EUCLIDEAN GIBBS MEASURES OF QUANTUM ANHARMONIC CRYSTALS

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Abstract. A lattice system of interacting temperature loops, which is used in the Euclidean approach to describe equilibrium thermodynamic properties of an infinite system of interacting quantum particles performing ν-dimensional anharmonic oscillations (quantum anharmonic crystal), is considered. For this system, it is proven that: (a) the set of tempered Gibbs measures $\mathcal{G}^{\rm t}$ is non-void and weakly compact; (b) every $\mu \in \mathcal{G}^{\rm t}$ obeys an exponential integrability estimate, the same for the whole set $\mathcal{G}^{\rm t}$; (c) every $\mu \in \mathcal{G}^{\rm t}$ has a Lebowitz-Presutti type support; (d) $\mathcal{G}^{\rm t}$ is a singleton at high temperatures. In the case of attractive interaction and $\nu = 1$ we prove that at low temperatures the system undergoes a phase transition, i.e., $|\mathcal{G}^{\rm t}| > 1$. The uniqueness of Gibbs measures due to strong quantum effects (strong diffusivity) and at a nonzero external field are also proven in this case. Thereby, a complete description of the properties of the set $\mathcal{G}^{\rm t}$ has been done, which essentially extends and refines the results obtained so far for models of this type.

Contents

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1. Introduction

Concepts and methods of probability theory constitute an important part of the mathematical background of statistical mechanics (quantum and classical). A special connection between quantum statistical mechanics and probability theory arises from the fact that the imaginary time evolution of quantum systems, responsible for its thermodynamics, can be described in terms of stochastic processes[1]. The research presented in this article is intended to contribute into the mathematical theory of quantum anharmonic crystals based on the properties of the corresponding stochastic processes. Quantum anharmonic crystals are the models describing structural phase transitions in ionic crystals triggered by ordering of interacting localized light quantum particles. Each such a particle moves in a crystalline field created by steady heavy ions, which has at least two minima. These minima correspond to different equilibrium phases in which the system may exist at the same values of the parameters determining its macroscopic properties, e.g., temperature. A mathematical model of the mentioned particle is the quantum anharmonic oscillator with multiple minima of the potential energy. The quantum anharmonic crystal itself is a countable set of interacting quantum anharmonic oscillators labeled by the elements of a crystalline lattice $\mathbb{L}$.

A complete description of the equilibrium thermodynamic properties of infinite particle systems may be made by constructing their Gibbs states at a given temperature and given values of the model parameters. Then the phase transition occurs if the set of such states consists of more than one element. Gibbs states of quantum models are defined as positive normalized functionals on proper algebras of observables satisfying the Kubo-Martin-Schwinger (KMS) conditions, see [20]. But for the quantum anharmonic crystal, this way is impossible since the KMS conditions cannot be formulated for the model as a whole, see the corresponding discussion in [4]. An alternative way of constructing the Gibbs states of models of this type was initiated in [1, 34, 38]. It uses the fact that the Schrödinger operators, H's, of finite systems of quantum particles generate stochastic processes. Then the description of the Gibbs states of such systems, based on the properties of the semigroups $\exp(-\tau H)$, $\tau > 0$, is translated into 'a probabilistic language', which opens a possibility to apply here various techniques from this domain. In this language the model we consider is a spin lattice over $\mathbb{L} = \mathbb{Z}^d$, $d \in \mathbb{N}$, with the single-spin spaces equal to the space of ν-dimensional continuous loops $\nu \in \mathbb{N}$, indexed by $[0, \beta]$, $\beta^{-1} = T > 0$ being absolute temperature. Each single-spin space is equipped with

[1] See Introduction in [72].

the path measure of the β-periodic Ornstein-Uhlenbeck process, corresponding to the harmonic part of the Schrödinger operator of a single oscillator, multiplied by a density determined by the anharmonic term via the Feynman-Kac formula. Finite subsystems are described by conditional probability measures, which through the Dobrushin-Lanford-Ruelle (DLR) formalism [32, 65] determine the set of *Euclidean Gibbs measures* $\mathcal{G}^{\rm t}$. This approach in the theory of Gibbs states of quantum models is called Euclidean due to its conceptual analogy with the Euclidean quantum field theory, see [33, 71]. An extended presentation of this approach may be found in [4, 8, 10, 11]. Among the achievements of the Euclidean approach one can mention the settlement in [2, 3, 5, 6, 45] of a long standing problem of understanding the role of quantum effects in structural phase transitions in quantum anharmonic crystals, first discussed in [69], see also [64, 75, 76].

This article presents a detailed mathematical theory of the quantum anharmonic crystal. We consider a very general version of this model and give the most complete description of the set of its Euclidean Gibbs states. These results are obtained by means of a technique based on probabilistic methods, which we develop in the article. This our technique may be applied to other models of this type describing interesting physical phenomena, such as strong electron-electron correlations caused by the interaction of electrons with vibrating ions [28, 29] or effects related with the interaction of vibrating quantum particles with a radiation (photon) field [35, 37, 60]. We believe it can find applications also outside of mathematical physics.

In Section 2 we introduce the object of our study – a system of interacting quantum oscillators, finite subsystems of which are described by their Schrödinger operators. We do not suppose that the system is translation invariant or that the inter-particle interaction has finite range. In this section we also introduce the spaces of temperature loops and the probability measures on these spaces describing the stochastic processes generated by the corresponding Schrödinger operators. By means of such measures we define the set of tempered Gibbs measures $\mathcal{G}^{\rm t}$ as the set of probability measures on the loop spaces which obey the DLR condition. A preliminary study of the elements of $\mathcal{G}^{\rm t}$ is also performed in Section 2. At the very end of this section we give a brief overview of the basic elements of the Euclidean approach in which infinite systems of quantum particles are described as systems of interacting diffusion processes. In Section 3 we formulate the results of the paper, which fall into two groups. The first one consists of Theorems 3.1 – 3.4. They describe the general case where the inter-particle and self-interaction potentials obey natural stability conditions only. Theorem 3.1 states that the set $\mathcal{G}^{\rm t}$ is non-void and weakly compact. Theorem 3.2 gives an exponential integrability estimate for the elements of $\mathcal{G}^{\rm t}$. According to Theorem 3.3 the support of the elements of $\mathcal{G}^{\rm t}$ is of the same type as the one obtained in [17, 54] for systems of classical unbounded spins. We call it a Lebowitz-Presutti support. Theorem 3.4 gives a sufficient condition for $\mathcal{G}^{\rm t}$ to be a singleton formulated as an upper bound for the inverse temperature (high temperature uniqueness). The second group of theorems describe the case where the inter-particle interaction is attractive and $\nu = 1$, i.e., the oscillations of quantum particles and hence the temperature loops are one-dimensional. In this case one can set an order on $\mathcal{G}^{\rm t}$ – the FKG order, see [65]. Theorem 3.5 states that $\mathcal{G}^{\rm t}$ has unique maximal and minimal elements with respect to this order and describes a number of properties of these elements. Theorem 3.6 gives a sufficient condition for the existence of phase transitions, i.e., for $|\mathcal{G}^{\rm t}| > 1$. Theorem 3.7 describes the so called quantum stabilization, which may also be interpreted as a stabilization of the system of interacting diffusions by large diffusion intensity. The stabilization means that $|\mathcal{G}^{\rm t}| = 1$ at all temperatures; it holds under a condition, which involves the inter-particle interaction intensity and the spectral parameters of the Schrödinger operator of a single anharmonic oscillator. On a certain example we compare the conditions which guarantee the phase transition to occur with those of quantum stabilization. Finally, Theorem 3.8 states that $\mathcal{G}^{\rm t}$ is a singleton at nonzero values of the external field. It holds under certain additional conditions imposed on the model. Extended comments on these theorems, which include comparison with the results known for similar models, conclude Section 3. The main technical resources of proving the above theorems are developed in Section 4. They are based on moment estimates for conditional local Gibbs measures (these measures define the Euclidean Gibbs measures through the DLR equation). The proof of Theorems 3.1 – 3.4 is performed in Section 5. Theorems 3.5, 3.6, and 3.7 are proven in Section 6. The proof is mainly based on correlation inequalities, taken from [4] and presented at the beginning of the section. By means of these inequalities we compare our model with two reference models. One of them is translation invariant and with nearest-neighbor interactions. It is more stable than our model. We prove that this reference model undergoes a phase transition, which implies the same for the model considered and hence proves Theorem 3.6. Another reference model is less stable than the one we consider but is more regular in a certain sense. We prove that this reference model is stabilized by strong quantum effects, which implies the stabilization for our model and hence proves Theorem 3.7. Section 7 is devoted to the proof of Theorem 3.8. Here we employ analytic methods based on the Lee-Yang property of the version of our model studied in this section.

2. DLR Formalism for Euclidean Gibbs Measures

The infinite system we consider is defined on the lattice $\mathbb{L} = \mathbb{Z}^d$, $d \in \mathbb{N}$. Subsets of $\mathbb{L}$ are denoted by Λ. As usual, $|\Lambda|$ stands for the cardinality of Λ and Λ^c – for its complement $\mathbb{L} \setminus \Lambda$. We write $\Lambda \Subset \mathbb{L}$ if Λ is non-void and finite. By $\mathcal{L}$ we denote a cofinal (ordered by inclusion and exhausting the lattice) sequence of subsets of $\mathbb{L}$. Limits taken along such $\mathcal{L}$ are denoted by $\lim_{\mathcal{L}}$. We write $\lim_{\Lambda \nearrow \mathbb{L}}$ if the limit is taken along an unspecified sequence of this type. If we say that something holds for all ℓ, we mean that it holds for all $\ell \in \mathbb{L}$; expressions like $\sum_\ell$ mean $\sum_{\ell \in \mathbb{L}}$. By $(\cdot, \cdot)$ and $|\cdot|$, we denote the scalar product and norm in all Euclidean spaces like $\mathbb{R}^\nu$, $\mathbb{R}^d$, ect; $\mathbb{N}_0$ will stand for the set of nonnegative integers.

2.1. **Loop spaces.** Temperature loops are continuous functions defined on the interval $[0, \beta]$, taking equal values at the endpoints. Here $\beta^{-1} = T > 0$ is absolute temperature. One can consider the loops as functions on the circle $S_\beta \cong [0, \beta]$ being a compact Riemannian manifold with Lebesgue measure $\mathrm{d}\tau$ and distance

$$|\tau - \tau'|_\beta \stackrel{\rm def}{=} \min\{|\tau - \tau'| \ ; \ \beta - |\tau - \tau'|\}, \quad \tau, \tau' \in S_\beta. \tag{2.1}$$

As single-spin spaces at a given ℓ, we use the standard Banach spaces

$$C_\beta \stackrel{\rm def}{=} C(S_\beta \to \mathbb{R}^\nu), \qquad C_\beta^\sigma \stackrel{\rm def}{=} C^\sigma(S_\beta \to \mathbb{R}^\nu), \quad \sigma \in (0, 1),$$

of all continuous and Hölder-continuous functions $\omega_\ell : S_\beta \to \mathbb{R}^\nu$ respectively, which are equipped with the supremum norm $|\omega_\ell|_{C_\beta}$ and with the Hölder norm

$$|\omega_\ell|_{C_\beta^\sigma} = |\omega_\ell|_{C_\beta} + \sup_{\tau,\tau' \in S_\beta,\ \tau \neq \tau'} \frac{|\omega_\ell(\tau) - \omega_\ell(\tau')|}{|\tau - \tau'|_\beta^\sigma}. \tag{2.2}$$

Along with them we also use the real Hilbert space $L^2_\beta = L^2(S_\beta \to \mathbb{R}^\nu, \mathrm{d}\tau)$, the inner product and norm of which are denoted by $(\cdot,\cdot)_{L^2_\beta}$ and $|\cdot|_{L^2_\beta}$ respectively. By $\mathcal{B}(C_\beta)$, $\mathcal{B}(L^2_\beta)$ we denote the corresponding Borel σ-algebras. In a standard way one defines dense continuous embeddings $C_\beta^\sigma \hookrightarrow C_\beta \hookrightarrow L^2_\beta$, that by the Kuratowski theorem, page 21 of [62], yields

$$C_\beta \in \mathcal{B}(L^2_\beta) \quad \text{and} \quad \mathcal{B}(C_\beta) = \mathcal{B}(L^2_\beta) \cap C_\beta. \tag{2.3}$$

The space of Hölder-continuous functions C_β^σ is not separable, however, as a subset of C_β or L^2_β, it is measurable (page 278 of [66]). Given $\Lambda \subseteq \mathbb{L}$, we set

$$\Omega_\Lambda = \{\omega_\Lambda = (\omega_\ell)_{\ell\in\Lambda} \mid \omega_\ell \in C_\beta\}, \quad \Omega = \Omega_{\mathbb{L}} = \{\omega = (\omega_\ell)_{\ell\in\mathbb{L}} \mid \omega_\ell \in C_\beta\}. \tag{2.4}$$

These loop spaces are equipped with the product topology and with the Borel σ-algebras $\mathcal{B}(\Omega_\Lambda)$. Thereby, each Ω_Λ is a Polish space; its elements are called configurations in Λ. In particular, Ω is the configuration space for the whole system. For $\Lambda \subset \Lambda'$, one can decompose $\omega_{\Lambda'} = \omega_\Lambda \times \omega_{\Lambda'\setminus\Lambda}$, which defines an embedding $\Omega_\Lambda \hookrightarrow \Omega_{\Lambda'}$ by identifying $\omega_\Lambda \in \Omega_\Lambda$ with $\omega_\Lambda \times 0_{\Lambda'\setminus\Lambda} \in \Omega_{\Lambda'}$. By $\mathcal{P}(\Omega_\Lambda)$ and $\mathcal{P}(\Omega)$ we denote the sets of all probability measures on $(\Omega_\Lambda, \mathcal{B}(\Omega_\Lambda))$ and $(\Omega, \mathcal{B}(\Omega))$ respectively.

2.2. **Quantum oscillators and stochastic processes.** A ν-dimensional quantum harmonic oscillator of mass $m > 0$ and rigidity $a > 0$ is described by its Schrödinger operator

$$H_\ell^{\mathrm{har}} = -\frac{1}{2m}\sum_{j=1}^{\nu}\left(\frac{\partial}{\partial x_\ell^{(j)}}\right)^2 + \frac{a}{2}|x_\ell|^2, \tag{2.5}$$

acting in the complex Hilbert space $L^2(\mathbb{R}^\nu)$. The operator semigroup $\exp(-\tau H_\ell^{\mathrm{har}})$, $\tau \in [0,\beta]$, defines a Gaussian β-periodic Markov process, called periodic Ornstein-Uhlenbeck velocity process, see [42]. In quantum statistical mechanics it first appeared in R. Høegh-Krohn's paper [38]. The canonical realization of this process on $(C_\beta, \mathcal{B}(C_\beta))$ is described by the path measure which one introduces as follows. In L^2_β we define the following self-adjoint (Laplace-Beltrami type) operator

$$A = \left(-m\frac{\mathrm{d}^2}{\mathrm{d}\tau^2} + a\right) \otimes \mathbf{I}, \tag{2.6}$$

where $\mathbf{I}$ is the identity operator in $\mathbb{R}^\nu$ and m, a are as in (2.5). Its spectrum consisting of the eigenvalues

$$\lambda_k = m(2\pi k/\beta)^2 + a, \quad k \in \mathbb{Z}. \tag{2.7}$$

As the inverse A^{-1} is of trace class, the Fourier transform

$$\int_{L^2_\beta} \exp[\imath(\phi, \upsilon)_{L^2_\beta}]\chi(\mathrm{d}\upsilon) = \exp\left\{-\frac{1}{2}(A^{-1}\phi,\phi)_{L^2_\beta}\right\}, \quad \phi \in L^2_\beta. \tag{2.8}$$

defines a Gaussian measure χ on $(L^2_\beta, \mathcal{B}(L^2_\beta))$. Employing the eigenvalues (2.7) one can show (by Kolmogorov's lemma, page 43 of [72]) that

$$\chi(C^\sigma_\beta) = 1, \quad \text{for all } \sigma \in (0, 1/2). \tag{2.9}$$

Then $\chi(C_\beta) = 1$ and by (2.3) we can redefine χ as a probability measure on $(C_\beta, \mathcal{B}(C_\beta))$. An account of the properties of χ may be found in [4]. By Fernique's theorem (Theorem 1.3.24 in [22]) the support property (2.9) yields the following

Proposition 2.1. *For every $\sigma \in (0, 1/2)$, there exists $\lambda_\sigma > 0$ such that*

$$\int_{L^2_\beta} \exp\left(\lambda_\sigma |\upsilon|^2_{C^\sigma_\beta}\right) \chi(\mathrm{d}\upsilon) < \infty. \tag{2.10}$$

Given $\Lambda \Subset \mathbb{L}$, the system of interacting anharmonic oscillators located in Λ is described by the Schrödinger operator

$$\begin{aligned} H_\Lambda &= \sum_{\ell\in\Lambda} \left[H^{\mathrm{har}}_\ell + V_\ell(x_\ell)\right] - \frac{1}{2} \sum_{\ell,\ell'\in\Lambda} J_{\ell\ell'}(x_l, x_{\ell'}) \\ &= -\frac{1}{2m} \sum_{\ell\in\Lambda} \sum_{j=1}^{\nu} \left(\frac{\partial}{\partial x^{(j)}_\ell}\right)^2 + W_\Lambda(x_\Lambda), \quad x_\Lambda = (x_\ell)_{\ell\in\Lambda}. \end{aligned} \tag{2.11}$$

In the latter formula the first term is the kinetic energy; the potential energy is

$$W_\Lambda(x_\Lambda) = -\frac{1}{2} \sum_{\ell,\ell'\in\Lambda} J_{\ell\ell'}(x_\ell, x_{\ell'}) + \sum_{\ell\in\Lambda} \left[(a/2)|x_\ell|^2 + V_\ell(x_\ell)\right]. \tag{2.12}$$

The self-interaction potentials V_ℓ and the dynamical matrix $(J_{\ell\ell'})_{\mathbb{L}\times\mathbb{L}}$ with the entries

$$J_{\ell\ell} = 0, \quad J_{\ell\ell'} = J_{\ell'\ell} \in \mathbb{R}, \quad \ell, \ell' \in \mathbb{L}, \tag{2.13}$$

are subject to the following

Assumption 2.2. *All $V_\ell : \mathbb{R}^\nu \to \mathbb{R}$ are continuous and such that $V_\ell(0) = 0$; there exist $r > 1$, $A_V > 0$, $B_V \in \mathbb{R}$, and a continuous function $V : \mathbb{R}^\nu \to \mathbb{R}$, $V(0) = 0$, such that for all ℓ and $x \in \mathbb{R}^\nu$,*

$$A_V |x|^{2r} + B_V \le V_\ell(x) \le V(x). \tag{2.14}$$

We also assume that

$$\hat{J}_0 \stackrel{\mathrm{def}}{=} \sup_\ell \sum_{\ell'} |J_{\ell\ell'}| < \infty. \tag{2.15}$$

The lower bound in (2.14) is responsible for confining each particle in the vicinity of its equilibrium position. The upper bound is to guarantee that the oscillations of the particles located far from the origin are not suppressed. An example of V_ℓ to bear in mind is the polynomial

$$V_\ell(x) = \sum_{s=1}^{r} b^{(s)}_\ell |x|^{2s} - (h, x), \quad b^{(s)}_\ell \in \mathbb{R}, \quad r \ge 2, \tag{2.16}$$

in which $h \in \mathbb{R}^\nu$ is an external field and the coefficients $b^{(s)}_\ell$ vary in certain intervals, such that both estimates (2.14) hold.

Under Assumption 2.2 H_Λ is a self-adjoint below bounded operator in $L^2(\mathbb{R}^{\nu|\Lambda|})$ having discrete spectrum. It generates a positivity preserving semigroup, such that

$$\mathrm{trace}[\exp(-\tau H_\Lambda)] < \infty, \quad \text{for all } \tau > 0. \tag{2.17}$$

Thus, for every $\beta > 0$, one can define the associated stationary β-periodic Markov process possessing a canonical realization on $(\Omega_\Lambda, \mathcal{B}(\Omega_\Lambda))$. It is described by the measure $\mu_\Lambda \in \mathcal{P}(\Omega_\Lambda)$ which marginal distributions are given by the integral kernels of the operators $\exp(-\tau H_\Lambda)$, $\tau \in [0, \beta]$. This means that

$$\begin{aligned} & \mathrm{trace}[F_1 e^{-(\tau_2-\tau_1)H_\Lambda} F_2 e^{-(\tau_3-\tau_2)H_\Lambda} \cdots F_n e^{-(\tau_{n+1}-\tau_n)H_\Lambda}]/\mathrm{trace}[e^{-\beta H_\Lambda}] \\ & \quad = \int_{\Omega_\Lambda} F_1(\omega_\Lambda(\tau_1) \cdots F_n(\omega_\Lambda(\tau_n))\mu_\Lambda(\mathrm{d}\omega_\Lambda), \end{aligned} \tag{2.18}$$

for all $F_1, \ldots, F_n \in L^\infty(\mathbb{R}^{\nu|\Lambda|})$, $n \in \mathbb{N}$ and $\tau_1, \ldots, \tau_n \in S_\beta$ such that $\tau_1 \leq \cdots \leq \tau_n \leq \beta$, $\tau_{n+1} = \tau_1 + \beta$. And vice verse, the representation (2.18) uniquely, up to equivalence, defines H_Λ (see [41]). By means of the Feynman-Kac formula the measure μ_Λ is obtained as a Gibbs modification

$$\mu_\Lambda(\mathrm{d}\omega_\Lambda) = \exp\left\{-I_\Lambda(\omega_\Lambda)\right\} \chi_\Lambda(\mathrm{d}\omega_\Lambda)/Z_\Lambda, \tag{2.19}$$

of the 'free measure'

$$\chi_\Lambda(\mathrm{d}\omega_\Lambda) = \prod_{\ell\in\Lambda} \chi(\mathrm{d}\omega_\ell). \tag{2.20}$$

Here

$$I_\Lambda(\omega_\Lambda) = -\frac{1}{2} \sum_{\ell,\ell'\in\Lambda} J_{\ell\ell'}(\omega_\ell, \omega_{\ell'})_{L^2_\beta} + \sum_{\ell\in\Lambda} \int_0^\beta V_\ell(\omega_\ell(\tau))\mathrm{d}\tau \tag{2.21}$$

is the energy functional describing the system of interacting loops ω_ℓ, $\ell \in \Lambda$, whereas

$$Z_\Lambda = \int_{\Omega_\Lambda} \exp\left\{-I_\Lambda(\omega_\Lambda)\right\} \chi_\Lambda(\mathrm{d}\omega_\Lambda), \tag{2.22}$$

is the partition function. The measure μ_Λ will be called a local Gibbs measure, where *local* means corresponding to a $\Lambda \Subset \mathbb{L}$. Further details on the relations between stochastic processes and systems of quantum oscillators are given in subsection 2.5.

Thereby, our system of interacting anharmonic oscillators is described by the Schrödinger operators (2.11), defined for all $\Lambda \Subset \mathbb{L}$, or equivalently by the path measures (2.19). They involve the parameters of the harmonic oscillator m, a, the self-interaction potentials V_ℓ, and the dynamical matrix $(J_{\ell\ell'})_{\mathbb{L}\times\mathbb{L}}$ subject to Assumption 2.2. We refer to these objects, both the Schrödinger operators H_Λ and the measures μ_Λ, as to the model we consider. Its particular cases are indicated by the following

Definition 2.3. *The model is ferromagnetic if $J_{\ell\ell'} \geq 0$ for all ℓ, ℓ'. The interaction has finite range if there exists $R > 0$ such that $J_{\ell\ell'} = 0$ whenever $|\ell - \ell'|$ exceeds this R. The model is translation invariant if $V_\ell = V$ for all ℓ, and the matrix $(J_{\ell\ell'})_{\mathbb{L}\times\mathbb{L}}$ is invariant under translations of the lattice.*

If $V_\ell \equiv 0$ for all ℓ, the model is known as a quantum harmonic crystal. It is stable if $\hat{J}_0 < a$; in this case the set of Gibbs measures is always a singleton. Unstable harmonic crystals, i.e., the ones with $\hat{J}_0 > a$, have no Gibbs states at all, see [44].

