Nonlinear Semigroups

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Abstract

For a topological space E and a measurable submarkov semigroup $\mathbb{P}$, we consider the restriction on $E \times \mathbb{R}_+$ of the kernel associated with the space time semigroup $\mathbb{P} \otimes \mathbb{T}$. A local Kato-class K^t_{Loc} related to V and for functions φ from $E \times \mathbb{R}_+ \times \mathbb{R}$ to $\overline{\mathbb{R}}$, notions of locally Kato-bounded, continuous locally Kato-bounded, locally Kato-Lipschitzian and Kato-Lipschitzian, which are not necessarily (locally) bounded and (locally) Lipschitzian, are introduced. Nonlinear monotone semigroups $({}^{\varphi}Q_t)_{t>0}$, defined not only for positive but for bounded Borel measurable functions and their monotone limits, are constructed. In contrast to many earlier works, our construction method does not rely on Picard iteration .

Introduction

Let E be a topological space and B_b the set of bounded and Borel measurable functions on E. Let $\mathbb{P} = (P_t)_{t\geq 0}$ be a measurable submarkov semigroup of linear operators on B_b. Let us consider the kernel V defined on $E \times \mathbb{R}_+$ by

$$Vf(x,s) = \int_0^s P_t(f_{s-t})(x)\,dt, \qquad \text{where } f_s(x) = f(x,s)$$

for $(x,s) \in E \times \mathbb{R}_+$ and f is a bounded measurable function from $E \times \mathbb{R}_+$ to $\mathbb{R}$.

We then define the local Kato-class K^t_{Loc} related to V (the locality is considered only relative to time) as follows: A Borel measurable function g from $E \times \mathbb{R}$ to $\overline{\mathbb{R}}$ belongs to K^t_{Loc} if, for every $R > 0$,

$$\lim_{\varepsilon \downarrow 0} \Big[\sup_{t_0 < R} \sup_{(x,t) \in E \times \mathbb{R}_+} V\left(1_{[t_0, t_0+\varepsilon[}|g|\right)(x,t) \Big] = 0.$$

For functions φ from $E \times \mathbb{R}_+ \times \mathbb{R}$ to $\overline{\mathbb{R}}$ we define notions of locally Kato-bounded, continuous locally Kato-bounded, Kato-bounded, locally Kato-Lipschitzian and Kato-Lipschitzian (§1).

A function φ satisfying these properties need not be locally bounded or locally Lipschitzian. By means of these notions and nonlinear perturbations we construct for an admissible function φ from $E \times \mathbb{R}$ to $\mathbb{R}$ a nonlinear monotone

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semigroups $({}^{\varphi}Q_t)_{t>0}$ defined not only for positive functions but for bounded Borel measurable functions and their monotone limits.

Our methods does not rely on Picard iteration in contrast to most earlier work.

One motivation for this work is our paper [BS] where we investigate nonlinear semigroups, occurring in physical and biological sciences, with evolutionary law governed by an autonomous system of partial differential equation of parabolic type. Another motivation (see e.g. [Fi]) is the study of superprocesses which are measure-valued Markov branching processes associated with a branching mechanism given by a function ψ (here $\psi(x,y) = y\varphi(x,y)$ for $(x,y) \in E \times \mathbb{R}$) satisfying $\psi(x,\cdot)$ negative definite on $(\mathbb{R}_+)$.

In the next section we do not assume any topological structure on E. We consider $g \in K^t_{Loc}, T > 0$ and the kernel K_T defined by $K_T(f) = V(1_{[0,T]}fg)$ for $f \in B_b(E \times \mathbb{R}_+)$. Using an idea of G. Ritter [R] from parabolic potential theory, we prove among other results, the invertibility of $(I + \alpha K_T)$ for every $\alpha \in \mathbb{R}$.

In the third section we assume that E is a topological space possessing a covering by an increasing sequence of compact subsets and that V has the following property:

(*) For every $u \in B(E \times \mathbb{R}_+)$ such that $V(|u|) \in \mathcal{C}(\mathcal{E} \times \mathbb{R}_+)$, the set $\{V(f), f \in B(E \times \mathbb{R}_+) \text{ with } |f| \leq |u|\}$ is equicontinuous on $E \times \mathbb{R}_+$.

These conditions are satisfied if the space time semigroup yields a balayage space; see Hansen [H]. More generally, if E is a locally compact second countable metric space and V fulfills the hypothesis of absolute continuity (hypothesis L of P.A. Meyer) then V satisfies the property (*) by Mokobodzki [Me]. We then consider a continuous locally Kato-bounded function φ from $E \times \mathbb{R}_+ \times \mathbb{R}$ to $\overline{\mathbb{R}}$ such that $\varphi^- = \sup(0, -\varphi)$ is Kato bounded and $\varphi(z,\cdot)$ is continuous on $\mathbb{R}$ for every $z \in E \times \mathbb{R}_+$. If $\psi(z,y) = y\varphi(z,y)$ is admissible (see Definition 3.2), we prove the following: For every function $f \in B_b(E)$, there exists a unique locally bounded function u with $f = u + V(\psi(\cdot,u))$.

In the fourth section we prove the existence and uniqueness of a nonlinear semigroup satisfying

$$P_t f(x) = Q_t f(x) + \int_0^t P_s(x, \psi(\cdot, Q_{t-s}f))ds$$

for every $x \in E, t > 0$ and $f \in B_b(E)$. Here $\psi(x,y) = y\varphi(x,y)$ and φ is a function from $E \times \mathbb{R}$ to $\overline{\mathbb{R}}$ which verifies the same conditions as in the previous section. The function Qf is bounded on $E \times [0,T]$ for every $T > 0$ and $f \in B_b(E)$ and monotone (see Definition 4.3). We give applications to elliptic-parabolic partial differential operators of second order.

In the last section, we study the corresponding excessive functions.

