

# On the bifractional Brownian motion

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## Abstract

This paper is devoted to analyze several properties of the bifractional Brownian motion introduced by Houdré and Villa. This process is a self-similar Gaussian process depending on two parameters  $H$  and  $K$  and it constitutes a natural generalization of the fractional Brownian motion (which is obtained for  $K = 1$ ). We adopt the strategy of the stochastic calculus via regularization. Particular interest has for us the case  $HK = \frac{1}{2}$ . In this case, the process is a finite quadratic variation process with bracket equal to a constant times  $t$  and it has the same order of self-similarity as the standard Brownian motion. It is a short memory process even though it is neither a semimartingale nor a Dirichlet process.

**Key words and phrases:** Bifractional Brownian motion, Dirichlet processes, self-similar processes, calculus via regularization.

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## 1 Introduction

The paper is devoted to investigate the properties and to construct a stochastic calculus with respect to the *bifractional Brownian motion*. The "old" and somewhat restrictive theory of the stochastic integration with respect to Gaussian processes from the sixties-seventies (see e.g. [30] or [35]) has been recently reinforced and considered from a new, modern point of view, with a particular attention on the case of the fractional Brownian motion (fBm) due to the significant applications of this process in different phenomena. Let us briefly recall the principal techniques used in the Gaussian stochastic integration.

- The Malliavin calculus approach (and the white noise calculus) has been used in e.g. [2], [13], [15] or [31, 4] among others, to integrate stochastically with respect to Gaussian processes of the form  $\int_0^t K(t, s) dW_s$  where  $K$  is a deterministic kernel satisfying some regularity conditions on  $s$  and  $t$ . This approach allows to prove an Itô formula and (for the fractional Brownian motion) a Tanaka formula with Skorohod integral representation for the local time. The "natural" barrier for this approach in the fBm case is  $H = \frac{1}{4}$  ( $H$  is the Hurst parameter) but an extended divergence integral can be defined for every  $H \in (0, 1)$  (see [9, 36]). The limits of this theory are given by the fact that it depends on the form and the properties of the kernel  $K$ , and also, there are not existence results for stochastic equations with fBm in the Skorohod sense.
- The stochastic calculus via regularization has been developed starting from [43], [42] and continued by several authors. Among typical contributions we quote [21], [18], [25], [26], [24]. In that approach, stochastic integrals are defined through integrator smoothing. This method was preceded by the discretization approach which consists in discretizing the integrator process. These two approaches are *almost of pathwise type*. The first contribution in applying pathwise techniques (discretization) in stochastic calculus comes back to Föllmer [20]; relevant are the the works of Bertoin [6] and more recently of [19], in the context of finite quadratic variation processes. A monography [14] was also devoted to pure pathwise stochastic integration. Concerning the specific case of Gaussian integrators, and in particular for fBm, stochastic calculus via regularization was also used partially by [1] or [48]. A significant role in that framework is played by the symmetric integral  $\int_0^t Y d^\circ B$ , where  $B$  is a continuous process and  $Y$  is a locally integrable process. The exact definition is recalled at Section 5. When the integrator  $B$  is Gaussian, the regularization approach does not use in an essential way the form and the properties of the kernel of the process; it is based in principal on the properties of the covariance function. Given an fBm  $B = B^H$ , the "natural" barrier for the existence of stochastic integrals of the type  $\int_0^t g(B) d^\circ B$  for smooth functions  $g$  is  $H = \frac{1}{6}$ . For  $H \leq \frac{1}{6}$ , in general the symmetric integral  $\int_0^t g(B) d^\circ B$ , does not exist; however an extended symmetric integral may be defined, see [26]. Using calculus via regularization, stochastic differential equation driven by fBm can be solved by standard methods as Doss-Sussmann, see e.g. [1, 18, 37]. Note that the integral via regularization is equal to the Skorohod integral plus a trace term (see [1]).
- The rough paths analysis introduced by Lyons (see [34] or [40]) can be applied to the fBm situation. See [11] for the construction of the rough paths for fBm with parameter  $H > \frac{1}{4}$ . As a consequence, stochastic equations driven by fBm can be stated and solved.

We will focus now our attention to a Gaussian process that generalize the fractional Brownian motion, called *bifractional Brownian motion* and introduced in [29]. Recall that the fBm is the only self-similar Gaussian process with stationary increments starting from zero. For small increments, in models such as turbulence, fBm seems a good model but inadequate

for large increments. For this reason, in [29] the authors introduced an extension of the fBm keeping some properties (self-similarity, gaussianity, stationarity for small increments) but enlarged the modelling tool kit. Moreover, it happens that this process is a quasi-helix, as defined e.g by J.P. Kahane (see [32], [33]).

We refer to [29] for the notions presented in this section.

**Definition 1** *The bifractional Brownian motion  $(B_t^{H,K})_{t \geq 0}$  is a centered Gaussian process, starting from zero, with covariance*

$$R^{H,K}(t, s) := R(t, s) = \frac{1}{2^K} \left( (t^{2H} + s^{2H})^K - |t - s|^{2HK} \right) \quad (1)$$

with  $H \in (0, 1)$  and  $K \in (0, 1]$ .

Note that, if  $K = 1$  then  $B^{H,1}$  is a fractional Brownian motion with Hurst parameter  $H \in (0, 1)$ .

- If  $\sigma_\varepsilon^2(t) := \mathbb{E} \left( B_{t+\varepsilon}^{H,K} - B_t^{H,K} \right)^2$ , then

$$\lim_{\varepsilon \rightarrow 0} \frac{\sigma_\varepsilon^2(t)}{\varepsilon^{2HK}} = 2^{1-K}. \quad (2)$$

- Let  $T > 0$ . For every  $s, t \in [0, T]$ , we have

$$2^{-K} |t - s|^{2HK} \leq \mathbb{E} \left( B_t^{H,K} - B_s^{H,K} \right)^2 \leq 2^{1-K} |t - s|^{2HK}. \quad (3)$$

Inequality (3) shows that the process  $B^{H,K}$  is a quasi-helix in the sense of J.P. Kahane (see [32] and [33] for various properties and applications of quasi-helices).

- For every  $H \in (0, 1)$  and  $K \in (0, 1]$ ,

$$\lim_{\varepsilon \rightarrow 0} \sup_{t \in [t_0 - \varepsilon, t_0 + \varepsilon]} \left| \frac{B_t^{H,K} - B_{t_0}^{H,K}}{t - t_0} \right| = +\infty$$

with probability one for every  $t_0$ .

- The process is  $HK$ -self-similar.
- The process is Hölder continuous of order  $\delta$  for any  $\delta < HK$ . This follows from the Kolmogorov criterium.

In order to develop a stochastic calculus with respect to this process, the use of Malliavin calculus in this context seems to constitute an hard task since the form of the kernel of  $B^{H,K}$  is not explicitly known. Therefore we will implement the stochastic calculus

via regularization with respect to  $B^{H,K}$ , examining the case of each possible  $H$  and  $K$ . Elements of the *discretization* approach (that is, the use of Riemann sums) are also present in the paper to study various properties of this process, such as the strong variation, the cubic variation, long-range dependence or local times. When  $HK \neq \frac{1}{2}$ , the process has a some kind of "fractional" behavior.

On the other hand, we will pay a special attention to the case  $HK = \frac{1}{2}$  (and  $K \neq 1$ ; if  $K = 1$  then  $H = \frac{1}{2}$  and we have a Brownian motion). In this case we will show that  $B^{H,K}$  admits a non-trivial quadratic variation equal to constant times  $t$ , thus different from the fractional situation. We will show that even in this case ( $HK = \frac{1}{2}$  and  $K \neq 1$ ) the process is not a semimartingale and it is not a Dirichlet process, although it is somewhat "closer" to the notion of semimartingale than the fBm is. In this special case, our process appears to have something in common with the fBm with parameter less, bigger or equal to one half. Let us summarize the results proved below.

- Although  $2HK = 1$  implies  $H > \frac{1}{2}$ , the process  $B^{H,K}$  seems in this case to have similar properties as the fBm with  $H < \frac{1}{2}$ : it is short-memory, it is not a Dirichlet process.
- Nevertheless, having finite energy, it is also linked to the fBm with parameter bigger than  $\frac{1}{2}$ .
- Finally, there are elements placing this process on the "Brownian motion scale", of order  $\frac{1}{2}$ ; for example, if one consider occupation integrals  $X_t = \int_0^t f(B_u^{H,K}) du$ , this quantity has to be renormalized by the factor  $t^{-\frac{1}{2}}$  to converge as  $t \rightarrow \infty$ ; moreover, the local time of  $B^{H,K}$  belongs to the same Sobolev-Watanabe space as the local time of the Wiener process.

We organized the paper as follows. Section 2 contains some preliminaries on the stochastic calculus via regularization. In Section 3 we study the strong variation of the bifractional Brownian motion  $B^{H,K}$  and we discuss its immediate consequences. Section 4 presents a detailed study of the process  $B^{H,K}$  when  $HK = \frac{1}{2}$ : in particular it is neither a semimartingale, nor a Dirichlet process, nor Markovian, but it is a short memory process. In Section 5 we derive an Itô formula for the same process.

## 2 Preliminaries on the stochastic calculus via regularization

We will present here some notions of the stochastic calculus via regularization intervening along this paper. Other elements of this theory will be recalled in Section 5. We refer to [43], [42] or [18] for a more complete exposition. Throughout the paper by *ucp*-convergence we mean the *uniform convergence in probability on each compact interval*. A stochastic process  $(X_t)_{t \geq 0}$  will be extended by  $X_0$  for  $t \leq 0$ , to the real line.

We will use the concept of  $\alpha$ - *strong variation*: that is, we say that the continuous

process  $X$  has a  $\alpha$ -variation ( $\alpha > 0$ ) if

$$ucp - \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_0^t |X_{s+\varepsilon} - X_s|^\alpha ds \text{ exists.} \quad (4)$$

The limit is denoted by  $[X]_t^{(\alpha)}$ .

