Existence, uniqueness and regularity w.r.t. the initial condition of mild solutions of SPDE's driven by Poisson noise

Claudia Ingrid Prévôt

Fakultät für Mathematik der Universität Bielefeld, Postfach 100131, D-33501 Bielefeld, cknoche@math.uni-bielefeld.de

1 Introduction

SPDE's driven by Gaussian noise are well studied (see [32], [26], [10] and references therein) whereas SPDE's with Poisson noise are little less well known. But within the last years SPDE's driven for example by a compensated Poisson random measure or a Lévy noise draw more attention, one reason for which may be the prospect of numerous applications, e.g. in biology (cf. [31],[20]), climatology (cf. [19]) or financial market theory (cf. [4], [13], [28]).

Apart from applications SPDE's with Poisson noise are of independent interest and basic investigations and a better understanding of stochastic integrals w.r.t. a compensated Poisson random measure and of SPDE's with Poisson noise is an important step for the study of SPDE's with Lévy noise. There is quite a substantial amount of work that has been done in this field. In [18], [1], [24], [2], [3] the authors analyze, among other things, SPDE's with Poisson noise in one dimension and show existence and uniqueness of solutions in H^2 (see below). Moreover, in [22] the authors deal with stochastic integral equations driven by non Gaussian noise on separable Banach spaces of M-type 1 and 2 and prove existence and uniqueness of pathwise solutions in H^1 and H^2.

In *Section 1* we give an introduction to the theory of Poisson random measures and Poisson point processes where we shall follow largely the organization of [18]. Moreover, we present the scheme of the construction of the stochastic integral of Hilbert space valued integrands w.r.t. a compensated Poisson random measure. Detailed proofs can be found in [21, Chapter 2.3]. In the style of the definition of the integral w.r.t. a Wiener process (cf. [9]) or w.r.t. a square-integrable martingale (cf. [23]) we define the integral by an L^2-isometry, which, in the case of the Wiener process, is just the classical Itô isometry. Independently, stochastic integration in Banach spaces of M-type p, $1 \leq p \leq 2$ was done in [30].
We also present some useful properties of the stochastic integral, but without proofs. For the interested reader we refer to [21, Chapter 3].

In *Section 2* we proof the existence of the unique mild solution in H^2 and analyze its dependence on the initial condition. We proof the Gâteaux differentiability of the mild solution as a mapping from L^2 to H^2.

As a consequence, in *Section 3* we obtain a gradient estimate for the Gâteaux derivative ∂X of X and for the resolvent (R_α) associated to the mild solution. Under the additional assumptions that $S(t)$, $t \geq 0$, is quasicontractive, $\nu(U) <$

∞, B is constant and F is dissipative we get that

$$\|\partial X(x)h(t)\| \leq e^{\omega_0 t}$$

for all $x, h \in H$ and $t \geq 0$. Moreover, for all $f \in C_b^1(H, \mathbb{R})$, $R_\alpha f : H \to \mathbb{R}$ is Gâteaux differentiable for all $\alpha \geq 0$ and

$$\|\partial R_\alpha f(x)\|_{L(H,\mathbb{R})} \leq \frac{1}{\alpha - \omega_0} \sup_{x \in H} \|Df(x)\|_{L(H)} \text{ for all } \alpha > \omega_0,\, x \in H$$

2 Poisson random measures, Poisson point processes and stochastic integration

2.1 Poisson random measures

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and $(E, \mathcal{S})$ a measurable space. Let $\mathbb{M}$ be the space of $\mathbb{Z}_+ \cup \{+\infty\}$-valued measures on $(E, \mathcal{S})$ and

$$\mathcal{B}_{\mathbb{M}} := \sigma(\mathbb{M} \ni \mu \mapsto \mu(B) \,|\, B \in \mathcal{S}).$$

A *Poisson random measure* on $(E, \mathcal{S})$ (and $(\Omega, \mathcal{F}, P)$) is a random variable $\Pi : (\Omega, \mathcal{F}) \to (\mathbb{M}, \mathcal{B}_{\mathbb{M}})$ such that for all $B \in \mathcal{S}$: $\Pi(B) : \Omega \to \mathbb{Z}_+ \cup \{+\infty\}$ is Poisson distributed with parameter $E[\Pi(B)]$ and such that for all pairwise disjoint $B_1, \ldots, B_m \in \mathcal{S}$, $\Pi(B_1), \ldots, \Pi(B_m)$ are independent.

It is shown in [18, I. Theorem 8.1, p.42] and a little bit more detailed in [21, Theorem 2.5, p.20] that given a σ-finite measure m on $(E, \mathcal{S})$ there exists a complete probability space $(\Omega, \mathcal{F}, P)$ such that there exists a Poisson random measure Π on $(E, \mathcal{S})$ and $(\Omega, \mathcal{F}, P)$ with $E[\Pi(B)] = m(B)$ for all $B \in \mathcal{S}$. m is then called the *mean measure* or *intensity measure* of the Poisson random measure Π.

2.2 Poisson point processes

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with a normal filtration $\mathcal{F}_t$, $t \geq 0$ (, i.e. $\mathcal{F}_t$, $t \geq 0$, is right-continuous such that $\mathcal{F}_0$ contains all P-nullsets of $\mathcal{F}$) and $(U, \mathcal{B})$ a measurable space.

A *point function p on U* is a mapping $p : D_p \subset (0, \infty) \to U$ where the domain D_p is countable.

p defines a measure $N_p(dt, dy)$ on $([0, \infty) \times U, \mathcal{B}([0, \infty)) \otimes \mathcal{B})$ in the following way. Define $\bar{p} : D_p \to (0, \infty) \times U$, $t \mapsto (t, p(t))$ and denote by c the counting measure on $(D_p, \mathcal{P}(D_p))$, i.e. $c(A) := \#A$ for all $A \in \mathcal{P}(D_p)$.

For $\bar{B} \in \mathcal{B}([0, \infty)) \otimes \mathcal{B}$ define

$$N_p(\bar{B}) := c(\bar{p}^{-1}(\bar{B})).$$

Then, in particular, we have for all $A \in \mathcal{B}([0, \infty))$ and $B \in \mathcal{B}$

$$N_p(A \times B) = \#\{t \in D_p | t \in A, p(t) \in B\}.$$

Notation: $N_p(t,B) := N_p(]0,t] \times B)$, $t \geq 0$, $B \in \mathcal{S}$.

Let $\mathcal{P}_U$ be the space of all point functions on U and

$$\mathcal{B}_{\mathcal{P}_U} := \sigma(\mathcal{P}_U \ni p \mapsto N_p(t,B) \,|\, t > 0, B \in \mathcal{B})$$

A *point process on* U (and $(\Omega, \mathcal{F}, P)$) is a random variable $p : (\Omega, \mathcal{F}) \to (\mathcal{P}_U, \mathcal{B}_{\mathcal{P}_U})$.

A point process p on U is called *Poisson point process* if there exists a Poisson random measure Π on $((0,\infty) \times U, \mathcal{B}(0,\infty) \otimes \mathcal{B})$ such that there exists a P-nullset $N \in \mathcal{F}$ such that for all $\omega \in N^c$ and for all $\bar{B} \in \mathcal{B}(0,\infty) \otimes \mathcal{B}$: $N_{p(w)}(\bar{B}) = \Pi(\omega)(\bar{B})$. By e.g. [18, Chapter I.9, II.3] it is known that given a σ-finite measure ν on $(U, \mathcal{B})$ there exists a so called *stationary* $(\mathcal{F}_t)$*-Poisson point process(with characteristic measure* ν*).* We formulate this existence statement in a more precise way in the following theorem. For a bit more detailed proof we refer to [21, Theorem 2.11, p.27; Remark 2.14, p. 30].

