Weak Dirichlet processes with a stochastic control perspective.

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Abstract. The motivation of this paper is to prove verification theorems for stochastic optimal control of finite dimensional diffusion processes without control in the diffusion term, in the case that the value function is assumed to be continuous in time and once differentiable in the space variable ($C^{0,1}$) instead of once differentiable in time and twice in space ($C^{1,2}$), like in the classical results. For this purpose, the replacement tool of the Itô formula will be the Fukushima-Dirichlet decomposition for weak Dirichlet processes. Given a fixed filtration, a weak Dirichlet process is the sum of a local martingale M plus an adapted process A which is orthogonal, in the sense of covariation, to any continuous local martingale. The mentioned decomposition states that a $C^{0,1}$ function of a weak Dirichlet process with finite quadratic variation is again a weak Dirichlet process. That result is established in this paper and it is applied to the strong solution of a Cauchy problem with final condition.

Applications to the proof of verification theorems will be addressed in a companion paper.

1 Introduction

In this paper we prepare a framework of stochastic calculus via regularization in order to apply it to the proof of verification theorems in stochastic optimal control in finite dimension. The application part will be implemented in the companion paper [16].

This paper has an interest in itself and its most significant result is a generalized time-dependent Fukushima-Dirichlet decomposition which is proved in section 3. This will be the major tool for applications.

The proof of verification theorems for stochastic control problems under classical conditions is an application of Itô formula. In fact, under good assumptions the value function $V : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ associated with a stochastic control problem is of class C^1 in time and C^2 in space ($C^{1,2}$ in symbols). This allows to apply it to the solution of a corresponding state equation (S_t) and differentiate $V(t, S_t)$ through the classical Itô formula, see e.g. [9, pp. 140, 163, 172]. The substitution tool of that formula will be a time-dependent Fukushima - Dirichlet decomposition which will hold for functions $u : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ that are $C^{0,1}$ in symbols; so, our verification theorem will have the advantage of requiring less regularity on the value function V than the classical ones.

It is also possible to prove a verification theorem in the case when V is only continuous (see e.g. [20], [35, Section 5.2], [17]) in the framework of viscosity solutions: however such result applied to our cases is weaker than ours, as it requires more assumptions on the candidate optimal strategy; see on this the last section of the companion paper [16] where also a comparison with other nonsmooth verification theorems is performed.

We come back to the Fukushima-Dirichlet decomposition as replacement of Itô formula. Roughly speaking, given a function u of class $C^{1,2}$, classical Itô formula gives a decomposition of $u(t, S_t)$ in a martingale part, say M (which is thrown away taking expectation in the case of deterministic data and expected cost) plus an absolutely continuous process, say A. Then, in case of deterministic data and expected cost, one uses the fact that u is a classical solution of a partial differential equation (PDE), which is in fact the Hamilton-Jacobi-Bellman (HJB) equation, to represent A in term of the Hamiltonian function. If one wants to repeat the above arguments when u is not $C^{1,2}$, a natural way is to try to extend the decomposition $u(\cdot, S) = M + A$ and the representation of A via the HJB PDE. This is what we perform in this paper in the case when $u \in C^{0,1}$ using a point of view that can be also applied to problems with stochastic data and pathwise cost, so the HJB becomes a stochastic PDE, see e.g. [3, 21, 22, 25]).

We propose in fact an extension of the classical Fukushima-Dirichlet decomposition. That decomposition is inspired by the theory of Dirichlet forms. A classical monography concerning this theory is [12] where one can find classical references on the subject. In Theorem 5.2.2 of [12], given a "good" symmetric Markov process $(X_t^x)_{t\geq 0}$ and a function belonging to some suitable space (Dirichlet space), it is possible to write

$$u(X_t) = u(x) + M_t^u + A_t^u, \tag{1}$$

where M is a local martingale and $(A_t^u)_{t\geq 0}$ is a zero quadratic variation process, for quasi-everywhere x, i.e. for x belonging to a zero capacity set. For instance, if $X = B$ is a classical Brownian motion in $\mathbb{R}^n$ then the Dirichlet space is $H^1(\mathbb{R}^n)$ and $M^u = \int_0^t \nabla u(B_u) dB_u$. We call (1) a Fukushima-Dirichlet decomposition. Our point of view is pathwise: as in [10], a process Y, as $u(X)$, which is the sum of a local martingale and a zero quadratic variation process, even without any link to Dirichlet forms, is called a Dirichlet process.

The papers [29] and [11] reinterpret A in a pathwise way as the covariation process $[\nabla u(B), B]$ transforming (1) in a true Itô's formula; the first work considers $u \in C^1(\mathbb{R}^n)$ and it extends the framework to reversible continuous semimartingales; the second work is connected with Brownian motion and $u \in H^1(\mathbb{R}^n)$. The literature on Itô's formula for non-smooth functions of semimartingales or diffusion processes has known a lot of development in the recent years, see for instance [23] for non-degenerate Brownian martingales, [8] for non-degenerate 1-dimensional diffusions with bounded measurable drift or [6] in the jump case.

In our applications, the fact of identifying the remainder process A^u as a covariation is not so important since the goal is to give the representation of it via the data of the HJB PDE. So we come back to the spirit of the Fukushima-Dirichlet decomposition. Besides our "pathwise" approach to Dirichlet processes, the true novelty of this approach is the time-inhomogeneous version of the decomposition; this is in particular motivated by non-autonomous problems in control theory.

This is based on the theory, under construction, of weak Dirichlet processes with respect to some fixed filtration $(\mathcal{F}_t)$. A weak Dirichlet process is the sum of a local martingale M and a process A which is adapted and $[A, N] = 0$ for any continuous $(\mathcal{F}_t)$-local martingale. We will be able in particular to decompose $u(t, D_t)$ when $u \in C^{0,1}$ and D a weak Dirichlet process with finite quadratic variation process, so in particular if D is a semimartingale (even diffusion process). This will be our time-dependent Fukushima-Dirichlet decomposition: it will be the object of Proposition 3.9 and Corollary 3.10. In particular that result holds for semimartingales (and so for diffusion processes). The notion of weak Dirichlet process appears also in [5]. Our Fukushima-Dirichlet decomposition could be linked to the theory of "time-dependent Dirichlet forms" developed for instance by [24, 33, 34] but we have not investigated that direction.

The paper is organized as follows. In section 2 we introduce some notations on real analysis and we establish preliminary notions on calculus via regularization with some remarks on classical Dirichlet processes. Section 3 will be devoted to some basic facts about weak Dirichlet process and to the above mentioned Fukushima-Dirichlet decomposition of process $(u(t, D_t))$, with some sufficient condition to guarantee that the resulting process is a true Dirichlet process. Section 4 will be concerned with application to the case where u is a strong $C^{0,1}$ solution of a Cauchy parabolic problem with initial condition; C^1 solutions of an elliptic problem are also represented probabilistically.

2 Preliminaries

2.1 Notation

Throughout this paper we will denote by $(\Omega, \mathcal{F}, \mathbb{P})$ a given stochastic basis (where $\mathcal{F}$ stands for a given filtration $(\mathcal{F}_s)_{s\geq 0}$ satisfying the usual conditions). Given a finite dimensional real Hilbert space E, W will denote a cylindrical Brownian motion with values in E and adapted to $(\mathcal{F}_s)_{s\geq 0}$. Given $0 \leq t \leq T \leq +\infty$ and setting $\mathcal{T}_t = [t, T] \cap \mathbb{R}$ the symbol $\mathcal{C}_{\mathcal{F}}(\mathcal{T}_t \times \Omega; E)$, will denote the space of all continuous processes adapted to the filtration $\mathcal{F}$ with values in E. This is a Fréchet space if endowed with the topology of the uniform convergence in probability (*u.c.p.* from now on). To be more precise this means that, given a sequence $(X^n) \subseteq \mathcal{C}_{\mathcal{F}}(\mathcal{T}_t \times \Omega; E)$ and $X \in \mathcal{C}_{\mathcal{F}}(\mathcal{T}_t \times \Omega; E)$ we have

$$X^n \to X$$

if and only if for every $\varepsilon > 0$, $t_1 \in \mathcal{T}_t$

$$\lim_{n\to+\infty} \sup_{s\in[t,t_1]} \mathbb{P}\left(|X^n_s - X_s|_E > \varepsilon\right) = 0.$$

Given a random time $\tau \geq t$ and a process $(X_s)_{s\in\mathcal{T}_t}$, we denote by X^τ the stopped process defined by $X^\tau_s = X_{s\wedge\tau}$. The space of all processes in $[t, T]$, adapted to $\mathcal{F}$ and square integrable with values in E is denoted by $L^2_{\mathcal{F}}(t, T; E)$. S^n will denote the space of all symmetric matrices of dimension n. If Z is a vector or a matrix, then Z^* is its transposition.