2.3. **Tempered configurations.** Above we have translated the description of finite systems of quantum oscillators into the language of probability measures on the loop spaces Ω_Λ, $\Lambda \Subset \mathbb{L}$. In order to construct the Gibbs measures corresponding to the whole infinite system we use the Dobrushin-Lanford-Ruelle (DLR) approach, based on local conditional distributions. This approach is standard for classical (non-quantum) statistical mechanics, see the books [32, 65]. However, in our case the single-spin spaces are infinite-dimensional and hence their topological properties are much richer. This fact manifests itself in a more sophisticated structure of the DLR technique we develop here.

To go further we have to define functions on loop spaces Ω_Λ with infinite Λ, including the space Ω itself. Among others, we will need the energy functional $I_\Lambda(\cdot|\xi)$ describing the interaction of the loops inside $\Lambda \Subset \mathbb{L}$ between themselves and with a configuration $\xi \in \Omega$ fixed outside of Λ. In accordance with (2.11) it is

$$I_\Lambda(\omega|\xi) = I_\Lambda(\omega_\Lambda) - \sum_{\ell\in\Lambda,\ \ell'\in\Lambda^c} J_{\ell\ell'}(\omega_\ell, \xi_{\ell'})_{L^2_\beta}, \quad \omega \in \Omega, \tag{2.23}$$

where I_Λ is defined by (2.21). Recall that $\omega = \omega_\Lambda \times \omega_{\Lambda^c}$; hence,

$$I_\Lambda(\omega|\xi) = I_\Lambda(\omega_\Lambda \times 0_{\Lambda^c}|0_\Lambda \times \xi_{\Lambda^c}). \tag{2.24}$$

Clearly, the second term in (2.23) makes sense for all $\xi \in \Omega$ only if the interaction has finite range. Otherwise, one has to restrict ξ to a subset of Ω, naturally defined by the condition

$$\forall \ell \in \mathbb{L}: \qquad \sum_{\ell'} |J_{\ell\ell'}| \cdot |(\omega_\ell, \xi_{\ell'})_{L^2_\beta}| < \infty, \tag{2.25}$$

that can be rewritten in terms of growth restrictions on $\{|\xi_\ell|_{L^2_\beta}\}_{\ell\in\mathbb{L}}$, determined by the decay of $J_{\ell\ell'}$ (c.f., (2.15)). Configurations obeying such restrictions are called tempered. In one or another way tempered configurations always appear in the theory of system of unbounded spins, see [17, 21, 54, 61]. To define them we use weights.

Definition 2.4. *Weights are the maps* $w_\alpha : \mathbb{L} \times \mathbb{L} \to (0, +\infty)$*, indexed by*

$$\alpha \in \mathcal{I}{=}(\underline{\alpha}, \overline{\alpha}), \quad 0 \leq \underline{\alpha} < \overline{\alpha} \leq +\infty, \tag{2.26}$$

which satisfy the conditions:

(a) for any $\alpha \in \mathcal{I}$ *and* ℓ, $w_\alpha(\ell, \ell) = 1$*; for any* $\alpha \in \mathcal{I}$ *and* ℓ_1, ℓ_2, ℓ_3*,*

$$w_\alpha(\ell_1, \ell_2) \cdot w_\alpha(\ell_2, \ell_3) \leq w_\alpha(\ell_1, \ell_3) \quad \textit{(triangle inequality)}, \tag{2.27}$$

(b) for any $\alpha, \alpha' \in \mathcal{I}$*, such that* $\alpha < \alpha'$*, and arbitrary* ℓ, ℓ'*,*

$$w_{\alpha'}(\ell, \ell') \leq w_\alpha(\ell, \ell'), \qquad \lim_{|\ell-\ell'|\to+\infty} w_{\alpha'}(\ell, \ell')/w_\alpha(\ell, \ell') = 0. \tag{2.28}$$

The concrete choice of w_α depends on the decay of $J_{\ell\ell'}$. Here we distinguish two typical cases. In the first one

$$\sup_\ell \sum_{\ell'} |J_{\ell\ell'}| \cdot \exp\left(\alpha|\ell - \ell'|\right) < \infty, \quad \text{for a certain } \alpha > 0. \tag{2.29}$$

Then by $\overline{\alpha}$ we denote the supremum of α obeying (2.29) and set

$$w_\alpha(\ell, \ell') = \exp\left(-\alpha|\ell - \ell'|\right), \quad \alpha \in \mathcal{I} = (0, \overline{\alpha}). \tag{2.30}$$

In the second case (2.29) does not hold for any positive α. Instead, we suppose that

$$\sup_{\ell} \sum_{\ell'} |J_{\ell\ell'}| \cdot (1 + |\ell - \ell'|)^{\alpha d} < \infty, \tag{2.31}$$

for a certain $\alpha > 1$. Then $\overline{\alpha}$ is set to be the supremum of α obeying (2.31) and

$$w_\alpha(\ell, \ell') = (1 + \varepsilon|\ell - \ell'|)^{-\alpha d}, \quad \mathcal{I} = (1, \overline{\alpha}), \tag{2.32}$$

where the parameter $\varepsilon > 0$ will be chosen later. If $|J_{\ell\ell'}| \leq J(1 + |\ell - \ell'|)^{-d-\gamma}$, $\gamma > 0$, then $\overline{\alpha} = \gamma/d$, which implies $\gamma > d$. Thus, our construction does not cover an interesting case of $\gamma \in (0, d]$, which will be done in a separate work. In both cases we have the following properties of the weights.

Proposition 2.5. *For all $\alpha \in \mathcal{I}$,*

$$\sup_{\ell} \sum_{\ell'} \log(1 + |\ell - \ell'|) \cdot w_\alpha(\ell, \ell') < \infty; \tag{2.33}$$

$$\hat{J}_\alpha \stackrel{\mathrm{def}}{=} \sup_{\ell} \sum_{\ell'} |J_{\ell\ell'}| \cdot [w_\alpha(\ell, \ell')]^{-1} < \infty. \tag{2.34}$$

Given $q = (q_\ell)_{\ell \in \mathbb{L}} \in \mathbb{R}^{\mathbb{L}}$ and $\alpha \in \mathcal{I}$, we set

$$|q|_{l^1(w_\alpha)} = \sum_{\ell} |q_\ell| w_\alpha(0, \ell), \qquad |q|_{l^\infty(w_\alpha)} = \sup_{\ell} \left\{ |q_\ell| w_\alpha(0, \ell) \right\},$$

and introduce the Banach spaces

$$l^p(w_\alpha) = \left\{ q \in \mathbb{R}^{\mathbb{L}} \ \middle| \ |q|_{l^p(w_\alpha)} < \infty \right\}, \quad p = 1, +\infty. \tag{2.35}$$

Remark 2.6. By (2.28) the embedding $l^1(w_\alpha) \hookrightarrow l^1(w_{\alpha'})$ is compact whenever $\alpha < \alpha'$. By (2.34), for every $\alpha \in \mathcal{I}$, the operator $q \mapsto Jq$, defined as $(Jq)_\ell = \sum_{\ell'} J_{\ell\ell'} q_{\ell'}$, is bounded in both $l^p(w_\alpha)$, $p = 1, +\infty$. Its norm does not exceed $\hat{J}_\alpha$.

For $\alpha \in \mathcal{I}$, let us consider

$$\Omega_\alpha = \left\{ \omega \in \Omega \ \middle| \ \|\omega\|_\alpha \stackrel{\mathrm{def}}{=} \left[\sum_{\ell} |\omega_\ell|^2_{L^2_\beta} w_\alpha(0, \ell) \right]^{1/2} < \infty \right\}, \tag{2.36}$$

and endow this set with the metric

$$\rho_\alpha(\omega, \omega') = \|\omega - \omega'\|_\alpha + \sum_{\ell} 2^{-|\ell|} \cdot \frac{|\omega_\ell - \omega'_\ell|_{C_\beta}}{1 + |\omega_\ell - \omega'_\ell|_{C_\beta}}, \tag{2.37}$$

which turns Ω_α into a Polish space. Then the set of tempered configurations is defined to be

$$\Omega^{\mathrm{t}} = \bigcap_{\alpha \in \mathcal{I}} \Omega_\alpha. \tag{2.38}$$

Equipped with the projective limit topology Ω^{t} becomes a Polish space as well. For any $\alpha \in \mathcal{I}$, we have continuous dense embeddings $\Omega^{\mathrm{t}} \hookrightarrow \Omega_\alpha \hookrightarrow \Omega$. Then by the Kuratowski theorem it follows that $\Omega_\alpha, \Omega^{\mathrm{t}} \in \mathcal{B}(\Omega)$ and the Borel σ-algebras of all these Polish spaces coincide with the ones induced on them by $\mathcal{B}(\Omega)$. Now we are at a position to complete the definition of the function (2.23).

Lemma 2.7. *For every $\alpha \in \mathcal{I}$ and $\Lambda \Subset \mathbb{L}$, the map $\Omega_\alpha \times \Omega_\alpha \ni (\omega, \xi) \mapsto I_\Lambda(\omega|\xi)$ is continuous. Furthermore, for every ball $B_\alpha(R) = \{\omega \in \Omega_\alpha \mid \rho_\alpha(0, \omega) < R\}$, $R > 0$, it follows that*

$$\inf_{\omega \in \Omega,\ \xi \in B_\alpha(R)} I_\Lambda(\omega|\xi) > -\infty, \qquad \sup_{\omega, \xi \in B_\alpha(R)} |I_\Lambda(\omega|\xi)| < +\infty. \tag{2.39}$$

Proof. As the functions $V_\ell : \mathbb{R}^\nu \to \mathbb{R}$ are continuous, the map $(\omega, \xi) \mapsto I_\Lambda(\omega_\Lambda)$ is continuous and locally bounded. Furthermore,

$$\begin{aligned}
&\left| \sum_{\ell \in \Lambda,\ \ell' \in \Lambda^c} J_{\ell\ell'} (\omega_\ell, \xi_{\ell'})_{L^2_\beta} \right| \leq \sum_{\ell \in \Lambda,\ \ell' \in \Lambda^c} |J_{\ell\ell'}| \cdot |\omega_\ell|_{L^2_\beta} \cdot |\xi_{\ell'}|_{L^2_\beta} \\
&= \sum_{\ell \in \Lambda} |\omega_\ell|_{L^2_\beta} [w_\alpha(0, \ell)]^{-1/2} \\
&\times \sum_{\ell' \in \Lambda^c} |J_{\ell\ell'}| \, [w_\alpha(0, \ell)/w_\alpha(0, \ell')]^{1/2} \cdot |\xi_{\ell'}|_{L^2_\beta} [w_\alpha(0, \ell')]^{1/2} \\
&\leq \sum_{\ell \in \Lambda} |\omega_\ell|_{L^2_\beta} [w_\alpha(0, \ell)]^{-1/2} \sum_{\ell' \in \Lambda^c} |J_{\ell\ell'}| \cdot [w_\alpha(\ell, \ell')]^{-1/2} \cdot |\xi_{\ell'}|_{L^2_\beta} [w_\alpha(0, \ell')]^{1/2} \\
&\leq \hat{J}_\alpha \|\omega\|_\alpha \|\xi\|_\alpha \sum_{\ell \in \Lambda} [w_\alpha(0, \ell)]^{-1}, \qquad (2.40)
\end{aligned}$$

where we used the triangle inequality (2.27). This yields the continuity stated and the upper bound in (2.39). To prove the lower bound we employ the super-quadratic growth of V_ℓ assumed in (2.14). Then for any $\varkappa > 0$ and $\alpha \in \mathcal{I}$, one finds $C > 0$ such that for any $\omega \in \Omega$ and $\xi \in \Omega^{\mathrm{t}}$,

$$\begin{aligned}
I_\Lambda(\omega|\xi) \quad &\geq \quad A_V \beta |\Lambda| + B_V \beta^{1-r} \sum_{\ell \in \Lambda} |\omega_\ell|^{2r}_{L^2_\beta} - \frac{1}{2} \sum_{\ell, \ell' \in \Lambda} J_{\ell\ell'} (\omega_\ell, \omega_{\ell'})_{L^2_\beta} \\
&- \quad \sum_{\ell \in \Lambda,\ \ell' \in \Lambda^c} J_{\ell\ell'} (\omega_\ell, \xi_{\ell'})_{L^2_\beta} \geq C|\Lambda| + \varkappa \sum_{\ell \in \Lambda} |\omega_\ell|^2_{L^2_\beta} - \varkappa^{-1} \|\xi\|^2_\alpha.
\end{aligned} \tag{2.41}$$

To get the latter estimate we used the Minkowski inequality and (2.40). □

Now for $\Lambda \Subset \mathbb{L}$ and $\xi \in \Omega^{\mathrm{t}}$, we introduce the partition function (c.f., (2.24))

$$Z_\Lambda(\xi) = \int_{\Omega_\Lambda} \exp\left[-I_\Lambda(\omega_\Lambda \times 0_{\Lambda^c}|\xi)\right] \chi_\Lambda(\mathrm{d}\omega_\Lambda). \tag{2.42}$$

An immediate corollary of the estimates (2.10) and (2.41) is the following

Proposition 2.8. *For every $\Lambda \Subset \mathbb{L}$, the function $\Omega^{\mathrm{t}} \ni \xi \mapsto Z_\Lambda(\xi) \in (0, +\infty)$ is continuous. Moreover, for any $R > 0$,*

$$\inf_{\xi \in B_\alpha(R)} Z_\Lambda(\xi) > 0, \qquad \sup_{\xi \in B_\alpha(R)} Z_\Lambda(\xi) < \infty. \tag{2.43}$$

2.4. **Local Gibbs specification.** In the DLR formalism Gibbs measures are determined by means of local Gibbs specifications. In our context it is the family $\{\pi_\Lambda\}_{\Lambda \Subset \mathbb{L}}$ of measure kernels

$$\mathcal{B}(\Omega) \times \Omega \ni (B, \xi) \mapsto \pi_\Lambda(B|\xi) \in [0, 1]$$

which we define as follows. For $\xi \in \Omega^{\rm t}$, $\Lambda \Subset \mathbb{L}$, and $B \in \mathcal{B}(\Omega)$, we set

$$\pi_\Lambda(B|\xi) = \frac{1}{Z_\Lambda(\xi)} \int_{\Omega_\Lambda} \exp\left[-I_\Lambda(\omega_\Lambda \times 0_{\Lambda^c}|\xi)\right] \mathbb{I}_B(\omega_\Lambda \times \xi_{\Lambda^c}) \chi_\Lambda(\mathrm{d}\omega_\Lambda), \tag{2.44}$$

where $\mathbb{I}_B$ stands for the indicator of B. We also set

$$\pi_\Lambda(\cdot|\xi) \equiv 0, \quad \text{for} \quad \xi \in \Omega \setminus \Omega^{\rm t}. \tag{2.45}$$

From these definitions one readily derives a consistency property

$$\int_\Omega \pi_\Lambda(B|\omega)\pi_{\Lambda'}(\mathrm{d}\omega|\xi) = \pi_{\Lambda'}(B|\xi), \quad \Lambda \subset \Lambda', \tag{2.46}$$

which holds for all $B \in \mathcal{B}(\Omega)$ and $\xi \in \Omega$. Furthermore, by (2.41) it follows that for any $\xi \in \Omega$, $\sigma \in (0, 1/2)$, and $\varkappa > 0$,

$$\int_\Omega \exp\left\{\sum_{\ell\in\Lambda}\left(\lambda_\sigma|\omega_\ell|^2_{C^\sigma_\beta} + \varkappa|\omega_\ell|^2_{L^2_\beta}\right)\right\}\pi_\Lambda(\mathrm{d}\omega|\xi) < \infty, \tag{2.47}$$

where λ_σ is the same as in Proposition 2.1.

By $C_{\rm b}(\Omega_\alpha)$ (respectively, $C_{\rm b}(\Omega^{\rm t})$) we denote the Banach spaces of all bounded continuous functions $f : \Omega_\alpha \to \mathbb{R}$ (respectively, $f : \Omega^{\rm t} \to \mathbb{R}$) equipped with the supremum norm. For every $\alpha \in \mathcal{I}$, one has a natural embedding $C_{\rm b}(\Omega_\alpha) \hookrightarrow C_{\rm b}(\Omega^{\rm t})$.

Lemma 2.9 (Feller Property)**.** *For every $\alpha \in \mathcal{I}$, $\Lambda \Subset \mathbb{L}$, and any $f \in C_{\rm b}(\Omega_\alpha)$, the function*

$$\begin{aligned} \Omega_\alpha \ni \xi \mapsto \pi_\Lambda(f|\xi) \\ &\stackrel{\rm def}{=} \frac{1}{Z_\Lambda(\xi)} \int_{\Omega_\Lambda} f(\omega_\Lambda \times \xi_{\Lambda^c}) \exp\left[-I_\Lambda(\omega_\Lambda \times 0_{\Lambda^c}|\xi)\right] \chi_\Lambda(\mathrm{d}\omega_\Lambda), \end{aligned} \tag{2.48}$$

belongs to $C_{\rm b}(\Omega_\alpha)$. The linear operator $f \mapsto \pi_\Lambda(f|\cdot)$ is a contraction on $C_{\rm b}(\Omega_\alpha)$.

Proof. By Lemma 2.7 and Proposition 2.8 the integrand

$$G^f_\Lambda(\omega_\Lambda|\xi) \stackrel{\rm def}{=} f(\omega_\Lambda \times \xi_{\Lambda^c}) \exp\left[-I_\Lambda(\omega_\Lambda \times 0_{\Lambda^c}|\xi)\right]/Z_\Lambda(\xi)$$

is continuous in both variables. Moreover, by (2.39) and (2.43) the map

$$\Omega_\alpha \ni \xi \mapsto \sup_{\omega_\Lambda\in\Omega_\Lambda} |G^f_\Lambda(\omega_\Lambda|\xi)|$$

is locally bounded. This allows us to apply Lebesgue's dominated convergence theorem, which yields the continuity stated. Obviously,

$$\sup_{\xi\in\Omega_\alpha} |\pi_\Lambda(f|\xi)| \leq \sup_{\xi\in\Omega_\alpha} |f(\xi)|. \tag{2.49}$$

□

Note that by (2.44), for $\xi \in \Omega^{\rm t}$, $\alpha \in \mathcal{I}$, and $f \in C_{\rm b}(\Omega_\alpha)$,

$$\pi_\Lambda(f|\xi) = \int_\Omega f(\omega)\pi_\Lambda(\mathrm{d}\omega|\xi). \tag{2.50}$$

Definition 2.10. *A measure $\mu \in \mathcal{P}(\Omega)$ is called a tempered Euclidean Gibbs measure at inverse temperature $\beta > 0$ if it satisfies the Dobrushin-Lanford-Ruelle (equilibrium) equation*

$$\int_\Omega \pi_\Lambda(B|\omega)\mu(\mathrm{d}\omega) = \mu(B), \quad \textit{for all} \quad \Lambda \Subset \mathbb{L} \ \ \textit{and} \ \ B \in \mathcal{B}(\Omega). \tag{2.51}$$

By $\mathcal{G}^{\rm t}$ we denote the set of all tempered Euclidean Gibbs measures of our model. So far we do not know if $\mathcal{G}^{\rm t}$ is non-void; if it is, its elements are supported by $\Omega^{\rm t}$. Indeed, by (2.44) and (2.45) $\pi_\Lambda(\Omega \setminus \Omega^{\rm t}|\xi) = 0$ for every $\Lambda \Subset \mathbb{L}$. Then by (2.51),

$$\mu(\Omega \setminus \Omega^{\rm t}) = 0 \quad \Longrightarrow \quad \mu(\Omega^{\rm t}) = 1. \tag{2.52}$$

Furthermore,

$$\mu\left(\left\{\omega \in \Omega^{\rm t} \mid \forall \ell \in \mathbb{L}: \ \omega_\ell \in C^\sigma_\beta\right\}\right) = 1, \tag{2.53}$$

which follows from (2.47).

Given $\alpha \in \mathcal{I}$, by $\mathcal{W}_\alpha$ we denote the usual weak topology on the set of all probability measures $\mathcal{P}(\Omega_\alpha)$, defined by means of bounded continuous functions on Ω_α. By $\mathcal{W}^{\rm t}$ we denote the weak topology on $\mathcal{P}(\Omega^{\rm t})$. With these topologies the sets $\mathcal{P}(\Omega_\alpha)$ and $\mathcal{P}(\Omega^{\rm t})$ become Polish spaces (Theorem 6.5, page 46 of [62]). In general, the convergence of $\{\mu_n\}_{n\in\mathbb{N}} \subset \mathcal{P}(\Omega^{\rm t})$ in every $\mathcal{W}_\alpha$, $\alpha \in \mathcal{I}$, does not yet imply its $\mathcal{W}^{\rm t}$-convergence. In Lemma 4.5 and Corollary 5.1 below we show that the topologies induced on $\mathcal{G}^{\rm t}$ by $\mathcal{W}_\alpha$ and $\mathcal{W}^{\rm t}$ coincide.

Lemma 2.11. *For each $\alpha \in \mathcal{I}$, every $\mathcal{W}_\alpha$-accumulation point $\mu \in \mathcal{P}(\Omega^{\rm t})$ of the family $\{\pi_\Lambda(\cdot|\xi) \mid \Lambda \Subset \mathbb{L}, \ \xi \in \Omega^{\rm t}\}$ is a tempered Euclidean Gibbs measure.*

Proof. For each $\alpha \in \mathcal{I}$, $C_{\rm b}(\Omega_\alpha)$ is a measure defining class for $\mathcal{P}(\Omega^{\rm t})$. Then a measure $\mu \in \mathcal{P}(\Omega^{\rm t})$ solves (2.51) if and only if for any $f \in C_{\rm b}(\Omega_\alpha)$ and all $\Lambda \Subset \mathbb{L}$,

$$\int_{\Omega^{\rm t}} f(\omega)\mu({\rm d}\omega) = \int_{\Omega^{\rm t}} \pi_\Lambda(f|\omega)\mu({\rm d}\omega). \tag{2.54}$$

Let $\{\pi_{\Lambda_k}(\cdot|\xi_k)\}_{k\in\mathbb{N}}$ converge in $\mathcal{W}_\alpha$ to some $\mu \in \mathcal{P}(\Omega^{\rm t})$. For every $\Lambda \Subset \mathbb{L}$, one finds $k_\Lambda \in \mathbb{N}$ such that $\Lambda \subset \Lambda_k$ for all $k > k_\Lambda$. Then by (2.46), one has

$$\int_{\Omega^{\rm t}} f(\omega)\pi_{\Lambda_k}({\rm d}\omega|\xi_k) = \int_{\Omega^{\rm t}} \pi_\Lambda(f|\omega)\pi_{\Lambda_k}({\rm d}\omega|\xi_k).$$

Now by Lemma 2.9, one can pass to the limit $k \to +\infty$ and get (2.54). $\square$

2.5. **Euclidean approach and local quantum Gibbs states.** Here we outline the basic elements of the Euclidean approach in quantum statistical mechanics, its detailed presentation may be found in [4, 8].