The notion of  $n$ -covariation was introduced in [18]. It plays a significant role in the stochastic calculus via regularization, for example in the case of the fractional Brownian motion with small Hurst index. If  $(X_1, \dots, X_n)$  is a continuous vector, then the  $n$ -covariation  $[X_1, \dots, X_n]$  is given by

$$[X_1, \dots, X_n]_t = \text{prob} - \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_0^t (X_{u+\varepsilon}^1 - X_u^1) \dots (X_{u+\varepsilon}^n - X_u^n) du.$$

In this work we will actually only essentially *cubic variation*  $[X, X, X]$  of a process  $X$ .

We recall the basic properties and relationships of the above notions.

1. If  $X$  and  $Y$  are both semimartingales, then  $[X, Y]$  is the usual semimartingale bracket.
2. Clearly, for  $n$  even integer, when it exists,  $[X]^{(n)} = \underbrace{[X, X, \dots, X]}_{n \text{ times}}$ .
3. Also, if the  $n$ -strong variation exists, then for every  $m > n$  it holds  $\underbrace{[X, X, \dots, X]}_{m \text{ times}} = 0$ .

### 3 The study of the $\alpha$ -strong variation

We study in this section the existence of the strong variation of the bifractional Brownian motion  $B^{H,K}$  and we discuss some immediate consequences.

**Proposition 1** *Let  $(B_t^{H,K})_{t \in [0,T]}$  be a bifractional Brownian motion with parameters  $H \in (0, 1)$  and  $K \in (0, 1]$ . Then it holds*

$$[B^{H,K}]_t^{(\alpha)} = 0, \text{ if } \alpha > \frac{1}{HK}$$

and

$$[B^{H,K}]_t^{(\alpha)} = 2^{\frac{1-K}{HK}} \rho_{HK} t \text{ if } \alpha = \frac{1}{HK},$$

where  $\rho_{HK} = \mathbb{E}|N|^{1/HK}$ ,  $N$  being a standard normal random variable.

**Proof:** We will regard only the case  $2HK = 1$ ; the other case ( $2HK > 1$ ) can be treated similarly. Denote by

$$C_\varepsilon^\alpha(t) = \frac{1}{\varepsilon} \int_0^t |B_{s+\varepsilon}^{H,K} - B_s^{H,K}|^\alpha ds.$$

By Lemma 3.1 of [43] it suffices to show that  $C_\varepsilon^{\frac{1}{HK}}(t)$  converges in  $L^2(\Omega)$  as  $\varepsilon \rightarrow 0$  to  $2^{\frac{1-K}{2HK}} \rho_{HK} t$ . We have by (2)

$$\mathbb{E} \left| B_{s+\varepsilon}^{H,K} - B_s^{H,K} \right|^{\frac{1}{HK}} = \left( \mathbb{E} \left| B_{s+\varepsilon}^{H,K} - B_s^{H,K} \right|^2 \right)^{\frac{1}{2HK}} \approx 2^{\frac{1-K}{2HK}} \rho_{HK} \varepsilon$$

(here the symbol  $\approx$  means that the ratio of the two sides tends to 1) and therefore

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E} \left( C_\varepsilon^{\frac{1}{HK}}(t) \right) = 2^{\frac{1-K}{2HK}} \rho_{HK} t.$$

To obtain the conclusion it suffices to show that

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E} \left( C_\varepsilon^{\frac{1}{HK}}(t) \right)^2 = \left( 2^{\frac{1-K}{2HK}} \rho_{HK} \right)^2 t^2. \quad (5)$$

We have

$$\mathbb{E} \left( C_\varepsilon^{\frac{1}{HK}}(t) \right)^2 = \frac{2}{\varepsilon^2} \int_0^t \int_0^u \mu_\varepsilon(u, v) dv du$$

where  $\mu_\varepsilon(u, v) := \mathbb{E} \left| \left( B_{u+\varepsilon}^{H,K} - B_u^{H,K} \right) \left( B_{v+\varepsilon}^{H,K} - B_v^{H,K} \right) \right|^{\frac{1}{HK}}$ .

Recall that if  $(G_1, G_2)$  is a Gaussian couple, then we can write

$$G_2 = \frac{Cov(G_1, G_2)}{Var^2(G_1)} G_1 + \sqrt{Var^2(G_2) - \frac{Cov^2(G_1, G_2)}{Var^2(G_1)}} N_2 \quad (6)$$

where  $N_2$  is a standard normal random variable.

Using (3) and (6) we get

$$\frac{\mu_\varepsilon(u, v)}{\varepsilon^2} = \mathbb{E} \left( \left| N_1 \right|^{\frac{1}{HK}} \left| \frac{\theta_\varepsilon(u, v)}{C_1 \varepsilon^{2HK}} N_1 + 2^{1-K} N_2 \sqrt{1 - \left( \frac{\theta_\varepsilon(u, v)}{C_2 \varepsilon^{2HK}} \right)^2} \right|^{\frac{1}{HK}} \right) \quad (7)$$

where  $C_1, C_2$  are strictly positive constants and we denoted by

$$\theta_\varepsilon(u, v) = \mathbb{E} \left( B_{u+\varepsilon}^{H,K} - B_u^{H,K} \right) \left( B_{v+\varepsilon}^{H,K} - B_v^{H,K} \right). \quad (8)$$

We compute

$$\begin{aligned} \theta_\varepsilon(u, v) &= R(u + \varepsilon, v + \varepsilon) - R(u + \varepsilon, v) - R(v + \varepsilon, u) + R(u, v) \\ &= (a_\varepsilon(u, v) + b_\varepsilon(u, v)) \end{aligned}$$

where

$$a_\varepsilon(u, v) = \frac{1}{2^K} \left[ ((u + \varepsilon)^{2H} + (v + \varepsilon)^{2H})^K - ((u + \varepsilon)^{2H} + v^{2H})^K - ((v + \varepsilon)^{2H} + u^{2H})^K + (u^{2H} + v^{2H})^K \right] \quad (9)$$

and

$$b_\varepsilon(u, v) = [(u + \varepsilon - v)_+^{2HK} + (u - \varepsilon - v)_+^{2HK} - 2(u - v)^{2HK}]. \quad (10)$$

First, note that the term  $b_\varepsilon$  appears in the study of the standard fractional Brownian motion with parameter  $HK$  (see [43]). It was actually proved that

$$\lim_{\varepsilon \rightarrow 0} \frac{b_\varepsilon(u, v)}{\varepsilon^{2HK}} = 0 \text{ and } \left| \frac{b_\varepsilon(u, v)}{\varepsilon^{2HK}} \right| \leq C. \quad (11)$$

Let us analyze the function  $a_\varepsilon(u, v)$  as  $\varepsilon \rightarrow 0$ . Note that in the fractional Brownian motion case (when  $K = 1$ ) this term vanishes. Using Taylor expansion and noticing that

$$a_0(u, v) = 0, \quad \frac{da_\varepsilon(u, v)}{d\varepsilon} \Big|_{\varepsilon=0} = 0 \text{ for every } u, v$$

and

$$\frac{d^2 a_\varepsilon(u, v)}{d\varepsilon^2} \Big|_{\varepsilon=0} = \frac{H^2 K(K-1)}{2^{K-3}} (u^{2H} + v^{2H})^{K-2} u^{2H-1} v^{2H-1}$$

we obtain, for every  $u, v$

$$a_\varepsilon(u, v) = \frac{H^2 K(K-1)}{2^{K-3}} (u^{2H} + v^{2H})^{K-2} u^{2H-1} v^{2H-1} \varepsilon^2 + o(\varepsilon^2).$$

This shows that

$$\lim_{\varepsilon \rightarrow 0} \frac{a_\varepsilon(u, v)}{\varepsilon} = 0 \quad \text{for every } u, v. \quad (12)$$

To obtain (5) from (7), (11), (12) and by the dominated convergence, it suffices to bound the quantity  $\frac{a_\varepsilon(u, v)}{\varepsilon}$  by a function  $H(u, v)$  integrable on  $[0, T]^2$  uniformly in  $\varepsilon$ , for  $\varepsilon$  small. Since

$$\frac{a_\varepsilon(u, v)}{\varepsilon} = g\left(\frac{u}{\varepsilon}, \frac{v}{\varepsilon}\right) \quad (13)$$

where

$$|g(x, y)| = \left| ((x+1)^{2H} + (y+1)^{2H})^K - ((x+1)^{2H} + y^{2H})^K - (x^{2H} + (y+1)^{2H})^K + (x^{2H} + y^{2H})^K \right|,$$

it is enough to bound  $g(x, y)$  for  $x, y$  large. This is obtained by observing that

$$|g(x, y)| \leq 2 |(y+1)^{2H} - y^{2H}|^K \xrightarrow{y \rightarrow \infty} 2(2H)^K. \quad (14)$$

■

**Remark 1** One can similarly show that the process  $B^{H,K}$  admits the same variation in the “classical” sense. That is, if

$$V_t^{\pi, \frac{1}{HK}}(B^{H,K}) = \sum_{i=0}^{n-1} |B_{t_{i+1}} - B_{t_i}|^{\frac{1}{HK}} \quad (15)$$

with  $\pi : 0 = t_0 < \dots, t_n = t$  denoting a partition of  $[0, t]$ , then

$$L^1(\Omega) - \lim_{|\pi| \rightarrow 0} V_t^\pi = 0 \text{ (if } \alpha > \frac{1}{HK}), 2^{\frac{1-K}{HK}} \rho_{HK} t \text{ (if } \alpha = \frac{1}{HK}) \text{ and } +\infty \text{ (if } \alpha < \frac{1}{HK}).$$

**Remark 2** The above Proposition 1 distinguishes a special case that seems to be more interesting than the other cases: the case  $KH = \frac{1}{2}$ . If  $K = 1$ , then  $H = \frac{1}{2}$  and we deal with a Wiener process. If  $K \neq 1$ , we have an example of a Gaussian process, having non-trivial quadratic variation which equals  $2^{1-K}t$ , so, modulo a constant, the same as the Brownian motion. The next section will be devoted to a detailed study of the process in this case.