Let $k \in \mathbb{N}$. As usual $C^k(\mathbb{R}^n)$ is the space of all functions : $\mathbb{R}^n \to \mathbb{R}$ that are continuous together with their derivatives up to the order k. This is a Fréchet space equipped with the seminorms

$$\sup_{x\in K} |u(x)|_{\mathbb{R}} + \sup_{x\in K} |\partial_x u(x)|_{\mathbb{R}^n} + \sup_{x\in K} |\partial_{xx} u(x)|_{\mathbb{R}^{n\times n}}$$

for every compact set $K \subset\subset \mathbb{R}^n$. This space will be denoted simply by C^k when no confusion may arise. The symbol $C^k_b(\mathbb{R}^n)$ will denote the Banach space of all continuous and bounded functions from $\mathbb{R}^n$ to $\mathbb{R}$. This space is endowed with the usual sup norm. Passing to parabolic spaces we denote by $C^0(\mathcal{T}_t \times \mathbb{R}^n)$ the space of all functions

$$u : \mathcal{T}_t \times \mathbb{R}^n \to \mathbb{R}, \qquad (s, x) \mapsto u(s, x)$$

that are continuous. This space is a Fréchet space equipped with the seminorms

$$\sup_{(s,x)\in[t,t_1]\times K} |u(s, x)|_{\mathbb{R}}$$

for every $t_1 > 0$ and every compact set $K \subset\subset \mathbb{R}^n$). Moreover we will denote by $C^{1,2}(\mathcal{T}_t \times \mathbb{R}^n)$ (respectively $C^{0,1}(\mathcal{T}_t \times \mathbb{R}^n)$), the space of all functions

$$u : \mathcal{T}_t \times \mathbb{R}^n \to \mathbb{R}, \qquad (s,x) \mapsto u(s,x)$$

that are continuous together with their derivatives $\partial_t u$, $\partial_x u$, $\partial_{xx} u$ (respectively $\partial_x u$). This space is a Fréchet space equipped with the seminorms

$$\sup_{(s,x)\in[t,t_1]\times K} |u(s,x)|_{\mathbb{R}} + \sup_{(s,x)\in[t,t_1]\times K} |\partial_s u(s,x)|_{\mathbb{R}^n}$$
$$+ \sup_{(s,x)\in[t,t_1]\times K} |\partial_x u(s,x)|_{\mathbb{R}^n} + \sup_{(s,x)\in[t,t_1]\times K} |\partial_{xx} u(s,x)|_{\mathbb{R}^{n\times n}}$$

(respectively

$$\sup_{(s,x)\in[t,t_1]\times K} |u(s,x)|_{\mathbb{R}} + \sup_{(s,x)\in[t,t_1]\times K} |\partial_x u(s,x)|_{\mathbb{R}^n} \;)$$

for every $t_1 > 0$ and every compact set $K \subset\subset \mathbb{R}^n$. This space will be denoted simply by $C^{1,2}$ (respectively $C^{0,1}$) when no confusion may arise.

Similarly, for $\alpha, \beta \in [0,1]$ one defines $C^{\alpha,1+\beta}(\mathcal{T}_t \times \mathbb{R}^n)$ (or simply $C^{\alpha,1+\beta}$) as the subspace of $C^{0,1}(\mathcal{T}_t \times \mathbb{R}^n)$ of functions $u : \mathcal{T}_t \times \mathbb{R}^n \mapsto \mathbb{R}$ such that are $u(\cdot,x)$ is $\alpha-$Hölder continuous and $\partial_x u(s,\cdot)$ is $\beta-$Hölder continuous (with the agreement that 0-Hölder continuity means just continuity).

Similarly to $C_b^k(\mathbb{R}^n)$ we define the Banach spaces $C_b^0(\mathcal{T}_t \times \mathbb{R}^n)$ $C_b^{1,2}(\mathcal{T}_t \times \mathbb{R}^n)$, $C_b^{\alpha,1+\beta}(\mathcal{T}_t \times \mathbb{R}^n)$, $C_b^{0,1}(\mathcal{T}_t \times \mathbb{R}^n)$.

2.2 The calculus via regularization

We will follow here a framework of calculus via regularizations started in [27]. At the moment many authors have contributed to it and we suggest the reader to consult the recent survey paper [32] on it.

For simplicity, all the considered processes, excepted if we mention the contrary, will be continuous processes. We first recall some one dimensional consideration. For two processes $(X_s)_{s\geq 0}$, $(Y_s)_{s\geq 0}$, we define the forward integral and the covariation as follows

$$\int_0^s X_r d^- Y_r = \lim_{\varepsilon\to 0} \int_0^s X_r \frac{Y_{r+\varepsilon} - Y_r}{\varepsilon} dr, \tag{2}$$

$$[X,Y]_s = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_0^s (X_{r+\varepsilon} - X_r)(Y_{r+\varepsilon} - Y_r)\, dr, \tag{3}$$

if those quantities exist in the sense of $u.c.p$ with respect to s. This ensures that the forward integral defined in (2) and the covariation process defined in (3) are continuous processes. It can be seen that the covariation is a bilinear and symmetric operator.

We fix now, as above, $0 \leq t \leq T \leq +\infty$ and set $\mathcal{T}_t = [t,T] \cap \mathbb{R}$. A process $(X_s)_{s\in\mathcal{T}_t}$ can always be extended (if $T < +\infty$) to a process indexed by $\mathbb{R}_+$ by continuity. The corresponding extension will always denoted by the same symbol. Given two processes $(X_s)_{s\in\mathcal{T}_t}$, $(Y_s)_{s\in\mathcal{T}_t}$, we define the corresponding stochastic integrals and covariations by the integrals and covariation of the corresponding extensions.

We define also integrals from t to s as follows.

$$\int_t^s X_r d^- Y_r = \int_0^s X_r d^- Y_r - \int_0^t X_r d^- Y_r.$$

If $\tau \geq t$ is a random (non necessarily) stopping time, the following equality holds:

$$[X,Y]^\tau = [X^\tau, Y^\tau]. \tag{4}$$

If $\left(X^1,...,X^n\right)$ is a vector of continuous processes we say that it has all its mutual covariations (brackets) if $\left[X^i,X^j\right]$ exist for any $1\leq i,j\leq n$. If $X^1,...,X^n$ have all their mutual covariations then by polarisation (i.e. writing a bilinear form as a sum/difference of quadratic forms) we know that $\left[X^i,X^j\right]$ $(1\leq i,j\leq n)$ are locally bounded variation processes.

If $[X,X]$ exists, then X is said to be a finite quadratic variation process; $[X,X]$ is called the quadratic variation of X. If $[X,X]=0$ then X is said to be a zero quadratic variation process. A bounded variation process is a zero quadratic variation process. If S^1, S^2 are $(\mathcal{F}_s)$-semimartingales then $\left[S^1,S^2\right]$ coincides with the classical bracket $\left\langle S^1,S^2\right\rangle$. If H is a $(\mathcal{F}_s)$-progressively measurable process then $\int_t^s H_r d^-S_r$ is the classical Itô integral $\int_t^s H_r dS_r$.

Remark 2.1 *Let X (respectively A) be a finite (respectively zero) quadratic variation process. Then (X,A) has all its mutual covariations and $[X,A]=0$.* ■

We recall now an easy extension of stability results, see [7, th. 2.9] that will be used in subsection 3.3.

Proposition 2.2 *Let $V=\left(V^1,...V^m\right)$ (respectively $X=\left(X^1,...X^n\right)$) be a vector of continuous processes on $\mathbb{R}_+$ with bounded variation processes (respectively having all its mutual covariations). Let $f,g\in C_{\text{loc}}^{\frac{1}{2}+\gamma,1}\left(\mathbb{R}^m\times\mathbb{R}^n\right)$ $(\gamma>0)$. Then $\forall s\geq 0$*

$$\left[f\left(V,X\right),g\left(V,X\right)\right]_s=\sum_{i,j=1}^{n}\int_t^s\partial_{x_i}f\left(V,X\right)\partial_{x_j}g\left(V,X\right)d\left[X^i,X^j\right]_r.$$

Proof. We give a sketch of the proof as the arguments are similar to the ones used in [7, th. 2.9]. By localisation $C_{\text{loc}}^{\frac{1}{2}+\gamma,1}$ can be replaced by $C^{\frac{1}{2}+\gamma,1}$. The case where f and g do not depend on V was treated for instance in [28, 30]. Since the covariation is a bilinear operation, using polarisation techniques we can take $g=f$. For simplicity we set here $m=n=1$. For given $\varepsilon>0$ we write, when $r\in\mathcal{T}_t$,

$$f\left(V_{r+\varepsilon},X_{r+\varepsilon}\right)-f\left(V_r,X_r\right)=J_1\left(r,\varepsilon\right)+J_2\left(r,\varepsilon\right),$$

where

$$\begin{aligned}J_1\left(r,\varepsilon\right)&=f\left(V_{r+\varepsilon},X_{r+\varepsilon}\right)-f\left(V_r,X_{r+\varepsilon}\right),\\ J_2\left(r,\varepsilon\right)&=f\left(V_r,X_{r+\varepsilon}\right)-f\left(V_r,X_r\right).\end{aligned}$$

