqueness of the strong solution to this forward rate equation, we obtain that $\tilde{r}_t = u_t = g(Z_t)$. Thus we have shown $\tilde{r}$-invariance, and $\tilde{r}$-invariance obviously implies invariance w.r.t. the stochastic differential equation (22) in the sense of Definition 5.8. □

Of course the Main Theorem 5.13 does not hold any longer since here the generalized Heath-Jarrow-Morton drift condition (11) is specifically used. However, in the following we will not need a corresponding theorem. Within this setting we can describe finite dimensional realizations in the sense of Definition 5.2 via invariant tangential manifolds as follows.

Assume that the Lie algebra generated by $\alpha(x)r$, $\beta(x)r$ and $\delta(x,y)r$ is well defined and finite dimensional for all r near an initial point $r^0 \in H$. In particular, we assume here that all coefficients are smooth vector fields, otherwise the Lie algebra would not be well defined. Since we now want to describe finite dimensional realizations for the whole interest rate curve r defined by the stochastic differential equation (21), with jumps at the stopping times $T_n(\omega)$ defined in the former subsection, we have to take care of the initial curves r^0 for each initial time s with $T_n(\omega) \le s < T_{n+1}(\omega)$. First of all we will have to consider a different initial curve. I.e. if we start with an initial curve r_0 at an initial time $s < T_1(\omega)$ we might have a finite dimensional realization up to time $\tau(r^0,s) > T_1(\omega)$ for the interest rate curve $\tilde{r}$ without jumps and thus also for the interest rate curve with jumps r up to time $T_1(\omega)-\epsilon$ for $\epsilon>0$ but no longer for the time $T_1(\omega)$ itself since there is a jump at this time. At this point we will have to change the initial curve by adding the jump at the time $T_1(\omega)$. This new initial curve r_1 at time $T_1(\omega)$ is given by

$$r_1=\tilde{r}_{T_1(\omega)}(x)+\int_E \delta(x,y)\cdot r_{T_1(\omega)}(x)\bar{\mu}(T_1(\omega),dy).$$

The obvious question now is how we will have to change our assumption of the Lie algebra to insure that the there also exist finite dimensional tangential manifolds for

$$\left\{\alpha(x)r,(\beta(x)r)(u),\delta(x,y)r;\ \ u\in U_0,\ \ x\geq 0,\ \ ,r\in H \text{ near } r^1\right\}$$

containing the initial point r_1.
To answer this question we will have to consider different stochastic differential equations without jumps with different initial curves. The notations are summarized in the following table.

Initial times	$s_i:=T_i(\omega)$
Initial curves	$r_i:=$ Jump at time s_i
Solutions	$\tilde{r}^i$ for $t\geq s_i$ for $i\in\mathbb{N}\cup\{0\}$.

for $i\in\mathbb{N}\cup\{0\}$.

Hence $\tilde{r}^n$ is the solution of the stochastic differential equation

$$(25)\qquad \left\{\begin{array}{rcl} d\tilde{r}^n_t(x) &=& \alpha(x)\tilde{r}^n_t(x)dt+\beta(x)\tilde{r}^n_t(x)\circ dW_t+\displaystyle\int_E\delta(x,y)\tilde{r}^n_t(x)\bar{\mu}(dt,dy)\\ && -\displaystyle\sum_{k\in\mathbb{N}}\sum_{(T_k(\omega);T_k(\omega)\leq t)}\int_E\delta(x,y)\tilde{r}_{T_k(\omega)}(x)\bar{\mu}(T_k(\omega),dy)\\ \tilde{r}^n_{T_n(\omega)}(x) &=& r_n(x)=\displaystyle\int_E\delta(x,y)\tilde{r}^n_{T_n(\omega)}(x)\bar{\mu}(T_n(\omega),dy), \end{array}\right\}$$

for $n\in\mathbb{N}\cup\{0\}$ and $t\geq s$, where we set $\tilde{r}^n_t(x)\equiv 0$ for all $t<s_n$.

Since the stochastic differential equation (21) is linear in the r-variable the solution r is given by

$$r=\sum_{i=0}^{\infty}\tilde{r}^i$$

where $\tilde{r}^n(x):=0$ for $x<s_n$. In the sum above only finitely many terms are non vanishing.

