Nonequilibrium Gas and Generalized Billiards

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January 13, 2005

Abstract

The description of a nonequilibrium gas based on the notion of generalized billiards is proposed. Generalized billiards can be considered both in the framework of Newtonian mechanics and of the relativity theory. In the Newtonian case, for a generalized billiard inside a ball and the corresponding nonequilibrium the invariant measure is equivalent to the phase volume, the Gibbs entropy are constructed. It is also proved that the Gibbs entropy is constant. The generalization of our previous results leads to the fact that, under some general conditions, the Gibbs entropy increases in the relativistic case.

1 Introduction

The description of a gas as a system of elastic balls moving inside a container goes back to Boltzmann and Poincaré [1]. For an ideal gas then balls representing its molecules are replaced by mass points (particles). The probability of interaction of two arbitrary particles is zero; so the particles move independently from each other. Behaviour of such a system is described by means of billiards that had been introduced by Birkhoff [2]: a particle moves linearly and uniformly inside a closed domain Π with the piece-wise smooth boundary $\Gamma = \partial\Pi$ and bounces off the boundary Γ in such a way that the normal component of its velocity changes its direction to the opposite, while the tangential component of its velocity is constant in time. Since the energy of the particle is preserved in time the description of a gas by Boltzmann's model has a physical sense only when the gas is in equilibrium.

In this paper we consider another model corresponding to the description of a nonequilibrium gas whose molecules move inside a container. This model is called a generalized billiards. Originally, it was considered and studied in special cases, where the domain was an interval or a parallelepiped ([3]–[5]) and then it was investigated in a general case, where the domain had an arbitrary shape [6]. This model was studied in the framework both of Newtonian ([4],[5]) and relativistic mechanics ([3]-[10]). From the physical point of view, a generalized billiards describes a gas consisting of finitely many particles moving in a container, while the container's walls either heat up or cool down. Here is the description of the generalization of the standard billiard model. As soon as the point hits the boundary Γ, the projection of its velocity to the direction normal to Γ transforms by means of a function $f(\gamma, t)$ defined on the direct product $\Gamma \times \mathbf{R}^1$ (where $\mathbf{R}^1$ is the real line, $\gamma \in \Gamma$ is a point of the boundary, and $t \in \mathbf{R}$ is time), according to the following law. Suppose that the trajectory of the particle that moves with a velocity v meets Γ at a point $\gamma \in \Gamma$ at time t^*. Then, at time t^*, the particle acquires the velocity v^*, as if it underwent an elastic collision with the infinitely-heavy plane Γ^* tangent to the boundary Γ at the point γ, and at the time t it moves with the velocity $\frac{\partial f}{\partial t}(\gamma, t^*)$ along the normal to Γ at the point γ. The positive direction of the motion of the plane Γ^* is taken towards the interior of the domain Π. We emphasize that the position of the boundary Γ does not change in time while its action to the particle is determined by the function $f(\gamma, t)$.

This generalized reflection law is very natural. First, it reflects an obvious fact that the walls of the vessel with gas are at rest. Second the action of the wall on the particle is still the classical elastic push. In the essence, we consider infinitesimally moving boundaries with given velocities.

Generalized billiards were introduced and studied because of their importance for problems of thermodynamics and nonequilibrium statistical mechanics (Loschmidt reversibility paradox and the justification the second law of thermodynamics) [5]. This model is a generalization and a concrete definition of the Poincarè model [1]. Poincarè studied one-dimensional and three-dimensional gases that consist of many particles, moving on the interval and in the parallelepiped respectively under the influence of external forces. These forces are caused by an external (hot) body that approaches and recedes from the vessel, thus acting on the particles. The feature peculiar to the model considered in our paper is that the influence of this external body is modelled by the action of the vessel boundary on the particles according to the generalized reflection law.