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1 Space time Kato class conditions and related properties

Let E be a topological space and B_b denote the set of bounded Borel measurable functions on E. We consider a submarkov semigroup $\mathbb{P}$ of kernels on E, i.e., a family $(P_t)_{t>0}$ such that $P_{t+s} = P_t P_s$ for all $s, t > 0$ and $P_t 1 \leq 1$ for every $t > 0$. We will suppose that $\mathbb{P}$ is measurable, i.e, such that for every $f \in B_b$, the mapping from $E \times \mathbb{R}_+^*$ in $\mathbb{R}_+$ which to (x,t) associates $P_t f(x)$ is measurable, where $\mathbb{R}_+^* := \mathbb{R}_+ \setminus \{0\}$. For every bounded measurable $f \in B_b(E \times \mathbb{R})$ we define $f_t \in B_b$ by $f_t(x) = f(x,t)$ for (x,t) in $E \times \mathbb{R}$ .We then consider $\widetilde{\mathbb{P}} = \mathbb{P} \otimes \mathbb{T}$, the space time semigroup on $E \times \mathbb{R}$ given by $\widetilde{\mathbb{P}}_t f(x,s) = P_t(f_{s-t})(x)$, where $\mathbb{T}$ is the translation semigroup on $\mathbb{R}$.

The potential kernel $\widetilde{V}$ of $\widetilde{\mathbb{P}}$ is then given by:

$$\widetilde{V} f(x,s) = \int_0^{+\infty} P_t(f(x, x-t))\, dt = \int_0^{\infty} P_t(f_{(s-t)})(x)\, dt.$$

We set $g(s) = 1_{[0,+\infty[}(s)$ and consider the kernel V defined by

$$\begin{aligned} V(f)(x,s) &= \widetilde{V}(gf)(x,s) = \textstyle\int_0^{+\infty} P_t(1_{[0,+\infty[}(s-t) f_{s-t})\, dt \\ &= \textstyle\int_0^s P_t(f_{s-t})(x)\, dt \text{ for } s \geq 0. \end{aligned}$$

and $V(f)(x,s) = 0$ for $s \leq 0$. Furthermore we have $V(1)(x,s) \leq s$ for every $x \in E$ and $s \geq 0$.

In the sequel, we consider V restricted to $E \times \mathbb{R}_+$. It is then easy to see (by e.g. [BH, p.76]) that V satisfies on $E \times \mathbb{R}_+$, the complete maximum principle and every $\widetilde{\mathbb{P}}$-excessive function (and hence $\mathbb{P}$-excessive) is V-dominant.

We now introduce the following Kato notions:

Definition 1.1. *A function g in $B(E \times \mathbb{R}_+)$ is in the local Kato-class relative to time and uniformly in x, denoted by $K^t_{Loc}(E \times \mathbb{R}_+)$ or K^t_{Loc} , if and only if for every $R > 0, d$ we have:*

$$\lim_{\varepsilon \downarrow 0} \Big[\sup_{t_0 < R} \sup_{(x,t) \in E \times \mathbb{R}_+} V\left(1_{[t_0, t_0+\varepsilon[} |g|\right)(x,t) \Big] = 0.$$

Example 1.2. *(i) If $g \in B_b(E \times \mathbb{R}_+)$, then $g \in K^t_{Loc}$.*

(ii) If $g \in B(E \times \mathbb{R})$ satisfies $\sup_{t \leq T} \sup_{x \in E, s \leq t} P_t |g_{t-s}|(x) < +\infty$ for every $T > 0$, then $g \in K^t_{Loc}$.

Let $\mathbb{P} = (P_t)_{t>0}$ be the Brownian semigroup on $\mathbb{R}^d$ $(d \geq 1)$ and $g \in B^+(E)$, i.e.,

$$P_t(g)(x) = \int_{\mathbb{R}^d} p(t,x,y) g(y) dy, \quad \text{ where } p(t,x,y) = (2\pi t)^{-d/2} \exp[-\frac{|x-y|^2}{4t}].$$

From [AS], we recall the following definition: g is in the Kato-class K_d if and only if $\lim\limits_{\alpha \downarrow 0} \sup\limits_{x \in \mathbb{R}^d} \int_{|x-y| \leq \alpha} \frac{g(y)\, dy}{|x-y|^{d-2}} = 0$. We have the following characterization:

Theorem 1.3. $g \in K^t_{Loc}$ *if and only if* $g \in K_d$.

Proof. By [AS, Thm. 4.5], it is enough to prove that $g \in K^t_{Loc}$ if and only if

$$\lim_{s \downarrow 0} \sup_{x \in \mathbb{R}^d} \int_0^s \big(\int_{\mathbb{R}}^d p(t,x,y) g(y)\, dy \big)\, ds = 0.$$

From the definition of the kernel V, we have for all $0 < s < \varepsilon$

$$\int_0^s (\int_{\mathbb{R}^d} p(t,x,y) g(y) dy) ds = \int_0^s p_t g(y)\, dt =$$
$$\int_0^s p_t g(y) 1_{[0,s[}(s-t)\, dt \leq \sup_{(x,s) \in E \times \mathbb{R}_+} V(1_{[t_0, t_0+\varepsilon[} g)(x,s).$$

We then obtain

$$\int_0^s p_t g(y)\, dt \leq \sup_{t_0 < R} \sup_{(x,s) \in E \times \mathbb{R}_+} V(1_{[t_0, t_0+\varepsilon[} g)(x,s)).$$

Hence $g \in K_d$. Conversely let $g \in K_d$. Choose $\delta > 0$ small enough such that:
$\sup\limits_{x \in \mathbb{R}^d} \int_{|x-y| \leq \delta} \frac{g(y) dy}{|x-y|^{d-2}} \leq 1 \; holds$. Then
$\sup\limits_{x \in \mathbb{R}^d} \int_{!x-y! \leq \delta} g(y) dy \leq \sup\limits_{x \in \mathbb{R}^d} \int_{|x-y| \leq \delta} \frac{\delta^{d-2} \; g(y) dy}{|x-y|^{d-2}} \leq \delta^{d-2}$, therefore
M$= \sup\limits_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} \exp[-\frac{|x-y|^2}{4}] g(y) dy \; < \infty$. For fixed $\alpha > 0$, we remark that
$\sup\limits_{x \in \mathbb{R}^d} \int_{|y-x| > \alpha} p(t,x,y) g(y) dy \; \leq (2\pi t)^{-\frac{d}{2}} \exp[-\frac{\alpha^2}{4t}] \int_{\mathbb{R}^d} \exp[-\frac{|x-y|^2}{4}] g(y) dy$.