Theorem 1. *Given a* σ*-finite measure* ν *on* $(U,\mathcal{B})$ *there exists a complete probability space* $(\Omega, \mathcal{F}, P)$ *with a normal filtration* $\mathcal{F}_t$, $t \geq 0$*, such that there exists a Poisson point process* p *on* U *and* $(\Omega, \mathcal{F}, P)$ *such that*

(i) *for every* $t \geq 0$ *and* $B \in \mathcal{B}$ $N_p(t,B)$ *is* $\mathcal{F}_t$*-measurable,*

(ii) $\{N_p(]t, t+h] \times B) \,|\, h > 0, B \in \mathcal{B}\}$ *is independent of* $\mathcal{F}_t$ *for all* $t \geq 0$*,*

(iii) $E[N_p(dt, dy)] = \lambda(dt) \otimes \nu(dy)$ *where* λ *denotes the Lebesgue-measure on* $(0,\infty)$*,*

(iv) *for all* $B \in \mathcal{B}$ *with* $\nu(B) < \infty$*,* $q(t,B) := N_p(t,B) - t\nu(B)$*,* $t \geq 0$*, is an* $(\mathcal{F}_t)$*-martingale.*

We call such a point process a stationary $(\mathcal{F}_t)$*-Poisson point process with characteristic measure* ν*.*

The definition of a stationary $(\mathcal{F}_t)$-Poisson point process (with characteristic measure ν) is covered by the more general definition of an $(\mathcal{F}_t)$-Poisson point process of class (QL) with compensator $\hat{N}_p$ (see e.g. [18, Chapter II.3]). A stationary $(\mathcal{F}_t)$-Poisson point process (with characteristic measure ν) is a $(\mathcal{F}_t)$-Poisson point process of class (QL) with compensator $\hat{N}_p(t,B) = t\nu(B)$, $t \geq 0$, $B \in \mathcal{B}$. In this paper our main focus is laid on the stationary $(\mathcal{F}_t)$-Poisson point processes and for that reason we do not go into the particulars of $(\mathcal{F}_t)$-Poisson point process of class (QL).

If p is a stationary $(\mathcal{F}_t)$-Poisson point process with characteristic measure ν, then by definition, for all $B \in \mathcal{B}$ with $\nu(B) < \infty$, $q(t,B) := N_p(t,B) - t\nu(B)$, $t \geq 0$, is an $(\mathcal{F}_t)$-martingale. It is even an L^2-martingale, i.e. $E[q(t,B)^2] < \infty$ for all $t \geq 0$. Then there exists the quadratic variation of $q(t,B)$, $t \geq 0$, i.e. a P-unique, integrable, increasing, predictable process $A(t)$, $t \geq 0$, (i.e. $A : [0,\infty[\times\Omega \to \mathbb{R}$ is measurable w.r.t. the predictable σ-field on $[0,T] \times \Omega$) such that $q(t,B)^2 - A(t)$, $t \geq 0$, is an $(\mathcal{F}_t)$-martingale. A is denoted by $\langle q(\cdot, B)\rangle$. One gets that the quadratic variation of $q(\cdot, B)$ is the compensator of p, i.e. for $B_1, B_2 \in \mathcal{B}$ with $\nu(B_i) < \infty$, $i = 1, 2$,

$$\langle q(\cdot, B_1), q(\cdot, B_2)\rangle(t) = t\nu(B_1 \cap B_2), \; t \geq 0.$$

(see e.g. [18, II. Theorem 3.1, p.60]).

From now on let $(H, \langle\ ,\ \rangle)$ be a separable Hilbert space, $(\Omega, \mathcal{F}, P)$ be a complete probability space with a normal filtration $\mathcal{F}_t$, $t \geq 0$, $(U, \mathcal{B}, \nu)$ a σ-finite measure space and p a stationary $(\mathcal{F}_t)$-Poisson point process on U with characteristic measure ν.
In the following two subsections we give schemes of the (stochastic) integration w.r.t. N_p and q. For this purpose we need the predctable σ-field

$$\begin{aligned}\mathcal{P}_T(U) :=& \,\sigma(g : [0,T] \times \Omega \times U \to \mathbb{R} \,|\, g \text{ is } (\mathcal{F}_t \otimes \mathcal{B}) - \text{adapted and left-continuous})\\ =& \,\sigma(\, \{]s,t] \times F_s \times B \,|\, 0 \leq s \leq t \leq T, F_s \in \mathcal{F}_s, B \in \mathcal{B}\}\\ & \quad \cup \{\{0\} \times F_0 \times B \,|\, F_0 \in \mathcal{F}_0, B \in \mathcal{B}\})\end{aligned}$$

2.3 Scheme of the construction of the stochastic integral w.r.t. q

In the **first step** we define the stochastic integral for elementary processes. For this purpose define the set

$$\Gamma_p := \{B \in \mathcal{B} \,|\, \nu(B) < \infty\}.$$

The class $\mathcal{E}$ of all elementary processes is determined by the following definition.

Definition 2. An H-valued process $\Phi(t) : \Omega \times U \to H$, $t \in [0,T]$, on $(\Omega \times U, \mathcal{F} \otimes \mathcal{B})$ is said to be *elementary* if there exists a partition $0 = t_0 < t_1 < \cdots < t_k = T$ of $[0,T]$ and for $m \in \{0, \ldots, k-1\}$ there exist $B_1^m, \ldots, B_{I(m)}^m \in \Gamma_p$, pairwise disjoint, such that

$$\Phi = \sum_{m=0}^{k-1} \sum_{i=1}^{I(m)} \Phi_i^m 1_{]t_m, t_{m+1}] \times B_i^m}$$

where $\Phi_i^m \in L^2(\Omega, \mathcal{F}_{t_m}, P; H)$, $1 \leq i \leq I(m)$, $0 \leq m \leq k-1$.
$\mathcal{E}$ is a linear space.

For $\Phi = \sum_{m=0}^{k-1} \sum_{i=1}^{I(m)} \Phi_i^m 1_{]t_m, t_{m+1}] \times B_i^m} \in \mathcal{E}$ define the stochastic integral process by

$$\begin{aligned}\mathrm{Int}(\Phi)(t) :=& \int_0^{t+} \int_U \Phi(s,y)\, q(ds, dy) := \int_{]0,t]} \int_U \Phi(s,y)\, q(ds, dy)\\ :=& \sum_{m=0}^{k-1} \sum_{i=1}^{I(m)} \Phi_i^m (q(t_{m+1} \wedge t, B_i^m) - q(t_m \wedge t, B_i^m)),\end{aligned}$$

$t \in [0,T]$.
Then $\mathrm{Int}(\Phi)$ is P-a.s. well-defined and Int is linear in $\Phi \in \mathcal{E}$.