Lemma 5.18 *Suppose there exist finite dimensional realizations of the stochastic differential equations without jumps (25) at initial curves $\tilde{r}^n_t(x)$ for all times $t\geq 0$ for all $n\in\mathbb{N}$, respectively. Then there exists a locally finite dimensional tangential manifold for*

$$\{\alpha(x)r,(\beta(x)r)(u),\delta(x,y)r;\ \ u\in U_0,y\in E,\ \ x\geq 0,\ \ r\in H\ \ \textit{near}\ \ r_t(x)\}$$

containing the initial point r_t for all $t\geq 0$ where r is given as

$$r=\sum_{i=0}^{\infty}\tilde{r}^i.$$

Proof. Since we assume that there exist finite dimensional realizations at an initial curve $\tilde{r}^n_t$ for all times $t\geq 0$ for the interest rate curves $\tilde{r}^n$ without jumps for all $n\in\mathbb{N}\cup\{0\}$ there exists in particular a finite dimensional realization for an initial curve $r_n=\tilde{r}^n_{s_n}(x)$

at the initial times s_n for the stochastic differential equations (25) for $n \in \mathbb{N} \cup \{0\}$. These realizations are determined by mappings g_n and finite dimensional processes Z_n as defined in (19) and (20) and g_n satisfies the Assumption 5.5, i.e. in a neighborhood of the initial time s_1 we have a mapping

$$g_0 : V_0 \longrightarrow H$$

and a finite dimensional process Z_0 such that $\tilde{r}^0 = g_0(Z_0)$. Hence we can apply Proposition 5.10. Thus there exists a finite dimensional tangential manifold G_0 for

$$\left\{\alpha(x)\tilde{r}^0, (\beta(x)\tilde{r}^0)(u), \delta(x,y)\tilde{r}^0 : u \in U, y \in E, \ \ |t - s_1| \text{ small}, \ \ x \geq 0, \tilde{r}^0 \in H \text{ near } r_1\right\}$$

containing the initial point r_1 and G_0 is given by $G_0 = \mathrm{Im}[g_0]$, as is shown in the proof of Proposition 5.10. Moreover there exists a $z_0 \in V_0$ where V_0 is an open finite dimensional subset of H such that $r_1 = g_0(z_0)$. This means that

$$\alpha(x)\tilde{r}^0_t(x) \in T_{G_0}(\tilde{r}^0_t(x))$$

$$\beta(x)\tilde{r}^0_t(x)(u) \in T_{G_0}(\tilde{r}^0_t(x))$$

$$\delta(x,y)\tilde{r}^0_t(x) - \sum_{n\in\mathbb{N}} \sum_{(T_n(\omega);T_n(\omega)\leq t)} \delta(x,y)\tilde{r}^0_{T_n(\omega)}(x) \in T_{G_0}(\tilde{r}^0_t(x))$$

for all $\tilde{r}^0_t(x)$ near r_1.

Similarly we have in a neighborhood of s_2 a mapping

$$g_1 : V_1 \longrightarrow H$$

defining a manifold $G_1 = \mathrm{Im}[g_1]$ by the above construction and a finite dimensional process Z_1 such that $\tilde{r}^1 = g_1(Z_1)$, where V_1 is a finite dimensional subset of H. Then the mapping

$$g_0 + g_1 : V_0 \times V_1 \to H \ \ ; \ \ (z_0, z_1) \mapsto g_0(z_0) + g_1(z_1)$$

defines a realization of $\tilde{r}^0 + \tilde{r}^1$ near the initial point r_{s_1} by

$$g_0(Z_0) + g_1(Z_1) = \tilde{r}^0 + \tilde{r}^1.$$

Furthermore the manifold $G := \mathrm{Im}[g_0 + g_1]$ is locally finite dimensional and tangential for

$$\{\alpha(x)r, (\beta(x)r)(u), \delta(x,y)r : u \in U, y \in E, \ |t - s_1| \text{ small}, \ \ x \geq 0, r \in H \text{ near } r_{s_1}\}$$