Therefore, for $s\geq 0$

$$\frac{1}{\varepsilon}\int_0^s\left[f\left(V_{r+\varepsilon},X_{r+\varepsilon}\right)-f\left(V_r,X_r\right)\right]^2dr\leq\frac{2}{\varepsilon}\int_0^s J_1^2\left(r,\varepsilon\right)dr+\frac{2}{\varepsilon}\int_0^s J_2^2\left(r,\varepsilon\right)dr.$$

Now

$$\frac{2}{\varepsilon}\int_0^s J_1^2\left(r,\varepsilon\right)dr\leq c^2\left(s,f,V\right)\frac{2}{\varepsilon}\int_0^s\left(\left|V_{r+\varepsilon}-V_r\right|^{\frac{1}{2}+\gamma}\right)^2dr,$$

where $c\left(s,f,V\right)$ is a (random) Hölder constant of f such that

$$\left|f\left(v_1,x\right)-f\left(v_2,x\right)\right|\leq c\left(s,f,V\right)\left|v_1-v_2\right|^{\frac{1}{2}+\gamma},$$

$$\forall v_1,v_2\in\left[\inf_{r\in[0,s]}V_r,\ \sup_{r\in[0,s]}V_r\right],\quad\forall x\in\left[\inf_{r\in[0,s]}X_r,\ \sup_{r\in[0,s]}X_r\right].$$

Then we get

$$\frac{2}{\varepsilon}\int_0^s J_1^2(r,\varepsilon)\,dr \le c^2(s,f,V)\,\frac{2}{\varepsilon}\int_0^s |V_{r+\varepsilon}-V_r|^{1+2\gamma}\,dr.$$

Since V is a bounded variation process, this term converges to zero in probability.

On the other hand,

$$J_2(r,\varepsilon) = \partial_x f(V_r,X_r)(X_{r+\varepsilon}-X_r) + J_3(r,\varepsilon),$$

where $J_3(r,\varepsilon)$ converges *u.c.p.* to zero, as in [7, th. 2.9]. Therefore, similarly as in [28] we have

$$\frac{2}{\varepsilon}\int_0^t J_2^2(s,\varepsilon)\,ds \to \sum_{i,j=1}^{n}\int_0^t (\partial_x f(V_s,X_s))^2\,d[X,X]_s$$

and so the claim. ■

For our purposes we need to express integrals and covariation in a multidimensional setting, in the spirit of [5].

If $X = (X^1,...,X^n)^*$ is a vector of continuous processes in $\mathbb{R}_+$, Y is a $m\times n$ matrix of continuous processes in $\mathbb{R}_+$, $(Y^{i,j})_{1\le i\le m,1\le j\le n}$ then the symbol $\int_0^s Y d^- X$ denotes, whenever it exists, the *u.c.p.* limit of the integral $\int_0^s Y_r \frac{X_{r+\varepsilon}-X_r}{\varepsilon}dr$ where the product is intended in the matrix sense. Similarly, il A is a $n\times d$ matrix $(A^{j,k})_{1\le j\le n,1\le k\le d}$ then $[Y,A]_s$ is the $m\times d$ real matrix constituted by the following *u.c.p.* limit (if it exists)

$$\frac{1}{\varepsilon}\int_0^s (Y_{r+\varepsilon}-Y_r)(A_{r+\varepsilon}-A_r)\,dr.$$

Clearly the matrix operation cannot be commutative in general. Let now A, X, Y, C be real matrix valued processes which are successively compatible for the matrix product. We define

$$\int_0^s A_r d[X,Y]_r C_r = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^s A_r(X_{r+\varepsilon}-X_r)(Y_{r+\varepsilon}-Y_r)C_r dr,$$

where the limit is intended in the *u.c.p.* sense. The previous stability transformations (Proposition 2.2 above) can be extended to the case of vector valued functions. This is pointed out in next remark.

Remark 2.3 *Let $f\in C_{\text{loc}}^{\frac{1}{2}+\gamma,1}(\mathbb{R}_+\times\mathbb{R}^n;\mathbb{R}^p)$, $g\in C_{\text{loc}}^{\frac{1}{2}+\gamma,1}(\mathbb{R}_+\times\mathbb{R}^m;\mathbb{R}^q)$, $X=(X^1,...,X^n)^*$, $Y=(Y^1,...,Y^m)^*$ such that (X,Y) has all its mutual covariations. Let V^1, V^2 be bounded variation processes. Then*

$$\left[f(V^1,X),g(V^2,Y)\right]_s = \int_0^s \partial_x f(V^1,X)\,d[X,Y^*]\,\partial_x g(V^2,Y)^*$$

for every $s\ge 0$. ■

One refined result is the vector Itô formula whose proof follows similarly as in [28, 7], where the involved stochastic integrals were scalar.

Proposition 2.4 *Let $f \in C^{1,2}(\mathcal{T}_t \times \mathbb{R}^n)$. Let $X = \left(X^1, ..., X^n\right)^*$, having all its mutual covariations, V be a bounded variation process indexed by $\mathcal{T}_t$. Then*

$$\begin{aligned} f(V_s, X_s) &= f(V_t, X_t) + \int_t^s \partial_x f(V_r, X_r)\, d^- X_r \\ &\quad + \int_t^s \partial_v f(V_r, X_r)\, dV_r \\ &\quad + \frac{1}{2}\int_t^s \partial_{xx} f(V_r, X_r)\, d[X, X^*]. \end{aligned}$$

Remark 2.5 *From the above statement it follows that, in particular, the integral $\int_t^s \partial_x f(V_r, X_r)\, d^- X_r$ automatically exists.* ■

Remark 2.6 *Let $W = \left(W^1, ..., W^n\right)^*$ be an $(\mathcal{F}_s)$-Brownian motion. Then $[W, W^*]_s = (\delta_{i,j})\, s$.* ■

3 Fukushima-Dirichlet decomposition

3.1 Definitions and remarks

Throughout all this section we fix, as above, $0 \le t \le T \le +\infty$ and set $\mathcal{T}_t = [t, T] \cap \mathbb{R}$. Recall that all processes we consider are continuous except when explicitely stated.

Definition 3.1 *A real process D is called an $(\mathcal{F}_s)$-Dirichlet process in $\mathcal{T}_t$ if it is $(\mathcal{F}_s)$-adapted and can be written as*

$$D = M + A, \tag{5}$$

where

(i) M is an $(\mathcal{F}_s)$- local martingale,

(ii) A is a zero quadratic variation process such that (for convenience) $A_0 = 0$.

A vector $D = \left(D^1, ..., D^n\right)$ is said to be Dirichlet if it has all its mutual covariations and every D^i is Dirichlet.

Remark 3.2 *An $(\mathcal{F}_s)$-semimartingale is an $(\mathcal{F}_s)$-Dirichlet process.* ■

Remark 3.3 *The decomposition (5) is unique, see for instance [31].* ■

The concept of Dirichlet process can be weakened for our purposes. We will make use of an extension of such processes, called *weak Dirichlet processes*, introduced parallely in [5] and implicitely in [4]. Recent developments about the subject appears in [2] and [32]. 3.9). Weak Dirichlet processes are not Dirichlet processes but they preserve a sort of orthogonal decomposition. In the mentioned paper however one deals with one-dimensional weak Dirichlet processes while here we treat the multidimensional case.

Definition 3.4 *A real process D is called $(\mathcal{F}_s)$-**weak Dirichlet** process in $\mathcal{T}_t$ if it can, be written as*

$$D = M + A, \tag{6}$$

where

(i) M is an $(\mathcal{F}_s)$-local martingale,

(ii) A is a process such that $[A,N]=0$ for every $(\mathcal{F}_s)$-continuous local martingale N. (For convenience $A_0=0$)

*A will be said **weak zero energy** process.*

Remark 3.5 *The decomposition (6) is unique. In fact, let*

$$D = M^1 + A^1 = M^2 + A^2,$$

where M^1, M^2, A^1, A^2 fulfill properties (i) and (ii) of Definition 3.4. Then we have $M + A = 0$ where $M = M^1 - M^2$ and $A = A^1 - A^2$ is such that $[A,N]=0$ for every $(\mathcal{F}_s)$- local martingale N.

It is now enough to evaluate the covariation of both members with M to get $[M,M]=0$. Since $A_0=0$ then $M_0=0$ and consequently $M\equiv 0$.

Example 3.6 *A simple example of weak Dirichlet process is given by a process Z which is independent of $\mathcal{F}$, for instance a deterministic one! Clearly if Z is not at least a finite quadratic variation process, it cannot be Dirichlet. However it is possible to show that $[Z,N]=0$ for any local $\mathcal{F}$-martingale. In fact*

$$\begin{aligned}\int_0^s (Z_{r+\varepsilon}-Z_r)(N_{r+\varepsilon}-N_r)dr &= \int_0^s dr(Z_{r+\varepsilon}-Z_r)\int_r^{r+\varepsilon}\frac{1}{\varepsilon}dN_\lambda \\ &= \int_0^s \frac{dN_\lambda}{\varepsilon}\int_{(\lambda-\varepsilon)\vee 0}^{\lambda}(Z_{r+\varepsilon}-Z_r)dr \\ &= \int_0^s \frac{dN_\lambda}{\varepsilon}\int_{(\lambda-\varepsilon)\vee 0}^{\lambda} Z_{r+\varepsilon}dr - \int_0^s \frac{dN_\lambda}{\varepsilon}\int_{(\lambda-\varepsilon)\vee 0}^{\lambda} Z_r dr.\end{aligned}$$

Previous expression converges u.c.p. to zero since the two last terms converge u.c.p. to the Itô integral $\int_0^s Z dN$ since N is also a local martingale with respect to the filtration $\mathcal{F}$ enlarged with Z.