In the **second step** we first show that for $\Phi \in \mathcal{E}$, $\mathrm{Int}(\Phi)$ is an element of

$\mathcal{M}_T^2(H)$, the space of all H-valued $(\mathcal{F}_t)$-martingales $M(t)$, $t \in [0,T]$, such that $E[\|M(T)\|^2] < \infty$. Then, defining the seminorm $\| \, \|_T$ on $\mathcal{E}$ by $\|\Phi\|_T^2 := E[\int_0^T \int_U \|\Phi(s,y)\|^2 \, \nu(dy) \, ds]$, we prove that

$$\mathrm{Int} : (\mathcal{E}, \| \, \|_T) \to (\mathcal{M}_T^2(H), \| \, \|_{\mathcal{M}_T^2})$$

is an isometry, where

$$\|M\|^2_{\mathcal{M}_T^2} := sup_{t\in[0,T]} E[\|M(t)\|^2] = E[\|M(T)\|^2], \, M \in \mathcal{M}_T^2(H).$$

, e.g. we show that

$$\|\mathrm{Int}(\Phi)\|_{\mathcal{M}_T^2} = sup_{t\in[0,T]} E[\| \int_0^{t+} \int_U \Phi(s,y) \, q(ds,dy)\|^2] = E[\int_0^T \int_U \|\Phi(s,y)\|^2 \, \nu(dy) \, ds]. \tag{1}$$

To get a norm on $\mathcal{E}$ one has to consider equivalence classes of elementary processes with respect to $\| \, \|_T$. For simplicity, the space of equivalence classes will be denoted by $\mathcal{E}$, too.
Since $\mathcal{E}$ is dense in the abstract completion $\bar{\mathcal{E}}^{\| \, \|_T}$ of $\mathcal{E}$ w.r.t. $\| \, \|_T$ it is clear that there is a unique isometric extension of Int to $\bar{\mathcal{E}}^{\| \, \|_T}$.

In the **third step** we characterize $\bar{\mathcal{E}}^{\| \, \|_T}$.

$\bar{\mathcal{E}}^{\| \, \|_T}$ can be characterized by

$$\mathcal{N}_q^2(T,U,H) = L^2([0,T] \times \Omega \times U, P_T(U), \lambda \otimes P \otimes \nu; H).$$

2.4 Properties of the stochastic integral w.r.t. q

Proposition 3. *Assume that $\Phi \in \mathcal{N}_q^2(T,U,H)$ and that τ is an $(\mathcal{F}_t)$-stopping time such that $P(\tau \le T) = 1$. Then $1_{]0,\tau]}\Phi \in \mathcal{N}_q^2(T,U,H)$ and*

$$\int_0^{t+} \int_U 1_{]0,\tau]}(s)\Phi(s,y) \, q(ds,dy) = \int_0^{(t\wedge\tau)+} \int_U \Phi(s,y) \, q(ds,dy) \tag{2}$$

for all $t \in [0,T]$ P-a.s.

Proposition 4. *Let $\Phi \in \mathcal{N}_q^2(T,U,H)$, $(\tilde{H}, \langle \, , \, \rangle_{\tilde{H}})$ a further Hilbert space and $L \in L(H,\tilde{H})$. Then $L(\Phi) \in \mathcal{N}_q^2(T,U,\tilde{H})$ and*

$$L\Big(\int_0^{t+} \int_U \Phi(s,y) \, q(ds,dy) \Big) = \int_0^{t+} \int_U L(\Phi(s,y)) \, q(ds,dy)$$

for all $t \in [0,T]$ P-a.s.

Proposition 5. *Let $\Phi \in \mathcal{N}_q^2(T,U,H)$ and define $X(t) := \int_0^{t+} \int_U \Phi(s,y) \, q(ds,dy)$, $t \in [0,T]$. Then X is càdlàg and $X(t) = X(t-)$ P-a.s. for all $t \in [0,T]$.*

Before we can formulate the next proposition we need the notion of the square bracket of a real local $(\mathcal{F}_t)$-martingale.

Definition 6. Let M, N be real càdlàg local $(\mathcal{F}_t)$-martingales. The *bracket process* of M, N, also called simply the *bracket* of M, N, is defined by

$$[M, N]_t := M(t)N(t) - \int_0^t M(s-)\, dN(s) - \int_0^t N(s-)\, dM(s).$$

$[M, M]$ will be denoted by $[M]$ and called the *square bracket* of M.

Theorem 7. *Let M, N be real càdlàg, locally square integrable local $(\mathcal{F}_t)$-martingales. The bracket $[M, N]$ of M is the P-unique, $(\mathcal{F}_t)$-adapted, càdlàg process $A(t)$, $t \geq 0$, with paths of finite variation on compacts such that*

(i) $MN - A$ *is a local* $(\mathcal{F}_t)$*-martingale,*

(ii) $\Delta A(t) = \Delta M(t) \Delta N(t)$ *for all* $t \geq 0$ P*-a.s.*

Proof. [27, II.6 Corollary 2, p.65] □ □

Proposition 8. *Let* $\Phi \in \mathcal{N}_q^2(T, U, \mathbb{R})$*. Then*
$\big(X(t)\big)_{t\geq 0} := \big(\int_0^{(t\wedge T)+} \int_U \Phi(s, y)\, q(ds, dy)\big)_{t\geq 0} \in \mathcal{M}^2(\mathbb{R})$ *and*

$$\Big[\int_0^{(\cdot\wedge T)+} \int_U \Phi(s, y)\, q(ds, dy)\Big] = \int_{]0,\cdot\wedge T]} \int_U |\Phi(s, y)|^2\, N_p(ds, dy).$$

Proposition 9. *Let* $\Phi \in \mathcal{N}_q^2(T, U, \mathbb{R})$*. Denote by X the integral process* $(X(t))_{t\geq 0} := \big(\int_{]0,t\wedge T]} \int_U \Phi(s, y)\, q(ds, dy)\big)_{t\geq 0} \in \mathcal{M}^2(\mathbb{R})$.
Moreover, let Y be an $(\mathcal{F}_t)$-adapted, left continuous, bounded process ($|Y(t, \omega)| \leq K < \infty$ for all $t \geq 0$ and $\omega \in \Omega$).
Then

(i) $Y \in \mathcal{L}_{ucp}$ *and* $Y\Phi \in \mathcal{N}_q^2(T, U, \mathbb{R})$,

(ii) $\displaystyle\int_{]0,t]} Y(s)\, dX(s) = \int_0^{t+} \int_U Y(s)\Phi(s, y)\, q(ds, dy)$ *for all* $t \in [0, T]$ P*-a.s.*

In all following sections let $(\Omega, \mathcal{F}, P)$ be a complete probability space with a normal filtration $\mathcal{F}_t$, $t \geq 0$, U a separable Banach space with Borel-σ-field $\mathcal{B}$, ν a σ-finite measure on $(U, \mathcal{B})$ and p a stationary $(\mathcal{F}_t)$-Poisson point process with characteristic measure ν.

3 Existence and uniqueness of a mild solution

Let $T > 0$ and consider the following type of stochastic differential equation in H

$$\begin{cases} dX(t) & = [AX(t) + F(X(t))]\, dt + B(X(t), y)\, q(dt, dy) \\ X(0) & = \xi \end{cases} \tag{3}$$

where we always assume that

1. $A : D(A) \subset H \to H$ is the infinitesimal generator of a C_0-semigroup $S(t)$, $t \geq 0$, of linear, bounded operators on H.
2. $F : H \to H$ is $\mathcal{B}(H)/\mathcal{B}(H)$-measurable.
3. $B : H \times U \to H$ is $\mathcal{B}(H) \otimes \mathcal{B}/\mathcal{B}(H)$-measurable s.t. $B(x, \cdot) \in L(U, H)$.
4. ξ is an H-valued, $\mathcal{F}_0$-measurable random variable.

Remark 10. If we set $M_T := \sup_{t \in [0,T]} \|S(t)\|_{L(H)}$ then $M_T < \infty$ by [25, Theorem 2.2, p.4].

We interpret (3) as an integral equation:

Definition 11 (Mild solution). An H-valued predictable process $X(t)$, $t \in [0, T]$, is called a *mild solution of equation (3)* if

$$\begin{aligned} X(t) = S(t)\xi + \int_0^t S(t-s)F(X(s))\, ds \\ + \int_0^{t+} \int_U S(t-s)B(X(s), y)\, q(ds, dy) \quad P\text{-a.s.} \end{aligned} \tag{4}$$

for all $t \in [0, T]$. In particular, the appearing integrals have to be well defined.