For $\Lambda \Subset \mathbb{L}$, the Schrödinger operator H_Λ, defined by (2.11), acts in the physical Hilbert space $\mathcal{H}_\Lambda \stackrel{\rm def}{=} L^2(\mathbb{R}^{\nu|\Lambda|})$. In view of (2.17), one can introduce

$$\mathfrak{C}_\Lambda \ni A \mapsto \varrho_\Lambda(A) \stackrel{\rm def}{=} \frac{{\rm trace}(Ae^{-\beta H_\Lambda})}{{\rm trace}(e^{-\beta H_\Lambda})}, \tag{2.55}$$

which is a positive normalized functional on the algebra $\mathfrak{C}_\Lambda$ of all bounded linear operators (observables) on $\mathcal{H}_\Lambda$. It is the Gibbs state of the system of quantum oscillators located in Λ (local Gibbs state). The mappings

$$\mathfrak{C}_\Lambda \ni A \mapsto \mathfrak{a}^\Lambda_t(A) \stackrel{\rm def}{=} e^{itH_\Lambda}Ae^{-itH_\Lambda}, \ \ t \in \mathbb{R}, \quad t \ - \ {\rm time} \tag{2.56}$$

constitute the group of time automorphisms which describes the dynamics of the system in Λ. The state ϱ_Λ satisfies the KMS (thermal equilibrium) condition relative to the dynamics $\mathfrak{a}^\Lambda_t$, see Definition 1.1 in [41]. Multiplication operators by bounded continuous functions act as

$$(F\psi)(x) = F(x)\cdot\psi(x), \quad \psi \in \mathcal{H}_\Lambda, \quad F \in C_{\rm b}(\mathbb{R}^{\nu|\Lambda|}).$$

One can prove, see [38, 49], that the linear span of the products

$$\mathfrak{a}^{\Lambda}_{t_1}(F_1)\cdots\mathfrak{a}^{\Lambda}_{t_n}(F_n), \tag{2.57}$$

with all possible choices of $n \in \mathbb{N}$, $t_1, \ldots, t_n \in \mathbb{R}$ and $F_1, \ldots, F_n \in C_{\rm b}(\mathbb{R}^{\nu|\Lambda|})$, is σ-weakly dense in $\mathfrak{C}_\Lambda$. As a σ-weakly continuous functional (page 65 of [20]), the state (2.55) is fully determined by its values on (2.57), i.e., by *the Green functions*

$$G^{\Lambda}_{F_1,\ldots,F_n}(t_1,\ldots,t_n) \stackrel{\rm def}{=} \varrho_\Lambda\left[\mathfrak{a}^{\Lambda}_{t_1}(F_1)\cdots\mathfrak{a}^{\Lambda}_{t_n}(F_n)\right]. \tag{2.58}$$

As was shown in [1, 4, 41], the Green functions can be considered as restrictions of functions $G^{\Lambda}_{F_1,\ldots,F_n}(z_1,\ldots,z_n)$ analytic in the domain

$$\mathcal{D}^n_\beta = \{(z_1,\ldots z_n)\in\mathbb{C}^n \mid 0<\Im(z_1)<\Im(z_2)<\cdots<\Im(z_n)<\beta\}, \tag{2.59}$$

and continuous on its closure $\bar{\mathcal{D}}^n_\beta \subset \mathbb{C}^n$. The 'imaginary time' subset

$$\{(z_1,\ldots,z_n)\in\mathcal{D}^n_\beta \mid \Re(z_1)=\cdots=\Re(z_n)=0\}$$

is an inner set of uniqueness for functions analytic on $\mathcal{D}^n_\beta$ (see pages 101 and 352 of [70]). Therefore, the Green functions (2.58), and hence the states (2.55), are completely determined by *the Matsubara functions*

$$\begin{aligned} \varGamma^{\Lambda}_{F_1,\ldots,F_n}(\tau_1,\ldots,\tau_n) &\stackrel{\rm def}{=} G^{\Lambda}_{F_1,\ldots,F_n}(\imath\tau_1,\ldots,\imath\tau_n) \\ &= {\rm trace}[F_1 e^{-(\tau_2-\tau_1)H_\Lambda}F_2 e^{-(\tau_3-\tau_2)H_\Lambda}\cdots F_n e^{-(\tau_{n+1}-\tau_n)H_\Lambda}]/{\rm trace}[e^{-\beta H_\Lambda}] \end{aligned} \tag{2.60}$$

taken at 'temperature ordered' arguments $0\le\tau_1\le\cdots\le\tau_n\le\tau_1+\beta\stackrel{\rm def}{=}\tau_{n+1}$, with all possible choices of $n\in\mathbb{N}$ and $F_1,\ldots,F_n\in C_{\rm b}(\mathbb{R}^{\nu|\Lambda|})$. Their extensions to the whole $[0,\beta]^n$ are defined as

$$\varGamma^{\Lambda}_{F_1,\ldots,F_n}(\tau_1,\ldots,\tau_n) = \varGamma^{\Lambda}_{F_{\sigma(1)},\ldots,F_{\sigma(n)}}(\tau_{\sigma(1)},\ldots,\tau_{\sigma(n)}),$$

where σ is the permutation of $\{1,2,\ldots,n\}$ such that $\tau_{\sigma(1)}\le\tau_{\sigma(2)}\le\cdots\le\tau_{\sigma(n)}$. One can show that for every $\theta\in[0,\beta]$,

$$\varGamma^{\Lambda}_{F_1,\ldots,F_n}(\tau_1+\theta,\ldots,\tau_n+\theta) = \varGamma^{\Lambda}_{F_1,\ldots,F_n}(\tau_1,\ldots,\tau_n), \tag{2.61}$$

where addition is modulo β. This periodicity along with the analyticity of the Green functions is equivalent to the KMS property of the state (2.56).

The central element of the Euclidean approach which links the local Gibbs states (2.55) and the local Gibbs measures (2.19) is the representation (c.f., (2.18))

$$\varGamma^{\Lambda}_{F_1,\ldots,F_n}(\tau_1,\ldots,\tau_n) = \int_{\Omega_\Lambda} F_1(\omega_\Lambda(\tau_1))\ldots F_n(\omega_\Lambda(\tau_n))\mu_\Lambda({\rm d}\omega_\Lambda). \tag{2.62}$$

The Gibbs state (2.55) corresponds to a finite $\Lambda \Subset \mathbb{L}$. Thermodynamic properties of the underlying physical model are described by the Gibbs states corresponding to the whole lattice $\mathbb{L}$. Such states should be defined on the C^*-algebra of quasi-local observables $\mathfrak{C}$, being the norm-completion of the algebra of local observables $\cup_{\Lambda\Subset\mathbb{L}}\mathfrak{C}_\Lambda$. Here each $\mathfrak{C}_\Lambda$ is considered, modulo embedding, as a subalgebra of $\mathfrak{C}_{\Lambda'}$ for any Λ' containing Λ. The dynamics of the whole system is to be defined by the limits, as $\Lambda \nearrow \mathbb{L}$, of the time automorphisms (2.56), which would allow one to define Gibbs states on $\mathfrak{C}$ as KMS states. This 'algebraic' way can be realized for models described by bounded local Hamiltonians H_Λ, for which the limiting time automorphisms exist, see section 6.2 of [20]. For the model considered here, such automorphisms do not exist and hence there is no canonical way to define Gibbs states of the whole infinite system. Therefore, the Euclidean approach based on the one-to-one correspondence between the local states and measures arising from the representation (2.62) seems to be the only way of developing a mathematical theory of the equilibrium thermodynamic properties of such models. For certain model of quantum crystals, the limiting states $\lim_{\Lambda \nearrow \mathbb{L}} \varrho_\Lambda$ were constructed by means of path measures, see [16, 58, 59]. The set of all Euclidean Gibbs measures $\mathcal{G}^{\mathrm{t}}$ we study in this article certainly includes all limiting points of this type. Furthermore, there exist axiomatic methods, see [19, 31], analogous to the Osterwalder-Schrader reconstruction theory [33, 71], by means of which KMS states are constructed on certain von Neumann algebras from a complete set of Matsubara functions. In our case such a set of functions constitute

$$\Gamma^{\mu}_{F_1,\ldots,F_n}(\tau_1,\ldots,\tau_n) = \int_{\Omega} F_1(\omega(\tau_1))\cdots F_n(\omega(\tau_n))\mu(\mathrm{d}\omega), \quad \mu \in \mathcal{G}^{\mathrm{t}}, \tag{2.63}$$

defined for all bounded local multiplication operators $F_1,\ldots,F_n$. Therefore, the theory of $\mathcal{G}^{\mathrm{t}}$ developed in the article may be used to constructing such algebras and states, which we leave as an important task for the future.

3. The Results

In the first subsection below we present the statements describing the general case, whereas the second subsection is dedicated to the case of $\nu = 1$ and $J_{\ell\ell'} \geq 0$.

3.1. **Euclidean Gibbs measures in the general case.**

We begin by establishing existence of tempered Euclidean Gibbs measures and compactness of their set $\mathcal{G}^{\mathrm{t}}$. For models with non-compact spins, here they are even infinite-dimensional, such a property is far from being evident.

Theorem 3.1. *For every $\beta > 0$, the set of tempered Euclidean Gibbs measures $\mathcal{G}^{\mathrm{t}}$ is non-void and $\mathcal{W}^{\mathrm{t}}$- compact.*

The next theorem gives an exponential moment estimate similar to (2.10). Recall that the Hölder norm $|\cdot|_{C^\sigma_\beta}$ was defined by (2.2).

Theorem 3.2. *For every $\sigma \in (0, 1/2)$ and $\varkappa > 0$, there exists a positive constant $C_{3.1}$ such that, for any ℓ and for all $\mu \in \mathcal{G}^{\mathrm{t}}$,*

$$\int_{\Omega} \exp\left(\lambda_\sigma |\omega_\ell|^2_{C^\sigma_\beta} + \varkappa |\omega_\ell|^2_{L^2_\beta}\right) \mu(\mathrm{d}\omega) \leq C_{3.1}, \tag{3.1}$$

where λ_σ is the same as in (2.10).

According to (3.1), the one-site projections of each $\mu \in \mathcal{G}^{\mathrm{t}}$ are sub-Gaussian. The bound $C_{3.1}$ does not depend on ℓ and is the same for all $\mu \in \mathcal{G}^{\mathrm{t}}$. Estimates of this type are important also in the study of the Dirichlet operators H_μ associated with the measures $\mu \in \mathcal{G}^{\mathrm{t}}$, see [10, 11].

The set of tempered configurations Ω^{t} was introduced in (2.36), (2.38) by means of rather slack restrictions (c.f., (2.25)) imposed on the L^2_β-norms of ω_ℓ. The elements of $\mathcal{G}^{\mathrm{t}}$ are supported by this set, see (2.52). It turn out that they have a much smaller support (a kind of the Lebowitz-Presutti one). Given $b > 0$ and $\sigma \in (0, 1/2)$, we define

$$\begin{aligned} \Xi(b,\sigma) \quad = \quad & \{\xi \in \Omega \mid (\forall \ell_0 \in \mathbb{L})\ (\exists \Lambda_{\xi,\ell_0} \Subset \mathbb{L})\ (\forall \ell \in \Lambda^c_{\xi,\ell_0}) : \\ & \qquad |\xi_\ell|^2_{C^\sigma_\beta} \leq b \log(1 + |\ell - \ell_0|)\}, \end{aligned} \tag{3.2}$$

which in view of (2.33) is a Borel subset of Ω^{t}.

Theorem 3.3. *For every $\sigma \in (0, 1/2)$, there exists $b > 0$, which depends on σ and on the parameters of the model only, such that for all $\mu \in \mathcal{G}^{\rm t}$,*

$$\mu(\Xi(b, \sigma)) = 1. \tag{3.3}$$

The last result in this group is a sufficient condition for $\mathcal{G}^{\rm t}$ to be a singleton, which holds for high temperatures (small β). It is obtained by controlling the 'non-convexity' of the potential energy (2.12). Let us decompose

$$V_\ell = V_{1,\ell} + V_{2,\ell}, \tag{3.4}$$

where $V_{1,\ell} \in C^2(\mathbb{R}^\nu)$ is such that

$$-a \leq b \stackrel{\rm def}{=} \inf_{\ell} \inf_{x,y \in \mathbb{R}^\nu,\ y \neq 0} \left(V''_{1,\ell}(x)y, y\right)/|y|^2 < \infty. \tag{3.5}$$

As for the second term, we set

$$0 \leq \delta \stackrel{\rm def}{=} \sup_{\ell} \left\{ \sup_{x \in \mathbb{R}^\nu} V_{2,\ell}(x) - \inf_{x \in \mathbb{R}^\nu} V_{2,\ell}(x) \right\} \leq \infty. \tag{3.6}$$

Its role is to produce multiple minima of the potential energy responsible for eventual phase transitions. Clearly, the decomposition (3.4) is not unique; its optimal realizations for certain types of V_ℓ are discussed in section 6 of [12].

Theorem 3.4. *The set $\mathcal{G}^{\rm t}$ is a singleton if*

$$e^{\beta\delta} < (a + b)/\hat{J}_0. \tag{3.7}$$

One observes that the latter condition surely holds at all β if

$$\delta = 0 \quad \text{and} \quad \hat{J}_0 < a + b. \tag{3.8}$$

In this case the potential energy W_Λ given by (2.12) is convex. The conditions (3.8) and (3.7) do not contain the particle mass m; hence, the uniqueness stated holds also in the quasi-classical limit[2] $m \to +\infty$.

3.2. **Scalar ferromagnetic models.** Here we study in more detail the case $\nu = 1$ and $J_{\ell\ell'} \geq 0$, that is tacitly assumed in this subsection. Recall that the components ω_ℓ of $\omega \in \Omega$ are continuous functions on $S_\beta \cong [0, \beta]$. For $\omega, \tilde{\omega} \in \Omega$, we set $\omega \leq \tilde{\omega}$ if for all ℓ and $\tau \in [0, \beta]$, one has $\omega_\ell(\tau) \leq \tilde{\omega}_\ell(\tau)$. This allows one to define an order on $\mathcal{P}(\Omega)$, called stochastic domination or FKG order, see [65]. A function $f : \Omega \to \mathbb{R}$ is called increasing if $\omega \leq \tilde{\omega}$ implies $f(\omega) \leq f(\tilde{\omega})$. Clearly, the subset of $C_{\rm b}(\Omega^{\rm t})$ consisting of all increasing functions is a measure determining class. For $\mu \in \mathcal{P}(\Omega^{\rm t})$ and $f \in C_{\rm b}(\Omega^{\rm t})$, we write (c.f., (2.48))

$$\mu(f) = \int_{\Omega^{\rm t}} f(\omega)\mu({\rm d}\omega).$$

Then for $\mu_1, \mu_2 \in \mathcal{P}(\Omega^{\rm t})$, we set

$$\mu_1 \leq \mu_2 \quad \text{if} \quad \mu_1(f) \leq \mu_2(f), \qquad \text{for all increasing} \quad f \in C_{\rm b}(\Omega^{\rm t}). \tag{3.9}$$

A measure $\mu \in \mathcal{G}^{\rm t}$ is called *shift invariant* if its Matsubara functions (2.63) have the property (2.61).

[2]For details on this limit see [4].

Theorem 3.5. *The set $\mathcal{G}^{\mathrm{t}}$ has unique maximal μ_+ and minimal μ_- elements in the sense of the order (3.9). These elements are extreme and shift invariant; they are translation invariant if the model is so. If $V_\ell(x) = V_\ell(-x)$ for all ℓ, then $\mu_+(B) = \mu_-(-B)$ for all $B \in \mathcal{B}(\Omega)$.*

Phase transitions correspond to $\mathcal{G}^{\mathrm{t}}$ possessing more than one element. In the underlying physical systems phase transitions manifest themselves in the macroscopic displacements of particles from their equilibrium positions (a long-range order). For translation invariant ferromagnetic models with $\nu = 1$ and $V_\ell = V$ obeying certain conditions, the appearance of the long-range order at low temperatures was proven in [18, 23, 36, 43, 63]. Thus, one can expect that also in our case $|\mathcal{G}^{\mathrm{t}}| > 1$ at big β. We prove this under certain conditions imposed on d, $J_{\ell\ell'}$ and V_ℓ. First we suppose that the interaction between the nearest neighbors is uniformly nonzero

$$\inf_{|\ell-\ell'|=1} J_{\ell\ell'} \stackrel{\mathrm{def}}{=} J > 0. \tag{3.10}$$

Next we suppose that V_ℓ are even continuous functions and the upper bound in (2.14) can be chosen as

$$V(x_\ell) = \sum_{s=1}^{r} b^{(s)} x_\ell^{2s}; \quad 2b^{(1)} < -a; \quad b^{(s)} \geq 0, \ s \geq 2, \tag{3.11}$$

where a is the same as in (2.5) or in (2.12), and $r \geq 2$ is either a positive integer or infinite. For $r = +\infty$, we assume that the series

$$\Phi(t) = \sum_{s=2}^{+\infty} \frac{(2s)!}{2^{s-1}(s-1)!} b^{(s)} t^{s-1}, \tag{3.12}$$

converges at some $t > 0$. Since $2b^{(1)} + a < 0$, the equation

$$a + 2b^{(1)} + \Phi(t) = 0, \tag{3.13}$$

has a unique solution $t_* > 0$. Finally, we suppose that for every ℓ,

$$V(x_\ell) - V_\ell(x_\ell) \leq V(\tilde{x}_\ell) - V_\ell(\tilde{x}_\ell), \quad \text{whenever} \quad x_\ell^2 \leq \tilde{x}_\ell^2. \tag{3.14}$$

By these assumptions all V_ℓ are 'uniformly double-welled'. If $V_\ell(x_\ell) = v_\ell(x_\ell^2)$ and v_ℓ are differentiable, the condition (3.14) may be formulated as an upper bound for v'_ℓ. Recall that $\mathbb{L} = \mathbb{Z}^d$; for $d \geq 2$, we set

$$\theta_d = \frac{1}{(2\pi)^d} \int_{(-\pi,\pi]^d} \frac{\mathrm{d}p}{\sqrt{E(p)}}, \quad E(p) = \sum_{j=1}^{d} [1 - \cos p_j]. \tag{3.15}$$

Theorem 3.6. *Let $d \geq 3$ and the above assumptions hold. Then under the condition*

$$J > \theta_d^2 / 8 m t_*^2, \tag{3.16}$$

there exists $\beta_* > 0$ such that $|\mathcal{G}^{\mathrm{t}}| > 1$ whenever $\beta > \beta_*$. The bound β_* is the unique solution of the following equation

$$t_* = \frac{1}{(2\pi)^d} \int_{(-\pi,\pi]^d} \frac{1}{\sqrt{8mJE(p)}} \coth\left(\frac{\beta^2 J E(p)}{2m}\right)^{1/2} \mathrm{d}p. \tag{3.17}$$

As was shown in [2, 6, 45], strong quantum effects, corresponding in particular to small values of the particle mass m, can suppress abnormal fluctuations and hence phase transitions. Therefore, one can expect that they can yield $|\mathcal{G}^{\rm t}| = 1$. In the probabilistic interpretation our model describes a system of interacting diffusion processes, in which strong quantum effects correspond to large diffusion intensity. The most general result in this domain – the uniqueness at all β due to strong quantum effects – was proven in [5]. In the present paper we essentially extend the class of self-interaction potentials for which this result holds as well as make precise the bounds of the uniqueness regime. Furthermore, unlike to the mentioned papers we do not suppose that the interaction has finite range and the model is translation invariant. In Theorem 3.7 below we assume that the potentials V_ℓ are even continuous functions possessing the following property. There exists a convex function $v : [0, +\infty) \to \mathbb{R}$ such that (c.f., (3.14))

$$V_\ell(x_\ell) - v(x_\ell^2) \leq V_\ell(\tilde{x}_\ell) - v(\tilde{x}_\ell^2) \quad \text{whenever} \quad x_\ell^2 < \tilde{x}_\ell^2. \tag{3.18}$$

In typical cases of V_ℓ, like (2.16), as such a v one can take a convex polynomial of degree $r \geq 2$. Then (3.18) is a stronger version of the lower bound in (2.14). Now let us introduce the following one-particle Schrödinger operator (c.f., (2.5), (2.11))

$$\tilde{H}_\ell = -\frac{1}{2m}\left(\frac{\partial}{\partial x_\ell}\right)^2 + \frac{a}{2}x_\ell^2 + v(x_\ell^2). \tag{3.19}$$

It has purely discrete non-degenerate spectrum $\{E_n\}_{n\in\mathbb{N}_0}$. Thus, one can define the parameter

$$\varDelta = \min_{n\in\mathbb{N}} \left(E_n - E_{n-1}\right), \tag{3.20}$$

which depends on m, a, and on the choice of v. Recall, that $\hat{J}_0$ was defined by (2.15).

Theorem 3.7. *Let the above assumptions regarding the potentials V_ℓ hold. Then the set of tempered Euclidean Gibbs measures is a singleton if*

$$m\varDelta^2 > \hat{J}_0. \tag{3.21}$$

Note that the above result holds for all $\beta > 0$. Thus, (3.21) is a stability condition like (3.8), where the parameter $m\varDelta^2$ appears as the oscillator rigidity (or diffusion intensity). If it holds, a stability-due-to-quantum-effects occurs, see [6, 44, 45, 48]. If v is a polynomial of degree $r \geq 2$, the rigidity $m\varDelta^2$ is a continuous function of the particle mass m; it gets small in the quasi-classical limit $m \to +\infty$, see [48]. At the same time, for $m \to 0+$, one has $m\varDelta^2 = O(m^{-(r-1)/(r+1)})$, see [2, 48]. Hence, (3.21) certainly holds in the small mass limit, c.f., [3, 5]. To compare the latter result with Theorem 3.6 let us assume that $J_{\ell\ell'} = J$ iff $|\ell - \ell'| = 1$ and all V_ℓ coincide with the function given by (3.11). Then the parameter (3.20) obeys the estimate $\varDelta < 1/2mt_*$, see [48], where t_* is the same as in (3.16), (3.17). In this case the condition (3.21) can be rewritten as

$$J < 1/8dmt_*^2. \tag{3.22}$$

One can show that $\theta_d > 1/d$ and $d\theta_d^2 \to 1$ as $d \to +\infty$, which indicates that the estimates (3.16) and (3.22) become precise for sufficiently large dimensions.

Now we consider a translation invariant version of our model and impose further conditions on the self-interaction potential. Set

$$\mathcal{F}_{\mathrm{Laguerre}} = \left\{ \varphi : \mathbb{R} \to \mathbb{R} \;\middle|\; \varphi(t) = \varphi_0 \exp(\gamma_0 t) t^n \prod_{i=1}^{\infty} (1 + \gamma_i t) \right\}, \tag{3.23}$$

where $\varphi_0 > 0$, $n \in \mathbb{N}_0$, $\gamma_i \geq 0$ for all $i \in \mathbb{N}_0$, and $\sum_{i=1}^{\infty} \gamma_i < \infty$. Each $\varphi \in \mathcal{F}_{\mathrm{Laguerre}}$ can be extended to an entire function $\varphi : \mathbb{C} \to \mathbb{C}$, which has no zeros outside of $(-\infty, 0]$. These are Laguerre entire functions, see [39, 46, 51]. In the next theorem the parameter a is the same as in (2.5).

Theorem 3.8. *Let the model we consider be translation invariant with the self-interaction potentials V being of the form*

$$V(x) = v(x^2) - hx, \quad h \in \mathbb{R}, \tag{3.24}$$

with $v(0) = 0$ and such that for a certain $b \geq a/2$, the derivative v' obeys the condition $b + v' \in \mathcal{F}_{\mathrm{Laguerre}}$. Then the set $\mathcal{G}^{\mathrm{t}}$ is a singleton if $h \neq 0$.

3.3. **Comments.** Here we comment the theorems and compare them with the corresponding results known for similar models.