Remark 3.7 *[5] provides an example of weak Dirichlet process coming from convolutions of local martingales. If for every $s>0$, $(G(s,\cdot))$ is a continuous random field such that $(G(s,\cdot))$ is $(\mathcal{F}_r)$-progressively measurable and M is an $(\mathcal{F}_r)$-local martingale, then $X_s = \int_t^s G(s,r)dM_r$ defines a weak Dirichlet process.*

Definition 3.8 *A vector $D = \left(D^1,...,D^n\right)$ is said to be a $(\mathcal{F}_s)$-**weak Dirichlet** process if every D^i is $(\mathcal{F}_s)$- weak Dirichlet process. A vector $A = \left(A^1,...,A^n\right)$ is said to be a $(\mathcal{F}_s)$-**weak zero energy** process if every A^i is $(\mathcal{F}_s)$- weak zero energy process.*

The aim now is to study what happens to a Dirichlet process after a $C^{0,1}$ type transformation. It is well known, see for instance [31], that a C^1 function of a finite quadratic variation process (respectively Dirichlet process) is a finite quadratic variation (respectively Dirichlet) process. Here, motivated by applications to optimal control, see the introduction, section 1, we look at the (possibly) inhomogeneous case showing two different results: the first result (Proposition 3.9 with Corollary 3.10) states that a function $C^{0,1}\left(\mathbb{R}_+\times\mathbb{R}^n\right)$ of a weak Dirichlet (vector) process having all its mutual covariations is again a weak Dirichlet process; the second (Proposition 3.13) gives stability for Dirichlet processes in the inhomogeneous case for functions in $C^{\frac{1}{2}+\gamma,1}_{\mathrm{loc}}\left(\mathcal{T}_t\times\mathbb{R}^n\right)$.

The first result comes from the need of treating optimal control problems where the state process is a semimartingale that solves a classical SDE's, which is the case we treat in this paper.

3.2 The decomposition for $C^{0,1}$ functions

We now go on with a result concerning weak Dirichlet processes. Suppose $(D_s)_{s\in\mathcal{T}_t}$ to be an $(\mathcal{F}_s)$-Dirichlet process with decomposition (5) where A is a zero quadratic variation process. Given a $C^{0,1}$ function u of D, we cannot expect that $Z = u(\cdot, D)$ is a Dirichlet process. However one can hope that it is at least a weak Dirichlet process. Indeed this result is true even if D is a weak Dirichlet process with finite quadratic variation.

Proposition 3.9 *Suppose $(D_s)_{s\geq 0}$ to be an $(\mathcal{F}_s)$- weak Dirichlet (vector) process having all its mutual covariations. For every $u \in C^{0,1}(\mathbb{R}_+ \times \mathbb{R}^n)$ we have, for $s \geq 0$,*

$$u(s, D_s) = u(t, D_t) + \int_t^s \partial_x u(r, D_r)\, dM_r + \mathcal{B}^D(u)_s - \mathcal{B}^D(u)_t, \tag{7}$$

where $\mathcal{B}^D : C^{0,1}(\mathbb{R}_+ \times \mathbb{R}^n) \to \mathcal{C}_{\mathcal{F}}(\mathbb{R}_+ \times \Omega; \mathbb{R}^n)$ is a linear map having the following properties:

a) $\mathcal{B}^D$ is continuous;

b) if $u \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n)$ then

$$\mathcal{B}_t^D(u)_s = \int_t^s \partial_s u(r, D_r)\, dr + \int_t^s \partial_{xx} u(r, D_r)\, d[M, M]_r + \int_0^s \partial_x u(r, D_r)\, d^- A_r;$$

c) if $u \in C^{0,1}(\mathbb{R}_+ \times \mathbb{R}^n)$ then $\left((\mathcal{B}^D(u)_s\right)$ is a weak zero energy process.

Corollary 3.10 *Suppose $(D_t)_{t\geq 0}$ to be an $(\mathcal{F}_t)$- weak Dirichlet (vector) process having all its mutual covariations. For every $u \in C^{0,1}(\mathbb{R}_+ \times \mathbb{R}^n)$, $(u(t, D_t)$ is a $(\mathcal{F}_t)$-weak Dirichlet process with martingale part $\tilde{M}_t = \int_0^t \partial_x u(s, D_s) dM_s$.*

Remark 3.11 *Given a bounded stopping time τ with values in $\mathcal{T}_t$, it is easy to see that decomposition (7) still holds for the stopped process D^τ. In fact given a $(\mathcal{F}_s)$- martingale N, and an $(\mathcal{F}_s)$- weak zero energy process A, we have $[N, A^\tau] = [N, A]^\tau = 0$.*

Proof (of the Proposition). Without restriction of generality we will set $t = 0$.

Property a) follows simply by writing

$$\mathcal{B}^D(u)_s = u(s, D_s) - u(0, D_0) - \int_0^s \partial_x u(r, D_r)\, dM_r$$

and observing that the process defined on the right hand side has the required continuity property.

Property b) follows from Proposition 2.4 applied reversely. Indeed, given $u \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n)$, Proposition 2.4 can be applied. In particular

$$\int_0^s \partial_x u(r, D_r)\, d^- D_r$$

exists; this implies that also

$$\int_0^s \partial_x u(r, D_r)\, d^- A_r$$

exists since

$$\int_0^s \partial_x u(r, D_r)\, d^- M_r = \int_0^s \partial_x u(r, D_r)\, dM_r$$

is the classical Itô integral.

It remains to prove point c)

$$\left[u(\cdot, D) - \int_0^{\cdot} \partial_x u(r, D_r)\, dM_r, N\right] = 0$$

for every one dimensional $(\mathcal{F}_r)$- local martingale N.

For simplicity of notations, we will suppose that D is one-dimensional. Therefore D will be therefore a finite quadratic variation proces. Since the covariation of semimartingales coincides with the classical covariation

$$\left[\int_0^{\cdot} \partial_x u(r, D_r)\, dM_r, N\right] = \int_0^{\cdot} \partial_x u(r, D_r)\, d[M, N]_r,$$

it remains to check that, for every $s \in [0, T]$,

$$[u(\cdot, D), N]_s = \int_0^s \partial_x u(r, D_r)\, d[M, N]_r.$$

For this, we have to evaluate the *u.c.p.* limit of

$$\int_0^s [u(r+\varepsilon, D_{r+\varepsilon}) - u(r, D_r)] \frac{N_{r+\varepsilon} - N_r}{\varepsilon} dr$$

in probability. This can be written as the sum of two terms:

$$I_1(s, \varepsilon) = \int_0^s (u(r+\varepsilon, D_{r+\varepsilon}) - u(r+\varepsilon, D_r)) \frac{N_{r+\varepsilon} - N_r}{\varepsilon} dr,$$

$$I_2(s, \varepsilon) = \int_0^s (u(r+\varepsilon, D_r) - u(r, D_r)) \frac{N_{r+\varepsilon} - N_r}{\varepsilon} dr.$$

First we prove that $I_1(s, \varepsilon)$ goes to $\int_0^t \partial_x u(r, D_r)\, d[M, N]_r$. In fact

$$\begin{aligned} I_1(s, \varepsilon) &= \int_0^s (u(r+\varepsilon, D_{r+\varepsilon}) - u(r+\varepsilon, D_r)) \frac{N_{r+\varepsilon} - N_r}{\varepsilon} dr \\ &= \int_0^s \partial_x u(r+\varepsilon, D_r)(D_{r+\varepsilon} - D_r) \frac{N_{r+\varepsilon} - N_r}{\varepsilon} dr + R_1(s, \varepsilon), \end{aligned} \tag{8}$$

where $R_1(s, \varepsilon) \to 0$ *u.c.p.* as $\varepsilon \to 0$. Indeed

$$\begin{aligned} R_1(s, \varepsilon) &= \int_0^s \left[\int_0^1 [\partial_x u(r+\varepsilon, D_r + \lambda(D_{r+\varepsilon} - D_r)) - \partial_x u(r+\varepsilon, D_r)]\, d\lambda \right. \\ &\quad \left. \frac{N_{r+\varepsilon} - N_r}{\varepsilon}(D_{r+\varepsilon} - D_r)\right] dr \end{aligned}$$

and the claim follows by the continuity of $\partial_x u$ and from the estimate

$$\begin{aligned} &\frac{1}{\varepsilon}\int_0^T (N_{r+\varepsilon} - N_r)(D_{r+\varepsilon} - D_r)\, dr \\ \le\ &\left[\frac{1}{\varepsilon}\int_0^T (N_{r+\varepsilon} - N_r)^2 dr \cdot \frac{1}{\varepsilon}\int_0^T (D_{r+\varepsilon} - D_r)^2 dr\right]^{\frac{1}{2}} \overset{\varepsilon \to 0}{\longrightarrow} ([N] \cdot [D])^{\frac{1}{2}}. \end{aligned} \tag{9}$$

On the other hand the first term in (8) can be rewritten as

$$\int_0^s \partial_x u\left(r, D_r\right)\left(D_{r+\varepsilon} - D_r\right) \frac{N_{r+\varepsilon} - N_r}{\varepsilon} dr + R_2\left(s, \varepsilon\right), \tag{10}$$

where $R_2(s,\varepsilon) \to 0$ *u.c.p.* arguing as for $R_1(s,\varepsilon)$. The integral in (10) goes then *u.c.p.* to $\int_0^s \partial_x u\left(r, D_r\right) d\left[M, N\right]_r$, since by (9) the measures $\frac{(N_{r+\varepsilon}-N_r)(D_{r+\varepsilon}-D_r)}{\varepsilon}dr$ weakly converge to $d[N,D]$ as $\varepsilon \to 0$.