Let $R > 0$, o $\leq t_0 < R$,$\varepsilon > 0$ and
$I = V(1_{[t_0,t_0+\varepsilon[} g)(x,s)) \; = \int_0^s p_t g(y) 1_{[t_0,t_0+\varepsilon[}(s-t)\, dt$
$= \int_0^s p_{(s-t)} g(y) 1_{[t_0,t_0+\varepsilon[}(t)\, dt = \int_0^s (\int_{\mathbb{R}^d} p(s-t,x,y) dy) 1_{[t_0,t_0+\varepsilon[}(t)\, dt$.

Let $I_1 \; = \int_0^s (\int_{|y-x| > \alpha} p(s-t,x,y) dy) 1_{[t_0,t_0+\varepsilon[}(t)\, dt$ and $I_2 = I - I_1$.

$$I_1 \leq \; M \int_0^s [(2\pi(s-t))^{-\frac{d}{2}} \exp(-\frac{\alpha^2}{4(t-s)})] 1_{[t_0,t_0+\varepsilon[}(t)\, dt.$$

$$\leq M \int_0^s [(2\pi(t))^{-\frac{d}{2}} \exp(-\frac{\alpha^2}{4t})] 1_{[t_0,t_0+\varepsilon[}(s-t)\, dt.$$

The uniform integrability of $f_\alpha(t) = (2\pi(t))^{-\frac{d}{2}} \exp(-\frac{\alpha^2}{4t})$ on $]0,+\infty[$ gives the following:

For every $\alpha > 0$ and $\varepsilon_1 > 0$,there exists $\varepsilon > 0$ such that $I_1 \leq \frac{\varepsilon_1}{2}$ for every $(x,s) \in E \times \mathbb{R}_+$.On the other hand we have
$I_2 = \int_0^s (\int_{|y-x| \leq \alpha} p(s-t,x,y) g(y) dy) 1_{[t_0,t_0+\varepsilon[}(t)\, dt$
$=\int_0^s (\int_{|y-x| \leq \alpha} p(t,x,y) g(y) dy) 1_{[t_0,t_0+\varepsilon[}(s-t)\, dt$
$\leq \int_0^\infty (\int_{|y-x| \leq \alpha} p(t,x,y) g(y) dy)\, dt \leq C_{|x-y| \leq \alpha} \frac{g(y) dy}{|x-y|^{d-2}}$,
where C is the constant giving the relation between the Newtonian kernel and the Brownian semigroup. The hypothesis gives the existence of $\alpha_0 > 0$ such that $I_2 \leq \frac{\varepsilon_1}{2}$ for every $(x,s) \in E \times \mathbb{R}_+$, this yields the required statement. □

Corollary 1.4. *Let* $g \in K_d^{loc}$*(the local Kato-class introduced in [AS]), then for every relatively compact subset* $A \in \mathbb{R}^d$, $g_1 = g1_A$ *is in* K_{Loc}^t.

Let φ be a function from $(E \times \mathbb{R}_+) \times \mathbb{R}$ to $\overline{\mathbb{R}}$ Borel measurable. As in [BBM], we introduce the following:

Definition 1.5. *(1) We shall say that* φ *is locally Kato bounded , if for every* $c \in \mathbb{R}_+^*$, *there exists* $g_c \in K_{Loc}^t$ *such that* $|\varphi(z, y)| \leq g_c(z)$ *for every* $z \in E \times \mathbb{R}_+$ *and* $y \in [-c, c]$.

(2) φ *is called continuous locally Kato-bounded if it is locally Kato bounded and if for every* $c \in \mathbb{R}_+^*$, $V(g_c)$ *is continuous in* x *for every* $t \in \mathbb{R}_+$.

(3) φ *is called Kato-bounded (resp. continuous Kato bounded) if there exist* $g \in K_{Loc}^t$ *(resp.* $g \in K_{Loc}^t$*and* $V(g)$ *continuous in* x *for every* $t \in \mathbb{R}_+$ *) such that*

$$|\varphi(z, y)| \leq g(z) \text{ for every } z \in E \times \mathbb{R}_+ \text{ and } y \in \mathbb{R}_+.$$

Example 1.6. *Let* $\varphi(z, y) = g(z)P(y)$ *or* $\varphi(z, y) = g(z)f(y)$ *where* $g \in K_{Loc}^t$, P *is a polynomial and* f *is locally bounded on* $\mathbb{R}$. *Then* φ *is locally-Kato bounded. If moreover* $V(g)$ *is continuous in* x, *then* φ *is continuous locally Kato bounded.*

Definition 1.7. *We shall say that* φ *is locally Kato-Lipschitzian if for every* $c \in \mathbb{R}_+^*$, *there exists* $g_c \in K_{Loc}^t$ *such that:*

$$|\varphi(z, y) - \varphi(z, y')| \leq g_c(z)|y - y'| \text{ for every } z \in E \times \mathbb{R}_+ \text{ and } y, y' \text{ in} [-c, c].$$

Kato -Lipschitzian functions φ *are defined in the same way as Kato bounded, the function* g_c *in (1.5) does not depend on* c.

Example 1.8. $\varphi(z, y) = g(z)f(y)$ *with* f *locally Lipschitzian on* $\mathbb{R}$ *and* $g \in K_{Loc}^t$. *Kato -Lipschitzian functions* φ *are defined in the same way as Kato bounded, the function* g_c *in (1.5) does not depend on* c.

Remark 1.9. *It is easy to see by the corollary 1.5 that* $g \in K_{Loc}^t$ *does not yield* g *locally bounded. We hence obtain by the previous Examples (1.4) and (1.6) that a function* φ *which is locally Kato-bounded or locally Kato-Lipschitzian need not be locally bounded and locally-Lipschitzian.*

2 Invertibility of Kernels

In this section we do not assume any topological structure on E. We consider a measurable space (E, B) and will denote also by B the set of real measurable functions. We consider a measurable submarkov semigroup and the same kernel V as in the previous section. The following simple lemma is important in the sequel.

Lemma 2.1. *Let $f \in B_b(E \times \mathbb{R}_+)$ and $T > 0$. Then for every $s < T$ we have:*

$$Vf(x,s) = V(1_{[0,T]}f)(x,s)$$

where

$$(1_{[0,T[}f)(x,t) = 1_{[0,T]}(t)f(x,t).$$

The following result will be the key to the perturbation of semigroups which will be investigated in the next section (3). Let $g \in K^t_{Loc}$, $T > 0$ and $K = K_T$ the kernel on $E \times R_+$ defined by $K(f) = V(1_{[0,T]}fg)$ for $f \in B_b(E \times \mathbb{R}_+)$.