Using similar arguments as in the proof of Proposition 1, we can show the following result, that will imply that the process is not a semimartingale when  $2HK \neq 1$ .

**Proposition 2** For any  $p, q > 0$ , we have

1)

$$n^{pHK-1} \sum_{i=0}^{n-1} \left| B_{\frac{i+1}{n}T}^{H,K} - B_{\frac{i}{n}T}^{H,K} \right|^p \rightarrow_{n \rightarrow \infty} 2^{\frac{1-K}{HK}} \mathbb{E} \left| B_T^{H,K} \right|^p \text{ in } L^1,$$

2)

$$n^{pHK-1-q} \sum_{i=0}^{n-1} \left| B_{\frac{i+1}{n}T}^{H,K} - B_{\frac{i}{n}T}^{H,K} \right|^p \rightarrow_{n \rightarrow \infty} 0 \text{ in } L^1,$$

3)

$$n^{pHK-1+q} \sum_{i=0}^{n-1} \left| B_{\frac{i+1}{n}T}^{H,K} - B_{\frac{i}{n}T}^{H,K} \right|^p \rightarrow_{n \rightarrow \infty} \infty \text{ in probability}$$

i.e. for all  $L > 0$  there is a  $n_0$  such that for all  $n \geq n_0$

$$P \left( n^{pHK-1+q} \sum_{j=0}^{n-1} \left| B_{\frac{j+1}{n}} - B_{\frac{j}{n}} \right|^p < L \right) < \frac{1}{L}.$$

The following result is a consequence of results in Cheridito [7] and of the above Proposition.



**Proposition 3** *The process  $B^{H,K}$  is not a semimartingale if  $HK \neq \frac{1}{2}$ .*

**Remark 3** *In fact, [7], pag. 20, introduces the notion of weak semimartingales which generalizes the concept of semimartingales. Proposition 1.9 and 1.11 of [7], and the above Proposition 2 imply that  $B^{H,K}$  is not a weak semimartingale when  $HK \neq \frac{1}{2}$ .*

In [8], the author introduced the so-called *mixed processes*, the sum of a Brownian motion and an independent Gaussian process and study the equivalence in law of this process to a Wiener process. Denote by  $W$  a standard Wiener processes. Mixing it with a bifractional Brownian motion we get the following

**Corollary 1** *The process  $B^{H,K} + W$ , restricted to each compact interval  $[0, T]$ , is equivalent in law with a Wiener process if  $HK > \frac{3}{4}$*

**Proof:** Recall Theorem 20 of Baudoin-Nualart [3]. If  $X$  is a Gaussian process with covariance  $R(t, s)$  such that  $\frac{\partial^2 R}{\partial s \partial t} \in L^2([0, T]^2)$ , the process  $Y_t = X_t + W_t$  is a semimartingale (in its own filtration) equivalent in law to a Wiener process.

Concerning the process  $B^{H,K}$ , note that for  $s \leq t$ ,

$$\frac{\partial^2 R}{\partial s \partial t}(s, t) = \frac{1}{2K} \left( 2HK(K-1)(t^{2H} + s^{2H})^{K-2} (st)^{2H-1} + 2HK(2HK-1)(t-s)^{2HK-2} \right).$$

Since  $(t^{2H} + s^{2H})^{K-2} \leq 2^{K-2}(st)^{H(K-2)}$ , the first part above belongs to  $L^2([0, T]^2)$  for  $HK > \frac{1}{2}$  and the second part for  $HK > \frac{3}{4}$ . ■

We finish this section with the study of the cubic variation. If  $B = B^H$  is a fbm with Hurst index  $H$ , the cubic variation  $[B, B, B]$  exists if and only if  $H > \frac{1}{6}$ : this fact provides an intuition on the natural barrier for the existence of the symmetric-Stratonovich integral  $\int_0^t g(B) d^\circ B$  for a smooth real function  $g$ . In fact, given a process  $X$ , it was observed in [25], taking  $g(x) = x^2$ , that  $\int_0^t X^2 d^\circ X$  exists if and only if  $[X, X, X]$  exists. Moreover the following inverse Itô formula holds:

$$\int_0^t X^2 d^\circ X = \frac{X^3}{3} - \frac{1}{6}[X, X, X]_t.$$

If  $X$  is a bifractional Brownian motion  $B^{H,K}$ , we can see that  $[X, X, X]$  exists if (and we think, as commented in Remark 4 below, only if)  $HK > \frac{1}{6}$ . A more complete discussion about the existence of symmetric integrals of the type  $\int_0^t g(B) d^\circ B$  will be provided in Section 5.

The next result show that the cubic variation of a bifractional Brownian motion  $B^{H,K}$  exists if  $HK > \frac{1}{6}$ .

**Proposition 4** *If  $HK > \frac{1}{6}$ ,*

$$[B^{H,K}, B^{H,K}, B^{H,K}]_t = 0 \tag{16}$$

*for every  $t \geq 0$ .*

**Proof:** See Appendix. ■

**Remark 4** *In [26], Theorem 4.1. 2b) the authors proved, for the standard fractional Brownian motion, that the cubic variation exists if and only if  $H > \frac{1}{6}$ . It seems that the arguments used along the proof of the fact that for  $H \leq \frac{1}{6}$  the cubic variation does not exist, can be quasi-immediately extended to the bifractional situation, except the following one which needs a particular attention: for fixed  $t > 0$ ,*

$$\frac{1}{\sqrt{\varepsilon}} \int_0^t \left( \frac{B_{u+\varepsilon} - B_u}{\varepsilon^{HK}} \right)^3 du$$

*converges in law to a centered Gaussian random variable. This statement has been proved in Theorem 2.4 of [24] for  $K = 1$ , the proof being long and rather technical. We believe the result is true also for  $K \neq 1$  and that the limit of the above expression worth to be the object of a further more detailed study.*

## 4 The study of the case $HK = \frac{1}{2}$

Throughout this Section (except the paragraph 4.6) we will assume that  $HK = \frac{1}{2}$  and  $K \neq 1$ . The covariance function is given by

$$R(t, s) = \frac{1}{2K} \left( \left( t^{\frac{1}{K}} + s^{\frac{1}{K}} \right)^K - |t - s| \right).$$

The Proposition 2.1 of [29] ensures that it is positive defined.

At a first look, the process appears in this case to enjoy special properties, different from the ones met in the study of the fractional Brownian motion: it has non-trivial quadratic variation equal to constant times  $t$  and it is  $\frac{1}{2}$ -self-similar. We will also investigate the following aspects of the process  $B^{H,K}$ : if it is a semimartingale or a Dirichlet process; having finite energy, it is known that it admits a Graversen-Rao decomposition (see [27]); we study if is a long or short-memory process; some remarks on the regularity of its local time are also given.

## 4.1 Not a Dirichlet process

**Definition 2** A process  $X$  will be said  $(\mathcal{F}_t)$   $L^1$ -**Dirichlet process** if it can be written as  $X = M + A$ , where  $M$  is local martingale with respect to  $(\mathcal{F}_t)$  and  $A$  is a zero quadratic variation process in the  $L^1$ - sense (i.e. the sequence  $V^{\pi,2}(A)$  given by (15) converges 0 zero as  $|\pi| \rightarrow 0$  in  $L^1(\Omega)$ ). If no filtration is mentioned, the underlying filtration will be the natural filtration.  $X$  will simply be called  $L^1$ -**Dirichlet process**.

We will need the following lemma.

**Lemma 1** Let us consider, for every  $i, l \geq 1$ , the following function on  $[0, \infty)$

$$\begin{aligned} f(\alpha) &= ((i + \alpha)^{2H} + (l + \alpha)^{2H})^K - (i^{2H} + (l + \alpha)^{2H})^K \\ &\quad - ((i + \alpha)^{2H} + l^{2H})^K + (i^{2H} + l^{2H})^K. \end{aligned} \quad (17)$$

Then the function  $f$  is strictly decreasing on  $[0, \infty)$  and we have that  $f(x) \leq 0$  for every  $x \geq 0$ .

**Proof:** We have  $f(0) = 0$  and, since  $2HK = 1$ ,

$$\begin{aligned} f'(\alpha) &= \left( \frac{(i + \alpha)^{2H} + (l + \alpha)^{2H}}{(i + \alpha)^{2H}} \right)^{K-1} + \left( \frac{(i + \alpha)^{2H} + (l + \alpha)^{2H}}{(l + \alpha)^{2H}} \right)^{K-1} \\ &\quad - \left( \frac{(i + \alpha)^{2H} + l^{2H}}{(i + \alpha)^{2H}} \right)^{K-1} - \left( \frac{i^{2H} + (l + \alpha)^{2H}}{(l + \alpha)^{2H}} \right)^{K-1} < 0. \end{aligned}$$

As a consequence  $f$  is decreasing and negative. ■

**Definition 3** A process  $X$  will be called  $(\mathcal{F}_t)$ - **quasi-Dirichlet process** if for any  $T > 0$ ,

$$S^\pi := \sum_{j=0}^{n-1} \mathbb{E} \left| \mathbb{E} (X_{t_{i+1}} - X_{t_i} / \mathcal{F}_{t_i}) \right|^2 \xrightarrow{|\pi| \rightarrow 0} 0, \quad (18)$$

where  $\pi : 0 = t_0 < \dots t_n = 1$  is a partition of  $[0, T]$ . Again, if no filtration is mentioned, the underlying filtration will be the natural filtration and  $X$  will simply be called **quasi-Dirichlet process**.

The next result can be easily established.

**Lemma 2** An  $(\mathcal{F}_t)$   $L^1$ - Dirichlet process is also an  $(\mathcal{F}_t)$ - **quasi-Dirichlet process**.