near the initial curve r_{s_1}. Indeed

$$\begin{aligned} \alpha(x) \cdot r_t(x) &= \alpha(x) \cdot \left(\tilde{r}^0_t(x) + \tilde{r}^1_t(x)\right) \\ &= \underbrace{\alpha(x) \cdot \tilde{r}^0_t(x)}_{\in \ T_{G_0}(\tilde{r}^0_t(x))} + \underbrace{\alpha(x) \cdot \tilde{r}^1_t(x)}_{\in \ T_{G_1}(\tilde{r}^1_t(x))} \ \in \ T_G(r_{s_1}(x)) \end{aligned}$$

for all $s_1 - \epsilon \leq t < s_1(\omega) + \epsilon$ for some $\epsilon > 0$. Similarly one can show that $\beta(x)r_t(x)$ and $\displaystyle\int_E \delta(x,y)r_t(x)\bar{\mu}(\cdot, dx)$ are in the tangent space of G near $r_{s_1} \in D(A^\infty)$.

Hence the jump at the time $s_1 = T_1(\omega)$ simply means a parallel shift which of course does not change the fact that G_0 is tangential for

$$\{\alpha(x)r_t(x), (\beta(x)r_t(x))(u), \delta(x,y)r_t(x) : \ \ u \in U_0, y \in E\}$$

for r_t near r_{s_1} and $\{T_1(\omega) - \epsilon < t < T_1(\omega)\} \cap \{T_1(\omega) < t < T_1(\omega) + \epsilon\}$, with $\epsilon > 0$. Then we say that G_0 is also tangent for

$$\{\alpha(x) r_t(x), (\beta(x) r_t(x))(u), \delta(x,y) r_t(x): \ u \in U_0, y \in E\}$$

at time s_1. Remark that because of the jump at time $s_1 = T_1(\omega)$ tangency cannot be defined at this time in the usual way. Thus we have shown that there exists a finite dimensional tangential manifold G for

$$\{\alpha(x) r, (\beta(x) r)(u), \delta(x,y) r; \ u \in U_0, y \in E, \ x \geq 0, \ , r \in H \text{ near } r_{s_1}(x)\}$$

near the initial curve r_{s_1}. Moreover G contains the initial curve r_{s_1} since

$$r_{s_1}(x) = \underbrace{\tilde{r}^0_{s_1}(x)}_{\in G_0} + \underbrace{\int_E \delta(x,y) \tilde{r}^1_{s_1}(x) \bar{\mu}(s_1, dy)}_{\in G_1}.$$

The other jumps can be handled similarly. $\square$

Theorem 5.19 *The following statements are equivalent:*

(i) The Lie algebras

$$\{\alpha(x)\tilde{r}^n_t(x), \beta(x)\tilde{r}^n_t(x), \delta(x,y)\tilde{r}^n_t(x)\}_{LA}$$

are finite dimensional near the initial curves $\tilde{r}^n_t$ *for all* $t \geq 0$ *for all* $n \in \mathbb{N} \cup \{0\}$.

(ii) There exists a finite dimensional realization of the stochastic differential equation (21) at an initial point r_t *for all* $t \geq 0$.

Proof. $\Rightarrow$ Theorem 5.13 states that if the Lie algebras above are all finite dimensional then there exist finite dimensional realizations of the stochastic differential equations without jumps (25) at initial curves $\tilde{r}^n_t$ for all times $t \geq 0$ for all $n \in \mathbb{N}$, respectively. Applying Lemma 5.18 yields that there exists a locally finite dimensional tangential manifold for

$$\{\alpha(x) r, (\beta(x) r)(u), \delta(x,y) r; \ u \in U_0, y \in E, \ x \geq 0, \ , r \in H \text{ near } r_t\}$$

containing the initial point r_t for all $t \geq 0$ where r is given as

$$r = \sum_{i=0}^{\infty} \tilde{r}^i.$$

Then a finite dimensional realization of the stochastic differential equation (21) is defined by the mapping

$$g := \sum_{i=0}^{\infty} g_i,$$

and by the sequence of finite dimensional processes

$$Z = (Z_i)_{i \in \mathbb{N} \cup \{0\}},$$

where g_k and Z_k are defined as in the proof of the Lemma (5.18). By construction we have

$$r_t = g(Z_t) = \sum_{i=0}^{\infty} g_i(Z_i).$$

The dimension of the finite dimensional realization of the solution r_t of (21) equals the supremum of the dimensions of the finite dimensional realizations of the stochastic differential equations (25). $\square$