Proposition 2.2. *For every $s > 0$, there exists a natural number $N > 0$ such that:*

$$\left|K^{(n)}1\right| \leq \left(\begin{array}{c} n \\ n+N \end{array}\right) s^n \text{ for every } n \in \mathbb{N}.$$

where $K^{(n)} = KoKo \ldots oK$ (n times) and $\binom{k}{p} = \frac{p!}{k!(p-k)!}$ for natural numbers $p \geq k$.

Proof. We shall use an idea of G. Ritter [Ri].let $s > 0$. By the definition of K^t_{Loc}, there exists $t_1, \ldots, t_{N-1} \in [0,T]$ such that $0 < t_1 < t_2 < \cdots < T_{N-1} < T$ and :

$$V(|g|1_{A_i}) < s \text{ on } E \times \mathbb{R}_+ \text{ for every } i \in \{1, \ldots, N-1\}, \text{where } \; A_i = E \times [t_{i-1}, t_i[,$$

we set $t_0 = 0, T_N = T$, $A_N = [T_{N-1}, T]$, $B_\ell = \bigcup\limits_{i=1}^{\ell} A_i$ and K_ℓ the kernel on $E \times \mathbb{R}_+$ defined by $K_\ell(f) := V(f|g|1_{B_\ell})$. We shall prove the required inequality by induction over $n \geq 0$ and $\ell \, (1 \leq \ell \leq N-1)$. For $\ell = 1$ and $n \in \mathbb{N}$ we have:

$$K_1^{(n)}(1) = V(|g|1_{A_1}V(|g|1_{A_1} \ldots V(|g|1_{A_1}))) \leq s^n \leq \begin{pmatrix} 1 \\ n+1 \end{pmatrix} s.$$

Let $\ell \geq 1$. For $n = 0$, we have $K_\ell^{(0)}(1) = 1 \leq \left(\begin{array}{c} \ell \\ \ell+0 \end{array}\right) s^0 = 1$. We now assume that the inequality $K_j^{(n)}(1) \leq \left(\begin{array}{c} j \\ n+j \end{array}\right) s^n$ is true for n and $j \in \{1, \ldots \ell\}$. We have by Lemma1 in section 2.

$$\begin{aligned} K_\ell^{(n+1)}(1) &= V(1_{A_\ell}|g|V(1_{A_\ell}|g| \ldots V(1_{A_\ell}|g|)) \\ &= V(1_{A_1}|g|V(1_{A_1}|g| \ldots V(1_{A_1}|g|) + V(1_{A_2}|g|(V^{(n)}1_{B_2}|g|) \\ &+V(1_{A_3}|g|V^{(n)}(1_{B_3}|g|) + \cdots + V(1_{A_\ell}|g|V^{(n)}(1_{B_\ell}|g|). \end{aligned}$$

We hence obtain by the induction hypothesis that:

$$\begin{aligned} K_\ell^{(n+1)}(1) &\leq s^{n+1} + \left(\begin{array}{c} 2 \\ n+2 \end{array}\right) s^{n+1} + \cdots + \left(\begin{array}{c} \ell \\ n+\ell \end{array}\right) s^{n+1} \\ &\leq \sum_{i=1}^{\ell} \left(\begin{array}{c} i \\ n+i \end{array}\right) s^{n+1} \leq \sum_{i=0}^{\ell} \left(\begin{array}{c} i \\ n+i \end{array}\right) s^{n+1} \\ &= \left(\begin{array}{c} \ell \\ n+1+\ell \end{array}\right) s^{n+1}. \end{aligned}$$

Let us now suppose that the inequality is true for $1 \le \ell \le N-1$ and every $n \in N$. We consider the kernel L_ℓ on $E \times \mathbb{R}_+$ defined by $L_\ell(f) = V(1_{A_{\ell+1}} f|g|)$. we then have:

$$K_{\ell+1}^{(n)} = L_\ell^{(n)} 1 + L_\ell^{(n-1)}(K_\ell^1) + L_\ell^{(n-1)}(K_\ell^{(2)}(1)) + \cdots + L_\ell^1(K_\ell^{(n-1)} 1) + K_\ell^{(n)} 1,$$

by the previous assumptions we get:

$$\begin{aligned} K_{\ell+1}^{(n)} 1 &\le s^n + \begin{pmatrix} 1 \\ \ell+1 \end{pmatrix} s^n + \begin{pmatrix} 2 \\ \ell+2 \end{pmatrix} + \cdots + \begin{pmatrix} \ell \\ n+\ell \end{pmatrix} s^n \\ &= \sum_{i=0}^{n} \begin{pmatrix} i \\ \ell+i \end{pmatrix} s^n = \begin{pmatrix} n \\ \ell+1+n \end{pmatrix} s^n. \end{aligned}$$

and therefore $K_\ell^{(n)} 1 \le \begin{pmatrix} n \\ \ell+n \end{pmatrix} s^n$ for every $n \ge 0$ and $\ell \le N$. Since $|K^{(n)} 1| \le K_N^{(n)} 1$, we hence obtain

$$|K^{(n)} 1| \le \begin{pmatrix} n \\ N+n \end{pmatrix} s^n$$

for every $n \in \mathbb{N}$ which yields the requested inequality. □

Corollary 2.3. *For every $T > 0$ and every $\alpha \in \mathbb{R}$, the operator $(I + \alpha K_T)$ is invertible from $B_b(E \times \mathbb{R}_+)$ into $B_b(E \times \mathbb{R}_+)$; in particular $(I - K_T)$ and $(I + K_T)$ are invertible.*

Proof. By the previous theorem, for every $s > 0$, there exists $N \in I\!N$ such that $|K_T^{(n)} 1| \le \begin{pmatrix} N \\ N+n \end{pmatrix} s^n$ for every $n \in I\!N$ and hence $|\alpha|^n K_T^{(n)} 1 \le \begin{pmatrix} N \\ N+n \end{pmatrix} (|\alpha| s)^n$. We then choose $s > 0$ such that $|\alpha| s < 1$, the series $\alpha_n := \begin{pmatrix} N \\ N+n \end{pmatrix} (|\alpha| s)^n$ is convergent and we have the statement. □

3 Nonlinear perturbation of semigroups

In the following we consider a topological space E and a Kernel V defined on $E \times \mathbb{R}_+$ in the same way as in the first section. We assume that E and V satisfy the following properties:

(1) There exists an increasing sequence (U_n) of compact subsets of E such that $E = \bigcup_{n \in I\!N} U_n$.