- **Theorem 3.1.** A classical tool for proving existence of Gibbs measures is the celebrated Dobrushin criterion, Theorem 1 in [24]. To apply it in our case one should find a compact positive function h defined on the single-spin space C_β such that for all ℓ and $\xi \in \Omega$,

$$\int_{\Omega} h(\omega_\ell) \pi_\ell(\mathrm{d}\omega|\xi) \leq A + \sum_{\ell'} I_{\ell\ell'} h(\xi_{\ell'}), \tag{3.25}$$

  where

$$A > 0; \quad I_{\ell\ell'} \geq 0 \quad \text{for all} \quad \ell, \ell', \quad \text{and} \quad \sup_{\ell} \sum_{\ell'} I_{\ell\ell'} < 1.$$

  Here and in the sequel to simplify notations we denoted $\pi_{\{\ell\}}$ by π_ℓ . Then the estimate (3.25) would yield that for any $\xi \in \Omega$, such that $\sup_\ell h(\xi_\ell) < \infty$, the family $\{\pi_\Lambda(\cdot|\xi)\}_{\Lambda \Subset \mathbb{L}}$ is relatively compact in the weak topology on $\mathcal{P}(\Omega)$ (but not yet in $\mathcal{W}_\alpha$, $\mathcal{W}^{\mathrm{t}}$). Next one would have to show that any accumulation point of $\{\pi_\Lambda(\cdot|\xi)\}_{\Lambda \Subset \mathbb{L}}$ is a Gibbs measure, which is much stronger than the fact established by our Lemma 2.11. Such a scheme was used in [17, 21, 74] where the existence of Gibbs measures for lattice systems with the single-spin space $\mathbb{R}$ was proven. Those proofs heavily utilized the specific properties of the models, e.g., attractiveness and translation invariance. The direct extension of this scheme to quantum models seems to be impossible. The scheme we employ to proving Theorem 3.1 is based on compactness arguments in the topologies $\mathcal{W}_\alpha$, $\mathcal{W}^{\mathrm{t}}$. After obvious modifications it can be applied to models with much more general types of inter-particle interaction potentials. Additional comments on this matter follow Corollary 4.2.
- **Theorem 3.2.** This statement gives a uniform integrability estimate for tempered Euclidean Gibbs measures in terms of model parameters, which in principle can be proven before establishing the existence. For systems of classical unbounded spins, the problem of deriving such estimates was first

posed in [17] (see the discussion following Corollary 4.2). For quantum anharmonic systems, similar estimates were obtained in the so called analytic approach, which is an equivalent alternative to the approach based on the DLR equation, see [7, 8, 9, 14]. In this analytic approach $\mathcal{G}^{\rm t}$ is defined as the set of probability measures satisfying an integration-by-parts formula, determined by the model. This gives additional tools for studying $\mathcal{G}^{\rm t}$ and provides a background for the stochastic dynamics method in which the Gibbs measures are treated as invariant distributions for certain infinite-dimensional stochastic evolution equations, see [15]. In both analytic and stochastic dynamics methods one imposes a number of technical conditions on the interaction potentials and uses advanced tools of stochastic analysis. The method we employ to proving Theorem 3.2 is much more elementary. At the same time, Theorem 3.2 gives an improvement of the corresponding results of [8, 9] because: (a) the estimate (3.1) gives a much stronger bound; (b) we do not suppose that the functions V_ℓ are differentiable – an important assumption of the analytic approach.

- **Theorem 3.3.** As might be clear from the proof of this theorem, every $\mu \in \mathcal{P}(\Omega^{\rm t})$ obeying the estimate (3.1) possesses the support property (3.3). For Gibbs measures of classical lattice systems of unbounded spins, a similar property was first established in [54]; hence, one can call $\Xi(b, \sigma)$ a Lebowitz-Presutti type support. This result of [54] was obtained by means of Ruelle's superstability estimates [68], applicable to translation invariant models only. The generalization to translation invariant quantum model was done in [61], where superstable Gibbs measures were specified by the following support property

$$\sup_{N\in\mathbb{N}} \left\{ (1+2N)^{-d} \sum_{\ell : |\ell|\leq N} |\omega_\ell|^2_{L^2_\beta} \right\} \leq C(\omega), \quad \mu - \text{a.s..}$$

Here we note that by the Birkhoff-Khinchine ergodic theorem, for any translation invariant measure $\mu \in \mathcal{P}(\Omega^{\rm t})$ obeying (3.1), it follows that for every $\sigma \in (0, 1/2)$, $\varkappa > 0$, and μ-almost all ω,

$$\sup_{N\in\mathbb{N}} \left\{ (1+2N)^{-d} \sum_{\ell : |\ell|\leq N} \exp\left(\lambda_\sigma |\omega_\ell|^2_{C^\sigma_\beta} + \varkappa |\omega_\ell|^2_{L^2_\beta} \right) \right\} \leq C(\sigma, \varkappa, \omega).$$

In particular, every periodic Euclidean Gibbs measure constructed in subsection 6.5 below has the above property.

- **Theorem 3.4** establishes a sufficient uniqueness condition, holding at high-temperatures (small β). Here we follow the papers [12, 13], where a similar uniqueness statement was proven for translation invariant ferromagnetic scalar version of our model. This was done by means of another renown Dobrushin result, Theorem 4 in [24], which gives a sufficient condition for the uniqueness of Gibbs measures. The main tool used in [12, 13] to estimating the elements of the Dobrushin matrix was the logarithmic Sobolev inequality for the kernels π_ℓ.
- **Theorem 3.5** is an extension of the corresponding statement proven in [17] for classical lattice models. The extreme elements mentioned in Theorem 3.5 play an important role in proving Theorems 3.6, 3.7, 3.8.

- **Theorem 3.6.** For translation invariant lattice models, phase transitions are established by showing the existence of nonergodic (with respect to the group of lattice translations) Gibbs measures. This mainly was being done by means of the infrared estimates, see [18, 23, 36, 43, 63]. Here we use a version of the technique developed in those papers and the corresponding correlation inequalities which allow us to compare the model considered with its translation invariant version (reference model).
- **Theorem 3.7.** For translation invariant models with finite range interactions and with the anharmonic self-interaction potential possessing special properties, the uniqueness by strong quantum effects was proven in [5] (see also [3]). With the help of the extreme elements $\mu_{\pm} \in \mathcal{G}^{\rm t}$ we essentially extend the results of those papers. As in the case of Theorem 3.6 we employ correlation inequalities to compare the model with a proper reference model.
- **Theorem 3.8.** For classical lattice models, the uniqueness at nonzero h was proven in [17, 52, 54] under the condition that the potential (3.24) possesses the property which we establish below in Definition 7.1. The novelty of Theorem 3.8 is that it describes a quantum model and gives an explicit sufficient condition for V to possess such a property[3]. This theorem is valid also in the quasi-classical limit $m \to +\infty$, in which it covers all the cases considered in [17, 52, 54]. For $(\phi^4)_2$ Euclidean quantum fields, a similar statement was proven in [30].

4. Properties of the Local Gibbs Specification

Here we develop our main tools based on the properties of the kernels (2.44).

4.1. Moment estimates.

4.1. Moment estimates. Moment estimates for the kernels (2.44) allow one to prove the $\mathcal{W}^{\rm t}$-relative compactness of the set $\{\pi_\Lambda(\cdot|\xi)\}_{\Lambda \Subset \mathbb{L}}$, which by Lemma 2.11 guaranties that $\mathcal{G}^{\rm t} \neq \emptyset$. Integrating them over $\xi \in \Omega^{\rm t}$ we get by the DLR equation (2.51) the corresponding estimates also for the elements of $\mathcal{G}^{\rm t}$. Recall that π_ℓ stands for $\pi_{\{\ell\}}$.

Lemma 4.1. *For any $\varkappa$, $\vartheta > 0$, and $\sigma \in (0, 1/2)$, there exists $C_{4.1} > 0$ such that for all $\ell \in \mathbb{L}$ and $\xi \in \Omega^{\rm t}$,*

$$\int_\Omega \exp\left\{\lambda_\sigma |\omega_\ell|^2_{C^\sigma_\beta} + \varkappa |\omega_\ell|^2_{L^2_\beta}\right\} \pi_\ell(\mathrm{d}\omega|\xi) \leq \exp\left\{C_{4.1} + \vartheta \sum_{\ell'} |J_{\ell\ell'}| \cdot |\xi_{\ell'}|^2_{L^2_\beta}\right\}. \tag{4.1}$$

Here $\lambda_\sigma > 0$ is the same as in (3.1).

Proof. Note that by (2.47) the left-hand side is finite and the second term in $\exp\{\cdot\}$ on the right-hand side is also finite since $\xi \in \Omega^{\rm t}$. For any $\vartheta > 0$, one has (see (2.15))

$$\left|\sum_{\ell'} J_{\ell\ell'}(\omega_\ell, \xi_{\ell'})_{L^2_\beta}\right| \leq \frac{\hat{J}_0}{2\vartheta}|\omega_\ell|^2_{L^2_\beta} + \frac{\vartheta}{2}\sum_{\ell'} |J_{\ell\ell'}| \cdot |\xi_{\ell'}|^2_{L^2_\beta}, \tag{4.2}$$

[3]Examples follow Proposition 7.2.

which holds for all $\omega, \xi \in \Omega^{\rm t}$. By these estimates and (2.21), (2.23), (2.42), (2.44) we get

$$\begin{aligned}(4.3)\qquad \mathrm{LHS}(4.1) \leq & [1/Y_\ell(\vartheta)] \cdot \exp\left\{\vartheta \sum_{\ell'} |J_{\ell\ell'}| \cdot |\xi_{\ell'}|^2_{L^2_\beta}\right\} \\ & \times \int_\Omega \exp\left\{\lambda_\sigma |\omega_\ell|^2_{C^\sigma_\beta} + \left(\varkappa + \hat{J}_0/2\vartheta\right)|\omega_\ell|^2_{L^2_\beta} - \int_0^\beta V_\ell(\omega_\ell(\tau))\mathrm{d}\tau\right\}\chi(\mathrm{d}\omega_\ell),\end{aligned}$$

where

$$Y_\ell(\vartheta) = \int_\Omega \exp\left\{-\frac{\hat{J}_0}{2\vartheta} \cdot |\omega_\ell|^2_{L^2_\beta} - \int_0^\beta V_\ell(\omega_\ell(\tau))\mathrm{d}\tau\right\}\chi(\mathrm{d}\omega_\ell).$$

Now we use the upper bound (2.14) to estimate $\inf_\ell Y_\ell(\vartheta)$, the lower bound (2.14) to estimate the integrand in (4.3), take into account Proposition 2.1, and arrive at (4.1). □

By Jensen's inequality we readily get from (4.1) the following Dobrushin-like bound.

Corollary 4.2. *For all ℓ and $\xi \in \Omega^{\rm t}$, the kernels $\pi_\ell(\cdot|\xi)$, obey the estimate*

$$(4.4)\qquad \int_\Omega h(\omega_\ell)\pi_\ell(\mathrm{d}\omega|\xi) \leq C_{4.1} + (\vartheta/\varkappa)\sum_{\ell'}|J_{\ell\ell'}| \cdot h(\xi_{\ell'}),$$

with

$$(4.5)\qquad h(\omega_\ell) = \lambda_\sigma |\omega_\ell|^2_{C^\sigma_\beta} + \varkappa |\omega_\ell|^2_{L^2_\beta},$$

which is a compact function $h : C_\beta \to \mathbb{R}$.

For translation invariant classical lattice systems of unbounded one-dimensional spins with ferromagnetic pair interactions, integrability estimates like

$$\log\left\{\int_{\mathbb{R}^{\mathbb{L}}} \exp(\lambda|x_\ell|)\pi_\ell(\mathrm{d}x|y)\right\} < A + \sum_{\ell'} I_{\ell\ell'}|y_{\ell'}|,$$

were first obtained by J. Bellissard and R. Høegh-Krohn, see Proposition III.1 and Theorem III.2 in [17]. Dobrushin type estimates like (3.25) were also proven in [21, 74]. The methods used there essentially employed the above mentioned properties of the model and hence cannot be of use in our situation. Instead, we develop the approach to getting such estimates which is much simpler but is applicable to both cases – classical and quantum. Its peculiarities are: (a) first we prove the exponential integrability (4.1) and then derive the Dobrushin bound (4.4) rather than prove it directly; (b) the function (4.5) consists of two additive terms, the first of which is to guarantee the compactness while the second one controls the inter-particle interaction.

Now by means of (4.1) we obtain moment estimates for the kernels π_Λ with arbitrary $\Lambda \Subset \mathbb{L}$. Let the parameters σ, $\varkappa$, and λ_σ be the same as in (4.1). For $\ell \in \Lambda \Subset \mathbb{L}$, we define

$$(4.6)\qquad n_\ell(\Lambda|\xi) = \log\left\{\int_\Omega \exp\left(\lambda_\sigma|\omega_\ell|^2_{C^\sigma_\beta} + \varkappa|\omega_\ell|^2_{L^2_\beta}\right)\pi_\Lambda(\mathrm{d}\omega|\xi)\right\},$$

which is finite by (2.47).

Lemma 4.3. *For every $\alpha \in \mathcal{I}$, there exists $C_{4.7}(\alpha) > 0$ such that for all ℓ_0 and $\xi \in \Omega^{\rm t}$,*

$$\limsup_{\Lambda \nearrow \mathbb{L}} \sum_{\ell \in \Lambda} n_\ell(\Lambda|\xi) w_\alpha(\ell_0, \ell) \leq C_{4.7}(\alpha); \tag{4.7}$$

hence,

$$\limsup_{\Lambda \nearrow \mathbb{L}} n_{\ell_0}(\Lambda|\xi) \leq C_{4.7}(\alpha), \quad for \quad any \quad \alpha \in \mathcal{I}. \tag{4.8}$$

Thereby, there exists $C_{4.9}(\ell, \xi) > 0$ such that for all $\Lambda \Subset \mathbb{L}$ containing ℓ,

$$n_\ell(\Lambda|\xi) \leq C_{4.9}(\ell, \xi). \tag{4.9}$$

Proof. Given $\varkappa > 0$ and $\alpha \in \mathcal{I}$, we fix $\vartheta > 0$ such that

$$\vartheta \sum_{\ell'} |J_{\ell\ell'}| \leq \vartheta \hat{J}_0 \leq \vartheta \hat{J}_\alpha < \varkappa. \tag{4.10}$$

By Jensen's inequality,

$$n_\ell(\Lambda|\xi) \geq \int_\Omega \left(\lambda_\sigma |\omega_\ell|^2_{C^\sigma_\beta} + \varkappa |\omega_\ell|^2_{L^2_\beta} \right) \pi_\Lambda({\rm d}\omega|\xi), \quad \ell \in \Lambda. \tag{4.11}$$

Integrating both sides of the exponential bound (4.1) with respect to the measure $\pi_\Lambda({\rm d}\omega|\xi)$ and taking into account the consistency condition (2.46) we get

$$\begin{aligned} n_\ell(\Lambda|\xi) \ & \leq \ C_{4.1} + \vartheta \sum_{\ell' \in \Lambda^c} |J_{\ell\ell'}| \cdot |\xi_{\ell'}|^2_{L^2_\beta} \\ & + \ \log \left\{ \int_\Omega \exp \left(\vartheta \sum_{\ell' \in \Lambda} |J_{\ell\ell'}| \cdot |\omega_{\ell'}|^2_{L^2_\beta} \right) \pi_\Lambda({\rm d}\omega|\xi) \right\} \\ & \leq \ C_{4.1} + \vartheta \sum_{\ell' \in \Lambda^c} |J_{\ell\ell'}| \cdot |\xi_{\ell'}|^2_{L^2_\beta} + \vartheta \sum_{\ell' \in \Lambda} |J_{\ell\ell'}| \cdot \int_\Omega |\omega_{\ell'}|^2_{L^2_\beta} \pi_\Lambda({\rm d}\omega|\xi) \\ & \leq \ C_{4.1} + \vartheta \sum_{\ell' \in \Lambda^c} |J_{\ell\ell'}| \cdot |\xi_{\ell'}|^2_{L^2_\beta} + \vartheta/\varkappa \sum_{\ell' \in \Lambda} |J_{\ell\ell'}| \cdot n_{\ell'}(\Lambda|\xi). \end{aligned} \tag{4.12}$$

Here we have used (4.10), (4.11), and the multiple Hölder inequality

$$\int \left(\prod_{i=1}^n \varphi_i^{\alpha_i} \right) {\rm d}\mu \leq \prod_{i=1}^n \left(\int \varphi_i {\rm d}\mu \right)^{\alpha_i},$$

in which μ is a probability measure, $\varphi_i \geq 0$ (respectively, $\alpha_i \geq 0$), $i = 1, \ldots, n$, are functions (respectively, numbers such that $\sum_{i=1}^n \alpha_i \leq 1$). Then (4.12) yields

$$\begin{aligned} n_{\ell_0}(\Lambda|\xi) & \leq \sum_{\ell \in \Lambda} n_\ell(\Lambda|\xi) w_\alpha(\ell_0, \ell) \\ & \leq \frac{1}{1 - \vartheta \hat{J}_\alpha/\varkappa} \left[C_{4.1} \sum_{\ell' \in \Lambda} w_\alpha(\ell_0, \ell') + \vartheta \hat{J}_\alpha \sum_{\ell' \in \Lambda^c} |\xi_{\ell'}|^2_{L^2_\beta} w_\alpha(\ell_0, \ell') \right]. \end{aligned} \tag{4.13}$$

Therefrom, for all $\xi \in \Omega^{\rm t}$, we get

$$\begin{aligned} \limsup_{\Lambda \nearrow \mathbb{L}} n_{\ell_0}(\Lambda|\xi) & \leq \limsup_{\Lambda \nearrow \mathbb{L}} \sum_{\ell \in \Lambda} n_\ell(\Lambda|\xi) w_\alpha(\ell_0, \ell) \\ & \leq \frac{C_{4.1}}{1 - \vartheta \hat{J}_\alpha/\varkappa} \sum_\ell w_\alpha(\ell_0, \ell) \ \stackrel{\rm def}{=} \ C_{4.7}(\alpha), \end{aligned} \tag{4.14}$$

which gives (4.7) and (4.8). The proof of (4.9) is straightforward. □

Recall that the norm $\|\cdot\|_\alpha$ was defined by (2.36). Given $\alpha \in \mathcal{I}$ and $\sigma \in (0,1/2)$, we set

$$\|\xi\|_{\alpha,\sigma} = \left[\sum_{\ell} |\xi_\ell|^2_{C^\sigma_\beta} w_\alpha(0,l)\right]^{1/2}. \tag{4.15}$$

Lemma 4.4. *Let the assumptions of Lemma 4.1 be satisfied. Then for every $\alpha \in \mathcal{I}$ and $\xi \in \Omega^{\mathrm{t}}$, one finds a positive $C_{4.16}(\xi)$ such that for all $\Lambda \Subset \mathbb{L}$,*

$$\int_\Omega \|\omega\|^2_\alpha \pi_\Lambda(\mathrm{d}\omega|\xi) \leq C_{4.16}(\xi). \tag{4.16}$$

Furthermore, for every $\alpha \in \mathcal{I}$, $\sigma \in (0,1/2)$, and $\xi \in \Omega^{\mathrm{t}}$ for which the norm (4.15) is finite, one finds a $C_{4.17}(\xi) > 0$ such that for all $\Lambda \Subset \mathbb{L}$,

$$\int_\Omega \|\omega\|^2_{\alpha,\sigma} \pi_\Lambda(\mathrm{d}\omega|\xi) \leq C_{4.17}(\xi). \tag{4.17}$$

Proof. For any fixed $\xi \in \Omega^{\mathrm{t}}$, by (4.11), (4.13) one has

$$\begin{aligned} &\limsup_{\Lambda \nearrow \mathbb{L}} \int_\Omega \|\omega\|^2_\alpha \pi_\Lambda(\mathrm{d}\omega|\xi) \\ &\quad \leq \limsup_{\Lambda \nearrow \mathbb{L}} \left[\frac{1}{\varkappa}\sum_{\ell\in\Lambda} n_\ell(\Lambda|\xi) w_\alpha(0,\ell) + \sum_{\ell \in \Lambda^c} |\xi_\ell|^2_{L^2_\beta} w_\alpha(0,\ell)\right] \\ &\quad \leq C_{4.7}/\varkappa. \end{aligned} \tag{4.18}$$

Hence, the set consisting of the left-hand sides of (4.16) indexed by $\Lambda \Subset \mathbb{L}$ is bounded in $\mathbb{R}$. The proof of (4.17) is analogous. □

4.2. **Weak convergence of tempered measures.** Recall that $f : \Omega \to \mathbb{R}$ is called a local function if it is measurable with respect to $\mathcal{B}(\Omega_\Lambda)$ for a certain $\Lambda \Subset \mathbb{L}$.

Lemma 4.5. *Let a sequence $\{\mu_n\}_{n\in\mathbb{N}} \subset \mathcal{P}(\Omega^{\mathrm{t}})$ have the following properties: (a) for every $\alpha \in \mathcal{I}$, each its element obeys the estimate*

$$\int_{\Omega^{\mathrm{t}}} \|\omega\|^2_\alpha \mu_n(\mathrm{d}\omega) \leq C_{4.19}(\alpha), \tag{4.19}$$

with one and the same $C_{4.19}(\alpha) > 0$; (b) for every local $f \in C_{\mathrm{b}}(\Omega^{\mathrm{t}})$, the sequence $\{\mu_n(f)\}_{n\in\mathbb{N}} \subset \mathbb{R}$ is fundamental. Then $\{\mu_n\}_{n\in\mathbb{N}}$ converges in $\mathcal{W}^{\mathrm{t}}$ to a certain $\mu \in \mathcal{P}(\Omega^{\mathrm{t}})$.