The idea to interpret (3) by (4) can be justified in the following way.

If $X(t)$, $t \in [0, T]$, is a mild solution of (3) and if we assume that $\int_0^t S(t-s)F(X(s))\, ds$ and $\int_0^{T+} \int_U 1_{]0,t]}(s) S(t-s)B(X(s), y)\, q(ds, dy)$, $t \in [0, T]$, have predictable versions and that for all $\zeta \in D(A^*)$

$$\begin{aligned} &\int_0^T \|F(X(s))\|\, ds < \infty \text{ and} \\ &\int_0^T E\Big[\int_0^t \int_U |\langle S(t-s)B(X(s), y), A^*\zeta\rangle|^2\, \nu(dy)\, ds\Big]\, dt < \infty \end{aligned}$$

then by the fundamental theorem for Bochner integrals, Fubini's theorem and a stochastic Fubini theorem for the integral w.r.t. q (see [5, Theorem 5]) X is a weak solution, i.e.

$$\begin{aligned}\langle X(t),\zeta\rangle = \langle \xi,\zeta\rangle + \int_0^t \langle X(s), A^*\zeta\rangle + \langle F(X(s)),\zeta\rangle\, ds \\ + \int_0^t \langle B(X(s),y),\zeta\rangle\, q(ds,dy) \quad P\text{-a.s.}\end{aligned}$$

for all $t \in [0,T]$and $\zeta \in D(A^*)$.

3.1 Existence and uniqueness of the mild solution in $H^2(T,H)$

Before stating the theorems about existence and uniqueness of a mild solution we give some notations and present the idea of the proof.
First, we introduce the space where we want to find the mild solution of the above problem. We define

$$\begin{aligned}\mathcal{H}^2(T,H) := \{Y(t), t\in[0,T] \mid Y \text{ has an } H\text{-predictable version,} \\ Y(t) \in L^2(\Omega,\mathcal{F}_t,P;H) \text{ and } \sup_{t\in[0,T]} E[\|Y(t)\|^2] < \infty\}\end{aligned}$$

and define a seminorm on $\mathcal{H}^2(T,H)$ by

$$\|Y\|_{\mathcal{H}^2} := \sup_{t\in[0,T]} \big(E[\|Y(t)\|^2]\big)^{\frac{1}{2}},\ Y \in \mathcal{H}^2(T,H).$$

We also consider the seminorms $\|\ \|_{2,\lambda,T}$, $\lambda \geq 0$, on $\mathcal{H}^2(T,H)$ given by

$$\|Y\|_{2,\lambda,T} := \sup_{t\in[0,T]} e^{-\lambda t}\big(E[\|Y(t)\|^2]\big)^{\frac{1}{2}}.$$

Then $\|\ \|_{\mathcal{H}^2} = \|\ \|_{2,0,T}$ and all seminorms $\|\ \|_{p,\lambda,T}$, $\lambda \geq 0$, are equivalent.
Let $\zeta \in \mathcal{L}_0^2 := \mathcal{L}^2(\Omega,\mathcal{F}_0,P;H)$ and $Z \in \mathcal{H}^2(T,H)$. Then Z has at least one predictable version which we denote again by Z. Define

$$\begin{aligned}\mathcal{F}(\zeta,Z) := \Big(S(t)\zeta + \int_0^t S(t-s)F(Z(s))\, ds \\ + \int_0^{t+}\int_U S(t-s)B(Z(s),y)\, q(ds,dy)\Big)_{t\in[0,T]}.\end{aligned} \tag{5}$$

Later we will prove that under certain conditions on F and B the appearing integrals are well-defined and the processes on the right hand side of (5) are elements of $\mathcal{H}^2(T,H)$. Moreover, under the assumption that all integrals are well-defined, $\mathcal{F}$ is well-defined in the sense of version, i.e. taking another $\tilde{\zeta}$ such that $\tilde{\zeta} = \zeta$ P-a.s. and another predictable version $\tilde{Z}$ of Z, then $\mathcal{F}(\zeta,Z)$ is a version of $\mathcal{F}(\tilde{\zeta},\tilde{Z})$ since $\mathcal{F}(\zeta,Z)(t) = \mathcal{F}(\tilde{\zeta},\tilde{Z})(t)$ in $\|\ \|_{L^2}$.

A mild solution of problem (3) with initial condition $\xi \in \mathcal{L}_0^2$ is by Definition 11 an H-predictable process $X(\xi)$ such that $\mathcal{F}(\xi,X(\xi)) = X(\xi)$ in the sense of versions.

The idea to prove the existence and uniqueness of a mild solution is to use Banach's fixed point theorem. This approach requires that $\mathcal{H}^2(T,H)$ is a Banach space. For this purpose we consider equivalence classes in $\mathcal{H}^2(T,H)$ w.r.t. $\| \|_{2,\lambda,T}$, $\lambda \geq 0$. We denote the space of equivalence classes by $H^2(T,H)$.
For simplicity we use the following notations

$$H^2(T,H) := (H^2(T,H), \| \|_{\mathcal{H}^2})$$

and

$$H^{2,\lambda}(T,H) := (H^2(T,H), \| \|_{2,\lambda,T}),\ \lambda > 0.$$

Now we define for $\xi \in L_0^2 := L^2(\Omega, \mathcal{F}_0, P; H)$ and $Y \in H^2(T,H)$, $\bar{\mathcal{F}}(\xi, Y)$ as the equivalence class of $\mathcal{F}(\zeta, Z)$ w.r.t. $\| \|_{\mathcal{H}^2}$ for an arbitrary $\zeta \in \xi$ and an arbitrary predictable representative $Z \in Y$. By the above considerations $\mathcal{F}(\zeta, Z)$ is independent of the representatives ζ and Z.
To get the existence of an implicit function $X : L_0^2 \to H^2(T,H)$ such that $\bar{\mathcal{F}}(\xi, X(\xi)) = X(\xi)$ in $H^2(T,H)$ for all $\xi \in L_0^2$ we prove that $\bar{\mathcal{F}}$ as a mapping from $L_0^2 \times H^2(T,H)$ to $H^2(T,H)$ is well-defined and that there exists $\lambda_{T,2} =: \lambda \geq 0$ such that

$$\bar{\mathcal{F}} : L_0^2 \times H^{2,\lambda}(T,H) \to H^{2,\lambda}(T,H)$$

is a contraction in the second variable. Then the existence and uniqueness of the mild solution $X(\xi) \in H^{2,\lambda}(T,H)$ of (3) with initial condition $\xi \in L_0^2$ follows by Banach's fixed point theorem.
Since the norms $\| \|_{2,\lambda,T}$, $\lambda \geq 0$, are equivalent we may consider $X(\xi)$ then as an element of $H^2(T,H)$ and get the existence of the implicit function $X : L_0^2 \to H^2(T,H)$ such that $\bar{\mathcal{F}}(\xi, X(\xi)) = X(\xi)$.

To get the existence of a mild solution on $[0,T]$ in $H^2(T,H)$ we make the following assumptions.

Hypothesis H.0

1. $F : H \to H$ is Lipschitz-continuous, i.e. there exists a constant $C > 0$ such that

$$\|F(x) - F(y)\| \leq C\|x - y\|$$
$$\|F(x)\| \leq C(1 + \|x\|) \quad \text{for all } x, y \in H.$$

2. $B : H \times U \to H$ $\mathcal{B}(H) \otimes \mathcal{B}/\mathcal{B}(H)$-measurable.