(2) For every $g \in B(E \times \mathbb{R}_+)$ such that $V(|g|) \in \mathcal{C}(E \times \mathbb{R}_+)$ the set $\{V(f),\ f \in B(E \times \mathbb{R}_+) \text{ with } |f| \le |g|\}$ is equicontinuous on $E \times \mathbb{R}$.

These two properties on E and V are satisfied if $(E \times \mathbb{R}, \mathcal{E}_{\widetilde{\mathbb{P}}})$ is a balayage space in the sense of [BH] (see [H]). $\mathcal{E}_{\widetilde{\mathbb{P}}}$ is the set of excessive functions for the space time semigroup corresponding to the semigroup $\mathbb{P}$ which defines the kernel V. More generally if E is a locally compact second countable metric space and V satisfies the hypothesis of absolute continuity (hypothesis L of P. A. Meyer), we have (by [Me]) the property (2) (the property 1 is trivial).

Fix a function φ from $(E \times \mathbb{R}_+) \times \mathbb{R}$ to $\overline{\mathbb{R}}$ with the following properties:

(a) φ is continuous locally Kato bounded and $\varphi^- = \sup(-\varphi, 0)$ is Kato bounded.

(b) For every $z \in E \times \mathbb{R}_+, \varphi(z, \cdot)$ is continuous.

We shall prove an existence theorem for the nonlinear perturbation by $V\psi$ where ψ is the function from $(E \times \mathbb{R}_+) \times \mathbb{R}$ to $\overline{\mathbb{R}}$ with $\psi(z, y) = y\varphi(z, y)$.

Theorem 3.1. *For every $f \in B_b(E \times \mathbb{R}_+)$ and every $T > 0$, there exists a function $u \in B_b(E \times \mathbb{R}_+)$ such that*

$$f = u + V(\psi(\cdot, u)1_{[0,T]}).$$

Proof. Let $f \in B_b(E \times \mathbb{R}_+)$. Let $\varphi^+(x, y) = \sup(\varphi(x, y), 0)$ and $\varphi^-(x, y) = \sup(-\varphi(x, y), 0)$, then $\varphi = \varphi^+ - \varphi^-$. Let $T > 0$ and $v \in B_b(E \times \mathbb{R}_+)$ .We consider the kernels $K^+$and K^- defined by

$K^+(h) = V(1_{[0,T]}\varphi^+(\cdot, v)h)$ and $K^-(h) = V(1_{[0,T]}\varphi^-(\cdot, v)h)$ for every $h \in B_b(E\times \mathbb{R}_+)$, where $1_{[0,T]} := 1_{E\times[0,T]}$. All these kernels depend on v, but for typographically reason we do not mention it . By the domination principle related to V, we have $(I+K^+)^{-1}1 \geq 0$ and since $(I + K^+)^{-1} = I - (I + K^+)^{-1}K^+$ we hence obtain

$|(\text{I+K}^+)^{-1}f| \leq 2\|f\|_\infty$. Again the domination principle related to V yields

$$|((I + K^+)^{-1}K^-)^{(n)}h| \leq (K^-)^{(n)}|h| \text{ for every } h \in B_b(E \times \mathbb{R})$$

thus

$$|((I + K^+)^{-1}K^-)^{(n)}(I + K^+)^{-1}f| \leq (K^-)^{(n)}(2\|f\|_\infty) = 2\|f\|_\infty(K^-)^{(n)}1.$$

By the proposition 2.1, the series $\sum\limits_{n=0}^{\infty}(K^-)^{((n)}1$ is convergent. Let

$$S(v) = \sum_{n\geq 0}((I + K^+)^{-1}K^-)^{(n)}(I + K^+)^{-}f.$$

Since φ^- is Kato bounded, then there exists $g \in K^t_{Loc}$ such that $(K^-)^{(n)}1 \leq K_g^{(n)}1$ with $K_g(h) = V(1_{[0,T]}gh)$ and also $\sum\limits_{n=0}^{\infty} K_g^{(n)}1$ is again by the proposition 2.1 convergent and bounded on $E \times \mathbb{R}_+$, let M_1 be its upper bound and $M = 2\|f\|_\infty M_1$. We set

$$A = \{v \in B_b(E \times \mathbb{R}_+) : \|v\|_\infty \leq M\},$$

we then have $\|S(v)\|_\infty \leq M$ and hence $S(v) \in A$. Moreover we can easily see that

$$f = S(v) + V(S(v)\varphi(\cdot, v)1_{[0,T]}).$$

Let $v \in A$ and $(v_n)_n \subset A$ such that (v_n) converges uniformly to v on $E \times \mathbb{R}_+$. Since S is bounded on A and φ is continuous locally Kato bounded, by the property 2 on V, the family $\{V(S(v)\varphi(\cdot, v)1_{[0,T]}, v \in A\}$ is equicontinuous on $E \times \mathbb{R}_+$ and hence relatively compact for the local uniform convergence and so is $(S(v_n))_n$. By the property 1 on E and a diagonal procedure, there exists a subsequence (ρ_n) of $(S(v_n))$ which is locally uniformly convergent. By the convergence theorem of Lebesgue, we then obtain

$$f = \lim \rho_n + V(\lim \rho_n \varphi(\cdot(v)1_{[0,T]}) = S(v) + V(S(v)\varphi(\cdot, v)1_{[0,T]}).$$

corollary 2.2 yields $\lim \rho_n = S(v)$ and $S(v_n)$ is then locally uniformly convergent to $S(v)$. Let $(U_n) \subset E \times \mathbb{R}_+$ be an increasing sequence of compact subsets with $E \times \mathbb{R}_+ = \bigcup\limits_n U_n$. Let $A_n = \{v \in B_b(U_n) : \|v\|_\infty \leq M\}$ and for every $v \in A_n$ we set $S_n(v) = S(\tilde{v})$ where $\tilde{v} = v$ on U_n and 0 outside. Let $T_n(v) = S_n(v)|_{U_n}$ and fix $n \in \mathbb{N}$.. We have $T_n(A_n) \subset A_n$, T_n is completely continuous on A_n and, by property 2 of V, is compact. By the fixed point theorem of Schauder, there exists then $v_n \in A_n$ such that $T_n(v_n) = v_n$. We hence obtain

$$f = S(\tilde{v}_n) + V(v_n\varphi(\cdot, v_n)1_{[0,T]}1_{U_n}) + V(S(\tilde{v}_n)\varphi(\cdot, \tilde{v}_n)1_{[0,T]}1_{CU_n}).$$