It makes sense to investigate the reflection from the boundary Γ both within the framework of Newtonian mechanics and the special theory of relativity ([3]–[10]). For classical billiards, when $\frac{\partial f}{\partial t}(\gamma, t) \equiv 0$, there is no difference between these two cases: it is the same dynamical system. However, the general billiard has the property $\frac{\partial f}{\partial t}(\gamma, t) \not\equiv 0$ that gives an enormous and principal difference: while the corresponding dynamical system is conservative in the Newtonian case, i.e. some measure equivalent to the volume in the phase space is preserved, the corresponding system in the relativistic case is dissipative. The drastic difference between the two approaches leads to the fact that the Gibbs entropy is constant in the Newtonian case, whereas it increases in the relativistic case ([4]–[7]). This observation resolves the well-known reversibility paradox. According to this paradox, the entropy of a system cannot increase: if the entropy increases, then, by changing the signs of the velocities of all molecules, one obtains a decreasing of the entropy, because of the same trajectories and of the same motion, in the opposite direction of the system. As it follows from [4] – [7], in our consideration such a phenomenon is impossible. This occurs because in the relativistic case, the general condition for the growth of the entropy depends on the absolute values of the velocities only. We remark that the velocities of the particles revers, whereas the moving boundary of the container does not: The reversibility of the moving walls is equivalent to the reversibility of time. In fact, time runs in one direction and it cannot be reversed. The proofs of the results mentioned were given in [4] and [5] for the case where the container is just a segment or a three-dimensional parallelepined. The aim of the current article is to deal with a nonequilibrium gas inside a ball; the case of a container in the shape of ball lies in base of the construction of a theory for the gas in a region of an arbitrary shape.

The generalized billiard is an approximation of the model with real moving boundaries if the initial velocities of the particles are sufficiently large. In the one-dimensional case such a model with periodically moving boundaries is the well-known Fermi-Ulam model which was proposed by Ulam [11] for rigorous justification of the Fermi acceleration mechanism [12]: an elastic ball moves between two parallel infinitely heavy walls which perform periodic oscillations. This system models the motion of particles between cosmic objects in the case when the particles interact with the object fields. The hypothesis of Fermi and Ulam was that the energy of the ball increases without limitation as the result of the collision with walls. The Fermi-Ulam problem in the framework of Newtonian mechanics was completely solved in the negative sense (the velocity and energy are always bounded) in [13]–[15]; some further results were obtained in [16] and [17].

In spite of these negative results, the Fermi conjecture on the possiblity of the unlimited increase of the energy of the particle in the Fermi-Ulam model is true (i.e., the energy can grow to infinity) if one considers a relativistic Fermi-Ulam model ([18], [3], [5]).

2 Invariant measure for a generalized Newtonian billiard in a ball

Suppose that Π is a ball in the three-dimensional space $\mathbf{R}^3$ equipped with the rectangle system of coordinates x, y, z; its boundary $\Gamma = \partial\Pi$ is a two-dimensional sphere. A point of the phase space is uniquely determined by its three coordinates x, y, z, and its three corresponding momentum, $p^{(x)}$, $p^{(y)}$, and $p^{(z)}$. To construct the invariant measure, we introduce a new phase space Ω. A point $\omega \stackrel{\text{def}}{=} (\gamma, \vec{p_\tau}, p_\nu, \Delta) \in \Omega$ of the space Ω is defined by the following four objects $\gamma, \vec{p_\tau}, p_\nu, \Delta$, where $\gamma \in \Gamma$ is a point on the sphere Γ, $\vec{p_\tau}$ is the projection of the momentum vector $\vec{p}$ onto the tangent plane to the sphere Γ at the point γ, p_ν is the normal component of the projection of the momentum vector $\vec{p}$ to the vector normal to the sphere Γ at the point γ, and Δ is length of the segment with the endpoints γ and $\bar{\gamma} = \gamma + \frac{\vec{p}}{m}\bar{t}$, where $\vec{p} = (\vec{p_\tau}, p_\nu)$, m is the mass of the point, and $\bar{t}$ is time which the point passes from the position γ to position (x, y, z) with velocity $\vec{v} = \frac{\vec{p}}{m}$, $\bar{\gamma}$ lies inside the ball Π. From physical point of view, the introduction of the phase space Ω means that, before hitting at the point $\bar{\gamma}$, the particle reflected last time from the boundary Γ at the point γ, so that the tangential and the normal components of its momentum $\vec{p}$ at this time are $\vec{p}_\tau$ and p_ν, respectively. The dimension of the space Ω is 6. However, due to the fact that, according to the definition of the generalized billiards, the component $\vec{p}_\tau$ of the vector $\vec{p}$ is not changed after reflecting from the boundary Γ, the phase space Ω is fibered by the invariant subspaces $\Omega_{\vec{p}_\tau}$ each of dimension 4, such that the tangential component of the momentum vector $\vec{p}$ at every reflection point is $\vec{p}_\tau$. Using the phase space Ω, one can prove that the invariant measure μ of the generalized billiards in the ball has the following form:

$$d\mu = \frac{d\gamma d\vec{p}_\tau dp_\nu d\Delta}{\sqrt{|\vec{v}_\tau|^2 + v_\nu^2}} \ , \tag{1}$$