**Proof:** Let  $X$  be an  $L^1$ - Dirichlet process with canonical decomposition of  $X = M + A$ . It holds that

$$\begin{aligned} \sum_{j=0}^{n-1} \mathbb{E} \left| \mathbb{E} (X_{t_{i+1}} - X_{t_i} / \mathcal{F}_{t_i}) \right|^2 &= \sum_{j=0}^{n-1} \mathbb{E} \left| \mathbb{E} (A_{t_{i+1}} - A_{t_i} / \mathcal{F}_{t_i}) \right|^2 \\ &\leq \sum_{j=0}^{n-1} \mathbb{E} (A_{t_{i+1}} - A_{t_i})^2 \rightarrow 0. \blacksquare \end{aligned}$$

The above Lemma 2 will help to establish the next result.

**Proposition 5** *When  $HK = \frac{1}{2}$  and  $K \neq 1$ , the process  $B^{H,K}$  is not an  $L^1$ - quasi-Dirichlet process.*

Using previous Lemma 2, we obtain the following.

**Corollary 2** *When  $HK = \frac{1}{2}$  and  $K \neq 1$ , the process  $B^{H,K}$  is not a  $L^1$ - Dirichlet process.*

**Proof** (of hte Porposition 5): Let  $n \geq 1$  and  $t_i = \frac{i}{n}$ , for  $i = 0 \dots n$ . We use the notation  $\Delta_k^n = B_{\frac{k}{n}} - B_{\frac{k-1}{n}}$ . Let us put

$$S_n = \sum_{j=0}^{n-1} \left\| \mathbb{E} \left( \Delta_{j+1}^n / \mathcal{F}_{\frac{j}{n}} \right) \right\|_2^2. \quad (19)$$

we will show that

$$\overline{\lim}_{n \rightarrow \infty} S_n \geq C > 0.$$

Since the norm of the conditional expectation is a contraction, we have

$$S_n \geq \sum_{j=0}^{n-1} \left\| \mathbb{E} \left( \Delta_{j+1}^n / \Delta_j^n, \dots, \Delta_1^n \right) \right\|_2^2$$

and using the fact that  $(\Delta_{j+1}^n, \Delta_j^n, \dots, \Delta_1^n)$  is a Gaussian vector, we obtain

$$\mathbb{E} \left( \Delta_{j+1}^n / \Delta_j^n, \dots, \Delta_1^n \right) = \sum_{k=1}^j b_k \Delta_k^n$$

where  $b = A^{-1}m$  with  $m$  the vector

$$m = \left( Cov \left( \Delta_{j+1}^n, \Delta_l^n \right) \right)_{l=1, \dots, j}$$

and  $A$  is the matrix

$$A = \left( Cov \left( \Delta_i^n, \Delta_l^n \right) \right)_{i, l=1, \dots, j}.$$

We get

$$\begin{aligned} & \|\mathbb{E}(\Delta_{j+1}^n/\Delta_j^n, \dots, \Delta_1^n)\|_2^2 \\ &= b^T A b = m^T A^{-1} m \geq \frac{\|m\|_2^2}{\lambda} \end{aligned} \quad (20)$$

$\lambda$  being the largest eigenvalue of  $A$ . We find first an upper bound for  $\lambda$ . The Gersghorin Circle Theorem (see [23], Theorem 8.1.3, pag. 395) and Lemma 1 implies that

$$\begin{aligned} \lambda &\leq \max_{i=1, \dots, j} \sum_{l=1}^j |A_{il}| \\ &= \frac{1}{2^K n} \max_{i=0, \dots, j-1} \sum_{l=0}^{j-1} \left| \left[ ((i+1)^{2H} + (l+1)^{2H})^K - (i^{2H} + (l+1)^{2H})^K \right. \right. \\ &\quad \left. \left. - ((i+1)^{2H} + l^{2H})^K + (i^{2H} + l^{2H})^K \right] \right| \\ &= \frac{1}{2^K} \frac{1}{n} \max_{i=0, \dots, j-1} \left[ (j^{2H} + i^{2H})^K - (j^{2H} + (i+1)^{2H})^K + 1 \right]. \end{aligned}$$

Let us define the function  $g : [0, j-1] \rightarrow \mathbb{R}$ ,

$$g(x) = (j^{2H} + x^{2H})^K - (j^{2H} + (x+1)^{2H})^K + 1.$$

We have

$$\begin{aligned} g'(x) &= \left( \frac{x^{2H} + j^{2H}}{x^{2H}} \right)^{K-1} - \left( \frac{(x+1)^{2H} + j^{2H}}{(x+1)^{2H}} \right)^{K-1} \\ &= \left( 1 + \left( \frac{j}{x} \right)^{2H} \right)^{K-1} - \left( 1 + \left( \frac{j}{x+1} \right)^{2H} \right)^{K-1} \leq 0 \end{aligned}$$

Thus  $g$  is decreasing and  $\max_{i=0, j-1} g(i) = g(0) = j - (1 + j^{2H})^K + 1$ . To summarize, we obtained

$$\lambda \leq \frac{1}{2^K} \frac{1}{n} h_1(j) \quad (21)$$

with  $h_1(j) = j - (1 + j^{2H})^K + 1$ .

On the other hand, by Lemma 1,

$$\begin{aligned} \|m\|_2^2 &= \sum_{l=1}^j (\text{Cov}(\Delta_{j+1}^n, \Delta_l^n))^2 \\ &= \frac{1}{2^{2K} n^2} \sum_{l=0}^{j-1} \left[ ((j+1)^{2H} + (l+1)^{2H})^K - (j^{2H} + (l+1)^{2H})^K \right. \\ &\quad \left. - ((j+1)^{2H} + l^{2H})^K + (j^{2H} + l^{2H})^K \right]^2 \\ &\geq \frac{1}{2^{2K} n^2} |f(\frac{1}{2})| h_2(j) \end{aligned} \quad (22)$$

where the function  $f$  is defined by (17) and we denoted by

$$\begin{aligned} h_2(j) &= - \sum_{l=0}^{j-1} \left[ ((j+1)^{2H} + (l+1)^{2H})^K - (j^{2H} + (l+1)^{2H})^K \right. \\ &\quad \left. - ((j+1)^{2H} + l^{2H})^K + (j^{2H} + l^{2H})^K \right] \\ &= 2^K j - ((j+1)^{2H} + j^{2H})^K + 1. \end{aligned}$$

Combining all the above estimations (21), (22) and (20) , we obtain

$$S_n = \sum_{j=0}^{n-1} \|\mathbb{E} \left( \Delta_{j+1}^n / \mathcal{F}_{\frac{j}{n}} \right)\|_2^2 \geq \frac{C}{n} \sum_{j=0}^{n-1} \frac{h_2(j)}{h_1(j)}.$$

By using the asymptotic behavior of the functions  $h_1$  and  $h_2$  we can see that

$$\lim_{j \rightarrow \infty} h_1(j) = \lim_{j \rightarrow \infty} h_2(j) = 1 - 2^{K-1} > 0.$$

Consequently,  $\frac{h_2(j)}{h_1(j)} > C > 0$  when  $j$  is large enough and the conclusion follows.  $\blacksquare$

A natural question is either our process is a semimartingale or not. Let us remember some important facts about Gaussian semimartingales.

**Remark 5** *In the Gaussian case, the notion of semimartingale is closely related to the notion of quasimartingale (see below the definition). That is, assuming that the Gaussian process  $X$  is continuous, then  $X$  is a semimartingale (for its natural filtration) if and only if it is a quasimartingale (for the natural filtration). We refer to C. Stricker [46], Proposition 1 and Emery [17], pag. 704, before Theorem 2 (see also Song [44]).*

We recall the definition of the quasimartingale.

**Definition 4** *A stochastic process  $(X_t)_{t \geq 0}$  is a **quasimartingale** if for every  $T > 0$ ,*

$$X_t \in L^1(\Omega) \text{ for every } t \in [0, T]$$

and

$$\sup_{\Delta} \sum_{j=0}^{n-1} \left\| \mathbb{E} \left( X_{t_{j+1}} - X_{t_j} / \mathcal{F}_{t_j}^X \right) \right\|_1 < \infty,$$

where  $\mathcal{F}^X$  denote the natural filtration of the process  $X$  and  $\Delta : 0 = t_0 < t_1 < \dots < t_n = 1$  is a partition of  $[0, T]$ .

An immediate consequence of the above result is that the process  $B^{H,K}$  is not a semimartingale.

**Proposition 6** *Let us suppose that  $HK = \frac{1}{2}$  and  $K \neq 1$ . Then the process  $B^{H,K}$  is not a semimartingale.*

**Proof:** Suppose that  $B^{H,K}$  is a semimartingale. Then, by the above Remark 5, it follows that  $B^{H,K}$  is a quasimartingale and this clearly implies that it is a quasi-Dirichlet process. The conclusion follows by Proposition 5.  $\blacksquare$

Of course, being not a semimartingale,  $B^{H,K}$  is not a quasimartingale. But it enjoys when  $HK = \frac{1}{2}$  (and only in this case) a special property:

$$\sup_{\pi} \sum_{j=0}^{n-1} \|\mathbb{E} \left( B_{t_{j+1}}^{H,K} - B_{t_j}^{H,K} / B_j^{H,K} - B_{j-1}^{H,K} \right)\|_1 < \infty \quad (23)$$

where as before  $0 = t_0 < \dots < t_n = 1$  is a partition of  $[0, T]$ .

Note that the relation (23) is not true for the fBm (it follows from the computations contained in [7], Section 4.3 but it can be also directly seen without difficulty). To interpret this, we will say that the process  $B^{H,K}$  with  $HK = \frac{1}{2}$  is somewhat "closer" to a semimartingale than the fBm (with parameter different from  $\frac{1}{2}$ ) is.