It remains to show that $I_2(s,\varepsilon) \to 0$ *u.c.p.* for every $s \in [0,T]$ as $\varepsilon \to 0$. By using suitable localization theorems (e.g. as usually done for instance in [26], section IV.1), it is enough to suppose u to be with compact support and N to be a square integrable martingale. Then we evaluate

$$\mathbb{E}\left(\sup_{0\le s\le T} |I_2(s,\varepsilon)|^2\right). \tag{11}$$

Now we have, exchanging integrals

$$\begin{aligned} I_2(s,\varepsilon) &= \int_0^s [u(r+\varepsilon, D_r) - u(r, D_r)]\frac{N_{r+\varepsilon} - N_r}{\varepsilon}dr \\ &= \int_0^s [u(r+\varepsilon, D_r) - u(r, D_r)]dr \int_r^{r+\varepsilon} \frac{1}{\varepsilon} dN_\lambda \\ &= \int_0^s \frac{dN_\lambda}{\varepsilon} \int_{(\lambda-\varepsilon)\vee 0}^{\lambda} [u(r+\varepsilon, D_r) - u(r, D_r)]dr. \end{aligned}$$

Doob inequality implies that (11) is smaller than

$$4\mathbb{E}\left\{\int_0^T d[N]_\lambda \left(\frac{1}{\varepsilon}\int_{(\lambda-\varepsilon)\vee 0}^{\lambda} [u(r+\varepsilon, D_r) - u(r, D_r)]dr\right)^2\right\}.$$

The fact that u is uniformly continuous on compact sets and the Lebesgue dominated convergence theorem imply the result. ■

A significant bracket evaluation, in the spirit of Proposition 2.2, but for $C^{0,1}-$ functions of semimartingales is the following. For simplicity we formulate the one-dimensional case, even if it extends to the multidimensional case.

Corollary 3.12 *Let S be a $(\mathcal{F}_s)$- semimartingale in $\mathbb{R}_+$, $u \in C^{0,1}(\mathcal{T}_t \times \mathbb{R})$. Then*

$$[u(\cdot, S), S]_t = \int_0^t \partial_x u(s, S_s) d[S]_s.$$

Proof. Let $S = M + V$ be the decomposition of S with M being a local martingale and V a finite variation process with $V_0 = 0$. Then

$$u(t, S_t) = u(0, S_0) + \int_0^t \partial_x u(s, S_s) dM_s + \tilde{A}_t,$$

where $\tilde{A}$ is a weak zero energy process. In particular, a classical localization argument shows that $[\tilde{A}, M] = 0$. On the other hand, obviously $[\tilde{A}, V] = 0$; consequently, by linearity and since the covariation of local martingales is the classical convolution, the result follows. ■

3.3 The decomposition for $C^{\frac{1}{2}+\gamma,1}$ functions

If, in Proposition 3.9, D is a Dirichlet process and u is of class $C^{\frac{1}{2}+\gamma,1}$, $\gamma > 0$, then the results of Proposition 3.9 and Corollary 3.10 can be better precised. In fact it is possible to show that Dirichlet processes are stable through $C^{\frac{1}{2}+\gamma,1}$, $\gamma > 0$ transformations.

Proposition 3.13 *Let $(D_s)_{s\geq 0}$ be an $(\mathcal{F}_s)$-Dirichlet process with decomposition (5). The statement of Proposition 3.9 holds with*

$$\mathcal{B}^D : C^{\frac{1}{2}+\gamma,1}\left(\mathbb{R}_+ \times \mathbb{R}^n\right) \to \mathcal{C}_{\mathcal{F}}\left(\mathbb{R}_+ \times \Omega; \mathbb{R}^n\right)$$

fulfilling properties a), b) and

c) if $u \in C^{\frac{1}{2}+\gamma,1}\left(\mathbb{R}_+ \times \mathbb{R}^n\right)$ then $\left(\mathcal{B}^D\left(u\right)_s\right)$ is a zero quadratic variation process.

Proof. Points a) and b) follow similarly as for the $C^{0,1}$ decomposition.

In order to establish Property c) we proceed using the bilinearity of the covariation. We will in fact show that $\mathcal{B}^D\left(u\right)$ is a zero quadratic variation process. We operate with the bilinearity of the covariation process and we evaluate

(i) $\left[u\left(\cdot, D\right), u\left(\cdot, D\right)\right]$,

(ii) $\left[u\left(\cdot, D\right), \int_0^{\cdot} \partial_x u\left(r, D_r\right) dM_r\right]$,

(iii) $\left[\int_0^{\cdot} \partial_x u\left(r, D_r\right) dM_r, \int_0^{\cdot} \partial_x u\left(r, D_r\right) dM_r\right]$,

as follows.

(i) We apply Proposition 2.2 to get that

$$\left[u\left(\cdot, D\right), u\left(\cdot, D\right)\right]_s = \int_0^s \partial_x u\left(s, D_r\right) d\left[D, D^*\right]_r \partial_x u\left(r, D_r\right)^*.$$

(ii) Setting $N_t = \int_0^s \partial_x u\left(r, D_r\right) dM_r$, Remark 2.1 implies that (N, D) has all its mutual brackets; therefore again Proposition 2.2 implies that

$$\left[u\left(\cdot, D\right), N^*\right]_s = \int_0^s \partial_x u\left(r, D_r\right) d\left[D, N^*\right]_r.$$

On the other hand, by Remark 2.1

$$\left[D, N^*\right]_t = \left[M, N^*\right]_s = \int_0^s \partial_x u\left(r, D_r\right) d\left[M, M^*\right]_r \partial_x u\left(r, D_r\right)^*.$$

(iii) The fact that the covariation of semimartingales coincides with the classical covariation gives

$$\begin{aligned} &\left[\int_0^{\cdot} \partial_x u\left(r, D_r\right) dM_r, \int_0^{\cdot} \partial_x u\left(r, D_r\right) dM_r\right]_s \\ =\ &\int_0^s \partial_x u\left(r, D_r\right) d\left[M, M^*\right]_r \partial_x u\left(r, D_r\right)^*. \end{aligned}$$

Finally by Remark 2.1 and the decomposition we get that

$$\left[D, D^*\right] = \left[M, M^*\right].$$

The bilinearity of the covariation allows now to conclude. ■

4 Representation of operator $\mathcal{B}$ when u solves a suitable PDE.

Here we want to develop the connection between suitable (deterministic) linear differential operators and our (stochastic) operators $\mathcal{B}$ introduced in the previous section. This connection is well known and obvious when u is the $C^{1,2}$ solution of a second order PDE and D is a diffusion process.Our aim is to extend the validity of such representation when u is only a $C^{0,1}$ solution (in a suitable sense that we will define below) and D is a weak Dirichlet process of a suitable kind (see below). This will be used as a key tool in the applications to optimal control.