By the convergence theorem of Lebesgue we have

$$\lim V(S(\tilde{v}_n)\varphi(\cdot, \tilde{v}_n)1_{[0,T]}1_{CU_n}) = 0.$$

by a diagonal procedure, there exist a subsequence (n_k) of (n) such that $S(\tilde{v}_{n_k})$ converges locally uniformly and let v be its limit . Since $S(\tilde{v}_{n_k})|U_{n_k} = v_{n_k}$, we hence obtain by the Lebesgue convergence theorem that:

$$f = v + V(v\varphi(\cdot, v)1_{[0,T]}).$$

□

In order to ensure a global existence and uniqueness theorem, we introduce the following definition:

Definition 3.2. *A locally Kato bounded function ψ from $(E \times \mathbb{R}_+) \times \mathbb{R}$ to $\overline{\mathbb{R}}$ will be called admissible if one of the following properties is satisfied:*

(I) ψ is increasing relatively to the last variable i.e. for every $z \in E \times \mathbb{R}_+, \psi(z, \cdot)$ is an increasing function from $\mathbb{R}$ to $\overline{\mathbb{R}}$.

(II) ψ is locally Kato-Lipschitzian.

We remark that the properties I and II are independent and if $\psi(x, y) = y\varphi(x, y)$, they are sufficient for the global uniqueness.

Proposition 3.3. *Let ψ be an admissible function, then the function $A(u) = u + V(\psi(\cdot, u))$ is injective from $B_b(E \times \mathbb{R}_+)$ into $B_b(E \times \mathbb{R}_+)$.*

Proof. We assume first that ψ is increasing relatively to the last variable. Let u, v in $B_b(E \times \mathbb{R}_+)$ such that $u + V(\psi(\cdot, u)) = v + V(\psi(\cdot, v))$. We have

$$V((\psi(\cdot, u) - \psi(\cdot, v))^+) \leq V((\psi(\cdot, u) - \psi(\cdot, v))$$

on $\{u - v \geq 0\}$, the assumption on ψ yields

$$\{(\psi(\cdot, v) - \psi(\cdot, v))^+ > 0\} \subset \{u - v \geq 0\},$$

the domination principle or the complete maximum principle implies then $V(\psi(\cdot, u) - \psi(\cdot, v)) \leq 0$ and $u \geq v$. The same proof gives the other inequality. Thus $u = v$.
We assume now that ψ is locally Kato-Lipschitzian. Let $u, v \in B_b(E \times \mathbb{R}_+)$ with $u + V(\psi(\cdot, u)) = v + V(\psi(\cdot, v))$. Let $c = \max\{\|u\|_\infty, \|v\|_\infty\}$, then there exists $g_c \in K^t_{Loc}$ such that $|\psi(\cdot, u) - \psi(\cdot, v)| \leq |u - v| g_c$, we then have:

$$|u - v| \leq V(|\psi(\cdot, u) - \psi(\cdot, v)|) \leq V(|u - v| g_c).$$

and for every $T > 0$ we obtain by Lemma 2.1.

$$|u - v| 1_{[0,T]} \leq V(|u - v| g_c 1_{[0,T]})|_{[0,T]}.$$

Let K_c be the kernel on $B_b(E \times \mathbb{R}_+)$ defined by $K_c h = V(g_c h 1_{[0,T]})$, lemma 2.1 yields $|u - v| 1_{[0,T]} \leq 2c(K_c^{(n)} 1) 1_{[0,T]}$. By the proposition 2.1, we have $\lim\limits_{n \mapsto +\infty} K_c^{(n)} 1 = 0$. Thus $u - v = 0$ on $[0, T]$, T being arbitrary on $\mathbb{R}_+$, we hence obtain $u = v$. $\square$

Remark 3.4. *In the proof of 3.3 and when ψ is increasing, we did not use that A is an application from B_b to B_b. The first part of the proof of the previous proposition is also valid for $u \in B(E \times [0, T])$ with $V(\psi(\cdot, u)(x, t) \in \mathbb{R}$ for every $(x, t) \in E \times [0, T]$.*

Theorem 3.5. *Let ψ defined by $\psi(x, y) = y\varphi(x, y)$ for $x \in E$ and $y \in \mathbb{R}$. If ψ is admissible, Then for every $f \in B_b(E \times \mathbb{R}_+)$, there exists a unique locally bounded function $u \in B_b(E \times \mathbb{R}_+)$ such that*

$$f = u + V(\psi(\cdot, u)).$$

Proof. By Theorem 3.1 and Proposition 3.3, there exists for every $T > 0$ a unique bounded function u_T on $E \times \mathbb{R}_+$ such that

$$f = u_T + V(\psi(\cdot, u_T) 1_{[0,T]}).$$

Let $T' < T$. We then have by Lemma 2.1

$$\begin{aligned} f 1_{[0,T]} &= u_T 1_{[0,T']} + V(\psi(\cdot, u_T) 1_{[0,T]}) 1_{[0,T']} \\ &= u_{T'} 1_{[0,T']} + V(\psi(\cdot, u_{T'} 1_{[0,T']}) 1_{[0,T']} \end{aligned}$$

hence

$$|u_T - u_{T'}| 1_{[0,T']} \leq V(|\psi(\cdot, u_T) - \psi(\cdot, u_{T'})| 1_{[0,T']}) 1_{[0,T']}$$