**Proof of (23):** Consider  $t_i = \frac{i}{n}$  for  $0 < i < n$  and recall the notation

$$\Delta_{j+1}^n = \Delta_{j+1}^n B^{H,K} = B_{\frac{j+1}{n}}^{H,K} - B_{\frac{j}{n}}^{H,K} \text{ for every } j.$$

Using the linear regression as in (6) we can write

$$\Delta_{j+1}^n = \alpha(j, n) \Delta_j^n + \beta(j, n) N$$

where  $N$  is a standard normal random variable independent of  $B_{\frac{j}{n}}^{H,K} - B_{\frac{j-1}{n}}^{H,K}$  and

$$\alpha(j, n) = \frac{\text{Cov}(\Delta_{j+1}^n, \Delta_j^n)}{\text{Var}(\Delta_j^n)}.$$

Therefore,

$$\mathbb{E} (\Delta_{j+1}^n / \Delta_j^n) = (\alpha(j, n) - 1) \Delta_j^n.$$

Using the fact that for a centered normal random variable  $Z$  we have

$$\|Z\|_{L^2} = \sqrt{\frac{\pi}{2}} \|Z\|_{L^1}$$

we will obtain

$$\begin{aligned}
T_n &:= \sum_{j=0}^{n-1} \left\| \mathbb{E} \left( B_{\frac{j+1}{n}}^{H,K} - B_{\frac{j}{n}}^{H,K} / \Delta_j^n \right) \right\|_1 \\
&= \sqrt{\frac{2}{\pi}} \sum_{j=0}^{n-1} \|(\alpha(j, n) - 1) \Delta_j^n\|_2 \\
&= \sqrt{\frac{2}{\pi}} \sum_{j=0}^{n-1} |\alpha(j, n) - 1| \left( \frac{1}{n} \right)^{HK}.
\end{aligned} \tag{24}$$

Therefore, we get

$$\begin{aligned}
T_n &= cst. \frac{1}{\sqrt{n}} \sum_{j=1}^{n-1} \left[ ((j+1)^{2H} + j^{2H})^K \right. \\
&\quad \left. - ((j+1)^{2H} + (j-1)^{2H})^K - 2^K j + ((j-1)^{2H} + j^{2H})^K \right] \\
&= \frac{1}{\sqrt{n}} \sum_{j=1}^{n-1} j f\left(\frac{1}{j}\right)
\end{aligned}$$

where

$$f(x) = \left[ ((1+x)^{2H} + 1)^K - ((1+x)^{2H} + (1-x)^{2H})^K - 2^K + (1 + (1-x)^{2H})^K \right].$$

We have that  $f(0) = 0$ ,  $f'(0) = 0$  and  $f''(0) = 2H(K-1)2^{K-1}$ . Thus, when  $n$  is large,  $T_n$  has the same behavior as

$$\frac{1}{\sqrt{n}} \sum_{j=1}^{n-1} \frac{1}{j}$$

which goes to 0 as  $n \rightarrow \infty$ . ■

## 4.2 Graversen-Rao decomposition

We discuss the Graversen-Rao decomposition for finite energy processes in the context of the discretization approach. Let us recall the main result of Graversen-Rao [27]. We say that a process  $X$  has finite energy if

$$\sup_{\pi} \mathbb{E} \left[ \sum_{i=0}^{n-1} (X_{t_{i+1}} - X_{t_i})^2 \right] < \infty. \tag{25}$$

The main result of [27] states if  $X$  has finite energy, then it can be decomposed as  $X = M + A$ , where  $M$  is a square integrable martingale and  $A$  is "orthogonal" along a subsequence of partitions, to any square integrable martingale. See also [10] for results in this context. More precisely, we have



**Theorem 1** *If  $X$  is a finite energy process, then we can decompose  $X$  as*

$$X = M + A \tag{26}$$

where  $M$  is a square integrable martingale and  $A$  is a predictable process such that, for any  $T > 0$ , there exists a subsequence  $\pi_{n_j}$  of partitions of  $[0, T]$  with the mesh tending to zero as  $j \rightarrow \infty$  satisfying

$$\mathbb{E} \left[ \sum_{t_i \in \pi_{n_j}, t_i \leq t} (A_{t_{i+1}} - A_{t_i})(N_{t_{i+1}} - N_{t_i}) \right] \rightarrow 0 \tag{27}$$

as  $j \rightarrow \infty$  for any square integrable martingale  $N$ .

**Remark 6** *The decomposition (26) is not unique. We have that (see [27]), if  $M' + A'$  is another decomposition, then  $A - A'$  is a continuous martingale. The continuity of the martingale  $A - A'$  can be obtained even if  $X$  is a càdlàg process.*

**Corollary 3** *The process  $B^{H,K}$  admits a decomposition  $B^{H,K} = M + A$ , where  $M$  is a (non identical zero) local martingale and  $A$  satisfies (27).*

Of course, the Graversen-Rao decomposition holds also for  $HK > \frac{1}{2}$  since it is a zero quadratic variation process; however in this case the martingale part is zero.

### 4.3 Short-memory process

We use the following definition.

**Definition 5** *We say that a stochastic process  $X$  has **long-memory** (resp. **short-memory**) if for any  $a > 0$  it holds*

$$\sum_{n \geq a} r(n) = \infty \text{ (resp. } \sum_{n \geq a} r(n) < \infty)$$

where

$$r(n) = \mathbb{E}((X_{n+1} - X_n)(X_{a+1} - X_a)). \tag{28}$$

**Proposition 7** *If  $2HK = 1$  and  $K \neq 1$ , the process  $B^{H,K}$  is a short-memory process.*

**Proof:** See Appendix. ■

**Remark 7** *We can also prove that if  $HK > \frac{1}{2}$  the process  $B^{H,K}$  has long-memory, and for  $HK < \frac{1}{2}$  it has short-memory.*

#### 4.4 Not a Markov process

**Proposition 8** *For every  $K \in (0, 1]$  and  $H \in (0, 1)$ , the process  $B^{H,K}$  is not a Markov process.*

**Proof:** Recall that (see Revuz-Yor [41]) a Gaussian process with covariance  $R$  is Markovian if and only if

$$R(s, u)R(t, t) = R(s, t)R(t, u)$$

for every  $s \leq t \leq u$ . It is straightforward to check that  $B^{H,K}$  does not satisfy this condition. ■

#### 4.5 Remarks on the local times

We provide in this subsection a brief study of the local time of the bifractional Brownian motion with a particular look at the case  $2HK = 1$ . Recall that, concerning the regularity of the local time of fBm, the following facts happen in general: it belongs to the Sobolev-Watababe space (see [38] or [49] for definitions)  $\mathbb{D}^{\alpha,2}$  with  $\alpha < \frac{1}{2H} - \frac{1}{2}$  and it has to be renormalized by the factor  $t^{-H}$  to converge to a nontrivial limit. We generalize this results to the bifractional case. As a general fact, the regularity "of order  $H$ " is replaced by the "order  $HK$ ". When  $HK = \frac{1}{2}$  the process  $B^{H,K}$ , from the point of view of the regularity of its local time, appears to belong to the same class as the standard Wiener process.

Let us define, for every  $t \geq 0$ , and  $x \in \mathbb{R}$ , the local time of  $B^{H,K}$  as

$$L(t, x) = L^2(\Omega) - \lim_{\varepsilon \rightarrow 0} \int_0^t p_\varepsilon(B_s - x) ds. \quad (29)$$

where  $p_\varepsilon(x) = \frac{1}{\sqrt{2\pi\varepsilon}} e^{-\frac{x^2}{2\varepsilon}}$  is the Gaussian kernel of variance  $\varepsilon > 0$ . The Wiener chaos expansion can be used to prove the existence and the regularity of  $L$ . By  $I_n$  we denote the Wiener-Itô multiple integral with respect to  $B$  (see [35] for details).

**Proposition 9** *For every  $t \geq 0$  and  $x \in \mathbb{R}$ , the local time  $L(t, x)$  exists and admits the following chaotic decomposition*

$$L(t, x) = \sum_{n \geq 0} \int_0^t \frac{p_{s^{2HK}}(x)}{s^{nHK}} H_n\left(\frac{x}{s^{2HK}}\right) I_n(1_{[0,s]}^{\otimes n}) ds. \quad (30)$$

where  $H_n$  is the  $n$ th Hermite polynomial defined for  $n \geq 1$  by

$$H_n(x) = \frac{(-1)^n}{n!} e^{x^2/2} \frac{d^n}{dx^n} \left( e^{-x^2/2} \right),$$

and  $H_0(x) = 1$ .

Moreover, it belongs to the space  $\mathbb{D}^{\alpha,2}$  for every  $\alpha < \frac{1}{2HK} - \frac{1}{2}$ .

**Proof:** The arguments of [12] can be applied to prove the existence and the chaotic expansion of  $L(t, x)$ .

Here we will just indicate how to evaluate the  $\mathbb{D}^{\alpha,2}$  norm. Similarly to [12] or [16], we have

$$\begin{aligned}
\|L(t, x)\|_{\alpha,2}^2 &= \sum_{n \geq 0} (1+n)^\alpha \mathbb{E} \left( \int_0^t \frac{p_{s^{2HK}}(x)}{s^{nHK}} H_n \left( \frac{x}{s^{2HK}} \right) I_n(1_{[0,s]}^{\otimes n}) ds \right) \\
&\leq C(t, H, K) \sum_{n \geq 0} \frac{(1+n)^\alpha}{\sqrt{n}} \int_0^t \int_0^u \frac{R(u, v)^n}{(uv)^{HKn} (uv)^H} dv du \\
&= C(t, H, K) \sum_{n \geq 0} \frac{(1+n)^\alpha}{\sqrt{n}} \int_0^t \int_0^u \frac{u^{2HKn} R(1, \frac{v}{u})^n}{(uv)^{HKn} (uv)^H} dv du \\
&= C(t, H, K) \sum_{n \geq 0} \frac{(1+n)^\alpha}{\sqrt{n}} \int_0^1 \left( \frac{R(1, z)}{z^{HK}} \right)^n \frac{dz}{z^{HK}},
\end{aligned}$$

where  $C(t, H, K)$  is a generic constant depending on  $t, H, K$  which may differ from line to line. We notice that

$$\frac{R(1, z)}{z^{HK}} \leq Q(z)^K$$

where  $Q(z) = \frac{1+z^{2H}-(1-z)^{2H}}{2z^H}$ . The behavior of the function  $Q$  has been studied in Lemma 2 of [16]. By applying the techniques used in [16], we get

$$\int_0^1 \left( \frac{R(1, z)}{z^{HK}} \right)^n \frac{dz}{z^{HK}} \leq c(H, K) n^{-\frac{1}{2HK}}$$

where the constant  $c(H, K)$  does not depend on  $n$ . This gives the conclusion.  $\blacksquare$

**Remark 8** *In particular, for  $HK = \frac{1}{2}$ , we find the same order of regularity as the standard Brownian motion (see [39]).*

We finish this section with a short result on the asymptotic behavior of the occupation integrals of  $B^{H,K}$ .