5.3 Example for Deterministic Coefficients

Consider the stochastic differential equation

$$
\begin{array}{rcl} dr_t &=& Ar_t dt + Br_t dN_t \\ r_0 &=& a \end{array} \tag{26}
$$

where $a \in \mathbb{R}^n$, $a \neq 0$, and $A, B \in \mathbb{R}^{n \times n}$. N_t is assumed to be a Poisson process with intensity λ. The solution of the stochastic differential equation (26) is given by the formula

$$
r_t = \exp\left\{At - \frac{1}{2}\lambda B^2 t + BN_t\right\} \cdot a
$$

what can be seen easily by Itô's formula. Thus we have by expanding the exponential function and using the binomial formula:

$$
\begin{array}{rcl} r_t &=& \exp\left\{At - \frac{1}{2}\lambda B^2 t + BN_t\right\} \cdot a \\ &=& \sum_{k=0}^{\infty} \left(At - \frac{1}{2}\lambda B^2 t + BN_t\right)^k \frac{1}{k!} \cdot a \\ &=& \sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{l=0}^{j} \left(\begin{array}{c} k \\ j \end{array}\right)\left(\begin{array}{c} j \\ l \end{array}\right) A^l B^{k+j-2l} \cdot a \cdot t^j (-\frac{\lambda}{2})^{j-l} \frac{1}{k!} \cdot N_t^{k-j}. \end{array}
$$

Since N_t^{k-j} is a scalar depending on ω, we get the following result. If the dimension

$$
\dim\left(\mathrm{span}\left\{A^j B^k \cdot a; j, k = 0, \ldots, \infty\right\}\right) < n
$$

then, according to 5.2 there is a lower dimensional version of (26).

Now consider the stochastic differential equation

$$
\begin{array}{rcl} dr_t &=& Ar_t dt + bdN_t \\ r_0 &=& a \end{array} \tag{27}
$$

where $A \in \mathbb{R}^{n \times n}$, $b \in \mathbb{R}^n$ and $c \in \mathbb{R}^n$, $c \neq 0$. N_t is again a Poisson process with intensity λ. The solution of the stochastic differential equation (27) is given by

$$
\begin{array}{rcl} r_t &=& e^{At} \cdot a + \int_0^t e^{A(t-s)} \cdot bdN_s \\ &=& \sum_{k=0}^{\infty} \frac{A^k t^k}{k!} \cdot a + \int_0^t \sum_{k=0}^{\infty} \frac{A^k (t-s)^k}{k!} \cdot bdN_s \\ &=& \sum_{k=0}^{\infty} \frac{A^k}{k!} \cdot \left(t^k \cdot a + b \cdot \int_0^t (t-s)^k dN_s\right) \end{array}
$$

If

$$
\dim\left(\mathrm{span}\left\{A^k \cdot a, A^k \cdot b; k = 0, 1, \ldots, \infty\right\}\right) < n
$$

then there is a lower dimensional version of (27).
If we choose for example

$$
A = \left(\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array}\right), \quad b = \left(\begin{array}{c} 1 \\ 0 \end{array}\right), \quad c = \left(\begin{array}{c} 1 \\ 0 \end{array}\right)
$$

for $n = 2$, then the solution of (27) can be written as

$$
\begin{array}{rcl} r_t &=& e^t \cdot A \cdot a + \int_0^t e^{t-s} \cdot A \cdot bdN_s \\ &=& \left(\begin{array}{c} 1 \\ 0 \end{array}\right) \cdot e^t + \left(\begin{array}{c} 1 \\ 0 \end{array}\right) \cdot \left(\int_0^t e^{t-s} dN_s\right) \end{array}
$$

and $\dim(\text{span}\{A^k \cdot a, A^k \cdot b; k = 0, 1, \ldots, \infty\}) = \dim(\text{span}\{b\}) = 1$. Thus r can also be described by the stochastic differential equation

$$\begin{aligned} dR_t &= r_t dt + dN_t \\ r_0 &= 1. \end{aligned}$$

For general A but such that $A \cdot a = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ it would not be possible to find such a finite dimensional realization.

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