3. There exists an integrable mapping $K : [0,T] \to [0,\infty[$ such that for all $t \in]0,T]$ and for all $x, z \in H$

$$\int_U \|S(t)(B(x,y) - B(z,y))\|^2 \, \nu(dy) \leq K(t)\|x - z\|^2$$
$$\int_U \|S(t)B(x,y)\|^2 \, \nu(dy) \leq K(t)(1 + \|x\|)^2.$$

Theorem 12. *Assume that the coefficients A, F and B fulfill the conditions of Hypothesis H.0 then for every initial condition $\xi \in L_0^2$ there exists a unique*

mild solution $X(\xi)(t)$, $t \in [0,T]$, of equation (3) in $H^2(T,H)$.
In addition, we even obtain that the mapping

$$X : L_0^2 \to H^2(T,H)$$

is Lipschitz continuous.

For the proof of Theorem 12 we need the following Lemma.

Lemma 13. *Let $Y(t)$, $t \geq 0$, be a process on $(\Omega, \mathcal{F}, P)$ with values in a separable Banach space E. If Y is adapted to $\mathcal{F}_t$, $t \in [0,T]$, and stochastically continuous then there exists a predictable version of Y.*
In particular, if Y is adapted to $\mathcal{F}_t$, $t \in [0,T]$, and continuous in the square mean then there exists a predictable version of Y.

Proof. [9, Proposition 3.6 (ii), p.76] □ □

Proof of Theorem 12. In the **first** part of the proof we show that $\bar{\mathcal{F}} : L_0^2 \times \mathcal{H}^2(T,H) \to \mathcal{H}^2(T,H)$ is well-defined and a contraction in the second variable.

Let $Y \in \mathcal{H}^2(T,H)$, predictable, then, obviously, $\Phi := \big(1_{]0,t]}(s)S(t-s)B(Y(s),\cdot)\big)_{s\in[0,T]}$ is an element of $\mathcal{N}_q^2(T,U,H)$ for all $t \in [0,T]$.
Next we show that there exists a predictable version of

$$(Z(t))_{t\in[0,T]} := \Big(\int_0^{t+}\int_U S(t-s)B(Y(s),y)\, q(ds,dy)\Big)_{t\in[0,T]}.$$

For this puprose we apply Lemma 13, i.e. we show that the process Z is adapted to $\mathcal{F}_t$, $t \in [0,T]$, and continuous as a mapping from $[0,T]$ to $L^2(\Omega,\mathcal{F},P;H)$.
Let $1 < \alpha < 2$ and define for $t \in [0,T]$

$$\begin{aligned} Z^\alpha(t) &:= \int_0^{(\frac{t}{\alpha})+}\int_U S(t-s)B(Y(s),y)\, q(ds,dy) \\ &= \int_0^{(\frac{t}{\alpha})+}\int_U S(t-\alpha s)S((\alpha-1)s)B(Y(s),y)\, q(ds,dy), \end{aligned}$$

where we used the semigroup property of $S(t)$, $t \geq 0$.
Set $\Phi^\alpha(s,\omega,y) := S((\alpha-1)s)B(Y(s,\omega),y)$ then Φ^α is $\mathcal{P}_T(U)/\mathcal{B}(H)$-measurable.
Moreover,

$$E\Big[\int_0^T\int_U \|\Phi^\alpha(s,y)\|^2\, \nu(dy)\, ds\Big] \leq 2(1+\|Y\|_{\mathcal{H}^2}^2)\frac{1}{\alpha-1}\int_0^{(\alpha-1)T} K(s)\, ds < \infty.$$

Now we show that the mapping $Z^\alpha : [0,T] \to L^2(\Omega,\mathcal{F},P;H)$ is continuous for all $\alpha > 1$. For this reason let $0 \leq u \leq t \leq T$. By Proposition 3 we get that

$$\begin{aligned} \Big(E\Big[\|&\int_0^{(\frac{t}{\alpha})+}\int_U S(t-\alpha s)\Phi^\alpha(s,y)\, q(ds,dy) \\ &-\int_0^{(\frac{u}{\alpha})+}\int_U S(u-\alpha s)\Phi^\alpha(s,y)\, q(ds,dy)\|^2\Big]\Big)^{\frac{1}{2}}, \end{aligned}$$

$$
\begin{aligned}
&\leq \Big(E\big[\| \int_0^{T+} \int_U 1_{]0,\frac{u}{\alpha}]}(s)(S(t-\alpha s) - S(u-\alpha s))\Phi^\alpha(s,y)\, q(ds,dy)\|^2\big]\Big)^{\frac{1}{2}} \\
&\quad + \Big(E\big[\| \int_0^{T+} \int_U 1_{]\frac{u}{\alpha},\frac{t}{\alpha}]}(s) S(t-\alpha s)\Phi^\alpha(s,y)\, q(ds,dy)\|^2\big]\Big)^{\frac{1}{2}} \\
&= \Big(E\big[\int_0^{T} \int_U 1_{]0,\frac{u}{\alpha}]}(s)\|(S(t-\alpha s) - S(u-\alpha s))\Phi^\alpha(s,y)\|^2\, \nu(dy)\, ds\big]\Big)^{\frac{1}{2}} \\
&\quad + \Big(E\big[\int_0^{T} \int_U 1_{]\frac{u}{\alpha},\frac{t}{\alpha}]}(s) M_T^2 \|\Phi^\alpha(s,y)\|^2\, \nu(dy)\, ds\big]\Big)^{\frac{1}{2}},
\end{aligned}
$$

by the isometric property of the stochastic integral.
The first summand converges to 0 as $u \uparrow t$ or $t \downarrow u$ by Lebesgue's dominated convergence theorem since the integrand converges pointwisely to 0 as $u \uparrow t$ or $t \downarrow u$ by the strong continuity of the semigroup and can be estimated independently of u and t by $4M_T^2\|\Phi^\alpha(s,\omega,y)\|^2$, $(s,\omega,y) \in [0,T]\times\Omega\times U$, where $E\big[\int_0^T \int_U \|\Phi^\alpha(s,y)\|^2\, \nu(dy)\, ds\big] < \infty$.
By analog arguments the second summand converges to 0 as $u \uparrow t$ or $t \downarrow u$
To obtain the continuity of $Z : [0,T] \to L^2(\Omega,\mathcal{F},P;H)$ we prove the uniform convergence of Z^{α_n}, $n \in \mathbb{N}$, to Z in $L^2(\Omega,\mathcal{F},P;H)$ for an arbitrary sequence α_n, $n \in \mathbb{N}$, with $\alpha_n \downarrow 1$ as $n \to \infty$.

$$
\begin{aligned}
\|Z(t) - Z^{\alpha_n}(t)\|_{L^2}^2 &= E\big[\| \int_0^{T+} \int_U 1_{]\frac{t}{\alpha_n},t]}(s) S(t-s) B(Y(s),y)\, q(ds,dy)\|^2\big] \\
&= E\big[\int_{\frac{t}{\alpha_n}}^{t} \int_U \|S(t-s)B(Y(s),y)\|^2\, \nu(dy)\, ds\big] \\
&\leq 2\big(1+\|Y\|_{\mathcal{H}^2}^2\big) \int_0^{\frac{\alpha_n-1}{\alpha_n}T} K(s)\, ds
\end{aligned}
$$

Moreover, we know for all $t \in [0,T]$ that

$$
\Big(\int_0^{u+} \int_U 1_{]0,t]}(s) S(t-s) B(Y(s),y)\, q(ds,dy)\Big)_{u\in[0,T]} \in \mathcal{M}_T^2(H)
$$

since $(1_{]0,t]}(s)S(t-s)B(Y(s),\cdot))_{s\in[0,T]} \in \mathcal{N}_q^2(T,U,H)$. In particular, this means that the process Z is $(\mathcal{F}_t)$-adapted.
Together with the continuity of $Z : [0,T] \to L^2(\Omega,\mathcal{F},P;H)$, by Lemma 13, this implies the existence of a predictable version of Z, which we denote by

$$
\Big(\int_0^{t-} \int_U 1_{]0,t]}(s) S(t-s) B(Y(s),y)\; q(ds,dy)\Big)_{t\in[0,T]}.
$$