Proof. The topology of the Polish space Ω^{t} is consistent with the following metric (c.f., (2.37))

$$\rho(\omega,\tilde{\omega}) = \sum_{k=1}^{\infty} 2^{-k} \frac{\|\omega - \tilde{\omega}\|_{\alpha_k}}{1 + \|\omega - \tilde{\omega}\|_{\alpha_k}} + \sum_{\ell} 2^{-|\ell|} \frac{|\omega_\ell - \tilde{\omega}_\ell|_{C_\beta}}{1 + |\omega_\ell - \tilde{\omega}_\ell|_{C_\beta}}, \tag{4.20}$$

where $\{\alpha_k\}_{k\in\mathbb{N}} \subset \mathcal{I} = (\underline{\alpha}, \overline{\alpha})$ is a monotone strictly decreasing sequence converging to $\underline{\alpha}$. Let us denote by $C^{\mathrm{u}}_{\mathrm{b}}(\Omega^{\mathrm{t}};\rho)$ the set of all bounded functions $f : \Omega^{\mathrm{t}} \to \mathbb{R}$, which are uniformly continuous with respect to (4.20). Thus, to prove the lemma it suffices to show that under its conditions the sequence $\{\mu_n(f)\}_{n\in\mathbb{N}}$ is fundamental for every $f \in C_{\rm b}^{\rm u}(\Omega^{\rm t};\rho)$. Given $\delta > 0$, we choose $\Lambda_\delta \Subset \mathbb{L}$ and $k_\delta \in \mathbb{N}$ such that

$$\sum_{\ell\in\Lambda_\delta^c} 2^{-|\ell|} < \delta/3, \qquad \sum_{k=k_\delta}^{\infty} 2^{-k} = 2^{-k_\delta+1} < \delta/3. \tag{4.21}$$

For this δ and a certain $R > 0$, we choose $\Lambda_\delta(R) \Subset \mathbb{L}$ to obey

$$\sup_{\ell\in\mathbb{L}\setminus\Lambda_\delta(R)} \left\{ w_{\alpha_{k_\delta-1}}(0,\ell)/w_{\alpha_{k_\delta}}(0,\ell)\right\} < \frac{\delta}{3R^2}, \tag{4.22}$$

which is possible in view of (2.28). Finally, for $R > 0$, we set

$$B_R = \{\omega \in \Omega^{\rm t} \mid \|\omega\|_{\alpha_{k_\delta}} \le R\}. \tag{4.23}$$

By (4.19) and the Chebyshev inequality, one has that for all $n \in \mathbb{N}$,

$$\mu_n\left(\Omega^{\rm t} \setminus B_R\right) \le C_{4.19}(\alpha_{k_\delta})/R^2. \tag{4.24}$$

Now for $f \in C_{\rm b}^{\rm u}(\Omega^{\rm t};\rho)$, $\Lambda \Subset \mathbb{L}$, and $n, m \in \mathbb{N}$, we have

$$\begin{aligned} |\mu_n(f) - \mu_m(f)| &\le |\mu_n(f_\Lambda) - \mu_m(f_\Lambda)| \\ &+ 2\max\{\mu_n(|f-f_\Lambda|); \mu_m(|f-f_\Lambda|)\}, \end{aligned} \tag{4.25}$$

where we set $f_\Lambda(\omega) = f(\omega_\Lambda \times 0_{\Lambda^c})$. By (4.24),

$$\begin{aligned} \mu_n(|f-f_\Lambda|) &\le 2C_{4.19}(\alpha_{k_\delta})\|f\|_\infty/R^2 \\ &+ \int_{B_R} |f(\omega) - f(\omega_\Lambda\times 0_{\Lambda^c})|\,\mu_n({\rm d}\omega). \end{aligned} \tag{4.26}$$

For chosen $f \in C_{\rm b}^{\rm u}(\Omega^{\rm t};\rho)$ and $\varepsilon > 0$, one finds $\delta > 0$ such that for all $\omega, \tilde{\omega} \in \Omega^{\rm t}$,

$$|f(\omega) - f(\tilde{\omega})| < \varepsilon/6, \quad \text{whenever} \quad \rho(\omega,\tilde{\omega}) < \delta.$$

For these f, ε, and δ, one picks up $R(\varepsilon,\delta) > 0$ such that

$$C_{4.19}(\alpha_{k_\delta})\|f\|_\infty / \left[R(\varepsilon,\delta)\right]^2 < \varepsilon/12. \tag{4.27}$$

Now one takes $\Lambda \Subset \mathbb{L}$, which contains both Λ_δ and $\Lambda_\delta[R(\varepsilon,\delta)]$ defined by (4.21), (4.22). For this Λ, $\omega \in B_{R(\varepsilon,\delta)}$, and $k = 1, 2, \ldots, k_\delta - 1$, one has

$$\begin{aligned} \|\omega - \omega_\Lambda \times 0_{\Lambda^c}\|^2_{\alpha_k} &= \sum_{\ell\in\Lambda^c} |\omega_\ell|^2_{L^2_\beta} w_{\alpha_{k_\delta}}(0,\ell)\left[w_{\alpha_k}(0,\ell)/w_{\alpha_{k_\delta}}(0,\ell)\right] \\ &\le \frac{\delta}{3\left[R(\varepsilon,\delta)\right]^2} \sum_{\ell\in\Lambda^c} |\omega_\ell|^2_{L^2_\beta} w_{\alpha_{k_\delta}}(0,\ell) < \frac{\delta}{3}, \end{aligned} \tag{4.28}$$

where (4.22), (4.23) have been used. Then by (4.20), (4.21), it follows that

$$\forall \omega \in B_{R(\varepsilon,\delta)}: \quad \rho(\omega, \omega_\Lambda \times 0_{\Lambda^c}) < \delta, \tag{4.29}$$

which together with (4.27) yields in (4.26)

$$\mu_n(|f - f_\Lambda|) < \frac{\varepsilon}{6} + \frac{\varepsilon}{6}\mu_n\left(B_{R(\varepsilon,\delta)}\right) \le \frac{\varepsilon}{3}.$$

By assumption (b) of the lemma, one finds N_ε such that for all $n, m > N_\varepsilon$,

$$|\mu_n(f_\Lambda) - \mu_m(f_\Lambda)| < \frac{\varepsilon}{3}.$$

Applying the latter two estimates in (4.25) we get that the sequence $\{\mu_n\}_{n\in\mathbb{N}}$ is fundamental in the topology $\mathcal{W}^{\rm t}$ in which the space $\mathcal{P}(\Omega^{\rm t})$ is complete. $\square$

5. Proof of Theorems 3.1 – 3.4

The existence of Euclidean Gibbs measures and the integrability estimate (3.1) can be proven independently. To establish the compactness of $\mathcal{G}^{\mathrm{t}}$ we will need the estimate (3.1). Thus, we first prove Theorem 3.2.

Proof of Theorem 3.2: Let us show that every $\mu \in \mathcal{P}(\Omega)$ which solves the DLR equation (2.51) ought to obey the estimate (3.1) with one and the same $C_{3.1}$. To this end we apply the bounds for the kernels $\pi_\Lambda(\cdot|\xi)$ obtained above. Consider the functions

$$G_N(\omega_\ell) \stackrel{\mathrm{def}}{=} \exp\left(\min\left\{\lambda_\sigma|\omega_\ell|^2_{C^\sigma_\beta} + \varkappa|\omega_\ell|^2_{L^2_\beta}; N\right\}\right), \quad N \in \mathbb{N}.$$

By (2.51), Fatou's lemma, and the estimate (4.8) with an arbitrarily chosen $\alpha \in \mathcal{I}$ we get

$$\begin{aligned}
\int_\Omega G_N(\omega_\ell)\mu(\mathrm{d}\omega) &= \limsup_{\Lambda \nearrow \mathbb{L}} \int_\Omega \left[\int_\Omega G_N(\omega_\ell)\pi_\Lambda(\mathrm{d}\omega|\xi)\right]\mu(\mathrm{d}\xi) \\
&\leq \limsup_{\Lambda \nearrow \mathbb{L}} \int_\Omega \left[\int_\Omega \exp\left(\lambda_\sigma|\omega_\ell|^2_{C^\sigma_\beta} + \varkappa|\omega_\ell|^2_{L^2_\beta}\right)\pi_\Lambda(\mathrm{d}\omega|\xi)\right]\mu(\mathrm{d}\xi) \\
&\leq \int_\Omega \left[\limsup_{\Lambda \nearrow \mathbb{L}} \int_\Omega \exp\left(\lambda_\sigma|\omega_\ell|^2_{C^\sigma_\beta} + \varkappa|\omega_\ell|^2_{L^2_\beta}\right)\pi_\Lambda(\mathrm{d}\omega|\xi)\right]\mu(\mathrm{d}\xi) \\
&\leq \exp C_{4.7}(\alpha) \stackrel{\mathrm{def}}{=} C_{3.1}.
\end{aligned}$$

In view of the support property (2.53) of any measure solving the equation (2.51) we can pass here to the limit $N \to +\infty$ and get (3.1). $\square$

Corollary 5.1. *For every $\alpha \in \mathcal{I}$, the topologies induced on $\mathcal{G}^{\mathrm{t}}$ by $\mathcal{W}_\alpha$ and $\mathcal{W}^{\mathrm{t}}$ coincide.*

Proof. Follows immediately from Lemma 4.5 and the estimate (3.1). $\square$

Proof of Theorem 3.1: Let us introduce the next scale of Banach spaces (c.f., (2.36))

$$\Omega_{\alpha,\sigma} = \{\omega \in \Omega \mid \|\omega\|_{\alpha,\sigma} < \infty\}, \quad \sigma \in (0, 1/2), \quad \alpha \in \mathcal{I}, \tag{5.1}$$

where the norm $\|\cdot\|_{\alpha,\sigma}$ was defined by (4.15). For any pair $\alpha, \alpha' \in \mathcal{I}$ such that $\alpha < \alpha'$, the embedding $\Omega_{\alpha,\sigma} \hookrightarrow \Omega_{\alpha'}$ is compact, see Remark 2.6. This fact and the estimate (4.17), which holds for any $\xi \in \Omega_{\alpha,\sigma}$, imply by Prokhorov's criterion the relative compactness of the set $\{\pi_\Lambda(\cdot|\xi)\}_{\Lambda \Subset \mathbb{L}}$ in $\mathcal{W}_{\alpha'}$. Therefore, the sequence $\{\pi_\Lambda(\cdot|0)\}_{\Lambda \Subset \mathbb{L}}$ is relatively compact in every $\mathcal{W}_\alpha$, $\alpha \in \mathcal{I}$. Then Lemma 2.11 yields that $\mathcal{G}^{\mathrm{t}} \neq \emptyset$. By the same Prokhorov criterion and the estimate (3.1), we get the $\mathcal{W}_\alpha$-relative compactness of $\mathcal{G}^{\mathrm{t}}$. Then in view of the Feller property (Lemma 2.9), the set $\mathcal{G}^{\mathrm{t}}$ is closed and hence compact in every $\mathcal{W}_\alpha$, $\alpha \in \mathcal{I}$, which by Corollary 5.1 completes the proof. $\square$.

Proof of Theorem 3.3: To some extent we shall follow the line of arguments used in the proof of Lemma 3.1 in [54]. Given ℓ, ℓ_0, $b > 0$, $\sigma \in (0, 1/2)$, and $\Lambda \subset \mathbb{L}$, we introduce

$$\begin{aligned}
\Xi_\ell(\ell_0, b, \sigma) &= \{\xi \in \Omega \mid |\xi_\ell|^2_{C^\sigma_\beta} \leq b\log(1 + |\ell - \ell_0|)\}, \\
\Xi_\Lambda(\ell_0, b, \sigma) &= \bigcap_{\ell \in \Lambda} \Xi_\ell(\ell_0, b, \sigma).
\end{aligned} \tag{5.2}$$

For a cofinal sequence $\mathcal{L}$, we set

$$\varXi(\ell_0, b, \sigma) = \bigcup_{\Lambda\in\mathcal{L}} \varXi_{\Lambda^c}(\ell_0, b, \sigma), \quad \varXi(b, \sigma) = \bigcap_{\ell_0\in\mathbb{L}} \varXi(\ell_0, b, \sigma). \tag{5.3}$$

The latter $\varXi(b,\sigma)$ is a subset of $\varOmega^{\mathrm{t}}$ and is the same as the one given by (3.2). To prove the theorem let us show that for any $\sigma \in (0, 1/2)$, there exists $b > 0$ such that for all ℓ_0 and $\mu \in \mathcal{G}^{\mathrm{t}}$,

$$\mu\left(\varOmega \setminus \varXi(\ell_0, b, \sigma)\right) = 0. \tag{5.4}$$

By (5.2) we have

$$\begin{aligned}\varOmega \setminus \varXi_{\Lambda^c}(\ell_0, b, \sigma) &= \{\xi \in \varOmega \mid (\exists \ell \in \Lambda^c): \ |\xi_\ell|^2_{C^\sigma_\beta} > b\log(1 + |\ell - \ell_0|)\} \\ &\subset \{\xi \in \varOmega \mid (\exists \ell \in \Delta^c): \ |\xi_\ell|^2_{C^\sigma_\beta} > b\log(1 + |\ell - \ell_0|)\},\end{aligned} \tag{5.5}$$

for any $\Delta \subset \Lambda$. Therefore,

$$\mu\left(\bigcap_{\Lambda\in\mathcal{L}} [\varOmega \setminus \varXi_{\Lambda^c}(\ell_0, b, \sigma)]\right) = \lim_{\mathcal{L}} \mu\left(\varOmega \setminus \varXi_{\Lambda^c}(\ell_0, b, \sigma)\right), \tag{5.6}$$

which holds for any cofinal sequence $\mathcal{L}$. By (5.5),

$$\begin{aligned}\mu\left(\varOmega \setminus \varXi_{\Lambda^c}(\ell_0, b, \sigma)\right) &= \mu\left(\bigcup_{\ell\in\Lambda^c} [\varOmega \setminus \varXi_\ell(\ell_0, b, \sigma)]\right) \\ &\leq \sum_{\ell\in\Lambda^c} \mu\left(\left\{\xi \mid |\xi_\ell|^2_{C^\sigma_\beta} > b\log(1 + |\ell - \ell_0|)\right\}\right) \\ &= \sum_{\ell\in\Lambda^c} \mu\left(\left\{\xi \mid \exp\left(\lambda_\sigma |\xi_\ell|^2_{C^\sigma_\beta}\right) > (1 + |\ell - \ell_0|)^{b\lambda_\sigma}\right\}\right).\end{aligned}$$

Applying here the Chebyshev inequality and the estimate (3.1) we get

$$\mu\left(\varOmega \setminus \varXi_{\Lambda^c}(\ell_0, b, \sigma)\right) \leq C_{3.1} \sum_{\ell\in\Lambda^c} (1 + |\ell - \ell_0|)^{-b\lambda_\sigma}.$$

As $\mathbb{L} = \mathbb{Z}^d$, the latter series converges for any $b > d/\lambda_\sigma$. In this case by (5.6)

$$\mu\left(\varOmega \setminus \varXi(\ell_0, b, \sigma)\right) = \lim_{\mathcal{L}} \mu\left([\varOmega \setminus \varXi_{\Lambda^c}(\ell_0, b, \sigma)]\right) = 0,$$

which yields (5.4). $\square$

By $\mathcal{E}(\varOmega^{\mathrm{t}})$ we denote the set of all continuous local functions $f : \varOmega^{\mathrm{t}} \to \mathbb{R}$, for each of which there exist $\sigma \in (0, 1/2)$, $\Delta_f \Subset \mathbb{L}$, and $D_f > 0$, such that

$$|f(\omega)|^2 \leq D_f \sum_{\ell\in\Delta_f} \exp\left(\lambda_\sigma |\omega_\ell|^2_{C^\sigma_\beta}\right), \quad \text{for all } \ \omega \in \varOmega^{\mathrm{t}}, \tag{5.7}$$

where λ_σ is as in (3.1). Let $\mathrm{ex}(\mathcal{G}^{\mathrm{t}})$ stand for the set of all extreme elements of $\mathcal{G}^{\mathrm{t}}$.

Lemma 5.2. *For every $\mu \in \mathrm{ex}(\mathcal{G}^{\mathrm{t}})$ and any cofinal sequence $\mathcal{L}$, it follows that: (a) the sequence $\{\pi_\Lambda(\cdot|\xi)\}_{\Lambda\in\mathcal{L}}$ converges in $\mathcal{W}^{\mathrm{t}}$ to this μ for μ-almost all $\xi \in \varOmega^{\mathrm{t}}$; (b) for every $f \in \mathcal{E}(\varOmega^{\mathrm{t}})$, one has $\lim_{\mathcal{L}} \pi_\Lambda(f|\xi) = \mu(f)$ for μ-almost all $\xi \in \varOmega^{\mathrm{t}}$.*

Proof. Claim (c) of Theorem 7.12, page 122 in [32], implies that for any local $f \in C_{\mathrm{b}}(\varOmega^{\mathrm{t}})$,

$$\lim_{\mathcal{L}} \pi_\Lambda(f|\xi) = \mu(f), \quad \text{for } \mu\text{−almost all } \xi \in \varOmega^{\mathrm{t}}. \tag{5.8}$$

Then the convergence stated in claim (a) follows from Lemmas 4.4 and 4.5. Given $f \in \mathcal{E}(\Omega^{\rm t})$ and $N \in \mathbb{N}$, we set $\Omega_N = \{\omega \in \Omega \mid |f(\omega)| > N\}$ and

$$f_N(\omega) = \begin{cases} f(\omega) & \text{if } |f(\omega)| \leq N; \\ Nf(\omega)/|f(\omega)| & \text{otherwise.} \end{cases}$$

Each f_N belongs to $C_{\rm b}(\Omega^{\rm t})$ and $f_N \to f$ point-wise as $N \to +\infty$. Then by (5.8) there exists a Borel set $\Xi_\mu \subset \Omega^{\rm t}$, such that $\mu(\Xi_\mu) = 1$ and for every $N \in \mathbb{N}$,

$$\lim_{\mathcal{L}} \pi_\Lambda(f_N|\xi) = \mu(f_N), \quad \text{for all } \xi \in \Xi_\mu. \tag{5.9}$$

Note that by (4.6), (4.9), and (5.7), for any $\xi \in \Xi_\mu$ one finds a positive $C_{5.10}(f, \xi)$ such that for all $\Lambda \Subset \mathbb{L}$, which contain Δ_f, it follows that

$$\int_\Omega |f(\omega)|^2 \pi_\Lambda(\mathrm{d}\omega|\xi) \leq C_{5.10}(f, \xi). \tag{5.10}$$

Hence

$$\begin{aligned} |\pi_\Lambda(f|\xi) - \pi_\Lambda(f_N|\xi)| &\leq 2\int_{\Omega_N} |f(\omega)|\pi_\Lambda(\mathrm{d}\omega|\xi) \\ &\leq \frac{2}{N} \cdot \int_\Omega |f(\omega)|^2 \pi_\Lambda(\mathrm{d}\omega|\xi) \leq \frac{2}{N} \cdot C_{5.10}(f, \xi). \end{aligned}$$

Similarly, by means of (5.7) and Theorem 3.2, one gets

$$|\mu(f) - \mu(f_N)| \leq \frac{2}{N} \cdot D_f C_{3.1}.$$

The latter two inequalities and (5.9) allow us to estimate $|\pi_\Lambda(f|\xi) - \mu(f)|$ and thereby to complete the proof. $\square$

Proof of Theorem 3.4: For the scalar translation invariant version of the model considered here, the high-temperature uniqueness was proven in [12, 13]. The proof given below is a modification of the arguments used there. Thus, we can be brief.

Given ℓ and $\xi, \xi' \in \Omega^{\rm t}$, we define the distance

$$R[\pi_\ell(\cdot|\xi); \pi_\ell(\cdot|\xi')] = \sup_{f \in \mathrm{Lip}_1(L^2_\beta)} \left| \int_\Omega f(\omega_\ell)\pi_\ell(\mathrm{d}\omega|\xi) - \int_\Omega f(\omega_\ell)\pi_\ell(\mathrm{d}\omega|\xi') \right|,$$

where $\mathrm{Lip}_1(L^2_\beta)$ stands for the set of Lipschitz-continuous functions $f : L^2_\beta \to \mathbb{R}$ with the Lipschitz constant equal to one. The proof is based upon the Dobrushin criterium (Theorem 4 in [24]), which employs the matrix

$$C_{\ell\ell'} = \sup \left\{ \frac{R[\pi_\ell(\cdot|\xi); \pi_\ell(\cdot|\xi')]}{|\xi_\ell - \xi_{\ell'}|_{L^2_\beta}} \right\}, \quad \ell \neq \ell', \quad \ell, \ell' \in \mathbb{L}, \tag{5.11}$$

where the supremum is taken over all $\xi, \xi' \in \Omega^{\rm t}$ which differ only at ℓ'. According to this criterium the uniqueness stated will follow from the fact

$$\sup_\ell \sum_{\ell' \in \mathbb{L} \setminus \{\ell\}} C_{\ell\ell'} < 1. \tag{5.12}$$

In view of (2.47) the map

$$L^2_\beta \ni \xi_{\ell'} \mapsto \Upsilon(\xi_{\ell'}) \stackrel{\rm def}{=} \int_\Omega f(\omega_\ell)\pi_\ell(\mathrm{d}\omega|\xi) \tag{5.13}$$

has the following derivative in direction $\zeta \in L^2_\beta$

$$(\nabla \Upsilon(\xi_{\ell'}), \zeta)_{L^2_\beta} = -J_{\ell\ell'} \left[\pi_\ell \left(f \cdot (\omega_\ell, \zeta)_{L^2_\beta} |\xi\right) - \pi_\ell \left(f|\xi\right) \cdot \pi_\ell \left((\omega_\ell, \zeta)_{L^2_\beta} |\xi\right)\right].$$

By Theorem 5.1 of [12] the measures $\pi_\ell(\cdot|\xi)$ obey the logarithmic Sobolev inequality with the constant

$$C_{\rm LS} = e^{\beta\delta}/(a+b), \tag{5.14}$$

which is independent of ξ. By standard arguments this yields the estimate

$$\left|(\nabla \Upsilon(\xi_{\ell'}), \zeta)_{L^2_\beta}\right| \leq C_{\rm LS} |J_{\ell\ell'}| \cdot |\zeta|^2_{L^2_\beta}. \tag{5.15}$$

Then with the help of the mean value theorem from (5.11) and (5.14) we get

$$C_{\ell\ell'} \leq |J_{\ell\ell'}| \cdot e^{\beta\delta}/(a+b).$$

Thereby, the validity of the uniqueness condition (5.12) is ensured by (3.7). $\square$

6. Proof of Theorems 3.5, 3.6

6.1. **Correlation Inequalities.** Recall that Theorems 3.5 – 3.8 describe ferromagnetic scalar models, i.e., the ones with $\nu = 1$ and $J_{\ell\ell'} \geq 0$. Thus, all the statements below refer to such models only. Their proofs are mainly based on correlation inequalities. For the Gibbs measures considered here, such inequalities were derived in [4] in the framework of the lattice approximation technique, analogous to that of Euclidean quantum fields [71]. We begin with the FKG inequality, Theorem 6.1 in [4].

Proposition 6.1. *For all $\Lambda \Subset \mathbb{L}$, $\xi \in \Omega^{\rm t}$ and any continuous increasing $\pi_\Lambda(\cdot|\xi)$-integrable functions f and g, it follows that*

$$\pi_\Lambda(f \cdot g|\xi) \geq \pi_\Lambda(f|\xi) \cdot \pi_\Lambda(g|\xi), \tag{6.1}$$

which yields that for all such functions,

$$\xi \leq \tilde{\xi} \quad \Longrightarrow \pi_\Lambda(f|\xi) \leq \pi_\Lambda(f|\tilde{\xi}). \tag{6.2}$$

Next, there follow the GKS inequalities, Theorem 6.2 in [4].

Proposition 6.2. *Let the self-interaction potential have the form*

$$V_\ell(x) = v_\ell(x^2) - h_\ell x, \quad h_\ell \geq 0 \ \text{ for all } \ell \in \mathbb{L}, \tag{6.3}$$

with v_ℓ being continuous. Let also the continuous functions $f_1, \dots, f_{n+m} : \mathbb{R} \to \mathbb{R}$ be polynomially bounded and such that every f_i is either an odd increasing function on $\mathbb{R}$ or an even positive function, increasing on $[0, +\infty)$. Then the following inequalities hold for all $\tau_1, \dots, \tau_{n+m} \in [0, \beta]$, and all $\ell_1, \dots, \ell_{n+m} \in \Lambda$,

$$\int_\Omega \left(\prod_{i=1}^n f_i(\omega_{\ell_i}(\tau_i))\right) \pi_\Lambda\left({\rm d}\omega|0\right) \geq 0; \tag{6.4}$$

$$\begin{aligned}\int_\Omega \left(\prod_{i=1}^n f_i(\omega_{\ell_i}(\tau_i))\right) \cdot \left(\prod_{i=n+1}^{n+m} f_i(\omega_{\ell_i}(\tau_i))\right) \pi_\Lambda\left({\rm d}\omega|0\right) \\ \geq \int_\Omega \left(\prod_{i=1}^n f_i(\omega_{\ell_i}(\tau_i))\right) \pi_\Lambda\left({\rm d}\omega|0\right) \cdot \int_\Omega \left(\prod_{i=n+1}^{n+m} f_i(\omega_{\ell_i}(\tau_i))\right) \pi_\Lambda\left({\rm d}\omega|0\right).\end{aligned} \tag{6.5}$$

Given $\xi \in \Omega^{\rm t}$, $\Lambda \Subset \mathbb{L}$, and ℓ, ℓ', $\tau, \tau' \in [0, \beta]$, the pair correlation function is

$$K^{\Lambda}_{\ell\ell'}(\tau, \tau'|\xi) = \int_{\Omega} \omega_\ell(\tau)\omega_{\ell'}(\tau')\pi_\Lambda({\rm d}\omega|\xi) - \int_{\Omega} \omega_\ell(\tau)\pi_\Lambda({\rm d}\omega|\xi) \cdot \int_{\Omega} \omega_{\ell'}(\tau')\pi_\Lambda({\rm d}\omega|\xi). \tag{6.6}$$

Then, by (6.2),

$$K^{\Lambda}_{\ell\ell'}(\tau, \tau'|\xi) \geq 0, \tag{6.7}$$

which holds for all ℓ, ℓ', τ, τ', and $\xi \in \Omega^{\rm t}$. The following result is a version of the estimate (12.129), page 254 of [27], which for the Euclidean Gibbs measures may be proven by means of the lattice approximation.