where $d\gamma$ is the element of the surface area on the sphere Γ, $d\vec{p}_\tau$ is the area in the two-dimensional tangent plane, $\vec{v}_\tau$ and v_ν are, respectively, the projections of the velocity vector to the tangent space and to the normal vector at the point γ, $|\vec{v}_\tau|$ is the magnitude of the vector $\vec{v}_\tau$. The measure μ is projected onto the invariant manifold $\Omega_{\vec{p}_\tau}$ to the invariant measure $\mu_{\vec{p}_\tau}$ of the form

$$d\mu_{\vec{p}_\tau} = \frac{d\gamma dp_\nu d\Delta}{\sqrt{|\vec{v}_\tau|^2 + v_\nu^2}} \ . \tag{2}$$

In the special case, where the dimension of the ball Π is 1 (Π is just a segment, $\Gamma = \partial\Pi$ is two endpoints, and $\vec{p}_\tau = \vec{v}_\tau = 0$), the invariant measures (1) and (2) have the form $d\mu = d\mu_{\vec{p}_\tau} = \frac{dqdp}{|v|}$ ([5], Chapter I, §2) where q, p, and v are, respectively, the coordinate, the momentum, and the velocity of the particle moving along the segment. The proof of the invariance of the measure (1) is carried out in the same way as that for the one-dimensional case in [5]. In the similar way, one can construct the invariant measure equivalent to the phase space volume for the generalized Newtonian billiards in the region Π having an arbitrary shape.

3 Gibbs entropy for a nonequilibrium gas in a ball

Here we study a three-dimensional gas, consisting of finitely many identical particles $P_1, \ldots, P_N$ that move in a container having the shape of the ball Π with boundary $\Gamma = \partial\Pi$. Denoting the point $(\gamma, \vec{p}_\tau, p_\nu, \Delta) \in \Omega$, velocities

$\vec{v}_\tau, v_\nu$ and the phase space Ω for the particle P_s by $(\gamma^{(s)}, \vec{p}_\tau^{(s)}, p_\nu^{(s)}, \Delta^{(s)}), \vec{v}_\tau^{(s)}, v_\nu^{(s)}$ and $\Omega^{(s)}$ $(s = 1, \ldots, N)$, let us introduce the statistical distribution function

$$\rho = \rho(t) = \rho(\gamma^{(1)}, \vec{p}_\tau^{(1)}, p_\nu^{(1)}, \Delta^{(1)}, \ldots, \gamma^{(N)}, \vec{p}_\tau^{(N)}, p_\nu^{(N)}, \Delta^{(N)}, t) \geq 0$$

for the particles $P_1, \ldots, P_N$ at time t. Since, in view of Section 2, the measure

$$d \text{ mes} = \frac{d\gamma^{(1)} d\vec{p}_\tau^{(1)} dp_\nu d\Delta^{(1)} \ldots d\gamma^{(N)} d\vec{p}_\tau^{(N)} dp_\nu^{(N)} d\Delta^{(N)}}{\prod_{s=1}^N \sqrt{|\vec{v}_\tau^{(s)}|^2 + (v_\nu^{(s)})^2}}$$

is invariant with respect to the classical dynamics, the Gibbs entropy $H(t)$ has the form

$$H(t) = -\int_K \rho(t) \ln \rho(t) d \text{ mes}(t) \ , \tag{3}$$

where $K = \Omega^{(1)} \times \ldots \times \Omega^{(N)}$ is the direct product of the spaces $\Omega^{(1)}, \ldots, \Omega^{(N)}$, and d mes(t) is the measure d mes at time t transformed from the measure d mes(t_0) at the initial moment t_0. The function ρ must satisfy the condition of normalization $\int_K d \text{ mes} = 1$, as well as the following condition which is the law of the mass conservation:

$$\rho(t) d \text{ mes}(t) = \rho(t_0) d \text{ mes}(t_0) \ , \tag{4}$$

where t and t_0 are two different moment of time such that $t > t_0$. Now we show that, in Newtonian mechanics, the Gibbs entropy is constant. Indeed, in view of (4), the invariance of mes and relation (3) imply $d \text{ mes}(t) = d \text{ mes}(t_0)$, $\rho(t) = \rho(t_0)$, and $H(t) = H(t_0)$. As it follows from [5] and [8]–[10] the generalized billiards in the relativistic case has no invariant measure equivalent to the phase volume under some general conditions. Moreover, the generalization of the proofs given in [4] and [5] allow us to claim that in the relativistic case and under some natural condition, the Gibbs entropy of a gas given by (3) increases!

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