**Proposition 10** *Let  $f$  be a continuous function with compact support and let us define, for every  $t \geq 0$ ,  $X_t = \int_0^t f(B_s^{H,K}) ds$ . Then we have*

$$t^{-HK} X_t \xrightarrow[t \rightarrow \infty]{d} \tilde{f} L(1, 0) \tag{31}$$

where  $\tilde{f} = \int_0^1 f(x) dx$  and " $\xrightarrow{d}$ " stands for the convergence in distribution.

**Proof:** Let us assume  $HK = \frac{1}{2}$ ; the general case is analogous. It holds, by the  $HK = \frac{1}{2}$ -self-similarity of the process and the occupation time formula

$$\begin{aligned} X_t &= t \int_0^1 f(B_{ut}) du \\ &\stackrel{d}{=} t \int_0^1 f(t^{\frac{1}{2}} B_u) du = t \int_{\mathbb{R}} f(t^{\frac{1}{2}} x) L(1, x) dx = t^{\frac{1}{2}} \int_0^t f(y) L(1, yt^{-\frac{1}{2}}) dy. \end{aligned}$$

This shows that

$$t^{-\frac{1}{2}} \int_0^t f(B_u) du \stackrel{d}{=} \int_{\mathbb{R}} f(y) L(1, t^{-\frac{1}{2}} y) dy.$$

Using the existence of a bicontinuous version for the local time (see [5] or [22]) one can see that the right side converges as  $t \rightarrow \infty$  to  $L(1, 0) \int_{\mathbb{R}} f(y) dy$ .  $\blacksquare$

## 5 Itô formula

We prove in this section an Itô formula for the bifractional Brownian motion with any parameter  $H \in (0, 1)$  and  $K \in (0, 1]$ . We first present some more elements of the stochastic calculus via regularization. Note that the notions are a little bit relaxed, the limits being considered in probability.

Let us consider two continuous processes  $X$  and  $Y$ . The *symmetric* integral of  $Y$  with respect to  $X$  is defined as

$$\int_0^t Y_u d^o X_u = \text{prob} - \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_0^t \frac{Y_{u+\varepsilon} + Y_u}{2} (X_{u+\varepsilon} - X_u) du, \quad t \geq 0.$$

More generally, if  $m \geq 1$  and  $Y$  is locally bounded, the *m-order symmetric integral* of  $Y$  with respect to  $X$  is given

$$\int_0^t Y_u d^{o(m)} X_u = \text{prob} - \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_0^t \frac{Y_u + Y_{u+\varepsilon}}{2} (X_{u+\varepsilon} - X_u)^m du, \quad t \geq 0.$$

We have

1. If  $X$  and  $Y$  are both semimartingales, then  $\int Y d^o X$  is the classical Fisk-Stratonovich integral.
2. We have

$$\int_0^t Y_s d^{o(1)} X_s = \int_0^t Y_s d^o X_s.$$

In the fractional Brownian motion case, the above more or less classical definitions are sufficient to develop a stochastic calculus if the parameter is strictly bigger than  $\frac{1}{6}$  (see [25], [26]). An extended notion of *m-order  $\nu$ -integral*, where  $\nu$  is a probability measure, is needed when the Hurst index is less than  $\frac{1}{6}$ . This extended approach has been introduced in [26]. We recall the definition of *the m-order  $\nu$ -integral* (see [26] and [47]).

**Definition 6** Let  $m \geq 1$  and  $\nu$  a probability measure on  $[0, 1]$ . For a locally bounded function  $g : \mathbb{R} \rightarrow \mathbb{R}$ , the  $m$ -order  $\nu$ -integral of  $g(X)$  with respect to  $X$  is given by

$$\int_0^t g(X_u) d^{\nu, m} X_u = \lim_{\varepsilon \rightarrow 0} \text{prob} \frac{1}{\varepsilon} \int_0^t du (X_{u+\varepsilon} - X_u)^m \int_0^1 g(X_u + \alpha(X_{u+\varepsilon} - X_u)) \nu(d\alpha). \quad (32)$$

This integral with respect to  $X$  is in general defined only for integrands of the type  $g(X)$ . Note that

- If  $\mu = \delta_0$  and  $m \in \mathbb{N}^*$ , then  $\int_0^t g(X_u) d^{\nu, m} X_u$  is the  $m$ -order forward integral (see [25]).
- If  $\mu = \frac{\delta_0 + \delta_1}{2}$ , then  $\int_0^t g(X_u) d^{\nu, m} X_u$  is the  $m$ -order symmetric integral defined above.

The following Itô's formula was proved in [26].

**Theorem 2** Let  $n, l \geq 1$  integers and let  $\nu$  be a symmetric probability measure on  $[0, 1]$  verifying

$$m_{2j} := \int_0^1 \alpha^{2j} \nu(d\alpha) = \frac{1}{2j+1} \text{ for } j = 1, \dots, l-1.$$

If  $f \in C^{2n}(\mathbb{R})$  and  $X$  is a continuous process with  $(2n)$ -variation it holds that

$$f(X_t) = f(X_0) + \int_0^t f'(X_u) d^{\nu, 1} X_u + \sum_{j=l}^{n-1} k_{l,j} \int_0^t f^{(2j+1)}(X_u) d^{\delta_{1/2}, 2j+1} X_u \quad (33)$$

provided that all the integrals except one exist. Here  $k_{l,j}$  denote universal constants. The above sum is by convention zero if  $l > n-1$ .

In particular, when  $\nu = \frac{\delta_0 + \delta_1}{2}$ , we have

$$f(X_t) = f(X_0) + \int_0^t f'(X_u) d^\circ X_u + \sum_{j=l}^{n-1} k_{l,j} \int_0^t f^{(2j+1)}(X_u) d^{\delta_{1/2}, 2j+1} X_u. \quad (34)$$

**Remark 9** When  $\nu$  is the Lebesgue measure on  $[0, 1]$  one can prove an Itô's formula with very few assumptions (see Proposition 3.5 in [26]): if  $f \in C^1(\mathbb{R})$ , then  $\int_0^t f'(X_u) d^{\nu, 1} X_u$  exists and

$$f(X_t) = f(X_0) + \int_0^t f'(X_u) d^{\nu, 1} X_u.$$

However this formula is almost a tautology; the interesting case arises when  $\nu$  is discrete.

We treat now the particular case of the bifractional Brownian motion. The proof of the below theorem will be postponed to the Appendix.

**Theorem 3** *Let  $g$  be a locally bounded real function. If  $n$  is a positive integer such that  $(2n + 1)HK > \frac{1}{2}$ , the integral  $\int_0^t g(B_u^{H,K}) d^{\delta_{1/2, 2n+1}} B_u^{H,K}$  exists and vanishes.*

**Remark 10** *As a consequence of the above theorem, if  $(2n + 1)HK > \frac{1}{2}$  the integrals  $\int_0^t g(B_u^{H,K}) d^{\delta_{1/2, 2l+1}} B_u^{H,K}$  exist and vanish for all integers  $l \geq n$ .*

*In particular, if  $HK > \frac{1}{6}$ , the integrals  $\int_0^t g(B_u^{H,K}) d^{\delta_{1/2, l}} B_u^{H,K}$  exist and vanish for  $l \geq 3$ .*

As a consequence of Theorem 2 and 3, we obtain the following Itô's formula for the bifractional Brownian motion.

**Theorem 4** *a. Let  $\nu$  be a symmetric probability measure. If  $HK > \frac{1}{6}$  and  $f \in C^6(\mathbb{R})$ , the integral  $\int_0^t f'(B_u^{H,K}) d^{\nu, 1} B_u^{H,K}$  exists and we have*

$$f(B_t^{H,K}) = f(0) + \int_0^t f'(B_u^{H,K}) d^{\nu, 1} B_u^{H,K}. \quad (35)$$

*b. Let  $\mu$  a probability measure satisfying*

$$m_{2j} := \int_0^1 \alpha^{2j} \mu(d\alpha) = \frac{1}{2j+1} \quad j = 1, \dots, r-1$$

*and let us assume  $r \geq 2$ . If  $2(2r + 1)H > 1$  and  $f \in C^{4r+2}(\mathbb{R})$ , then the integral  $\int_0^t f'(B_u^{H,K}) d^{\nu, 1} B_u^{H,K}$  exists and we have*

$$f(B_t^{H,K}) = f(0) + \int_0^t f'(B_u^{H,K}) d^{\nu, 1} B_u^{H,K}. \quad (36)$$

**Remark 11** *An example of a measure verifying point b. of Theorem 4 is given in [26], Remark 4.6.*

**Remark 12** • *if  $HK > \frac{1}{6}$ , then the stochastic calculus via regularization gives an Itô's formula of standard type for the bifractional Brownian motion*

$$f(B_t^{H,K}) = f(0) + \int_0^t f'(B_u^{H,K}) d^0 B_u^{H,K}.$$

*This follows from Theorem 4 b. and Theorem 2, formula (34).*

• *if  $HK \leq \frac{1}{6}$  one need an extended, relaxed way to integrate.*

**Remark 13** *Itô's formula given in Theorem 4 could be written for the case  $f(t, B_t^{H,K})$ . This fact can be used to solve stochastic differential equations driven by  $B^{H,K}$  of the type*

$$dX_t = \sigma(X_t) dB_t^{H,K} + b(X_t) dt, \quad X_0 = \alpha. \quad (37)$$

*where the stochastic integral is understood in the symmetric sense. We will not insist on this topic since standard arguments apply. We refer e.g. to [1] for the definition of the solution of (37) and for the method to solve this equation. Another approach, not related to Gaussian processes, is provided in [18].*