Proposition 6.3. *Let V_ℓ be of the form (6.3) with $h_\ell = 0$ and the functions v_ℓ being convex. Then for all ℓ, ℓ', τ, τ' and for any $\xi \in \Omega^{\rm t}$ such that $\xi \geq 0$, it follows that*

$$K^{\Lambda}_{\ell\ell'}(\tau, \tau'|\xi) \leq K^{\Lambda}_{\ell\ell'}(\tau, \tau'|0). \tag{6.8}$$

Let us consider

$$\begin{aligned} U^{\Lambda}_{\ell_1\ell_2\ell_3\ell_4}(\tau_1, \tau_2, \tau_3, \tau_4) = & \int_{\Omega} \omega_{\ell_1}(\tau_1)\omega_{\ell_2}(\tau_2)\omega_{\ell_3}(\tau_3)\omega_{\ell_4}(\tau_4)\pi_\Lambda({\rm d}\omega|0) \\ & - K^{\Lambda}_{\ell_1\ell_2}(\tau_1, \tau_2|0)K^{\Lambda}_{\ell_3\ell_4}(\tau_3, \tau_4|0) - K^{\Lambda}_{\ell_1\ell_3}(\tau_1, \tau_3|0)K^{\Lambda}_{\ell_2\ell_4}(\tau_2, \tau_4|0) \\ & - K^{\Lambda}_{\ell_1\ell_4}(\tau_1, \tau_4|0)K^{\Lambda}_{\ell_2\ell_3}(\tau_2, \tau_3|0), \end{aligned} \tag{6.9}$$

which is the Ursell function for the measure $\pi_\Lambda(\cdot|0)$. The next statement gives the Gaussian domination and Lebowitz inequalities, see [4].

Proposition 6.4. *Let V_ℓ be of the form (6.3) with $h_\ell = 0$ and the functions v_ℓ being convex. Then for all $n \in \mathbb{N}$, $\ell_1, \ldots, \ell_{2n} \in \Lambda \Subset \mathbb{L}$, $\tau_1, \ldots, \tau_{2n} \in [0, \beta]$, it follows that*

$$\begin{aligned} & \int_{\Omega} \omega_{\ell_1}(\tau_1)\omega_{\ell_2}(\tau_2)\cdots\omega_{\ell_{2n}}(\tau_{2n})\pi_\Lambda({\rm d}\omega|0) \\ & \leq \sum_{\sigma} \prod_{j=1}^{n} \int_{\Omega} \omega_{\ell_{\sigma(2j-1)}}(\tau_{\sigma(2j-1)})\omega_{\ell_{\sigma(2j)}}(\tau_{\sigma(2j)})\pi_\Lambda({\rm d}\omega|0), \end{aligned} \tag{6.10}$$

where the sum runs through the set of all partitions of $\{1, \ldots, 2n\}$ onto unordered pairs. In particular,

$$U^{\Lambda}_{\ell_1\ell_2\ell_3\ell_4}(\tau_1, \tau_2, \tau_3, \tau_4) \leq 0. \tag{6.11}$$

6.2. **Proof of Theorem 3.5.** Given ℓ_0 and $b > 0$, let $\xi^{\ell_0} = (\xi^{\ell_0}_\ell)_{\ell \in \mathbb{L}}$ be the following constant (with respect to $\tau \in S_\beta$) configuration

$$\xi^{\ell_0}_\ell(\tau) = [b \log(1 + |\ell - \ell_0|)]^{1/2}. \tag{6.12}$$

Proof of Theorem 3.5: The existence and uniqueness of $\mu_\pm$ can be proven by literal repetition of the arguments used in [17] to proving Theorem IV.3. These measures are extreme, see the proof of Proposition V.1 in [17]. Now let us show how to construct $\mu_\pm$. Fix $\sigma \in (0, 1/2)$ and b obeying the condition $b > d/\lambda_\sigma$ (see the proof of Theorem 3.3). In view of (2.33), for any ℓ_0, ξ^{ℓ_0} belongs to $\Omega^{\rm t}$. It also belongs to $\Xi(\ell_0, b, \sigma)$ and for all $\xi \in \Xi(b, \sigma)$, one finds $\Delta \Subset \mathbb{L}$ such that $\xi_\ell(\tau) \leq \xi^{\ell_0}_\ell(\tau)$ for all $\tau \in [0, \beta]$ and $\ell \in \Delta^c$. Therefore, for any cofinal sequence

$\mathcal{L}$ and $\xi \in \Xi(b, \sigma)$, one finds $\Delta \in \mathcal{L}$ such that for all $\Lambda \in \mathcal{L}$, $\Delta \subset \Lambda$, one has $\pi_\Lambda(\cdot|\xi) \leq \pi_\Lambda(\cdot|\xi^{\ell_0})$. As was established in the proof of Theorem 3.1, every sequence $\{\pi_\Lambda(\cdot|\xi)\}_{\Lambda\in\mathcal{L}}$, $\xi \in \Xi(b, \sigma) \subset \Omega^{\mathrm{t}}$, is relatively compact in any $\mathcal{W}_\alpha$, $\alpha \in \mathcal{I}$, which by Lemmas 4.4, 4.5 yields its $\mathcal{W}^{\mathrm{t}}$-relative compactness. For a cofinal sequence $\mathcal{L}$, let μ^{ℓ_0} be any of the accumulating points of $\{\pi_\Lambda(\cdot|\xi^{\ell_0})\}_{\Lambda\in\mathcal{L}}$. By Lemma 2.11 $\mu^{\ell_0} \in \mathcal{G}^{\mathrm{t}}$ and by Lemma 5.2 μ^{ℓ_0} dominates every element of $\mathrm{ex}(\mathcal{G}^{\mathrm{t}})$. Hence, $\mu^{\ell_0} = \mu_+$ since the maximal element is unique. The same is true for the remaining accumulation points of $\{\pi_\Lambda(\cdot|\xi)\}_{\Lambda\in\mathcal{L}}$; thus, for every cofinal sequence $\mathcal{L}$ and for every ℓ_0, we have

$$\lim_{\mathcal{L}} \pi_\Lambda(\cdot|\xi^{\ell_0}) = \mu_+. \tag{6.13}$$

As the configuration (6.12) is constant with respect to $\tau \in S_\beta$, the kernel $\pi_\Lambda(\cdot|\xi^{\ell_0})$ may be considered as the one $\tilde{\pi}_\Lambda(\cdot|0)$ corresponding to the Schrödinger operator (2.11) with an additional site-dependent 'external field', i.e., to the operator

$$H_\Lambda - \sum_{\ell\in\Lambda} x_\ell \cdot [b \log(1 + |\ell - \ell_0|)]^{1/2}.$$

Then the Matsubara functions have the property (2.61), which yields by (6.13) the same property for the functions Γ^{μ_+}. Analogously, one sets $\mu_- = \lim_{\mathcal{L}} \pi_\Lambda(\cdot|-\xi^{\ell_0}) \in \mathrm{ex}(\mathcal{G}^{\mathrm{t}})$ and proves its shift invariance. The remaining properties can be established by repetition of the arguments used in [17] to prove Proposition V.3. $\square$.

Lemma 6.5. *The set $\mathcal{G}^{\mathrm{t}}$ is a singleton if and only if*

$$\int_\Omega \omega_\ell(0)\mu_+(\mathrm{d}\omega) = \int_\Omega \omega_\ell(0)\mu_-(\mathrm{d}\omega), \quad \textit{for any} \ \ \ell. \tag{6.14}$$

Proof. Since the measures $\mu_\pm$ are shift invariant, we have

$$\int_\Omega \omega_\ell(\tau)\mu_\pm(\mathrm{d}\omega) = \int_\Omega \omega_\ell(0)\mu_\pm(\mathrm{d}\omega), \qquad \text{for all} \ \ \tau \in [0, \beta]. \tag{6.15}$$

Certainly, (6.14) holds if $|\mathcal{G}^{\mathrm{t}}| = 1$. Let us prove the converse. One observes that each local bounded continuous function on Ω may be written as

$$\Omega \ni \omega \mapsto f(\omega_{\ell_1}(\tau_1), \ldots, \omega_{\ell_n}(\tau_n)),$$

with certain $n \in \mathbb{N}$, $\ell_1, \ldots, \ell_n \in \mathbb{L}$, $\tau_1, \ldots, \tau_n \in [0, \beta]$, and $f \in C_{\mathrm{b}}(\mathbb{R}^n)$. Obviously, the set of all local bounded continuous functions is a defining class for $\mathcal{P}(\Omega^{\mathrm{t}})$. Thus, the proof will be done if we show that all finite-dimensional projections

$$\nu^{\tau_1,\ldots,\tau_n;\pm}_{\ell_1,\ldots,\ell_n} \stackrel{\mathrm{def}}{=} \nu_{n;\pm},$$

of the measures $\mu_\pm$ coincide whenever (6.14) holds. Hence, we have to prove that $\nu_{1;+} = \nu_{1;-}$ implies $\nu_{n;+} = \nu_{n;-}$ for all $n \in \mathbb{N}$. To this end we consider the Wasserstein distance between the pairs $\nu_{n;\pm}$

$$R\,[\nu_{n,+}, \nu_{n,-}] = \inf \int_{\mathbb{R}^{2n}} |x - x'| P(\mathrm{d}x, \mathrm{d}x'), \tag{6.16}$$

where the infimum is taken over the set $\Pi^{(2n)}_\pm$ of all probability measures on $(\mathbb{R}^{2n}, \mathcal{B}(\mathbb{R}^{2n}))$ which marginal distributions are $\nu_{n,\pm}$, see [25]. The relation (6.2) defines an order on $\mathcal{P}(\Omega^{\mathrm{t}})$ and hence on $\mathcal{P}(\mathbb{R}^n)$. In this sense $\nu_{n,-}$ is dominated by its counterpart. Then by Strassen's theorem (Theorem 2.4 in [56]), there exists $\tilde{P} \in \Pi^{(2n)}_\pm$ such that

$$\tilde{P}\left(\left\{(x, x') \in \mathbb{R}^{2n} \mid x \geq x'\right\}\right) = 1.$$

Thereby,

$$\begin{aligned} R\,[\nu_{n,+},\nu_{n,-}] &\leq \int_{\mathbb{R}^{2n}} |x-x'|\tilde{P}(\mathrm{d}x,\mathrm{d}x'), \\ &\leq \sum_{i=1}^{n}\int_{\mathbb{R}^n} x^{(i)}\,[\nu_{n,+}(\mathrm{d}x)-\nu_{n,-}(\mathrm{d}x)] \\ &= \sum_{i=1}^{n}\int_{\mathbb{R}^n} x^{(i)}\,[\nu_{1,+}(\mathrm{d}x)-\nu_{1,-}(\mathrm{d}x)]\,, \end{aligned}$$

which completes the proof. □

Theorem 3.5 and the lemma just proven have the following

Corollary 6.6. *If $V_\ell(x)=V_\ell(-x)$ for all ℓ, the set $\mathcal{G}^{\mathrm{t}}$ is a singleton if and only if $\mu_+(\omega_\ell(0))=0$ for all ℓ.*

6.3. **Reference models.** We shall prove Theorems 3.6, 3.7 by comparing our model with two reference models, defined as follows. Let J and V be the same as in (3.10) and (3.11) respectively. For $\Lambda \Subset \mathbb{L}$, we set (c.f., (2.11))

$$H_\Lambda^{\mathrm{low}} = \sum_{\ell\in\Lambda}\left[H_\ell^{\mathrm{har}} + V(x_\ell)\right] - \frac{1}{2}\sum_{\ell,\ell'\in\Lambda} J\epsilon_{\ell\ell'}x_\ell x_{\ell'}, \tag{6.17}$$

where H_ℓ^{har} is given by (2.5) and $\epsilon_{\ell\ell'}=1$ if $|\ell-\ell'|=1$ and $\epsilon_{\ell\ell'}=0$ otherwise. Next, for $\Lambda\Subset\mathbb{L}$, we set

$$H_\Lambda^{\mathrm{upp}} = \sum_{\ell\in\Lambda}\left[H_\ell^{\mathrm{har}} + v(x_\ell^2)\right] - \frac{1}{2}\sum_{\ell,\ell'} J_{\ell\ell'}x_\ell x_{\ell'} = \sum_{\ell\in\Lambda}\tilde{H}_\ell - \frac{1}{2}\sum_{\ell,\ell'} J_{\ell\ell'}x_\ell x_{\ell'}, \tag{6.18}$$

where $\tilde{H}_\ell$ is defined by (3.19) and the interaction intensities $J_{\ell\ell'}$ are the same as in (2.11). Since both these models are particular cases of the model we consider, their sets of Euclidean Gibbs measures have the properties established by Theorems 3.1 – 3.5. By $\mu_\pm^{\mathrm{low}}$, $\mu_\pm^{\mathrm{upp}}$ we denote the corresponding extreme elements.

Remark 6.7. The self-interaction potentials of both reference models have the form (6.3) with the zero external field $h_\ell=0$ and the functions v_ℓ being convex. Hence, they obey the conditions of all the statements of subsection 6.1. The *low*-reference model is translation invariant. The *upp*-reference model is translation invariant if $J_{\ell\ell'}$ are invariant with respect to the translations of $\mathcal{L}$.

Lemma 6.8. *For every ℓ, it follows that*

$$\mu_+^{\mathrm{low}}(\omega_\ell(0)) \leq \mu_+(\omega_\ell(0)) \leq \mu_+^{\mathrm{upp}}(\omega_\ell(0)). \tag{6.19}$$

Proof. By (6.13) we have that for any $\mathcal{L}$,

$$\int_\Omega \omega_\ell(\tau)\mu_\pm(\mathrm{d}\omega) = \lim_{\mathcal{L}}\int_\Omega \omega_\ell(\tau)\pi_\Lambda(\mathrm{d}\omega|\pm\xi^{\ell_0}), \quad \text{for all } \tau\in[0,\beta]. \tag{6.20}$$

Thus, the proof will be done if we show that for all $\Lambda\Subset\mathbb{L}$ and $\ell\in\Lambda$,

$$\pi_\Lambda^{\mathrm{low}}(\omega_\ell(0)|\xi^{\ell_0}) \leq \pi_\Lambda(\omega_\ell(0)|\xi^{\ell_0}) \leq \pi_\Lambda^{\mathrm{upp}}(\omega_\ell(0)|\xi^{\ell_0}), \tag{6.21}$$

where $\pi_\Lambda^{\rm low}$, $\pi_\Lambda^{\rm upp}$ are the kernels for the reference models. First we prove the left-hand inequality in (6.21). For given $\Lambda \Subset \mathbb{L}$ and $t, s \in [0,1]$, we introduce

$$\begin{aligned}(6.22)\qquad \varpi_\Lambda^{(t,s)}({\rm d}\omega_\Lambda) = \frac{1}{Y(t,s)} \exp\Bigg(\frac{1}{2}\sum_{\ell,\ell'\in\Lambda} J\epsilon_{\ell\ell'}(\omega_\ell,\omega_{\ell'})_{L^2_\beta} + \sum_{\ell\in\Lambda}(\omega_\ell, \eta_\ell^{\ell_0,s})_{L^2_\beta} \\ -\sum_{\ell\in\Lambda}\int_0^\beta V(\omega_\ell(\tau)){\rm d}\tau + \frac{s}{2}\sum_{\ell,\ell'\in\Lambda}\left[J_{\ell\ell'} - J\epsilon_{\ell\ell'}\right](\omega_\ell,\omega_{\ell'})_{L^2_\beta} \\ - t\sum_{\ell\in\Lambda}\int_0^\beta \left[V_\ell(\omega_\ell(\tau)) - V(\omega_\ell(\tau))\right]{\rm d}\tau\Bigg)\chi_\Lambda({\rm d}\omega_\Lambda),\end{aligned}$$

where, see (6.12),

$$\begin{aligned}(6.23)\qquad \eta_\ell^{\ell_0,s}(\tau) &\stackrel{\rm def}{=} \sum_{\ell'\in\Lambda^c} J\epsilon_{\ell\ell'}\xi_{\ell'}^{\ell_0}(\tau) \\ &+ \quad s\sum_{\ell'\in\Lambda^c}\left[J_{\ell\ell'} - J\epsilon_{\ell\ell'}\right]\xi_{\ell'}^{\ell_0}(\tau) \geq \sum_{\ell'\in\Lambda^c} J\epsilon_{\ell\ell'}\xi_{\ell'}^{\ell_0}(\tau) > 0,\end{aligned}$$

which in fact is independent of τ, and

$$\begin{aligned}Y(t,s) = \int_{\Omega_\Lambda}\exp\Bigg(\frac{1}{2}\sum_{\ell,\ell'\in\Lambda} J\epsilon_{\ell\ell'}(\omega_\ell,\omega_{\ell'})_{L^2_\beta} + \sum_{\ell\in\Lambda}(\omega_\ell, \eta_\ell^{\ell_0,s})_{L^2_\beta} \\ -\sum_{\ell\in\Lambda}\int_0^\beta V(\omega_\ell(\tau)){\rm d}\tau + \frac{s}{2}\sum_{\ell,\ell'\in\Lambda}\left[J_{\ell\ell'} - J\epsilon_{\ell\ell'}\right](\omega_\ell,\omega_{\ell'})_{L^2_\beta} \\ - t\sum_{\ell\in\Lambda}\int_0^\beta \left[V_\ell(\omega_\ell(\tau)) - V(\omega_\ell(\tau))\right]{\rm d}\tau\Bigg)\chi_\Lambda({\rm d}\omega_\Lambda).\end{aligned}$$

Since the site-dependent 'external field' (6.23) is positive, the moments of the measure (6.22) obey the GKS inequalities. Therefore, for any $\ell \in \Lambda$, the function

$$(6.24)\qquad \phi(t,s) = \varpi_\Lambda^{(t,s)}(\omega_\ell(0)), \quad t,s \in [0,1],$$

is continuous and increasing in both variables. Indeed, taking into account (3.10), (3.11), and (3.14), we get

$$\begin{aligned}\frac{\partial}{\partial s}\phi(t,s) &= \sum_{\ell'\in\Lambda}\left[J_{\ell\ell'} - J\epsilon_{\ell\ell'}\right]\xi_{\ell'}^{\ell_0}(0) \\ &\times \int_0^\beta\left\{\varpi_\Lambda^{(t,s)}\left[\omega_\ell(0)\omega_{\ell'}(\tau)\right] - \varpi_\Lambda^{(t,s)}\left[\omega_\ell(0)\right]\cdot\varpi_\Lambda^{(t,s)}\left[\omega_{\ell'}(\tau)\right]\right\}{\rm d}\tau \\ &+ \frac{1}{2}\sum_{\ell_1,\ell_2\in\Lambda}\left[J_{\ell_1\ell_2} - J\epsilon_{\ell_1\ell_2}\right]\Big\{\varpi_\Lambda^{(t,s)}\Big[\omega_\ell(0)(\omega_{\ell_1},\omega_{\ell_2})_{L^2_\beta}\Big] \\ &- \varpi_\Lambda^{(t,s)}\left[\omega_\ell(0)\right]\cdot\varpi_\Lambda^{(t,s)}\Big[(\omega_{\ell_1},\omega_{\ell_2})_{L^2_\beta}\Big]\Big\} \geq 0, \\ \frac{\partial}{\partial t}\phi(t,s) &= \sum_{\ell'\in\Lambda}\int_0^\beta\Big\{\varpi_\Lambda^{(t,s)}\left(\omega_\ell(0)\cdot\left[V(\omega_{\ell'}(\tau)) - V_{\ell'}(\omega_{\ell'}(\tau))\right]\right) \\ &- \varpi_\Lambda^{(t,s)}\left[\omega_\ell(0)\right]\cdot\varpi_\Lambda^{(t,s)}\left[V(\omega_{\ell'}(\tau)) - V_{\ell'}(\omega_{\ell'}(\tau))\right]\Big\}{\rm d}\tau \geq 0.\end{aligned}$$

But by (6.22) and (6.24)

$$\phi(0,0) = \pi_\Lambda^{\rm low}(\omega_\ell(0)), \quad \phi(1,1) = \pi_\Lambda(\omega_\ell(0)),$$

which proves the left-hand inequality in (6.21). To prove the right-hand one we have to take the measure (6.22) with $s = 1$ and $v(x_\ell^2)$ instead of $V(x_\ell)$ and repeat the above steps taking into account (3.18). □

Corollary 6.9 (Comparison Criterion)**.** *The model considered undergoes a phase transition if the* low*-reference model does so. The uniqueness of tempered Euclidean Gibbs measures of the* upp*-reference model implies that* $|\mathcal{G}^{\rm t}| = 1$*.*

6.4. **Estimates for pair correlation functions.** For $\Delta \subset \Lambda$, $\ell, \ell' \in \Lambda$, $\tau, \tau' \in [0, \beta]$, and $t \in [0,1]$, let us consider

$$Q^\Lambda_{\ell\ell'}(\tau,\tau'|\Delta, t) = \int_{\Omega_\Lambda} \omega_\ell(\tau)\omega_{\ell'}(\tau')\varpi^{(t)}_{\Lambda,\Delta}({\rm d}\omega_\Lambda), \tag{6.25}$$

where this time we have set

$$\begin{aligned}
\varpi^{(t)}_{\Lambda,\Delta}({\rm d}\omega_\Lambda) = \frac{1}{Y_{\Lambda,\Delta}(t)} \exp\Bigg\{ \frac{1}{2}\sum_{\ell_1,\ell_2\in\Lambda\setminus\Delta} J_{\ell_1\ell_2}(\omega_{\ell_1},\omega_{\ell_2})_{L^2_\beta} \qquad\qquad \\
+t\left(\sum_{\ell_1\in\Delta}\sum_{\ell_2\in\Lambda\setminus\Delta} J_{\ell_1\ell_2}(\omega_{\ell_1},\omega_{\ell_2})_{L^2_\beta} + \frac{1}{2}\sum_{\ell_1,\ell_2\in\Delta} J_{\ell_1\ell_2}(\omega_{\ell_1},\omega_{\ell_2})_{L^2_\beta}\right) \\
-\sum_{\ell\in\Lambda}\int_0^\beta V_\ell(\omega_\ell(\tau)){\rm d}\tau\Bigg\}\chi_\Lambda({\rm d}\omega_\Lambda), \qquad\qquad \\
Y_{\Lambda,\Delta}(t) = \int_{\Omega_\Lambda} \exp\Bigg\{ \frac{1}{2}\sum_{\ell_1,\ell_2\in\Lambda\setminus\Delta} J_{\ell_1\ell_2}(\omega_{\ell_1},\omega_{\ell_2})_{L^2_\beta} \qquad\qquad \\
+t\left(\sum_{\ell_1\in\Delta}\sum_{\ell_2\in\Lambda\setminus\Delta} J_{\ell_1\ell_2}(\omega_{\ell_1},\omega_{\ell_2})_{L^2_\beta} + \frac{1}{2}\sum_{\ell_1,\ell_2\in\Delta} J_{\ell_1\ell_2}(\omega_{\ell_1},\omega_{\ell_2})_{L^2_\beta}\right) \\
-\sum_{\ell\in\Lambda}\int_0^\beta V_\ell(\omega_\ell(\tau)){\rm d}\tau\Bigg\}\chi_\Lambda({\rm d}\omega_\Lambda). \qquad\qquad
\end{aligned} \tag{6.26}$$

By literal repetition of the arguments used for proving Lemma 6.8 one proves the following

Proposition 6.10. *The above* $Q^\Lambda_{\ell\ell'}(\tau,\tau'|\Delta,t)$ *is an increasing continuous function of* $t \in [0,1]$*.*

Corollary 6.11. *Let the conditions of Proposition 6.2 be satisfied. Then for any pair* $\Lambda \subset \Lambda' \Subset \mathbb{L}$ *and for all* τ *and* ℓ*, the functions (6.2) obey the estimate*

$$K^\Lambda_{\ell\ell'}(\tau,\tau'|0) \leq K^{\Lambda'}_{\ell\ell'}(\tau,\tau'|0), \tag{6.27}$$

which holds for all $\ell,\ell' \in \Lambda$ *and* $\tau,\tau' \in [0,\beta]$*.*

Now we obtain bounds for the correlation functions of the reference models for a one-point $\Lambda = \{\ell\}$. Set

$$K^{\rm upp}_\ell(\tau,\tau') = \pi^{\rm upp}_\ell(\omega_\ell(\tau)\omega_\ell(\tau')|0), \quad K^{\rm low}_\ell(\tau,\tau') = \pi^{\rm low}_\ell(\omega_\ell(\tau)\omega_\ell(\tau')|0), \tag{6.28}$$

Both these functions are independent of ℓ. We recall that the parameter $\varDelta$ was defined by (3.20).