4.1 Strong solutions of parabolic PDE's

Let $0 < T < +\infty$, consider two continuous functions

$$b : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n, \qquad \sigma : [0,T] \times \mathbb{R}^n \to \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$$

and the linear parabolic operator

$$\mathcal{L}_0 : D(\mathcal{L}_0) \subseteq C^0([0,T] \times \mathbb{R}^n) \longrightarrow C^0([0,T] \times \mathbb{R}^n),$$

$$D(\mathcal{L}_0) = C^{1,2}([0,T] \times \mathbb{R}^n),$$

$$\mathcal{L}_0 u(t,x) = \partial_t u(t,x) + \langle b(t,x), \partial_x u(t,x)\rangle + \frac{1}{2}\mathrm{Tr}\ [\sigma^*(t,x)\, \partial_{xx} u(t,x)\, \sigma(t,x)].$$

Defining

$$L_0(t)\, u(t,x) = \langle b(t,x), \partial_x u(t,x)\rangle + \frac{1}{2}\mathrm{Tr}\ [\sigma^*(t,x)\, \partial_{xx} u(t,x)\, \sigma(t,x)],$$

we can write

$$\mathcal{L}_0 u(t,x) = \partial_t u(t,x) + L_0(t)\, u(t,x).$$

Recall that an operator $\mathcal{M} : D(\mathcal{M}) \subseteq F \to G$ (F, G suitable Fréchet spaces) is closable if, given any sequence $(u_n)_{n\in\mathbb{N}} \subseteq D(\mathcal{M})$, we have

$$\left.\begin{array}{ll} u_n \longrightarrow 0, & \text{in } F \\ \mathcal{M}u_n \longrightarrow \eta & \text{in } G \end{array}\right\} \Longrightarrow \eta = 0.$$

When $\mathcal{L}_0$ is closable we denote by $\mathcal{L}$ its closure and recall that

$$u \in D(\mathcal{L}) \Longleftrightarrow \exists\, (u_n) \subset C^{1,2}([0,T] \times \mathbb{R}^n) : \left\{\begin{array}{l} u_n \longrightarrow u \\ \mathcal{L}_0 u_n \longrightarrow \mathcal{L}u \end{array}\right. \text{ in } C^0([0,T] \times \mathbb{R}^n).$$

Now, given $\phi \in C^0(\mathbb{R}^n)$ and $h \in C^0([0,T] \times \mathbb{R}^n)$ we consider the inhomogenous backward parabolic problem

$$\left\{\begin{array}{l} \partial_t u(t,x) + \langle b(t,x), \partial_x u(t,x)\rangle + \frac{1}{2}\mathrm{Tr}\ [\sigma^*(t,x)\, \partial_{xx} u(t,x)\, \sigma(t,x)] = h(t,x), \\ \qquad\qquad\qquad\qquad t \in [0,T], \qquad x \in \mathbb{R}^n, \\ u(T,x) = \phi(x), \qquad\qquad x \in \mathbb{R}^n, \end{array}\right. \tag{12}$$

that can be rewritten as

$$\partial_t u(t,x) + L_0(t)\, u(t,x) = h(t,x), \qquad u(T,x) = \phi(x),$$

or

$$\mathcal{L}_0 u(s,x) = h(s,x), \qquad u(T,x) = \phi(x).$$

Definition 4.1 *We say that $u \in C^0([0,T] \times \mathbb{R}^n)$ is a strict solution to the backward Cauchy problem (12) if $u \in D(\mathcal{L}_0)$ and (12) holds.*

Definition 4.2 *We say that $u \in C^0([0,T] \times \mathbb{R}^n)$ is a strong solution to the backward Cauchy problem (12) if there exists a sequence $(u_n) \subset D(\mathcal{L}_0)$ and two sequences $(\phi_n) \subseteq C^0(\mathbb{R}^n)$, $(h_n) \subseteq C^0([0,T] \times \mathbb{R}^n)$, such that*

1. *For every $n \in \mathbb{N}$ u_n is a strict solution of the problem*

$$\mathcal{L}_0 u_n(t,x) = h_n(t,x), \qquad u_n(T,x) = \phi_n(x).$$

2. *The following limits hold*

$$\begin{aligned} u_n &\longrightarrow u \text{ in } C^0([0,T] \times \mathbb{R}^n), \\ h_n &\longrightarrow h \text{ in } C^0([0,T] \times \mathbb{R}^n), \\ \phi_n &\longrightarrow \phi \text{ in } C^0(\mathbb{R}^n). \end{aligned}$$

4.2 The representation result

Let u be a strong solution of class $C^{0,1}$ of (12) and S be a weak Dirichlet process that can be written in the following form:

$$S_t = S_0 + \int_0^t \sigma(s, S_s) dW_s + A_t$$

where σ is as in the previous section and A is a weak zero energy process with finite quadratic variation.

We observe that for our applications to optimal control it would be enough to take S semimartingale. We deal with this more general case to prepare the field for a forthcoming paper in which we consider the optimal control of solutions of SDEs whre the drift b is the derivative in space of a continuous function β, therefore a Schwartz distribution. One would have in that case $A_t =$ "$\int_0^t \partial_x \beta(s,x) ds$" in some specific sense. Equations of that type, when there is no dependence in time appear for instance in [8]. Solutions are Dirichlet processes in the time-homogeneous case and weak Dirichlet in the general case.

We remark that the coefficient σ *must coincide* with the one appearing in the second order term of the operator.

We state first a technical lemma whose proof is elementary.

Lemma 4.3 *Let $T < +\infty$. Let $f_n, f : [0,T] \to \mathbb{R}$, $n \in \mathbb{N}$, continuous such that $f_n \to f$ uniformly. For a fixed constant $K > 0$, we define*

$$\tau_n = \inf\{t \in [0,T] : |f_n(t)| \geq K\}, \qquad \tau = \inf\{t \in [0,T] : |f(t)| \geq K\}$$

with the convention that $\inf \emptyset = T$. *Then*

$$\lim_{n \to +\infty} f_n^{\tau_n}(T) = f^{\tau}(T)$$

where f^τ (respectively $f_n^{\tau_n}$) is the stopped function defined by $f^\tau(t) = f^\tau(\tau \wedge t)$ (respectively $f_n^{\tau_n}(t) = f_n^{\tau_n}(\tau_n \wedge t)$).

¿From the above lemma we get the following Proposition.

Proposition 4.4 *The set $\mathcal{M}_{loc}$ of all $(\mathcal{F}_t)-$continuous local martigales is a closed subset of $\mathcal{C}_{\mathcal{F}}([0,T]\times\Omega;\mathbb{R})$ endowed with the u.c.p. topology.*

Proof. Let $(M_n(t)\,, \quad t\in[0,T])_{n\in\mathbb{N}}$ be a sequence of local continuous martingales converging *u.c.p.* to a continuous martingale M. For $K>0$ we define the following stopping times:

$$\tau^n = \inf\{t\in[0,T] : |M_n(t)| \geq K\}$$

$$\tau = \inf\{t\in[0,T] : |M(t)| \geq K\}$$

with the convention that $\inf\emptyset = T$.

In order to conclude, it suffices to show that M^{τ} is a square integrable martingale. Lemma 4.3 implies that

$$M_n^{\tau^n}(T) \longrightarrow M^{\tau}(T) \qquad a.e. \tag{13}$$

Using (13) above and Lebesgue dominated convergence theorem, we have

$$M_n^{\tau^n}(T) \longrightarrow M^{\tau}(T) \qquad \text{in } L^2(\Omega).$$

The fact that M^{τ} is a square integrable martingale follows then from Proposition 5.23, Ch. 1 of [18]. ■

We are now able to state a useful representation result. Below we fix $t\in[0,T]$ and, since now $T<+\infty$, we have $\mathcal{T}_t=[t,T]$.

Theorem 4.5 *Let $T<+\infty$ and*

$$b:[0,T]\times\mathbb{R}^n\to\mathbb{R}^n, \qquad \text{resp. } \sigma:[0,T]\times\mathbb{R}^n\to\mathcal{L}(\mathbb{R}^m,\mathbb{R}^n)$$

be continuous functions. Let $u\in C^{0,1}([0,T]\times\mathbb{R}^n)$ be a strong solution of the Cauchy problem (12).

Fix $t\in[0,T]$, $x\in\mathbb{R}^n$ and let $(S_s)_{s\in\mathcal{T}_t}$ be a process of the form

$$S_s = x + \int_t^s \sigma(s,S_r)\,dW_r + A_t - A_s$$

where $(A_s)_{s\in\mathcal{T}_t}$ is an $(\mathcal{F}_s)$-weak zero energy process having all its mutual covariations.

Then, provided that the following assumption is verified for every $s\in\mathcal{T}_t$:

$$\lim_{n\to+\infty}\int_t^s (\partial_x u_n(r,S_r) - \partial_x u(r,S_r))d^-A_r - \int_t^s \langle\partial_x u_n(r,S_r)-\partial_x u(r,S_r), b(r,S_r)\rangle\,dr = 0, \qquad u.c.p., \tag{14}$$

we have

$$u(s,S_s) = u(t,S_t) + \int_t^s \partial_x u(r,S_r)\,\sigma(r,S_r)dW_r + \mathcal{B}^S(u)_s - \mathcal{B}^S(u)_t\,, \tag{15}$$

where, for $s\in\mathcal{T}_t$

$$\mathcal{B}^S(u)_s = \int_0^s h(r,S_r)\,dr + \int_0^s \partial_x u(r,S_r)\,d^-A_r - \int_0^s \langle\partial_x u(r,S_r)\,, b(r,S_r)\rangle\,dr. \tag{16}$$

Proof. We set $t=0$ for simplicity. The general case is obtained by additivity of different integrals. Let $u_n \longrightarrow u$ in $C^0([0,T]\times\mathbb{R}^n)$ be a sequence such that $\mathcal{L}_0 u_n = h_n \longrightarrow h$ in $C^0([0,T]\times\mathbb{R}^n)$. By Proposition 2.4, we get

$$\begin{aligned} u_n(s,S_s) &= u_n(0,S_0) + \int_0^s \mathcal{L}_0 u_n(r,S_r)\,dr - \int_0^s \langle \partial_x u_n(r,S_r), b(r,S_r)\rangle\,dr \\ &+ \int_0^s \partial_x u_n(r,S_r)\,\sigma(r,S_r)\,dW_r + \int_0^s \partial_x u_n(r,S_r)\,d^- A_r. \end{aligned}$$