The Bochner integral $\displaystyle\int_0^t S(t-s)F(Y(s))\, ds$, $t \in [0,T]$, is well defined since $F(Y(t))$, $t \in [0,T]$, is predictable and the process $F(Y(t))$, $t \in [0,T]$, is P-a.s. Bochner integrable. Moreover, $\displaystyle\int_0^t S(t-s)F(Y(s))\, ds$, $t \in [0,T]$, as well as $S(t)\xi$, $t \in [0,T]$, $\xi \in \mathcal{L}_0^2$, are P-a.s. continuous and $(\mathcal{F}_t)$-adapted, hence predictable.
Concerning the $H^2(T,H)$-norm we obtain for all $\xi \in \mathcal{L}_0^2$ and all predictable

$Y \in \mathcal{H}^2(T,H)$ that

$$
\begin{aligned}
&\|S(\cdot)\xi + \int_0^{\cdot} S(\cdot - s)F(Y(s))\,ds + \int_0^{\cdot -} \int_U S(\cdot - s)B(Y(s),y)\; q(ds,dy)\|_{\mathcal{H}^2}\\
&\leq \|S(\cdot)\xi + \int_0^{\cdot} S(\cdot - s)F(Y(s))\,ds\|_{\mathcal{H}^2}\\
&\quad + \|\int_0^{\cdot +} \int_U S(\cdot - s)B(Y(s),y)\; q(ds,dy)\|_{\mathcal{H}^2}\\
&\leq M_T\|\xi\|_{L^2} + CTM_T(1+\|Y\|_{\mathcal{H}^2}) + C_p(1+\|Y\|_{\mathcal{H}^2})\Big(\int_0^T K(s)\,ds\Big)^{\frac{1}{2}} < \infty
\end{aligned}
$$

where we used the isometric property of the stochastic integral. It remains to check that there exists $\lambda_T =: \lambda \geq 0$ such that for all $\xi \in L_0^2$

$$
\bar{\mathcal{F}}(\xi,\cdot) : H^{2,\lambda}(T,H) \to H^{2,\lambda}(T,H)
$$

is a contraction where the contraction constant $L_{T,\lambda}$ does not depend on ξ. For this purpose let $\xi \in \mathcal{L}_0^2$, $Y, \tilde{Y} \in \mathcal{H}^2(T,H)$, predictable, and $\lambda \geq 0$, then

$$
\begin{aligned}
&\sup_{t\in[0,T]} e^{-\lambda t}\|\big(\mathcal{F}(\xi,Y) - \mathcal{F}(\xi,\tilde{Y})\big)(t)\|_{L^2}\\
&\leq \sup_{t\in[0,T]} e^{-\lambda t}\|\int_0^t S(t-s)[F(Y(s)) - F(\tilde{Y}(s))]\,ds\|_{L^2}\\
&\quad + \sup_{t\in[0,T]} e^{-\lambda t}\|\int_0^{t+}\int_U S(t-s)[B(Y(s),y) - B(\tilde{Y}(s),y)]\,q(ds,dy)\|_{L^2}.
\end{aligned}
$$

First we estimate the second summand. We use again equality (1) to obtain that

$$
\begin{aligned}
&E\big[\|\int_0^{t+}\int_U S(t-s)[B(Y(s),y) - B(\tilde{Y}(s),y)]\,q(ds,dy)\|^2\big]^{\frac{1}{2}}\\
&= \Big(\int_0^t\int_U E\big[\|S(t-s)[B(Y(s),y) - B(\tilde{Y}(s),y)]\|^2\big]\,\nu(dy)\,ds\Big)^{\frac{1}{2}}\\
&\leq \Big(\int_0^t K(t-s)\|Y(s) - \tilde{Y}(s)\|_{L^2}^2\,ds\Big)^{\frac{1}{2}}\\
&\leq \|Y - \tilde{Y}\|_{2,\lambda,T}e^{\lambda t}\Big(\int_0^T\int_U K(s,y)e^{-2\lambda s}\,\nu(dy)\,ds\Big)^{\frac{1}{2}}.
\end{aligned}
$$

Dividing both sides of the above inequality by $e^{\lambda t}$ provides that

$$
\begin{aligned}
&\|\int_0^{\cdot +}\int_U S(\cdot - s)[B(Y(s),y) - B(\tilde{Y}(s),y)]\,q(ds,dy)\|_{2,\lambda,T}\\
&\leq \underbrace{\Big(\int_0^T K(s)e^{-2\lambda s}\,ds\Big)^{\frac{1}{2}}}_{\to 0 \text{ as } \lambda\to\infty}\|Y - \tilde{Y}\|_{p,\lambda,T}.
\end{aligned}
$$

Using the Lipschitz continuity of F, we estimate the first summand in a similar way and obtain as upper bound for the first summand $M_TCT^{\frac{1}{2}}(\frac{1}{\lambda 2})^{\frac{1}{2}}\|Y - \tilde{Y}\|_{2,\lambda,T}$

where $M_T C T^{\frac{1}{2}}(\frac{1}{\lambda 2})^{\frac{1}{2}} \to 0$ as $\lambda \to \infty$.
Thus, we finally proved the existence of constants $\lambda_T =: \lambda$ and $L_{T,\lambda} < 1$ such that

$$\|\bar{\mathcal{F}}(\xi, Y) - \bar{\mathcal{F}}(\xi, \tilde{Y})\|_{2,\lambda,T} \le L_{T,\lambda}\|Y - \tilde{Y}\|_{2,\lambda,T}$$

for all $Y, \tilde{Y} \in H^{2,\lambda}(T, H)$ and $\xi \in L_0^2$. Hence the existence of a unique implicit function

$$\begin{aligned} X : L_0^2 &\to H^2(T, H) \\ \xi &\mapsto X(\xi) = \bar{\mathcal{F}}(\xi, X(\xi)) \end{aligned}$$

is verified.
X is even Lipschitz continuous by Theorem 26 (ii) since

$$\|\mathcal{F}(\xi, Y) - \mathcal{F}(\zeta, Y)\|_{\mathcal{H}^2} = \|S(\cdot)(\xi - \zeta)\|_{\mathcal{H}^2} \le M_T\|\xi - \zeta\|_{L^2}$$

and therefore

$$\bar{\mathcal{F}}(\cdot, Y) : L_0^2 \to H^2(T, H)$$

is Lipschitz continuous where the Lipschitz constant does not depend on Y. $\square$

3.2 Differentiability of the mild solution w.r.t. the initial condition

In this subsection we analyze the Gâteaux differentiability of the mild solution of equation (3) with respect to the initial condition $\xi \in L_0^2$. To this end we make the following assumptions.

Hypothesis H.1

- F is Gâteaux differentiable and

$$\partial F : H \times H \to H$$

  is continuous.

- For all $y \in U$ $B(\cdot, y) : H \to H$ is Gâteaux differentiable and for all $y \in U$, $z \in H$ and $t \in]0, T]$

$$S(t)\partial_1 B(\cdot, y)z : H \to H$$

  is continuous.

- For all $t \in]0, T]$ and $z \in H$ the mapping

$$\begin{aligned} S(t)\partial_1 B(\cdot, \cdot)z : H &\to L^2(U, \mathcal{B}, \nu; H) \\ x &\mapsto S(t)\partial_1 B(x, \cdot)z \end{aligned}$$

  is continuous.