We assume first that ψ is locally Kato-Lipschitzian. Let $c = \max(\|u_T\|_\infty, \|u_{T'}\|_\infty)$ and $g_c \in K^t_{Loc}$ such that

$$|\psi(\cdot, u_T) - \psi(\cdot, u_{T'})| \leq |u_T - u_{T'}| gc \leq 2c g_c.$$

Let $K_c h = V(g_c h 1_{[0,T']})$, we then obtain again by Lemma 2.1

$$|u_T - u_{T'}| 1_{[0,T']} \leq 2c K_c^{(n)} 1,$$

by Proposition 2.1 we get $u_T = u_{T'}$ on $E \times [0, T']$; $(u_T)_{T>0}$ is then locally constant. If ψ is increasing the proof of $u_T = u_{T'}$ on $E \times [0, T']$ follows from the dominations principle and Remark 3.4. Let $u = \lim\limits_{T \mapsto +\infty} u_T$. It is easy to see that $u 1_{[0,T]} = u_T 1_{[0,T]}$ for every $T > 0$. The uniqueness of u follows from Proposition 3.3 . $\square$

Remark 3.6. *The function u in the previous theorem is in general not bounded, as the following example shows: Let P_t be the Brownian semigroup on $\mathbb{R}^d$, $f = 1$ and $\varphi : \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R} \to \overline{\mathbb{R}}$ with $\varphi(z, y) = -1$, then it is easy to see that $u = e^t$.*

4 Nonlinear semigroups

We consider the same assumptions as in the previous section (§3) for the space E and the kernel V. We consider a function φ from $E \times \mathbb{R}$ to $\overline{I\!R}$ with the following properties:

(1) φ is continuous locally Kato bounded and φ^- is Kato bounded.
(2) For every $x \in E$ $\varphi(x, \cdot)$ is continuous .
(3) ψ is an admissible function. , where $\psi(x, y) = y\varphi(x, y)$.

Theorem 4.1. *There exists a unique nonlinear semigroup Q_t such that for every $f \in B_b(E)$,*

$$P_t f(x) = Q_t f(x) + \int_0^t P_s(x, \psi(\cdot, Q_{t-s}f))ds.$$

Furthermore Qf is bounded on $E \times [0, T]$ for every $T > 0$.

Proof. Let $f \in B_b(E)$ and $v(x, t) = P_t f(x)$. Since the semigroup P is submarkov, v is bounded and by the Theorem 3.5, there exists a unique function Qf, bounded on $E \times [o, T]$ for every $T > 0$ such that:

$$P_t f = Q_t f + \int_0^t P_s(\cdot, \psi(\cdot, Q_{t-s}f))ds.$$

We now have to prove the semigroup property. Fix $t' > 0$ then:

$$P_{t+t'} f(x) = Q_{t+t'} f(x) + \int_0^{t+t'} P_s(x, \psi(\cdot, Q_{t+t'-s}f))ds.$$

and

$$P_{t'}(P_t f)(x) = P_{t'}(Q_t f)(x) + \int_0^t P_{t'+s}(x, \psi(\cdot, Q_{t-s}f)ds.$$

Further

$$P_{t'}(Q_t f) = Q_{t'}(Q_t f)(x) + \int_0^{t'} P_s(x, \psi(\cdot, Q_{t'-s}(Q_t f))ds$$

we hence obtain

$$\begin{aligned} & Q_{t'+t} f(x) + \textstyle\int_0^{t+t'} P_s(x, \psi(\cdot, Q_{t+t'-s}f)ds \\ = \ & Q_{t'} Q_t f + \textstyle\int_0^{t'} P_s(x, \psi(\cdot, Q_{t'-s} Q_t f)ds + \int_0^t P_{t'+s}(x, \psi(\cdot, Q_{t-s}f)ds. \end{aligned}$$

Putting $u = t' + s$ we obtain

$$\begin{aligned} & \textstyle\int_0^t P_{t'+s}(x, \psi(\cdot, Q_{t-s}f))\, dt = \int_{t'}^{t+t'} P_u(x, \psi(\cdot, Q_{t+t'-u}f)du \\ & = \textstyle\int_0^{t+t'} P_s(x, \psi(\cdot, Q_{t+t'-s}f)ds - \int_0^{t'} P_s(x, \psi(\cdot, Q_{t+t'-s}f))ds. \end{aligned}$$

Henceforth we get

$$\begin{aligned}&Q_{t+t'}f(x) + \int_0^{t'} P_s(x, \psi(\cdot, Q_{t+t'-s}f)ds\\&= Q_{t'}Q_tf + \int_0^{t'} P_s(x, \psi(\cdot, Q_{t'-s}Q_tf)ds\end{aligned}$$

since ψ is admissible, by Proposition 3.3 we get

$$Q_{t'+t}f = Q_{t'}Q_tf = Q_tQ_{t'}f.$$

$\square$

Example 4.2. *a) Let L be a linear differential operator on $\mathbb{R}^d$ admitting a linear semigroup such that V satisfies (*). Let p be a polynomial on $\mathbb{R}$ such that $\lim\limits_{|x|\to+\infty} p(x) = +\infty$. Then there exists a nonlinear semigroup Q_t such that for every bounded $f \in B_b(\mathbb{R}^d)$ we have (formally)*

$$LQ_tf - \frac{\partial Q_tf}{\partial t} - Q_tfP(Q_tf) = 0.$$

Moreover if $\beta = \inf\{p(x), x \in \mathbb{R}\}$, we have from the proof of Theorem 3.1:
$|Q_tf(x)| \le e^{-\beta t}\|f\|_\infty$ for every $t > 0$ and $x \in E$. Hence if $\beta \ge 0$ we have $\|Q_tf\|_\infty \le \|f\|_\infty$ for every $f \in B_b(\mathbb{R}^d)$.

b)Let $L = \Delta$ and $\varphi : \mathbb{R} \to \mathbb{R}$ locally Lipschitzian (e.g. $\mathcal{C}^1$) and $g \in K^n_{loc}(\mathbb{R}^d)$ (see [AS]) . Let $c = g1_A$, where A is a measurable relatively compact set in $\mathbb{R}^d$. Then there exists a nonlinear semigroup Q_t such that $\Delta Q_tf - \frac{\partial Q_tf}{\partial t} = c(x)Q_tf|\varphi(Q_tf)|$ in the distributional sense for every $f \in B_b(E)$. Moreover we have $\|Q_tf\|_\infty \le \|f\|_\infty$ and if $f \le g$ we have $Q_tf \le Q_tg$.