¿From (14) we conclude that

$$\begin{aligned} M_s^n &= u_n(s,S_s) - u_n(0,S_0) - \int_0^s \mathcal{L}_0 u_n(r,S_r)\,dr \\ &+ \int_0^s \langle \partial_x u_n(r,S_r), b(r,S_r)\rangle\,dr - \int_0^s \partial_x u_n(r,S_r)\,d^- A_r \end{aligned}$$

converges *u.c.p.* to

$$\begin{aligned} M_s &= u(s,S_s) - u(0,S_0) - \int_0^s h(r,S_r)\,dr \\ &+ \int_0^s \langle \partial_x u(r,S_r), b(r,S_r)\rangle\,dr \int_0^s \partial_x u(r,S_r)\,d^- A. \end{aligned}$$

Using Proposition 4.4 above we get that M is an $(\mathcal{F}_s)$-local martingale. The result follows by Proposition 3.9, where $dM = \sigma(s,S_s)ds$ and $D=S$, by identification of the weak zero energy processes. ∎

For our applications in [16] we will need to consider a process A which is of bounded variation (so that S is solution of an SDE) but it is non-Markovian.

Corollary 4.6 *Let $T<+\infty$ and*

$$b_1 : \Omega \times [0,T]\times\mathbb{R}^n \to \mathbb{R}^n,$$

be a continuous progressively measurable field (continuous in (s,x)) and

$$b : [0,T]\times\mathbb{R}^n \to \mathbb{R}^n, \qquad \sigma : [0,T]\times\mathbb{R}^n \to \mathcal{L}(\mathbb{R}^m,\mathbb{R}^n)$$

be continuous functions. Let $u \in C^{0,1}([0,T]\times\mathbb{R}^n)$ be a strong solution of the Cauchy problem (12).

Fix $t\in[0,T]$, $x\in\mathbb{R}^n$ and let (S_s) be a solution to the SDE

$$dS_s = b_1(s,S_s)\,ds + \sigma(s,S_s)\,dW_s; \qquad S_t = x.$$

Then, provided that the following assumption be verified for every $s\in\mathcal{T}_t$

$$\lim_{n\to+\infty}\int_t^s \langle \partial_x u_n(r,S_r) - \partial_x u(r,S_r), b_1(r,S_r) - b(r,S_r)\rangle\,dr = 0, \qquad u.c.p., \tag{17}$$

15 holds with

$$\mathcal{B}^S(u)_s = \int_0^s h(r,S_r)\,dr + \int_0^s \langle \partial_x u(r,S_r)\,b_1(r,S_r)\rangle\,dr - \int_0^s \langle \partial_x u(r,S_r)\,b(r,S_r)\rangle\,dr.$$

Proof. The result follows setting

$$A_s = \int_0^s b_1(s, S_s)ds$$

in previous Theorem 4.5. ∎

The above result depends on the extra assumption (17) which is essential but not easy to check. We give first a special (but useful) case where it holds and then an improvement for the nondegenerate case. We have the following.

Remark 4.7 *If*

$$\lim_{n\to+\infty} \partial_x u_n = \partial_x u \qquad \text{in } C^0\left([0,T]\times\mathbb{R}^n\right)$$

then Assumption (14) is verified. This means that the result of Proposition 4.4 above applies if we know that u is a strong solution in a more restrictive sense, i.e. substituting the point 2 of Definition 4.2 with

$$\begin{aligned}
&u_n \longrightarrow u \text{ in } C^0\left([0,T]\times\mathbb{R}^n\right)\\
&\partial_x u_n \longrightarrow \partial_x u \text{ in } C^0\left([0,T]\times\mathbb{R}^n\right)\\
&h_n \longrightarrow h \text{ in } C^0\left([0,T]\times\mathbb{R}^n\right)\\
&\phi_n \longrightarrow \phi \text{ in } C^0\left(\mathbb{R}^n\right).
\end{aligned}$$

This is a particular case of our setting and it is the one used e.g in [13, 14, 15] to get the verification result. We can say that in these works a result like Theorem 4.5 is proved under the assumption that u is a strong solution in this more restrictive sense. It is worth to note that in such simplified setting, the proof of Theorem 4.5 follows simply by using standard convergence arguments. In particular there, one does not need to use the Fukushima-Dirichlet decomposition presented in Section 3. So, from the methodological point of view there is a serious difference with the result of Theorem 4.5, see [16], Section 8for comments. ∎

A more significant achievement concerns the nondegenerate case. It is illustrated in the following corollary.

Corollary 4.8 *We make the same assumption of Corollary 4.6 except (14) which we replace by the assumption that*

$$\sigma^{-1}\left(b_1 - b\right) \text{ is bounded.} \tag{18}$$

where σ^{-1} stands for the pseudo-inverse of σ. Then the same conclusion of Corollary 4.6 holds.

Proof. Setting $t = 0$ for simplicity we write

$$\beta_s = W_s + \int_0^s \left(\sigma^{-1}\left(b_1 - b\right)\right)(r, S_r)\, dr$$

and applying Girsanov Theorem, there is a probability $\mathbb{Q}$ equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{F}_T)$ such that $(\beta_s)_{s\in[0,T]}$ is an $(\mathcal{F}_s)$-standard Brownian motion.

So, under $\mathbb{Q}$ the process (S_s) fulfills the equation

$$dS_s = b\left(s, S_s\right) ds + \sigma\left(s, S_s\right) dW_s; \qquad S_0 = x.$$

Under $\mathbb{Q}$, Assumption (14) is trivially verified since $b = b_1$ and so we have (the tilde stands for operators under the new probability $\mathbb{Q}$)

$$\tilde{\mathcal{B}}^S(u)_s = \int_0^s h(r, S_r)\, dr$$

and

$$u(s, S_s) = u(0, S_0) + \int_0^s \partial_x u(r, S_r)\, \sigma(r, S_r)\, d\beta_r + \tilde{\mathcal{B}}^S(u)_s .$$

Expressing β in terms of W we obtain the result. ■

4.3 Some useful consequences

The previous results have some important consequences.

Remark 4.9 *From Remark 3.11 it follows that the conclusion of the above Corollary 4.8 holds also if we stop the processes $\mathcal{B}(u)$ with a stopping time $t \le \tau \le T$. More precisely we have*

$$\mathcal{B}^S(u)_{s\wedge\tau} = \int_0^{s\wedge\tau} h(r, S_r)\, dr - \int_0^{s\wedge\tau} \langle \partial_x u(r, S_r), b_1(r, S_r) - b(r, S_r)\rangle\, dr.$$

This fact will be useful in [16], Section 6. ■

Remark 4.10 *The results of the Corollary 4.6 and of Corollary 4.8 above still hold true with suitable modifications if we assume that, instead of having $u \in C^{0,1}([0,T] \times \mathbb{R}^n)$:*

(i) the strong solution u belongs to $C^0([0,T] \times \mathbb{R}^n) \cap C^{0,1}([\varepsilon,T] \times \mathbb{R}^n)$ for every small $\varepsilon > 0$;

(ii) for some $\beta \in (0,1)$ the map $(t,x) \to t^\beta \partial_x u(t,x)$ belong to $C^{0,1}([0,T] \times \mathbb{R}^n)$.

The proof of Corollary 4.6 in this case is a straightforward generalization of the one presented above: we do not give it here to avoid technicalities. In fact in proving the verification theorem in [16], Section 6, we will deal with initial data that are only continuous so with solutions u satisfying (i) and (ii) above and possibly not $C^{0,1}$. This difficulty will be faced directly in the proof of verification Theorem 6.19 in [16] by approximating the initial data with C^1 ones, using (16) and passing to the limit. ■

Remark 4.11 *From the above Corollary 4.6 it follows that the process $\mathcal{B}^S(u)_s$ is in fact a semimartingale (and also absolutely continuous).* ■

4.4 The elliptic case

We devote the last part of this subsection to apply the same setting above to elliptic problems. Consider the inhomogenous elliptic problem

$$\lambda u(x) + L_0 u(x) + h(x) = 0, \qquad \forall x \in \mathbb{R}^n. \tag{19}$$

where $D(L_0) = C^2(\mathbb{R}^n)$ and

$$L_0 u(x) = \langle b(x), \partial_x u(s,x)\rangle + \frac{1}{2}\mathrm{Tr}\,\left[\sigma^*(x)\, \partial_{xx} u(x)\, \sigma(x)\right].$$

Definition 4.12 *We say that u is a strict solution to the elliptic problem (19) if $u \in D(L_0)$ and (19) holds.*

Definition 4.13 *We say that u is a strong solution to the elliptic problem (19) if there exists a sequence $(u_n) \subseteq D(L_0)$ and a sequence $(h_n) \subseteq C^0(\mathbb{R}^n)$, such that*

1. *For every $n \in \mathbb{N}$ u_n is a strict solution of the problem*

$$\lambda u_n(x) - L_0 u_n(x) = h_n(x), \qquad \forall x \in \mathbb{R}^n.$$

2. *The following limits hold*

$$\begin{aligned} u_n &\longrightarrow u \text{ in } C^0(\mathbb{R}^n), \\ h_n &\longrightarrow h \text{ in } C^0(\mathbb{R}^n). \end{aligned}$$

Note that if L is the closure of L_0 in $C^0(\mathbb{R}^n)$ then a strong solution u, by construction, belongs to $D(L)$. We now exploit the above setting to show that, for functions $u \in C^1(\mathbb{R}^n)$ that are strong solutions of the elliptic problem (19) the following results holds; we omit the proof as it is completely similar (and even simpler) to the one of Theorem 4.5 for the parabolic case.