Theorem 14. *Assume that the coefficients A, F and B fulfill the conditions of hypothesis H.0 and H.1. Then the following statements hold.*

(i) *The mild solution of (3)*

$$\begin{aligned} X : L_0^2 &\to H^2(T, H) \\ \xi &\mapsto X(\xi) \end{aligned}$$

is Gâteaux differentiable and the mapping

$$\partial X : L_0^2 \times L_0^2 \to H^2(T, H)$$

is continuous.

(ii) *For all $\bar{\xi}, \bar{\zeta} \in L_0^2$ the Gâteaux derivative of X fulfills the following equation*

$$\begin{aligned} \partial X(\bar{\xi})\bar{\zeta} =& \Big(S(t)\bar{\zeta} + \int_0^t S(t-s)\partial F(X(\bar{\xi})(s))\partial X(\bar{\xi})\bar{\zeta}(s)\, ds \\ &+ \int_0^{t+} \int_U S(t-s)\partial B(X(\bar{\xi})(s), y)\partial X(\bar{\xi})\bar{\zeta}(s)\; q(ds, dy) \Big)_{t \in [0,T]} \end{aligned}$$

in $\mathcal{H}^2(T, H)$ where the right-hand side is defined as the equivalence class of

$$\begin{aligned} \Big(S(t)\zeta &+ \int_0^t S(t-s)\partial F(Y(s))Z(s)\, ds \\ &+ \int_0^{t+} \int_U S(t-s)\partial B(Y(s), y)Z(s)\; q(ds, dy) \Big)_{t \in [0,T]} \end{aligned}$$

w.r.t. $\| \, \|_{\mathcal{H}^2}$ for arbitrary $\zeta \in \bar{\zeta}$ and arbitrary predictable $Y \in X(\bar{\xi})$, $Z \in \partial X(\bar{\xi})\bar{\zeta}$.

(iii) *In addition, the following estimate is true*

$$\|\partial X(\xi)\zeta\|_{\mathcal{H}^2} \leq K_{T,2}\|\zeta\|_{L^2}$$

for all $\xi, \zeta \in L_0^2$ where $K_{T,2}$ denotes the Lipschitz constant of the mapping $X : L_0^2 \to H^2(T, H)$.

For the proof of the above theorem we need the following lemmas.

Lemma 15. *If $Y : [0, T] \times \Omega \times U \to H$ is $\mathcal{P}_T(U)/\mathcal{B}(H)$-measurable then the mapping*

$$[0, T] \times \Omega \times U \to H,\ (s, \omega, y) \mapsto 1_{]0,t]}(s)S(t-s)Y(s, \omega, y)$$

is $\mathcal{P}_T(U)/\mathcal{B}(H)$-measurable for all $t \in [0, T]$.

Proof. [21, Lemma 4.5] □

Lemma 16. *(i) If F satisfies H.0 and H.1 we obtain that $\|\partial F(x)\|_{L(H)} \leq C$ for all $x \in H$.*

(ii) If we assume that $B : H \times U \to H$ satisfies hypothesis H.0 and is Gâteaux differentiable in the first variable then we get for all $t \in]0,T]$ and $x \in H$ that $H \ni z \mapsto S(t)\partial_1 B(x,\cdot)z \in L(H, L^2(U,\mathcal{B},\nu;H))$ with

$$\|S(t)\partial_1 B(x,\cdot)\|_{L(H,L^2(U,\mathcal{B},\nu;H))} \leq \sqrt{K(t)}.$$

In particular, we obtain for all $t \in [0,T]$ and for all predictable $Y, Z \in \mathcal{H}^2(T,H)$ that the mapping

$$\begin{aligned} G_t : [0,T] \times \Omega \times U &\to H \\ (s,\omega,y) &\mapsto 1_{]0,t]}(s) S(t-s)\partial_1 B(Y(s,\omega),y)Z(s,\omega) \end{aligned}$$

is an element of $\mathcal{N}_q^2(T,U,H)$.

Proof. [21, Lemma 5.2] □

Lemma 17. *Assume that the mapping B satisfies the conditions of H.0 and H.1. Then for all $t \in]0,T]$ and $x, z \in H$*

$$\begin{aligned} &\|\frac{1}{h}\big(S(t)B(x+hz,\cdot) - S(t)B(x,\cdot)\big) - S(t)\partial_1 B(x,\cdot)z\|^2_{L^2(U,\mathcal{B},\nu;H)} \\ \leq &\frac{1}{h}\int_0^h \|S(t)\partial_1 B(x+sz,\cdot)z - S(t)\partial_1 B(x,\cdot)z\|^2_{L^2(U,\mathcal{B},\nu;H)}\, ds \end{aligned}$$

and therefore, in particular, one has that for all $t \in]0,T]$

$$\frac{S(t)B(x+hz,\cdot) - S(t)B(x,\cdot)}{h} \underset{h\to 0}{\longrightarrow} S(t)\partial_1 B(x,\cdot)z$$

in $L^2(U,\mathcal{B},\nu;H)$.

Proof. Let $t \in]0,T]$. Since $S(t)\partial_1 B(\cdot,y)z : H \to H$ is continuous we obtain by the fundamental theorem for Bochner integrals 31 that

$$\begin{aligned} &\int_U \|\frac{1}{h}\big(S(t)B(x+hz,y) - S(t)B(x,y)\big) - S(t)\partial_1 B(x,y)z\|^2\, \nu(dy) \\ = &\int_U \|\frac{1}{h}\int_0^h S(t)\partial_1 B(x+sz,y)z - S(t)\partial_1 B(x,y)z\, ds\|^2\, \nu(dy) \\ \leq &\int_U \frac{1}{h^2}\Big(\int_0^h \|S(t)\partial_1 B(x+sz,y)z - S(t)\partial_1 B(x,y)z\|\, ds\Big)^2\, \nu(dy) \\ \leq &\int_U \frac{1}{h}\int_0^h \|S(t)\partial_1 B(x+sz,y)z - S(t)\partial_1 B(x,y)z\|^2\, ds\, \nu(dy) \\ = &\frac{1}{h}\int_0^h \|S(t)\partial_1 B(x+sz,\cdot)z - S(t)\partial_1 B(x,\cdot)z\|^2_{L^2(U,\mathcal{B},\nu;H)}\, ds. \end{aligned}$$

Since

$$\begin{aligned} S(t)\partial_1 B(x+\cdot z,\cdot)z : [0,1] &\to L^2(U,\mathcal{B},\nu;H) \\ s &\mapsto S(t)\partial_1 B(x+sz,\cdot)z \end{aligned}$$

is uniformly continuous by hypothesis H.1 the second part of the assertion follows. □

Lemma 18. *Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space and let (E, d) be a polish space.*
Moreover, let $Y, Y_n, n \in \mathbb{N}$, be E-valued random variables on $(\Omega, \mathcal{F}, \mu)$ such that

$$Y_n \longrightarrow Y \quad \text{in measure as } n \to \infty.$$

Let $(\tilde{E}, \tilde{d})$ be an arbitrary metric space and $f : (E, d) \to (\tilde{E}, \tilde{d})$ a continuous mapping. Then

$$f \circ Y_n \longrightarrow f \circ Y \quad \text{in measure as } n \to \infty.$$

Proof. [15, Lemma 4.6, p.95] □

Proof of theorem 14:
In order to prove the stated differentiability of the mild solution X we apply theorem 27 (i) to the spaces $\Lambda = L_0^2$ and $E = H^{2,\lambda}(T, H)$ and to the mapping $G = \bar{\mathcal{F}}$, where $\lambda \geq 0$ is such that $\bar{\mathcal{F}} : L_0^2 \times H^{2,\lambda}(T, H) \to H^{2,\lambda}(T, H)$ is a contraction in the second variable. In this way we obtain that $X : L_0^2 \to H^{2,\lambda}(T, H)$ is Gâteaux differentiable. By the equivalence of the norms $\| \|_{2,\lambda,T}$, $\lambda \geq 0$, we then also get the Gâteaux differentiability of X as a mapping from L_0^2 to $H^2(T, H)$. For simplicity, we check that $\bar{\mathcal{F}} : L_0^2 \times H^2(T, H) \to H^2(T, H)$ fulfills the conditions of theorem 27 which implies, again by the equivalence of the norms $\| \|_{2,\lambda,T}$, $\lambda \geq 0$, that $\bar{\mathcal{F}} : L_0^2 \times H^{2,\lambda}(T, H) \to H^{2,\lambda}(T, H)$ satisfies them, too.