We recall that c can be chosen as follow: Let $a_1, \dots, a_n$ be a finite sequence in $\mathbb{R}^d$, $\alpha_i \in [-\infty, 2[$ for every $i \in \{1, \dots, n\}$, and $c(x) = \sum\limits_{i=1}^{n} \frac{1}{\|x-a_i\|^{\alpha_i}} 1_{B(a_i,1)}$, for every $x \in \mathbb{R}^d$.

Let $(E, \mathcal{B})$ be a measurable space and $Q = (Q_t)_{t>0}$ be a family of applications from $B_b(E)$ to $B(E)$. We will call Q measurable iff for every $f \in B_b(E)$ the mapping $(x, t) \to Q_tf(x)$ from $E \times \mathbb{R}^*_+$ to $\mathbb{R}$ is measurable.

Definition 4.3. *We shall say that the family Q is a monotone (nonlinear) semigroup, if the following properties are satisfied:*

(1) Q is measurable.

(2) $Q_t(Q_sf) = Q_{t+s}f$ for every $t, s > 0$ and $f \in B_b(E)$. (semigroup property)

(3) for every $t > 0, Q_t$ is increasingly continuous i.e. for every monotone sequence $(f_p) \subset B_b(E)$ which is convergent to a bounded measurable function $f, (Q_tf_p)_p$ is monotone in the same sense as f_p and converges to Q_tf .

Remark 4.4. *The property (3) implies that Q_t is increasing.*

Lemma 4.5. *Let Q be a monotone semigroup and (f_n) and (g_n) two sequences in $B_b(E)$ which are monotone in the same sense and $\lim_n f_n = \lim_n g_n$ then $\lim Q_t f_n = \lim_n Q_t g_n$ for every $t > 0$.*

Proof. Assume that (f_n) is increasing. Let $p \in I\!N$ and $u_n = inf(f_n, g_p)$, then u_n is monotone and converge to g_p and the property 3 of (4.3) yields $\lim_n Q_t u_n := Q_t g_p \leq \lim Q_t f_n$ for every $p \in I\!N$. Therefore $\lim Q_t g_n \leq \lim Q_t f_n$ and we obtain the statement.
We can also extend Q_t to every lower bounded or every upper bounded f by setting $Q_t f := \lim_P Q_t f_p$ for $(f_p) \subset B_b$ monotone and convergent to f. □

Proposition 4.6. *Let Q be any (nonlinear) semigroup constructed in the previous section, then Q is a monotone semigroup in the sense of (4.3).*

Proof. The measurability follow from the measurability of $\mathbb{P}$ and the construction of Q (theorem4.1). It is then enough to prove the property (3) of (4.3). let $f, g \in B_b(E)$ with $f \leq g$. We have:

$$P_t(g-f) = Q_t g - Q_t f + \int_0^t P_s(x, \psi(Q_{t-s}g) - \psi(Q_{t-s}f))ds.$$

If ψ is locally Kato-Lipschitzian, there exists $c \in K^t_{Loc}$ such that $\psi(Q_{t-s}g) - \psi(Q_{t-s}f)) = c(Q_{t-s}g - Q_{t-s}f)$.and the same proof as Theorem 3.1 yields the statement since $P_t(g-f) \geq 0$ and is V-dominant.If ψ is increasing the property (3) in (4,3) follows from the domination principle. □

5 Excessive functions

Let $(E, \mathcal{B})$ be a measurable space and Q be a monotone semigroup in the sense of the definition 4.3. let $u \in B^+(E)$. As in the linear case, u will be called Q excessive if and only if $\sup_{t>0} Q_t u = u$. We shall denote by $\mathcal{E}_Q$ these functions.
In the sequel, we consider the same conditions as in the previous section (§4) and we assume that $\varphi(x, y) \geq 0$ for every $x \in E$ and $y \in \mathbb{R}$. Let Q be the semigroup given by φ as in §4.

Proposition 5.1. *Let $u \in \mathcal{E}_{\mathbb{P}}$, then $u \in \mathcal{E}_Q$.*

Proof. For every $p \in \mathbb{N}$, let $u_p = inf(u, p)$. Then $u_p \in B^+_b(E)$ and $P_t u_p = Q_t u_p + \int_0^t P_s(x, \psi(\cdot, Q_{t-s}u_p))ds$ and hence $Q_t u_p \leq u_p$. Since Q is a monotone semigoup, we obtain that $(Q_t u_p)_t$ is decreasing. Hence $\sup_{t>0} Q_t u_p \leq u_p$. It is easy to see, since φ is continuous locally Kato bounded, that

$$\lim_{t \downarrow 0} \int_0^t P_s(x, \psi(\cdot, Q_{t-s}u_p))ds = 0.$$

Thus

$$u_p = \sup_{t>0} P_t u_p = \sup_{t>0} Q_t u_p$$

and

$$inf(u,p) \le \sup_{t>0} Q_t u \le u.$$

Passing p to infinity, we obtain the statement. $\square$

For every $u \in B^+(E)$ we set $Ku = \int_0^\infty P_s(\cdot, \psi(\cdot, u))ds$. For the converse of 5.1 we then have the following:

Proposition 5.2. *let $u \in \mathcal{E}_Q$ such that Ku is finite on E, then $v = u + Ku$ is in $\mathcal{E}_{\mathbb{P}}$.*

Proof. Let $v = u + Ku$. Then

$$\begin{aligned} P_t v &= P_t u + P_t K u = P_t u + \textstyle\int_t^\infty P_s(\cdot, \psi(\cdot, u))ds \\ &= Q_t u + \textstyle\int_0^t P_s(\cdot, \psi(\cdot, Q_{t-s}u))^t ds + \int_t^\infty P_s(\cdot, \psi(\cdot, u)ds \\ &\le v. \end{aligned}$$

On the other hand we have:

$$P_t v \ge Q_t u + \int_t^\infty P_s(\cdot, \psi(\cdot, u))ds.$$

The assumptions on u and K yields

$$\lim_{t \downarrow 0} P_t v = \sup_{t>0} P_t v \ge u + \int_0^\infty) P_s(\cdot, \psi(\cdot, u))ds = v.$$

Thus $\sup_{t>0} P_t v = v$.

$\square$

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