Lemma 6.12. *For every β, it follows that*

$$K_\ell^{\rm upp} \stackrel{\rm def}{=} \int_0^\beta K_\ell^{\rm upp}(\tau,\tau')\mathrm{d}\tau \leq 1/m\varDelta^2. \tag{6.29}$$

Proof. In view of (2.61) the above integral in independent of τ. By (2.60) and (2.62)

$$K_\ell^{\rm upp} = \frac{1}{\tilde{Z}_\ell}\int_0^\beta \mathrm{trace}\left\{x_\ell e^{-\tau\tilde{H}_\ell}x_\ell e^{-(\beta-\tau)\tilde{H}_\ell}\right\}\mathrm{d}\tau, \quad \tilde{Z}_\ell = \mathrm{trace}[e^{-\beta\tilde{H}_\ell}], \tag{6.30}$$

where the Schrödinger operator $\tilde{H}$ was defined in (3.19). Its spectrum $\{E_n\}_{n\in\mathbb{N}}$ determines by (3.20) the parameter $\varDelta$. Integrating in (6.30) we get

$$\begin{aligned} K_\ell^{\rm upp} &= \frac{1}{\tilde{Z}_\ell}\sum_{n,n'\in\mathbb{N}_0,\ n\neq n'} \left|(\psi_n, x_\ell\psi_{n'})_{L^2(\mathbb{R})}\right|^2 \frac{(E_n-E_{n'})(e^{-\beta E_{n'}}-e^{-\beta E_n})}{(E_n-E_{n'})^2} \\ &\leq \frac{1}{\tilde{Z}_\ell}\cdot\frac{1}{\varDelta^2}\sum_{n,n'\in\mathbb{N}_0} \left|(\psi_n, x_\ell\psi_{n'})_{L^2(\mathbb{R})}\right|^2 (E_n-E_{n'})(e^{-\beta E_{n'}}-e^{-\beta E_n}) \\ &= \frac{1}{\varDelta^2}\cdot\frac{1}{\tilde{Z}_\ell}\mathrm{trace}\left\{\left[x_\ell,\left[\tilde{H}_\ell,x_\ell\right]\right]e^{-\beta\tilde{H}_\ell}\right\} = \frac{1}{m\varDelta^2}, \end{aligned} \tag{6.31}$$

where ψ_n, $n\in\mathbb{N}_0$ are the eigenfunctions of $\tilde{H}_\ell$ and $[\cdot,\cdot]$ stands for commutator. □

For the functions $K_\ell^{\rm low}$, a representation like (6.30) is obtained by means of the following Schrödinger operator

$$\hat{H}_\ell = H_\ell^{\rm har} + V(x_\ell) = -\frac{1}{2m}\left(\frac{\partial}{\partial x_\ell}\right)^2 + \frac{a}{2}x_\ell^2 + V(x_\ell), \tag{6.32}$$

where m and a are the same as in (3.19) but V is given by (3.11). Thereby,

$$K_\ell^{\rm low}(0,0) = \mathrm{trace}[x_\ell^2\exp(-\beta\hat{H}_\ell)]/\mathrm{trace}[\exp(-\beta\hat{H}_\ell)] \stackrel{\rm def}{=} \hat{\varrho}(x_\ell^2). \tag{6.33}$$

Lemma 6.13. *Let t_* be the solution of (3.13). Then $K_\ell^{\rm low}(0,0)\geq t_*$.*

Proof. By Bogoliubov's inequality (see e.g., [73]), it follows that

$$\hat{\varrho}_\ell\left([p_\ell,[\hat{H}_\ell,p_\ell]]\right)\geq 0, \quad p_\ell = -\sqrt{-1}\frac{\partial}{\partial x_\ell},$$

which by (3.11), (3.12) yields

$$\begin{aligned} & a + 2b^{(1)} + \sum_{s=2}^r 2s(2s-1)b^{(s)}\hat{\varrho}\left[x_\ell^{2(s-1)}\right] \\ & = a + 2b^{(1)} + \sum_{s=2}^r 2s(2s-1)b^{(s)}\pi_\ell^{\rm low}\left[(\omega_\ell(0))^{2(s-1)}\right]\geq 0. \end{aligned}$$

Now we use the Gaussian domination inequality (6.10) and obtain $K_\ell^{\rm low}\geq t_*$. □

6.5. **Proof of Theorem 3.6.** In view of Corollary 6.9 we show that

$$\mu_{+}^{\mathrm{low}}(\omega_\ell(0)) > 0, \tag{6.34}$$

if the conditions of Theorem 3.6 are satisfied. Note that the left-hand side of (6.34) is independent of ℓ as the measure is translation invariant. The proof of (6.34) will be based on [43], see also Theorem 8.1 in [4]. Here the translation invariance and reflection positivity of the reference model are used to show that for $\beta > \beta_*$ the set of its Euclidean Gibbs measures contains a nonergodic element with respect to the group of translations of $\mathbb{L}$. This gives non-uniqueness and hence (6.34). To follow this line we construct periodic Euclidean Gibbs states by ntroducing (c.f., (2.21))

$$I_\Lambda^{\mathrm{per}}(\omega_\Lambda) = -\frac{J}{2}\sum_{\ell,\ell'\in\Lambda}\epsilon_{\ell\ell'}^{\Lambda}(\omega_\ell,\omega_{\ell'})_{L^2_\beta} + \sum_{\ell\in\Lambda}\int_0^\beta V\left(\omega_\ell(\tau)\right)\mathrm{d}\tau, \tag{6.35}$$

where

$$\Lambda = (-L, L]^d \bigcap \mathbb{L}, \quad L\in\mathbb{N}, \tag{6.36}$$

and $\epsilon_{\ell\ell'}^{\Lambda} = 1$ if $|\ell-\ell'|_\Lambda = 1$ and $\epsilon_{\ell\ell'}^{\Lambda} = 0$ otherwise. Here $|\ell-\ell'|_\Lambda = [|\ell_1-\ell_1'|_L^2 + \cdots + |\ell_d - \ell_d'|_L^2]^{1/2}$ and $|\ell_j - \ell_j'|_L = \min\{|\ell_j-\ell_j'|; L - |\ell_j-\ell_j'|\}$, $j=1,\ldots,d$. Clearly, I_Λ^{per} is invariant with respect to the translations of the torus which one obtains by identifying the opposite walls of the box (6.36). The energy functional I_Λ^{per} corresponds to the following periodic Schrödinger operator

$$H_\Lambda^{\mathrm{per}} = \sum_{\ell\in\Lambda}\left[H_\ell^{\mathrm{har}} + V(x_\ell)\right] - \frac{J}{2}\sum_{\ell,\ell'\in\Lambda}\epsilon_{\ell\ell'}^{\Lambda}x_\ell x_{\ell'}, \tag{6.37}$$

in the same sense as I_Λ given by (2.21) corresponds to H_Λ given by (2.11). Now we introduce the periodic kernels (c.f., (2.44))

$$\pi_\Lambda^{\mathrm{per}}(B) = \frac{1}{Z_\Lambda^{\mathrm{per}}}\int_\Omega \exp\left[-I_\Lambda^{\mathrm{per}}(\omega_\Lambda)\right]\mathbb{I}_B(\omega_\Lambda\times 0_{\Lambda^c})\chi_\Lambda(\mathrm{d}\omega_\Lambda), \quad B\in\mathcal{B}(\Omega), \tag{6.38}$$

$$Z_\Lambda^{\mathrm{per}} = \int_\Omega \exp\left[-I_\Lambda^{\mathrm{per}}(\omega_\Lambda)\right]\chi_\Lambda(\mathrm{d}\omega_\Lambda).$$

Thereby, for every box Λ, the above $\pi_\Lambda^{\mathrm{per}}$ is a probability measure on Ω^{t}. By $\mathcal{L}_{\mathrm{box}}$ we denote the sequence of boxes (6.36) indexed by $L\in\mathbb{N}$. For a given $\alpha\in\mathcal{I}$, let us choose $\vartheta, \varkappa > 0$ such that the estimate (4.14) holds.

Lemma 6.14. *For every box Λ, $\alpha\in\mathcal{I}$, and $\sigma\in(0,1/2)$, the measure $\pi_\Lambda^{\mathrm{per}}$ obeys the estimate*

$$\int_\Omega \|\omega\|_{\alpha,\sigma}^2 \pi_\Lambda^{\mathrm{per}}(\mathrm{d}\omega) \leq C_{6.39}. \tag{6.39}$$

Thereby, the sequence $\{\pi_\Lambda^{\mathrm{per}}\}_{\Lambda\in\mathcal{L}_{\mathrm{box}}}$ is $\mathcal{W}^{\mathrm{t}}$-relatively compact.

Proof. For $\ell\in\Lambda$ such that $\{\ell'\in\mathbb{L} \mid |\ell-\ell'| = 1\}\subset\Lambda$, we set $\Delta_\ell = \mathbb{L}\setminus\{\ell\}$. Then let ν_ℓ^Λ be the projection of $\pi_\Lambda^{\mathrm{per}}$ onto $\mathcal{B}(\Omega_{\Delta_\ell})$. Let also $\nu_\ell(\cdot|\xi)$, $\xi\in\Omega$ be the following probability measure on the single-spin space $\Omega_{\{\ell\}} = C_\beta$

$$\nu_\ell(\mathrm{d}\omega_\ell|\xi) = \frac{1}{N_\ell(\xi)}\exp\left\{J\sum_{\ell'}\epsilon_{\ell\ell'}(\omega_\ell,\xi_{\ell'})_{L^2_\beta} - \int_0^\beta V(\omega_\ell(\tau))\mathrm{d}\tau\right\}\chi(\mathrm{d}\omega_\ell). \tag{6.40}$$

Then (c.f., (2.46)) desintegrating $\pi_\Lambda^{\mathrm{per}}$ we get

$$\pi_\Lambda^{\mathrm{per}}(\mathrm{d}\omega) = \nu_\ell(\mathrm{d}\omega_\ell|\omega_{\Delta_\ell})\nu_\ell^\Lambda(\mathrm{d}\omega_{\Delta_\ell}). \tag{6.41}$$

As in Lemma 4.1 and Corollary 4.2 one proves that the measure $\nu_\ell(\cdot|\xi)$ obeys

$$\int_{C_\beta} \exp\left\{\lambda_\sigma|\omega_\ell|^2_{C^\sigma_\beta} + \varkappa|\omega_\ell|^2_{L^2_\beta}\right\}\nu_\ell(\mathrm{d}\omega_\ell|\omega_{\Delta_\ell}) \leq \exp\left\{C_{4.1} + \vartheta J\sum_{\ell'}\epsilon_{\ell\ell'}|\omega_{\ell'}|^2_{L^2_\beta}\right\},$$

where λ_σ, $\varkappa$, and ϑ are as in (4.1), (4.4). Now we integrate both sides of this inequality with respect to ν^Λ_ℓ and get, c.f., (4.13), (4.14)

$$n^{\mathrm{per}}_\ell(\Lambda) \stackrel{\mathrm{def}}{=} \log\left\{\int_\Omega \exp[\lambda_\sigma|\omega_\ell|^2_{C^\sigma_\beta} + \varkappa|\omega_\ell|^2_{L^2_\beta}]\pi^{\mathrm{per}}_\Lambda(\mathrm{d}\omega)\right\} \leq C_{4.7}.$$

Then the estimate (6.39) is obtained in the same way as (4.17) was proven. The relative $\mathcal{W}_\alpha$-compactness of $\{\pi^{\mathrm{per}}_\Lambda\}_{\Lambda\in\mathcal{L}_{\mathrm{per}}}$ follows from (6.39) and the compactness of the embeddings $\Omega_{\alpha,\sigma} \hookrightarrow \Omega_{\alpha'}$, $\alpha < \alpha'$. The $\mathcal{W}^{\mathrm{t}}$-compactness is a consequence of by Lemma 4.5. $\square$

Lemma 6.15. *Every $\mathcal{W}^{\mathrm{t}}$-accumulation point μ^{per} of the sequence $\{\pi^{\mathrm{per}}_\Lambda\}_{\Lambda\in\mathcal{L}_{\mathrm{per}}}$ is a Euclidean Gibbs measure of the* low-*reference model.*

Proof. Let $\mathcal{L} \subset \mathcal{L}_{\mathrm{per}}$ be the subsequence along which $\{\pi^{\mathrm{per}}_\Lambda\}_{\Lambda\in\mathcal{L}}$ converges to $\mu^{\mathrm{per}} \in \mathcal{P}(\Omega^{\mathrm{t}})$. Then $\{\nu^\Lambda_\ell\}_{\Lambda\in\mathcal{L}}$ converges to the projection of μ^{per} on $\mathcal{B}(\Omega_{\Delta_\ell})$. Employing the Feller property (Lemma 2.9) we pass in (6.41) to the limit along this $\mathcal{L}$ and apply both its sides to a function $f \in C_{\mathrm{b}}(\Omega^{\mathrm{t}})$. This yields that μ^{per} has the same one-point conditional distributions as the Euclidean Gibbs measures of the reference model. But according to Theorem 1.33 of [32], page 23, every Gibbs measure is uniquely defined by its conditional distributions corresponding to one-point sets $\Lambda = \{\ell\}$ only. $\square$

Now we are at a position to prove that (6.34) holds if $\beta > \beta_*$. As the *low*-reference model is translation invariant, in the case of uniqueness the unique element of the set of its Euclidean Gibbs states should be ergodic. Periodic states, which one obtains as accumulation points of the sequence $\{\pi^{\mathrm{per}}_\Lambda\}_{\Lambda\in\mathcal{L}_{\mathrm{box}}}$, are automatically translation invariant but they can be nonergodic. By the von Neumann ergodic theorem (page 244 of [73]), periodic Euclidean Gibbs states are nonergodic if

$$\liminf_{\mathcal{L}_{\mathrm{box}}} P_\Lambda(\beta) \stackrel{\mathrm{def}}{=} P(\beta) > 0, \tag{6.42}$$

where

$$P_\Lambda(\beta) = \int_\Omega \left(\frac{1}{|\Lambda|}\sum_{\ell\in\Lambda}\omega_\ell(0)\right)^2 \pi^{\mathrm{per}}_\Lambda(\mathrm{d}\omega). \tag{6.43}$$

Lemma 6.16. *Let the assumptions of Theorem 3.6 be satisfied and β_* be the solution of (3.17). Then for $\beta > \beta_*$ the order parameter $P(\beta)$ is positive.*

Proof. Here we mainly follow [43], see also Theorem 8.1 in [4]. By means of the infrared estimates one obtains (see equation (8.13) in [4]) that for every box Λ

$$\begin{aligned} P(\beta) \geq &\int_\Omega [\omega_\ell(0)]^2\pi^{\mathrm{per}}_\Lambda(\mathrm{d}\omega) \\ &- \frac{1}{2(2\pi)^d}\int_{(-\pi,\pi]^d}[2mJE(p)]^{-1/2}\coth\left(\beta\sqrt{JE(p)/2m}\right)\mathrm{d}p, \end{aligned} \tag{6.44}$$

where the function $E(p)$ is given by (3.15) whereas m and J are as in (3.17). Therefore, we have to find an appropriate bound for the first term in (6.44). For any ℓ, one can take the box Λ such that the Euclidean distance from this ℓ to Λ^c be greater than 1. Then by Corollary 6.11 and Lemma 6.13 one gets

$$\int_{\Omega} [\omega_\ell(0)]^2 \pi_\Lambda^{\rm per}({\rm d}\omega) \geq K_\ell^{\rm low}(0,0) \geq t_*.$$

Hence, for $\beta > \beta_*$, the right-hand side of (6.44) is positive. □

Proof of Theorem 3.6: By the latter lemma the periodic state of the *low*-reference model is nonergodic at $\beta > \beta_*$ yielding (6.34).□

6.6. **Proof of Theorem 3.7.** By Corollary 6.9 it is enough to prove the uniqueness for the *upp*-reference model, which by Lemma 6.5 is equivalent to

$$\mu_+^{\rm upp}(\omega_\ell(0)) = 0, \quad \text{for all } \ \beta > 0 \ \text{ and } \ \ell. \tag{6.45}$$

Given $\Lambda \Subset \mathbb{L}$, we introduce the matrix $(T^\Lambda_{\ell\ell'})_{\ell,\ell'\in\mathbb{L}}$ as follows. We set $T^\Lambda_{\ell\ell'} = 0$ if either of ℓ, ℓ' belongs to Λ^c. For $\ell, \ell' \in \Lambda$,

$$T^\Lambda_{\ell\ell'} = \sum_{\ell_1 \in \Lambda} J_{\ell\ell_1} \int_0^\beta \pi_\Lambda^{\rm upp}\left[\omega_{\ell_1}(\tau)\omega_{\ell'}(\tau')\,|0\right]{\rm d}\tau'. \tag{6.46}$$

By (2.61) the above integral is independent of τ.

Lemma 6.17. *If (3.21) is satisfied, there exists $\alpha \in \mathcal{I}$, such that for every $\Lambda \Subset \mathbb{L}$, the matrix $(T^\Lambda_{\ell\ell'})_{\ell,\ell'\in\mathbb{L}}$ defines a bounded operator in the Banach space $l^\infty(w_\alpha)$.*

Proof. The proof will be based on a generalization of the method used in [5] for proving Lemma 4.7. For $t \in [0,1]$, let $\varpi_\Lambda^{(t)} \in \mathcal{P}(\Omega_\Lambda))$ be defined by (6.26) with $\Delta = \Lambda$ and each $V_\ell(\omega_\ell(\tau))$ replaced by $v([\omega_\ell(\tau)]^2)$, where the latter function is the same as in (3.19). Then by (6.18)

$$\varpi_\Lambda^{(0)} = \prod_{\ell\in\Lambda} \pi_\ell^{\rm upp}(\cdot|0), \quad \varpi_\Lambda^{(1)} = \pi_\Lambda^{\rm upp}(\cdot|0), \quad \text{for any } \ \Lambda \Subset \mathbb{L}. \tag{6.47}$$

Thereby, we set

$$T^\Lambda_{\ell\ell'}(t) = \sum_{\ell_1} J_{\ell\ell_1} \int_0^\beta \varpi_\Lambda^{(t)}\left[\omega_{\ell_1}(\tau)\omega_{\ell'}(\tau')\right]{\rm d}\tau' \quad t \in [0,1]. \tag{6.48}$$

One can show that for every fixed ℓ, ℓ', the above $T^\Lambda_{\ell\ell'}(t)$ are differentiable on the interval $t \in (0,1)$ and continuous at its endpoints where (see (6.29))

$$T^\Lambda_{\ell\ell'}(0) = J_{\ell\ell'} K_{\ell'}^{\rm upp} \leq J_{\ell\ell'}/m\varDelta^2, \qquad T^\Lambda_{\ell\ell'}(1) = T^\Lambda_{\ell\ell'}. \tag{6.49}$$

Computing the derivative we get

$$\begin{aligned} \frac{\partial}{\partial t} T^\Lambda_{\ell\ell'}(t) &= \frac{1}{2} \sum_{\ell_1,\ell_2,\ell_3} J_{\ell\ell_1} J_{\ell_2\ell_3} \int_0^\beta \int_0^\beta U^\Lambda_{\ell\ell'\ell_2\ell_3}(t,\tau,\tau',\tau_1,\tau_1){\rm d}\tau'{\rm d}\tau_1 \\ &+ \sum_{\ell_1} T^\Lambda_{\ell\ell_1}(t) T^\Lambda_{\ell_1\ell'}(t), \end{aligned} \tag{6.50}$$

where $U^\Lambda_{\ell\ell'\ell_1\ell_2}(t,\tau,\tau',\tau_1,\tau_1)$ is the Ursell function which obeys the estimate (6.11) since the function v is convex. Hence, except for the trivial case $J_{\ell\ell'} \equiv 0$, the first term in (6.50) is strictly negative. Bearing in mind (3.21) let us show that there exists $\alpha \in \mathcal{I}$ such that

$$\hat{J}_0 < \hat{J}_\alpha < m\varDelta^2. \tag{6.51}$$

Here the cases of $J_{\ell\ell'}$ obeying (2.29) and the power-like decaying $J_{\ell\ell'}$ should be considered separately. In the first case we have $\mathcal{I} = (0, \overline{\alpha})$ and the above α exist in view of

$$\lim_{\alpha \to 0+} \hat{J}_\alpha = \hat{J}_0, \tag{6.52}$$

which readily follows from (2.29), (2.30). In the second case the weights are defined by (2.32) with a positive ε, which we are going to use to ensure (6.51). To indicate the dependence of $\hat{J}_\alpha$ on ε we write $\hat{J}^{(\varepsilon)}_\alpha$. Simple calculations yield

$$0 < \hat{J}^{(\varepsilon)}_\alpha - \hat{J}_0 \leq \varepsilon \alpha d \hat{J}^{(1)}_\alpha .$$

Thereby, we fix $\alpha \in \mathcal{I}$ and choose ε to obey $\varepsilon < m\varDelta^2/\alpha d \hat{J}^{(1)}_\alpha$. This yields (6.51).

Let us consider the following Cauchy problem

$$\frac{\partial}{\partial t} L_{\ell\ell'}(t) = \sum_{\ell_1} L_{\ell\ell_1}(t) L_{\ell_1\ell'}(t), \quad L_{\ell\ell'}(0) = \lambda J_{\ell\ell'}, \quad \ell, \ell' \in \mathbb{L}, \tag{6.53}$$

where $\lambda \in (1/m\varDelta^2, 1/\hat{J}_\alpha)$, with $\alpha \in \mathcal{I}$ chosen to obey (6.51). For such α, one can solve the problem (6.53) in the space $l^\infty(w_\alpha)$ (see Remark 2.6) and obtain

$$L(t) = \lambda J \left[I - \lambda t J \right]^{-1}, \quad \|L(t)\|_{l^\infty(w_\alpha)} \leq \frac{\lambda \hat{J}_\alpha}{1 - \lambda t \hat{J}_\alpha}. \tag{6.54}$$

where I is the identity operator. Now let us compare (6.50) and (6.53) considering the former expression as a differential equation subject to the initial condition (6.49). Since the first term in (6.50) is negative, one can apply Theorem V, page 65 of [77] and obtain $T^\Lambda_{\ell\ell'} < L_{\ell\ell'}(1)$, which in view of (6.54) yields the proof. $\square$

Proof of Theorem 3.7: For $\ell, \ell_0, \Lambda \Subset \mathbb{L}$, such that $\ell \in \Lambda$, and $t \in [0, 1]$, we set

$$\psi_\Lambda(t) = \int_\Omega \omega_\ell(0) \pi^{\rm upp}_\Lambda ({\rm d}\omega | t\xi^{\ell_0}), \tag{6.55}$$

where ξ^{ℓ_0} is the same as in (6.12). The function ψ_Λ is obviously differentiable on the interval $t \in (-1, 1)$ and continuous at its endpoints. Then

$$0 \leq \psi_\Lambda(1) \leq \sup_{t \in [0,1]} \psi'_\Lambda(t). \tag{6.56}$$

The derivative is

$$\psi'_\Lambda(t) = \sum_{\ell_1 \in \Lambda, \ \ell_2 \in \Lambda^c} J_{\ell\ell_1} \int_0^\beta \pi^{\rm upp}_\Lambda \left[\omega_{\ell_1}(0) \omega_{\ell_2}(\tau) \left| t\xi^{\ell_0} \right. \right] \eta_{\ell_2} {\rm d}\tau, \tag{6.57}$$

where the 'external field' $\eta_{\ell'} = [b \log(1 + |\ell' - \ell_0|)]^{1/2}$ is positive at each site. Thus, we may use (6.8) and obtain

$$\psi'_\Lambda(t) \leq \sum_{l' \in \Lambda^c} T^\Lambda_{\ell\ell'} \eta_{\ell'}. \tag{6.58}$$

By Assumption 2.5 (b), $\eta \in l^\infty(w_\alpha)$ with any $\alpha > 0$, then employing Lemma 6.17, the estimate (6.54) in particular, we conclude that the right-hand side of (6.58) tends to zero as $\Lambda \nearrow \mathbb{L}$, which by (6.20) and (6.55), (6.56) yields (6.45). $\square$

7. Uniqueness at Nonzero External Field

In statistical mechanics phase transitions may be associated with nonanalyticity of thermodynamic characteristics considered as functions of the external field h. In special cases one can oversee at which values of h this nonanaliticity can occur. The Lee-Yang theorem states that the only such value is $h = 0$; hence, no phase transitions can occur at nonzero h. In the theory of classical lattice models these arguments were applied in [52, 53, 54]. We refer also to sections 4.5, 4.6 in [33] and sections IX.3 – IX.5 in [71] where applications of such arguments in quantum field theory and classical statistical mechanics are discussed.