Theorem 4.14 *Let*

$$b_1 : \Omega \times \mathbb{R}^n \to \mathbb{R}^n,$$

be a continuous progressively measurable process (continuous in x) and

$$b : \mathbb{R}^n \to \mathbb{R}^n, \qquad \sigma : \mathbb{R}^n \to \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$$

be continuous functions. Let S_s be a solution to the SDE

$$dS_s = b_1(S_s)\, ds + \sigma(S_s)\, dW_s; \qquad S_0 = x.$$

Let $u \in C^1(\mathbb{R}^n)$ be a strong solution of the elliptic problem (19). Assume that

$$\lim_{n\to+\infty} \int_0^s \langle \partial_x u_n(S_r) - \partial_x u(S_r), b_1(S_r) - b(S_r) \rangle\, dr = 0, \qquad u.c.p., \tag{20}$$

or that (18) holds. Then we have

$$u(S_s) = u(S_t) + \int_t^s \partial_x u(S_r)\, \sigma(S_r) dW_r + \mathcal{B}^S(u)_s - \mathcal{B}^S(u)_t, \tag{21}$$

where

$$\mathcal{B}^S(u)_s = \int_0^s h(S_r)\, dr + \int_0^s \langle \partial_x u(S_r), b_1(S_r) - b(S_r) \rangle\, dr.$$

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References

[1] R. CHITASHVILI and M. MANIA, *Generalized Itô formula and derivation of Bellman's equation.* In K. Engelbert et alia. Stochastic processes and related topics (Siegmundsberg, 1994), 1–21, Stochastics Monogr., 10, Gordon and Breach, 1996.

[2] F. COQUET, A. JAKUBOWSKI, J. MEMIN and L. SLOMINSKI, *Natural decomposition of processes and weak Dirichlet processes*. Preprint.

[3] P. DAI PRA, G. DI MASI and B. TRIVELLATO, *Pathwise optimality in stochastic control.* SIAM J. Control Optim. **39** (2000), no. 5, 1540–1557

[4] M. ERRAMI and F. RUSSO, *Covariation de convolutions de martingales.* Comptes rendus de l'Académie des Sciences, t. **326**, Série 1, (1998), p. 601-609.

[5] M. ERRAMI and F. RUSSO, *n-covariation, generalized Dirichlet processes and calculus with respect to finite cubic variation processes.* Stochastic Processes and their Applications. **104**, (2003), 259–299.

[6] M. ERRAMI, F. RUSSO and P. VALLOIS, *Itô formula for $C^{1,\lambda}$-functions of a càdlàg semimartingale.* Probab. Theory Rel. Fields. **122**, (2002), 191-221.

[7] F. FLANDOLI, F. RUSSO, *Generalized Integration and Stochastic ODEs.* Annals of Probability. **Vol 30**, No 1 (2002), 270-292.

[8] F. FLANDOLI, F. RUSSO and J. WOLF, *Some SDEs with distributional drift. Part II: Lyons-Zheng structure, Itô's formula and semimartingale characterization.* Random Operators Stochastic Equations (ROSE), Vol. 12, No. 2, (2004), 145–184.

[9] W.H. FLEMING and H.M. SONER, Controlled Markov processes and viscosity solutions. Springer-Verlag, Berlin, New-York, 1993.

[10] H. FOELLMER, *Calcul d'Itô sans probabilités.* Sém. Prob. XV, Lect. Notes Math. **850**, (1981), 143-150.

[11] H. FOELLMER and P. PROTTER, *Quadratic covariation and an extension of Itô's formula.* Probab. Rel. Fields **16** (2000), 1–20.

[12] M. FUKUSHIMA, Y, OSHIMA and T. TAKEDA, Dirichlet forms and symmetric Markov processes. De Gruyter Studies in Mathematics, **19**, Walter de Gruyter & Co., Berlin, 1994.

[13] F. GOZZI, *Regularity of solutions of a second order Hamilton-Jacobi equation and application to a control problem.* Comm. Part. Diff. Eq. **20**, n.5 & 6 (1995), 775–826.

[14] F. GOZZI, *Global regular solutions of second order Hamilton-Jacobi equations in Hilbert spaces with locally Lipschitz nonlinearities.* J. Math. Anal. Appl. **198** (1996), 399–443.

[15] F. GOZZI and E. ROUY, *Regular solutions of second order stationary Hamilton-Jacobi equation. J. of Diff. Eq.* **130**, No. 1 (1996), 201–234.

[16] F. GOZZI and F. RUSSO, *Verification theorems for stochastic optimal control problems via a time-dependent Fukushima-Dirichlet decomposition.* Preprint.

[17] F. GOZZI, A. SWIECH and X. Y. ZHOU, *Stochastic verification theorems within the framework of viscosity solutions: a corrected proof.* SIAM Journal on Control and Optimization, **43**, 6, (2005), 2009-2019.

[18] I. KARATZAS and S. SHREVE, Brownian Motion and Stochastic Calculus. Graduate Texts in Mathematics **113**, 1995.

[19] N.V. KRYLOV, *Stochastic PDE's and Kolmogorov equations in infinite dimensions.* Lectures given at the 2nd C.I.M.E. Session held in Cetraro, August 24–September 1, 1998. Edited by G. Da Prato. Lecture Notes in Mathematics, (1999), **1715**.

[20] X. LI, J. YONG and X.Y. ZHOU, *Stochastic verification theorems within the framework of viscosity solutions.* SIAM J. Control Optim. **35** (1997), no. 1, 243–253.

[21] P.-L. LIONS and P.E. SOUGANIDIS, *Fully nonlinear stochastic partial differential equations: non-smooth equations and applications.* C. R. Acad. Sci. Paris Sér. I Math. **327** (1998), no. 8, 735–741.

[22] P.-L. LIONS and P.E. SOUGANIDIS, *Fully nonlinear stochastic partial differential equations.* C. R. Acad. Sci. Paris Sér. I Math. **326** (1998), no. 9, 1085–1092.

[23] S. MORET and D. NUALART, *Quadratic covariation and Itô's formula for smooth nondegenerate martingales.* J. Theoret. Probab. 13 (2000), no. 1, 193–224.

[24] Y. OSHIMA, *On a construction of Markov processes associated with time dependent Dirichlet spaces.* Forum Mathematicum **4**, (1992), 395-415.

[25] S.G. PENG, *Stochastic Hamilton-Jacobi-Bellman equations.* SIAM J. Control and Optim. **30**, 2, (1992), 247–261.

[26] D. REVUZ and M. YOR, Continuous martingales and Brownian motion. Second Edition, Springer-Verlag, 1994.

[27] F. RUSSO and P. VALLOIS, *Forward, backward and symmetric stochastic integration.* Probab. Theory Relat. Fields **97**, (1993), 403-421.

[28] F. RUSSO and P. VALLOIS, *The generalized covariation process and Itô formula.* Stochastic Processes and their Applications **59**, (1995), 81-104.

[29] F. RUSSO and P. VALLOIS, *Itô formula for C^1-functions of semimartingales.* Probab. Theory Relat. Fields **104**, (1996), 27-41.

[30] F. RUSSO and P. VALLOIS, *Stochastic calculus with respect to a finite variation process.* Stochastics and Stochastic Reports **70**, (2000), 1-40.

[31] F. RUSSO, P. VALLOIS and J. WOLF, *A generalized class of Lyons-Zheng processes.* Bernoulli **7 (2)**, (2001), 363-379.

[32] F. RUSSO and P. VALLOIS, *Elements of stochastic calculus via regularization.* Preprint LAGA-Paris 13 2004-28.

[33] W. STANNAT, *The theory of generalized Dirichlet forms and its applications in analysis and stochastics.* Mem. Amer. Math. Soc. **142**, (1999), no. 678.

[34] G. TRUTNAU, *Stochastic calculus of generalized Dirichlet forms and applications to stochastic differential equations in infinite dimensions.* Osaka J. Math. **37**, no. 2, (2000), 315-343.

[35] J. YONG and X.Y. ZHOU, Stochastic controls. Hamiltonian systems and HJB equations. Applications of Mathematics, **43**. Springer-Verlag, New York, 1999.