## 6 Appendix

**Proof of Proposition 4:** For every  $t \geq 0$ , denote by

$$I_t^\varepsilon = \mathbb{E} \left( \int_0^t \frac{(B_{u+\varepsilon}^{H,K} - B_u^{H,K})^3}{\varepsilon} du \right)^2.$$

We will prove that  $I_t^\varepsilon$  converges to zero as  $\varepsilon \rightarrow 0$  for every  $t$ . Using the relation

$$\mathbb{E}(G_3^3 G_4^3) = 6Cov^3(G_3, G_4) + 9Cov(G_3, G_4)Var(G_3)Var(G_4)$$

if  $G_3, G_4$  are two centered Gaussian random variables, it holds that

$$\begin{aligned} I_t^\varepsilon &= 2 \int_0^t \int_0^u \frac{\mathbb{E} \left( (B_{u+\varepsilon}^{H,K} - B_u^{H,K})^3 (B_{v+\varepsilon}^{H,K} - B_v^{H,K})^3 \right)}{\varepsilon^2} dv du \\ &= 12 \int_0^t \int_0^u \frac{\theta_\varepsilon(u, v)^3}{\varepsilon^2} dv du \\ &\quad + 9 \int_0^t \int_0^u \frac{\theta_\varepsilon(u, v) Var \left( B_{u+\varepsilon}^{H,K} - B_u^{H,K} \right) Var \left( B_{v+\varepsilon}^{H,K} - B_v^{H,K} \right)}{\varepsilon^2} dv du \\ &= 12 \int_0^t \int_0^u \frac{(a_\varepsilon(u, v) + b_\varepsilon(u, v))^3}{\varepsilon^2} dv du \\ &\quad + 9 \int_0^t \int_0^u \frac{(a_\varepsilon(u, v) + b_\varepsilon(u, v)) Var \left( B_{u+\varepsilon}^{H,K} - B_u^{H,K} \right) Var \left( B_{v+\varepsilon}^{H,K} - B_v^{H,K} \right)}{\varepsilon^2} dv du \\ &:= A_\varepsilon + B_\varepsilon. \end{aligned}$$

where  $\theta, a, b$  are given by (8), (9) and (10).

We estimate first the term  $B_\varepsilon$ . We have

$$\begin{aligned} B_\varepsilon &= 9 \int_0^t \int_0^u \frac{a_\varepsilon(u, v) Var \left( B_{u+\varepsilon}^{H,K} - B_u^{H,K} \right) Var \left( B_{v+\varepsilon}^{H,K} - B_v^{H,K} \right)}{\varepsilon^2} dv du \\ &\quad + 9 \int_0^t \int_0^u \frac{b_\varepsilon(u, v) Var \left( B_{u+\varepsilon}^{H,K} - B_u^{H,K} \right) Var \left( B_{v+\varepsilon}^{H,K} - B_v^{H,K} \right)}{\varepsilon^2} dv du \\ &:= B_\varepsilon^1 + B_\varepsilon^2. \end{aligned}$$

By (3) it holds that

$$B_\varepsilon^2 \leq cst. \varepsilon^{4HK} \int_0^t \int_0^u \frac{b_\varepsilon(u, v)}{\varepsilon^2} dv du$$

and this converges to zero since it has been already studied in [25], Proposition 3.8.

The  $B_\varepsilon^1$  can be bounded by

$$B_\varepsilon^1 \leq cst.\varepsilon^{4HK} \int_0^t \int_0^u \frac{a_\varepsilon(u,v)}{\varepsilon^2} dvdu$$

and this goes to zero by the (12) and (14).

Concerning the summand  $A_\varepsilon$ , we can write

$$\begin{aligned} A_\varepsilon &= 12 \int_0^t \int_0^u \frac{a_\varepsilon(u,v)^3}{\varepsilon^2} dvdu + 36 \int_0^t \int_0^u \frac{a_\varepsilon(u,v)^2 b_\varepsilon(u,v)}{\varepsilon^2} dvdu \\ &\quad + 36 \int_0^t \int_0^u \frac{a_\varepsilon(u,v) b_\varepsilon(u,v)^2}{\varepsilon^2} dvdu + 12 \int_0^t \int_0^u \frac{b_\varepsilon(u,v)^3}{\varepsilon^2} dvdu \\ &:= A_\varepsilon^1 + A_\varepsilon^2 + A_\varepsilon^3 + A_\varepsilon^4. \end{aligned}$$

The last term  $A_\varepsilon^4$  appears in the study of the fractional Brownian motion with parameter  $HK$ . When  $\varepsilon$  is close to zero, it behaves as (see [25], proof of Proposition 3.8)

$$cst.\varepsilon^{6HK-1} \int_0^\infty ((x+1)^{2HK} + (x-1)^{2HK} - 2x^{2HK})^3 dx$$

and this goes to zero if  $HK > \frac{1}{6}$ .

Treatment of the three remaining terms is similar. For example, concerning  $A_\varepsilon^1$ , using (12) we obtain that

$$A_\varepsilon^1 \leq cst. \int_0^t \int_0^u a_\varepsilon(u,v) dvdu$$

and this tends to zero as  $\varepsilon \rightarrow 0$  by (12), (13) and (14). ■

**Proof of Proposition 7 :** We have, for any  $a > 0$  and  $n \geq a$ ,

$$\begin{aligned} r(n) &= \mathbb{E}((B_{n+1} - B_n)(B_{a+1} - B_a)) \\ &= \frac{1}{2^K} \left[ ((n+1)^{2H} + (a+1)^{2H})^K - ((n+1)^{2H} + a^{2H})^K \right. \\ &\quad \left. - (n^{2H} + (a+1)^{2H})^K + (n^{2H} + a^{2H})^K \right] \\ &= \frac{1}{2^K} n f\left(\frac{1}{n}\right) \end{aligned}$$

where

$$\begin{aligned} f(x) &= ((1+x)^{2H} + ((a+1)x)^{2H})^K - (1 + ((a+1)x)^{2H})^K \\ &\quad - ((1+x)^{2H} + (ax)^{2H})^K + (1 + (ax)^{2H})^K. \end{aligned}$$

We study the behavior of  $f$  at the origin. We have  $f(0) = 0$  and

$$f'(x) = x^{2H-1} G_1(x) + G_2(x)$$



where

$$G_1(x) = (a+1)^{2H} \left( (1+x)^{2H} + ((a+1)x)^{2H} \right)^{K-1} - (a+1)^{2H} \left( 1 + ((a+1)x)^{2H} \right)^{K-1} - a^{2H} \left( (1+x)^{2H} + (ax)^{2H} \right)^{K-1} + a^{2H} \left( 1 + (ax)^{2H} \right)^{K-1}$$

and

$$G_2(x) = (1+x)^{2H-1} \left[ \left( (1+x)^{2H} + ((a+1)x)^{2H} \right)^{K-1} - \left( (1+x)^{2H} + (ax)^{2H} \right)^{K-1} \right].$$

We can show that  $G_1(x) = (1-2H) [(a+1)^{2H} - a^{2H}] x + o(x)$  and writing  $G_2'(x) = x^{2H-1} H_1(x) + H_2$ , we obtain  $G_2'(x) = (1-2H) [(a+1)^{2H} - a^{2H}] x^{2H} + o(x^{2H})$ . Consequently,

$$f(x) = 2(1-2H) [(a+1)^{2H} - a^{2H}] x^{2H+1} + o(x^{2H+1}).$$

Therefore,

$$\sum_{n \geq a} r(n) = 2^{-K} \sum_{n \geq a} n f\left(\frac{1}{n}\right)$$

has the same nature as the series

$$\sum_{n \geq a} \frac{1}{n^{2H}}$$

and this is finite since  $2HK = 1$  implies  $H > \frac{1}{2}$ . ■

**Proof of Theorem 3 :** After use of localization arguments, it will be enough to suppose  $g$  bounded. In that case, we will prove that, for  $m = 2n + 1$  with  $n \geq 1$ , the quantity

$$\frac{1}{\varepsilon} \int_0^1 g \left( \frac{B_{u+\varepsilon}^{H,K} - B_u^{H,K}}{2} \right) \left( B_{u+\varepsilon}^{H,K} - B_u^{H,K} \right)^m du$$

converges to zero in  $L^2(\Omega)$  as  $\varepsilon \rightarrow 0$  or equivalently

$$\begin{aligned} J_\varepsilon^{(m)} &:= \frac{1}{2\varepsilon} \int_D \int \mathbb{E} \left[ g \left( \frac{B_{u+\varepsilon}^{H,K} - B_u^{H,K}}{2} \right) g \left( \frac{B_{v+\varepsilon}^{H,K} - B_v^{H,K}}{2} \right) \right. \\ &\quad \left. \times \left( B_{u+\varepsilon}^{H,K} - B_u^{H,K} \right)^m \left( B_{v+\varepsilon}^{H,K} - B_v^{H,K} \right)^m \right] dv du \end{aligned} \quad (38)$$

tends to zero as  $\varepsilon \rightarrow 0$ , where  $D$  is the set  $(0 < u < v < 1)$ .