In the case of lattice models with the single-spin space $\mathbb{R}$ the validity of the Lee-Yang theorem depends on the properties of the self-interaction potentials. For the polynomials $V(x) = x^4 + ax^2$, $a \in \mathbb{R}$, the Lee-Yang theorem holds, see e.g., Theorem IX.15 on page 342 in [71]. But no other examples of this kind were known, see the discussion on page 71 in [33]. Below we give a sufficient condition for self-interaction potentials to have the corresponding property and discuss some examples. Here we use the family $\mathcal{F}_{\mathrm{Laguerre}}$ defined by (3.23). We also prove a number of lemmas, which allow us to employ the arguments based on the Lee-Yang theorem to our quantum model and hence to prove Theorem 3.8.

7.1. **The Lee-Yang property.** Recall that the elements of $\mathcal{F}_{\mathrm{Laguerre}}$ can be continued to entire functions $\varphi : \mathbb{C} \to \mathbb{C}$, which have no zeros outside of $(-\infty, 0]$.

Definition 7.1. *A probability measure ν on the real line is said to have the Lee-Yang property if there exists $\varphi \in \mathcal{F}_{\mathrm{Laguerre}}$ such that*

$$\int_{\mathbb{R}} \exp(xy)\nu(\mathrm{d}y) = \varphi(x^2).$$

In [46], see also Theorem 2.3 in [50], the following fact was proven.

Proposition 7.2. *Let the function $u : \mathbb{R} \to \mathbb{R}$ be such that for a certain $b \geq 0$, its derivative obeys the condition $b + u' \in \mathcal{F}_{\mathrm{Laguerre}}$. Then the probability measure*

$$\nu(\mathrm{d}y) = C \exp[-u(y^2)]\mathrm{d}y, \tag{7.1}$$

has the Lee-Yang property.

This statement gives a sufficient condition, the lack of which was mentioned on page 71 of the book [33]. The example of a polynomial given there for which the corresponding classical models undergo phase transitions at nonzero h, in our notations is $u(t) = t^3 - 2t^2 + (\alpha+1)t$, $\alpha > 0$. It certainly does not meet the condition of Proposition 7.2. Turning to the model described by Theorem 3.8 we note that, for $v(t) = t^3 + b^{(2)}t^2 + b^{(1)}t$, the function $u(t) = v(t) + at/2$ obeys the conditions of Proposition 7.2 if and only if $b^{(2)} \geq 0$ and $b^{(1)} + a/2 \leq [b^{(2)}]^2/3$. Therefore, according to Theorem 3.8 we have $|\mathcal{G}^{\mathrm{t}}| = 1$ at $h \neq 0$ and $2b^{(1)} + a < 0$, $b^{(2)} \geq 0$. On the other hand, for this model, by Theorem 3.6 one has a phase transition at $h = 0$ and the same coefficients of v.

Set

$$f(h^2) = \int_{\mathbb{R}^n} \exp\left[h\sum_{i=1}^n x_i + \sum_{i,j=1}^n M_{ij}x_i x_j \right] \prod_{i=1}^n \nu(\mathrm{d}x_i), \quad h \in \mathbb{R}. \tag{7.2}$$

By Theorem 3.2 of [55], we have the following

Proposition 7.3. *If in (7.2) $M_{ij} \geq 0$ for all $i, j = 1, \ldots, n$, and the measure ν is as in Proposition 7.2, then the function f, if exists, belongs to $\mathcal{F}_{\rm Laguerre}$. It certainly exists if u' is not constant.*

Now let the potential V obey the conditions of Theorem 3.8. Then the partition function $Z_\Lambda(0)$ (here 0 means $\xi = 0$) given by (2.42) is an even function of h. Define

$$p_\Lambda(h) = \frac{1}{|\Lambda|} \log Z_\Lambda(0), \quad \varphi_\Lambda(h^2) = p_\Lambda(h). \tag{7.3}$$

Lemma 7.4. *If V obeys the conditions of Theorem 3.8, the function* $\exp(|\Lambda|\varphi_\Lambda)$ *belongs to $\mathcal{F}_{\rm Laguerre}$.*

Proof. With the help of the lattice approximation technique the function $\exp(|\Lambda|\varphi_\Lambda)$ may be approximated by f_N, $N \in \mathbb{N}$, having the form (7.2) with the measures ν having of the form (7.1) with $u(t) = v(t) + at/2$, v is as in (3.24), and non-negative M_{ij} (see Theorem 5.2 in [4]). For every $h \in \mathbb{R}$, $f_N(h^2) \to \exp\big(|\Lambda|\varphi_\Lambda(h^2)\big)$ as $N \to +\infty$. The entire functions f_N are ridge, with the ridge $[0, +\infty)$. For sequences of such functions, their point-wise convergence on the ridge implies via the Vitali theorem (see e.g., [71]) the uniform convergence on compact subsets of $\mathbb{C}$, which yields the property stated (for more details, see [47, 51]). $\square$

7.2. **Existence of pressure.** The results obtained in this subsection are valid for any translation invariant model with $\nu = 1$, obeying Assumption 2.2 and hence possessing the properties described by Theorems 3.1 – 3.4.

Along with (7.3) we introduce

$$p_\Lambda(h, \xi) = \frac{1}{|\Lambda|} \log Z_\Lambda(\xi), \quad \xi \in \Omega^{\rm t}. \tag{7.4}$$

Then $p_\Lambda(h) = p_\Lambda(h, 0)$. For $\mu \in \mathcal{G}^{\rm t}$, we set

$$p_\Lambda^\mu(h) = \int_\Omega p_\Lambda(h, \xi)\mu({\rm d}\xi). \tag{7.5}$$

If for a cofinal sequence $\mathcal{L}$, the limit

$$p^\mu(h) \stackrel{\rm def}{=} \lim_{\mathcal{L}} p_\Lambda^\mu(h), \tag{7.6}$$

exists, we shall call it pressure in the Gibbs state μ. We shall also consider $\lim_{\mathcal{L}} p_\Lambda(h)$ with $p_\Lambda(h)$ given by (7.3). To obtain such limits we impose certain conditions on the sequences $\mathcal{L}$. Given $l = (l_1, \ldots l_d)$, $l' = (l_1', \ldots l_d') \in \mathbb{L} = \mathbb{Z}^d$, such that $l_j < l_j'$ for all $j = 1, \ldots, d$, we set

$$\Gamma = \{\ell \in \mathbb{L} \mid l_j \leq \ell_j \leq l_j', \ \ \text{for all } j = 1, \ldots, d\}. \tag{7.7}$$

For this parallelepiped, let $\mathfrak{G}(\Gamma)$ be the family of all pair-wise disjoint translates of Γ which covers $\mathbb{L}$. Then for $\Lambda \Subset \mathbb{L}$, we set $N_-(\Lambda|\Gamma)$ (respectively, $N_+(\Lambda|\Gamma)$) to be the number of the elements of $\mathfrak{G}(\Gamma)$ which are contained in Λ (respectively, which have non-void intersections with Λ). Then we introduce the following (see [67])

Definition 7.5. *Given $\mathcal{L}$ is a van Hove sequence if for every Γ,*

$$(a) \quad \lim_{\mathcal{L}} N_-(\Lambda|\Gamma) = +\infty; \qquad (b) \quad \lim_{\mathcal{L}} \left(N_-(\Lambda|\Gamma)/N_+(\Lambda|\Gamma)\right) = 1. \tag{7.8}$$

Given $R > 0$ and $\Lambda \Subset \mathbb{L}$, let $\partial_R^+ \Lambda$ be the set of all $\ell \in \Lambda^c$, such that $\mathrm{dist}(\ell, \Lambda) \leq R$. Then for a van Hove sequence $\mathcal{L}$ and any $R > 0$, one has $\lim_{\mathcal{L}} |\partial_R^+ \Lambda|/|\Lambda| = 0$, yielding

$$\lim_{\mathcal{L}} \frac{1}{|\Lambda|} \sum_{\ell \in \Lambda, \ell' \in \Lambda^c} J_{\ell\ell'} = 0. \tag{7.9}$$

The existence of van Hove sequences means amenability of the graph $(\mathbb{L}, E)$, E being the set of all pairs ℓ, ℓ', such that $|\ell - \ell'| = 1$. For nonamenable graphs, phase transitions with $h \neq 0$ are possible; hence, statements like Theorem 3.8 do not hold, see [40, 57]. Now we are at a position to prove the existence of the pressure for all $\mu \in \mathcal{G}^{\mathrm{t}}$. It will be done in two subsequent lemmas.

Lemma 7.6. *The limiting pressure* $p(h) \stackrel{\mathrm{def}}{=} \lim_{\mathcal{L}} p_\Lambda(h)$ *exists for every van Hove sequence* $\mathcal{L}$. *It is independent of the particular choice of* $\mathcal{L}$.

Proof. For $t \geq 0$, $\xi \in \Omega^{\mathrm{t}}$, and $\Delta \subset \Lambda$, let $\varpi^{(t)}_{\Lambda,\Delta}$, $Y_{\Lambda,\Delta}(t)$ be defined by (6.26) with the potentials $V_\ell = V$ having the form (3.24). Then we define

$$f_{\Lambda,\Delta}(t) = \frac{1}{|\Lambda|} \log Y_{\Lambda,\Delta}(t), \quad t \geq 0. \tag{7.10}$$

This function is differentiable and

$$\begin{aligned} g_{\Lambda,\Delta}(t) \stackrel{\mathrm{def}}{=} f'_{\Lambda,\Delta}(t) &= \frac{1}{2|\Lambda|} \sum_{\ell,\ell' \in \Delta} J_{\ell\ell'} \varpi^{(t)}_{\Lambda,\Delta}[(\omega_\ell, \omega_{\ell'})_{L^2_\beta}] \\ &+ \frac{1}{|\Lambda|} \sum_{\ell \in \Delta, \ell' \in \Lambda \setminus \Delta} J_{\ell\ell'} \varpi^{(t)}_{\Lambda,\Delta}[(\omega_\ell, \omega_{\ell'})_{L^2_\beta}] \geq 0. \end{aligned} \tag{7.11}$$

Here we used that $\varpi^{(t)}_{\Lambda,\Delta}[(\omega_\ell, \omega_{\ell'})_{L^2_\beta}] \geq 0$, which follows from the GKS inequality (6.4). The function $g_{\Lambda,\Delta}$ is also differentiable and

$$g'_{\Lambda,\Delta}(t) \geq 0, \tag{7.12}$$

which may be proven similarly by means of the GKS inequality (6.5). Therefore,

$$f_{\Lambda,\Delta}(0) \leq f_{\Lambda,\Delta}(1) \leq g_{\Lambda,\Delta}(1). \tag{7.13}$$

Now we take here $\Delta = \Lambda$ and obtain by (4.16) that for any $\alpha \in \mathcal{I}$,

$$\log Y_{\{\ell\},\{\ell\}}(0) \leq p_\Lambda(h) \leq \hat{J}_0 C_{4.16}(0)/2. \tag{7.14}$$

By the translation invariance the lower bound in (7.14) is independent of ℓ. Therefore, the set $\{p_\Lambda(h)\}_{\Lambda \Subset \mathbb{L}}$ has accumulation points. For one of them, $p(h)$, let $\{\Gamma_n\}_{n \in \mathbb{N}}$ be the sequence of parallelepipeds such that $p_{\Gamma_n}(h) \to p(h)$ as $n \to +\infty$. Let also $\mathcal{L}$ be a van Hove sequence. Given $n \in \mathbb{N}$ and $\Lambda \in \mathcal{L}$, let $\mathfrak{L}_n^-(\Lambda) \subset \mathfrak{G}(\Gamma_n)$ (respectively, $\mathfrak{L}_n^+(\Lambda) \subset \mathfrak{G}(\Gamma_n)$) consist of the translates of Γ_n which are contained in Λ (respectively, which have non-void intersections with Λ). Let also

$$\Lambda_n^\pm = \bigcup_{\Gamma \in \mathfrak{L}_n^\pm} \Gamma. \tag{7.15}$$

Now we take in (7.10) first $\Delta = \Lambda_n^-$, then $\Delta = \Lambda$, $\Lambda = \Lambda_n^+$ and obtain by (7.13)

$$\frac{|\Lambda_n^-|}{|\Lambda|} p_{\Lambda_n^-}(h) \leq p_\Lambda(h) \leq \frac{|\Lambda_n^+|}{|\Lambda|} p_{\Lambda_n^+}(h). \tag{7.16}$$

Let us estimate $p_{\Lambda_n^\pm}(h) - p_{\Gamma_n}(h)$. To this end we introduce for $t \geq 0$, c.f., (6.26),

$$
\begin{aligned}
(7.17) \qquad X_{\Lambda_n^-}(t) &= \int_\Omega \exp\left\{ \frac{1}{2} \sum_{\Gamma \in \mathfrak{L}_n^-} \sum_{\ell,\ell' \in \Gamma} J_{\ell\ell'} (\omega_\ell, \omega_{\ell'})_{L^2_\beta} \right. \\
&+ t \sum_{\Gamma,\Gamma' \in \mathfrak{L}_n^-,\ \Gamma \neq \Gamma'} \sum_{\ell \in \Gamma} \sum_{\ell' \in \Gamma'} J_{\ell\ell'} (\omega_\ell, \omega_{\ell'})_{L^2_\beta} \\
&+ \left. \sum_{\ell \in \Lambda_n^-} \int_0^\beta \left[h\omega_\ell(\tau) - v([\omega_\ell(\tau)]^2) \right] \mathrm{d}\tau \right\} \chi_\Lambda(\mathrm{d}\omega),
\end{aligned}
$$

and

$$
(7.18) \qquad f_{\Lambda_n^-}(t) = \frac{1}{|\Lambda_n^-|} \log X_{\Lambda_n^-}(t).
$$

Then

$$
(7.19) \qquad f_{\Lambda_n^-}(1) = p_{\Lambda_n^-}(h), \quad f_{\Lambda_n^-}(0) = \frac{|\Gamma_n|}{|\Lambda_n^-|} \sum_{\Gamma \in \mathfrak{L}_n^-} p_\Gamma(h) = p_{\Gamma_n}(h).
$$

Observe that $p_\Gamma(h) = p_{\Gamma_n}(h)$ for all $\Gamma \in \mathfrak{G}(\Gamma_n)$ follows from the translation invariance of the model. Thereby,

$$
\begin{aligned}
(7.20) \qquad 0 &\leq p_{\Lambda_n^-}(h) - p_{\Gamma_n}(h) \leq f'_{\Lambda_n^-}(1) \\
&= \frac{1}{|\Lambda_n^-|} \sum_{\Gamma,\Gamma' \in \mathfrak{L}_n^-,\ \Gamma \neq \Gamma'} \sum_{\ell \in \Gamma} \sum_{\ell' \in \Gamma'} J_{\ell\ell'} \pi_{\Lambda_n^-} \left((\omega_\ell, \omega_{\ell'})_{L^2_\beta} | 0 \right) \\
&\leq \frac{1}{|\Lambda_n^-|} \sum_{\Gamma \in \mathfrak{L}_n^-} \sum_{\ell \in \Gamma} \sum_{\ell' \in \Gamma^c} J_{\ell\ell'} \pi_{\Lambda_n^-} \left((\omega_\ell, \omega_{\ell'})_{L^2_\beta} | 0 \right) \\
&\leq \hat{J}(\Gamma_n) C_{4.16}(0),
\end{aligned}
$$

where we used the estimate (4.16) and set

$$
(7.21) \qquad \hat{J}(\Gamma_n) = \frac{1}{|\Gamma_n|} \sum_{\ell \in \Gamma_n} \sum_{\ell' \in \Gamma_n^c} J_{\ell\ell'} = \frac{1}{|\Gamma|} \sum_{\ell \in \Gamma} \sum_{\ell' \in \Gamma^c} J_{\ell\ell'}, \quad \text{for every } \Gamma \in \mathfrak{G}(\Gamma_n).
$$

In deriving (7.20) we took into account that the function (7.18) has positive first and second derivatives, c.f., (7.11) and (7.12). By literal repetition one proves that (7.20) holds also for $p_{\Lambda_n^+}(h) - p_{\Gamma_n}(h)$. In view of (7.9) the above $\hat{J}(\Gamma_n)$ may be made arbitrarily small by taking big enough Γ_n. Thereby, for any $\varepsilon > 0$, one can choose $n \in \mathbb{N}$ such that the following estimates hold (recall that $p_{\Gamma_n} \to p$ as $n \to +\infty$)

$$
(7.22) \qquad |p_{\Gamma_n}(h) - p(h)| < \varepsilon/3, \quad 0 \leq p_{\Lambda_n^-}(h) - p_{\Gamma_n}(h) \leq p_{\Lambda_n^+}(h) - p_{\Gamma_n}(h) < \varepsilon/3.
$$

As $\mathcal{L}$ is a van Hove sequence, one can pick up $\Lambda \in \mathcal{L}$ such that

$$
\max \left\{ \left(\frac{|\Lambda_n^+|}{|\Lambda|} - 1 \right) p_{\Lambda_n^+}(h); \left(1 - \frac{|\Lambda_n^-|}{|\Lambda|} \right) p_{\Lambda_n^+}(h) \right\} < \varepsilon/3,
$$

which is possible in view of (7.14). Then for the chosen n and $\Lambda \in \mathcal{L}$, one has

$$
\begin{aligned}
|p_\Lambda(h) - p(h)| &\leq |p_{\Gamma_n}(h) - p(h)| + p_{\Lambda_n^+}(h) - p_{\Gamma_n}(h) \\
&+ \max \left\{ \left(\frac{|\Lambda_n^+|}{|\Lambda|} - 1 \right) p_{\Lambda_n^+}(h); \left(1 - \frac{|\Lambda_n^-|}{|\Lambda|} \right) p_{\Lambda_n^+}(h) \right\} < \varepsilon,
\end{aligned}
$$

which obviously holds also for all $\Lambda' \in \mathcal{L}$ such that $\Lambda \subset \Lambda'$. $\square$

Lemma 7.7. *For every $\mu \in \mathcal{G}^{\rm t}$ and any van Hove sequence $\mathcal{L}$,*

$$\lim_{\mathcal{L}} p^{\mu}_{\Lambda}(h) = p(h).$$

Proof. By the Jensen inequality one obtains for $t_1, t_2 \in \mathbb{R}$, $\xi \in \Omega^{\rm t}$,

$$Z_\Lambda((t_1+t_2)\xi) \geq Z_\Lambda(t_1\xi)\exp\left\{ t_2 \sum_{\ell\in\Lambda, \ell'\in\Lambda^c} J_{\ell\ell'}\pi_\Lambda\left[(\omega_\ell, \xi_{\ell'})_{L^2_\beta}\,|t_1\xi\right]\right\}.$$

We set here first $t_1 = 0$, $t_2 = 1$, then $t_1 = -t_2 = 1$, and obtain after taking logarithm and dividing by $|\Lambda|$

$$\begin{aligned} p_\Lambda(h) \quad + \quad & \frac{1}{|\Lambda|}\sum_{\ell\in\Lambda,\ell'\in\Lambda^c} J_{\ell\ell'}\pi_\Lambda\left[(\omega_\ell,\xi_{\ell'})_{L^2_\beta}|0\right] \leq p_\Lambda(h,\xi) \\ & \leq p_\Lambda(h) + \frac{1}{|\Lambda|}\sum_{\ell\in\Lambda,\ell'\in\Lambda^c} J_{\ell\ell'}\pi_\Lambda\left[(\omega_\ell,\omega_{\ell'})_{L^2_\beta}|\xi\right], \end{aligned} \tag{7.23}$$

where we used that $\pi_\Lambda\left[(\omega_\ell,\omega_{\ell'})_{L^2_\beta}|\xi\right] = \pi_\Lambda\left[(\omega_\ell,\xi_{\ell'})_{L^2_\beta}|\xi\right]$, see (2.44). Thereby, we integrate (7.23) with respect to $\mu \in \mathcal{G}^{\rm t}$, take into account (2.51), and obtain after some calculations the following

$$\begin{aligned} p_\Lambda(h) \quad - \quad & \frac{1}{|\Lambda|}\sum_{\ell\in\Lambda,\ell'\in\Lambda^c} J_{\ell\ell'}\pi_\Lambda\left(|\omega_\ell|_{L^2_\beta}\,|0\right)\mu\left(|\xi_{\ell'}|_{L^2_\beta}\right) \leq p^\mu_\Lambda \\ & \leq p_\Lambda(h) + \frac{1}{|\Lambda|}\sum_{\ell\in\Lambda,\ell'\in\Lambda^c} J_{\ell\ell'}\mu\left((\omega_\ell,\omega_{\ell'})_{L^2_\beta}\right). \end{aligned} \tag{7.24}$$

By means of Theorem 3.2 (respectively, Lemma 4.4), one estimates $\mu\left((\omega_\ell,\omega_{\ell'})_{L^2_\beta}\right)$, $\mu\left(|\xi_{\ell'}|_{L^2_\beta}\right)$ (respectively, $\pi_\Lambda(|\omega_\ell|_{L^2_\beta}|0)$) by positive constants independent of ℓ, ℓ'. Thereby, the property stated follows from (7.9) and Lemma 7.6. $\square$

7.3. **Proof of Theorem 3.8.** By Lemma 7.4, for every $\Lambda \Subset \mathbb{L}$, $p_\Lambda(h)$ can be extended to a function of $h \in \mathbb{C}$, holomorphic in the right and left open half-planes. By standard arguments, see e.g., Lemma 39, page 34 of [47], and Lemma 7.6 it follows that the limit of such extensions $p(h)$ is holomorphic in certain subsets of those half-planes containing the real line, except possibly for the point $h = 0$. Therefore, $p(h)$ is differentiable at each $h \neq 0$. This yields by Lemma 7.7 that for every $\mu \in \mathcal{G}^{\rm t}$, the corresponding $p^\mu(h)$ enjoys this property and the derivatives of all these functions coincide at every $h \neq 0$. In particular,

$$\frac{\partial}{\partial h}p^{\mu_+}(h) = \frac{\partial}{\partial h}p^{\mu_-}(h). \tag{7.25}$$

For every $\mu \in \mathcal{G}^{\mathrm{t}}$ and $\Lambda \Subset \mathbb{L}$, one has

$$\begin{aligned}
\frac{\partial}{\partial h} p^{\mu}_{\Lambda}(h) &= \int_{\Omega} \frac{\partial}{\partial h} \left(p^{\mu}_{\Lambda}(h,\xi)\right) \mu(\mathrm{d}\xi) \\
&= \frac{1}{|\Lambda|} \sum_{\ell \in \Lambda} \int_0^{\beta} \int_{\Omega} \pi_{\Lambda}\left[\omega_{\ell}(\tau)|\xi\right] \mu(\mathrm{d}\xi)\mathrm{d}\tau \\
&= \frac{1}{|\Lambda|} \sum_{\ell \in \Lambda} \int_0^{\beta} \mu\left[\omega_{\ell}(\tau)\right] \mathrm{d}\tau
\end{aligned} \tag{7.26}$$

Both extreme measures $\mu_{\pm}$ are translation and shift invariant. Then combining (7.25) and (7.26) one obtains $\mu_{+}(\omega_{\ell}(0)) = \mu_{-}(\omega_{\ell}(0))$ for any $h \neq 0$. By Lemma 6.5 this gives the proof. $\square$

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