Using exactly the same arguments as in [26], we can consider the following reductions:

•

$$\frac{1}{2m} < HK \leq \frac{1}{m} \leq \frac{1}{2}$$

,

- The integration domain  $D$  can be replaced by the set

$$D_\varepsilon = \{\varepsilon^{1-\rho} < v < u < 1, \varepsilon^{1-\rho} < v - u < 1 \text{ with } \rho \text{ small enough } \}.$$

The next step is to do the linear regression on the Gaussian vector

$$(G_1, G_2, G_3, G_4) = (B_{u+\varepsilon} + B_u, B_{v+\varepsilon} + B_v, B_{u+\varepsilon} - B_u, B_{v+\varepsilon} - B_v)$$

with covariance matrix

$$\Lambda = \begin{pmatrix} \Lambda_{11} & \Lambda_{21}^* \\ \Lambda_{21} & \Lambda_{22} \end{pmatrix}$$

with

$$\Lambda_{11} = \begin{pmatrix} A_\varepsilon(u, u) & A_\varepsilon(u, v) \\ A_\varepsilon(u, v) & A_\varepsilon(v, v) \end{pmatrix}$$

where

$$\begin{aligned} A_\varepsilon(u, v) = & \frac{1}{2K} \left[ \left( ((u+\varepsilon)^{2H} + (v+\varepsilon)^{2H})^K - (u-v)^{2HK} \right) \right. \\ & + \left( ((u+\varepsilon)^{2H} + v^{2H})^K - (u+\varepsilon-v)^{2HK} \right) \\ & + \left( (u^{2H} + (v+\varepsilon)^{2H})^K - (u-\varepsilon-v)^{2HK} \right) \\ & \left. \left( (u^{2H} + v^{2H})^K - (u-v)^{2HK} \right) \right]. \end{aligned}$$

The matrix  $\Lambda_{21}$  is given by

$$\Lambda_{21} = \begin{pmatrix} Cov(G_3, G_1) & Cov(G_3, G_2) \\ Cov(G_4, G_1) & Cov(G_4, G_2) \end{pmatrix}.$$

Since the matrix  $\Lambda_{11}$  is symmetric and positive definite, we can write  $\Lambda_{11} = MM^*$  where

$$M = \begin{pmatrix} \sqrt{A_\varepsilon(u, u)} & 0 \\ \frac{A(u, v)}{\sqrt{A_\varepsilon(u, u)}} & \frac{\sqrt{D_\varepsilon(u, v)}}{\sqrt{A(u, u)}} \end{pmatrix}$$

and

$$(M^*)^{-1} = \begin{pmatrix} \frac{1}{\sqrt{A_\varepsilon(u, u)}} & -\frac{A(u, v)}{\sqrt{A_\varepsilon(u, u)D_\varepsilon(u, v)}} \\ 0 & \frac{\sqrt{A_\varepsilon(u, u)}}{\sqrt{D_\varepsilon(u, v)}} \end{pmatrix}.$$

By linear regression, we can write (see also [26])

$$\begin{aligned} \begin{pmatrix} G_3 \\ G_4 \end{pmatrix} &= A \begin{pmatrix} G_1 \\ G_2 \end{pmatrix} + \begin{pmatrix} Z_3 \\ Z_4 \end{pmatrix} \\ &= R \begin{pmatrix} N_1 \\ N_2 \end{pmatrix} + \begin{pmatrix} Z_3 \\ Z_4 \end{pmatrix} \end{aligned}$$

where  $R$  is given by  $R = \Lambda_{21}(M^*)^{-1}$ , the vector  $(Z_3, Z_4)$  is independent from  $(G_1, G_2)$  and the random variables  $N_1$  and  $N_2$  are independent.

Next, the term  $J_\varepsilon^{(m)}$  given by (38) can be divided into three summands as follows.

$$\begin{aligned}
J_\varepsilon^{(m)} &= \frac{1}{2\varepsilon} \int_{D_\varepsilon} \int \mathbb{E} \left[ g \left( \frac{G_1}{2} \right) g \left( \frac{G_2}{2} \right) (\Gamma_3 + Z_3)^m (\Gamma_4 + Z_4)^m \right] dvdu \\
&= \frac{1}{2\varepsilon} \int_{D_\varepsilon} \int \mathbb{E} \left[ g \left( \frac{G_1}{2} \right) g \left( \frac{G_2}{2} \right) \right] Z_3^m Z_4^m dvdu \\
&\quad + \frac{1}{2\varepsilon} \int_{D_\varepsilon} \int \mathbb{E} \left[ g \left( \frac{G_1}{2} \right) g \left( \frac{G_2}{2} \right) \right] (\Gamma_3 Z_3^{m-1} Z_4^m + \Gamma_4 Z_3^m Z_4^{m-1}) dvdu \\
&\quad + \frac{1}{2\varepsilon} \int_{D_\varepsilon} \int \mathbb{E} \left[ g \left( \frac{G_1}{2} \right) g \left( \frac{G_2}{2} \right) \right] \\
&\quad \times \sum_{j=0}^m \sum_{k=2}^m C_m^j C_m^k \left( \Gamma_3^j Z_3^{m-j} \Gamma_4^k Z_4^{m-k} + \Gamma_3^k Z_3^{m-k} \Gamma_4^j Z_4^{m-j} \right) dvdu \\
&:= J_1 + J_2 + J_3.
\end{aligned}$$

First, by Lemma 5.2 in [26], we note that  $J_2 = 0$ . We prove next that the term  $J_3$  goes to zero as  $\varepsilon \rightarrow 0$ . In order to do it, following the computations in [26], the key point that we have to check is to show that

$$\int_{D_\varepsilon} \int |r_{ij}|^l dudv \leq cst.\varepsilon^{1+lKH} \text{ for all } i, j \in \{1, 2\}$$

where  $r_{ij}$  are the coefficients of the matrix  $R$ . This follows since we prove the following bounds:

$$cst.u^{2HK} \leq A_\varepsilon(u, u) \leq cst.u^{2HK}, \quad A_\varepsilon(u, v) \leq cst.u^{HK}v^{HK} \quad (39)$$

and

$$D_\varepsilon(u, v) \geq cst.u^{2HK}(v-u)^{2HK} \quad (40)$$

where  $cst.$  denotes a generic positive constant. We will use the following inequalities, for  $x, y > 0$ ,

$$(x+y)^K \geq 2^{K-1}(x^K + y^K) \text{ and } (x+y)^{2HK} \geq 2^{2HK-1}(x^{2HK} + y^{2HK}) \geq \frac{1}{2}(x^{2HK} + y^{2HK}).$$

Concerning the lower bound of (39), we can write

$$\begin{aligned}
A_\varepsilon(u, u) &= \frac{1}{2^K} \left[ 2^K(u+\varepsilon)^{2HK} + 2^K u^{2HK} + 2((u+\varepsilon)^{2H} + u^{2H})^K - 2\varepsilon^{2H} + 2^K u^{2HK} \right] \\
&\geq \frac{1}{2^K} \left[ 2^{K+1}(u+\varepsilon)^{2HK} + 2^{K+1}u^{2HK} - 2\varepsilon^{2HK} \right] \\
&\geq (\text{since } u > \varepsilon) \frac{1}{2^K} (2^{K+1}u^{2HK} + (2^{K+1+2HK-2})\varepsilon^{2HK}) \\
&\geq cst.u^{2HK}
\end{aligned}$$

and for the lower bound in (39), it holds

$$\begin{aligned}
A_\varepsilon(u, u) &\leq 4(u + \varepsilon)^{2H} + u^{2H})^K - 2\varepsilon^{2HK} \\
&\leq 2((2u)^{2H} + u^{2H})^K + 2((u + \varepsilon)^{2H} + u^{2H} - \varepsilon^{2H})^K \\
&\leq \text{cst.} u^{2HK}.
\end{aligned}$$

Regarding the inequality (40), we note that

$$\frac{A_\varepsilon(u, u)}{u^{2HK}} = \varphi\left(\frac{\varepsilon}{u}\right), \quad \frac{A_\varepsilon(v, v)}{(v - u)^{2HK}} = \frac{\varphi\left(\frac{u}{\varepsilon} \frac{u}{v}\right)}{\left(1 - \frac{u}{v}\right)^{2HK}}$$

where

$$\varphi(x) = (1 + x)^{2HK} + 1 + 2^{1-K} [(1 + (1 + x)^{2H})^K - x^{2HK}]$$

and

$$\frac{A_\varepsilon(u, v)^2}{u^{2HK}(v - u)^{2HK}} = h\left(\frac{\varepsilon}{u}, \frac{v}{u}\right)$$

with

$$\begin{aligned}
h(x, y) &= \frac{y^{2HK}}{2^K(y - 1)^{2HK}} \left[ \left( \left( \frac{1}{y} + 2\frac{x}{y} + \frac{x^2}{y} \right)^H + \left( y + 2x + \frac{x^2}{y} \right)^H \right)^K - \left( \frac{1}{y} - 2 + y \right)^{HK} \right] \\
&+ \left[ \left( \frac{1}{y^H} + \left( y + 2x + \frac{x^2}{y} \right)^H \right)^K - \left( \frac{1}{y} + y + \frac{x^2}{y} - 2 - 2\frac{x}{y} + 2x \right)^{HK} \right] \\
&+ \left[ \left( \left( \frac{1}{y} + 2\frac{x}{y} + \frac{x^2}{y} \right)^H + xy^H \right)^K - \left( \frac{1}{y} + y + \frac{x^2}{y} - 2 + 2\frac{x}{y} - 2x \right)^{HK} \right] \\
&+ \left[ \left( \frac{1}{y^H} + y^H \right)^K - \left( \frac{1}{y} - 2 + y \right)^{HK} \right].
\end{aligned}$$

As a consequence we can write,

$$\frac{D_\varepsilon(v, v)}{u^{2HK}(v - u)^{2HK}} = \tau\left(\frac{\varepsilon}{u}, \frac{v}{u}\right)$$

where the function  $\tau : [0, 1] \times ]1, \infty[$  is given by

$$\tau(x, y) = \frac{y^{2HK} \varphi(x) \varphi\left(\frac{x}{y}\right)}{(y - 1)^{2HK}} - h(x, y).$$

Next, we note the following point: if  $v > u$ ,  $\tau(0, \frac{v}{u})$  is strictly positive since  $D_0(u, v)$  is the determinant of the covariance matrix of  $(B_u, B_v)$  and as in [26] it suffices to check that

$$\forall \varepsilon > 0, \forall (u, v) \in D_\varepsilon, \left| \varphi\left(\frac{\varepsilon}{u}, \frac{v}{u}\right) - \varphi\left(0, \frac{v}{u}\right) \right| \leq c \left(\frac{\varepsilon}{u}\right)^\alpha \quad \text{with } \alpha > 0$$

and it can be checked that every term appearing in the expression of  $T$  satisfies this property.

The proof of the fact that  $J_1$  tends to zero as  $\varepsilon \rightarrow 0$  is a straightforward generalization of Lemma 5.4 in [26]. ■

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