Proof of (i):

Step 1:

We show the existence of the directional derivatives of $\bar{\mathcal{F}}$. For this purpose let $\bar{\xi}, \bar{\zeta} \in L_0^2$ and $\bar{Y}, \bar{Z} \in H^2(T, H)$. We show that there exist the directional derivatives $\partial_1 \mathcal{F}(\xi, Y; \zeta)$ and $\partial_2 \mathcal{F}(\xi, Y; Z)$ in $\mathcal{H}^2(T, H)$ for $\xi \in \bar{\xi}$, $\zeta \in \bar{\zeta}$, $Y \in \bar{Y}$ and $Z \in \bar{Z}$, where Y and Z are predictable. Then there exist the directional derivatives of $\bar{\mathcal{F}}$ as the respective equivalence classes w.r.t. $\| \|_{\mathcal{H}^2}$.

(a) It is obvious that $\partial_1 \mathcal{F}(\xi, Y; \zeta) = S(\cdot)\zeta \in \mathcal{H}^2(T, H)$.
(b) The integrals

$$\int_0^t S(t-s)\partial F(Y(s))Z(s)\, ds,\ t \in [0, T],\ \text{and}$$
$$\int_0^{t+} \int_U 1_{]0,t]}(s) S(t-s)\partial_1 B(Y(s), y) Z(s)\, q(ds, dy),\ t \in [0, T],$$

are well defined by H.0, H.1 theorem 36 (i) and lemma 16 (ii). In the following we show that

$$\partial_2 \mathcal{F}(\xi, Y; Z) = \Big(\int_0^t S(t-s)\partial F(Y(s))Z(s)\, ds + \int_0^{t+} \int_U S(t-s)\partial_1 B(Y(s), y) Z(s)\, q(ds, dy) \Big)_{t \in [0,T]}$$

$$\in \mathcal{H}^2(T,H)$$

Let $t \in [0,T]$ and $h \neq 0$. Then we get that

$$\begin{aligned}
&\|\frac{\mathcal{F}(\xi, Y+hZ)(t) - \mathcal{F}(\xi, Y)(t)}{h} - \int_0^t S(t-s)\partial F(Y(s))Z(s)\, ds \\
&\quad - \int_0^{t+}\int_U S(t-s)\partial_1 B(Y(s),y)Z(s)\; q(ds,dy)\|_{L^2(\Omega,\mathcal{F},P;H)} \\
&\le \|\int_0^t S(t-s)\big(\frac{F(Y(s)+hZ(s)) - F(Y(s))}{h} - \partial F(Y(s))Z(s)\big)\, ds\|_{L^2} \\
&\quad + \|\int_0^{t+}\int_U S(t-s)\big(\frac{B(Y(s)+hZ(s),y) - B(Y(s),y)}{h} \\
&\qquad\qquad - \partial_1 B(Y(s),y)Z(s)\big)\; q(ds,dy)\|_{L^2}
\end{aligned}$$

The first summand can be estimated independently of $t \in [0,T]$ by

$$M_T T^{\frac{1}{2}} E\big[\int_0^T \|\frac{F(Y(s)+hZ(s)) - F(Y(s))}{h} - \partial F(Y(s))Z(s)\|^2\, ds\big]^{\frac{1}{2}}$$

and converges to 0 as $h \to 0$ by Lebesgue's dominated convergence theorem. To get the convergence to 0 of the second summand as $h \to 0$ we first fix $\alpha > 1$ and get by the isometric formula (1)

$$\begin{aligned}
&\big(E\big[\|\int_0^{t+}\int_U S(t-s)\big(\frac{B(Y(s)+hZ(s),y) - B(Y(s),y)}{h} \\
&\qquad\qquad - \partial_1 B(Y(s),y)Z(s)\big)\; q(ds,dy)\|^2\big]\big)^{\frac{1}{2}} \\
&= \Big(E\big[\int_0^{\frac{t}{\alpha}}\int_U \|S(t-\alpha s)S((\alpha-1)s)\big(\frac{B(Y(s)+hZ(s),y) - B(Y(s),y)}{h} \\
&\qquad\qquad - \partial_1 B(Y(s),y)Z(s)\big)\|^2\; \nu(dy)\; ds\big] \\
&\quad + E\big[\int_{\frac{t}{\alpha}}^t\int_U \|S(t-s)\big(\frac{B(Y(s)+hZ(s),y) - B(Y(s),y)}{h} \\
&\qquad\qquad - \partial_1 B(Y(s),y)Z(s)\big)\|^2\; \nu(dy)\; ds\big]\Big)^{\frac{1}{2}},
\end{aligned}$$

where we used the semigroup property of $S(t)$, $t \ge 0$.
The first integral can be estimated by

$$\begin{aligned}
&M_T^2 E\big[\int_0^T\int_U \|S((\alpha-1)s)\big(\frac{B(Y(s)+hZ(s),y) - B(Y(s),y)}{h} \\
&\qquad\qquad - \partial_1 B(Y(s),y)Z(s)\big)\|^2\; \nu(dy)\; ds\big].
\end{aligned}$$

If we fix $s \in]0,T]$ we know by lemma 17 that

$$\begin{aligned}
&\|\frac{1}{h}\big(S((\alpha-1)s)B(Y(s)+hZ(s),\cdot) - S((\alpha-1)s)B(Y(s),\cdot)\big) \\
&\quad - S((\alpha-1)s)\partial_1 B(Y(s),\cdot)Z(s)\|^2_{L^2(U,\mathcal{B},\nu;H)} \\
&\quad \to 0 \quad \text{ as } h \to 0.
\end{aligned}$$

Lemma 16 (ii), gives us the upper bound for the above sequence so that we can apply Lebesgue's dominated convergence theorem to obtain that

$$\begin{aligned} M_T^2 E\Big[\int_0^T \int_U \|S((\alpha-1)s)\Big(& \frac{B(Y(s)+hZ(s),y)-B(Y(s),y)}{h} \\ & - \partial_1 B(Y(s),y)Z(s)\Big)\|^2 \, \nu(dy)\, ds\Big]. \\ \to 0 \quad \text{as } h \to 0. & \end{aligned}$$

Again by lemma 16 (ii), the second integral can be estimated independently of $h \neq 0$ and $t \in [0,T]$ in the following way

$$\begin{aligned} & E\Big[\int_{\frac{t}{\alpha}}^t \int_U \|S(t-s)\Big(\frac{B(Y(s)+hZ(s),y)-B(Y(s),y)}{h} \\ & \qquad - \partial_1 B(Y(s),y)Z(s)\Big)\|^2 \, \nu(dy)\, ds\Big] \\ & \leq 4 \int_0^{\frac{(\alpha-1)T}{\alpha}} K(s)\, ds \|Z\|_{\mathcal{H}^2}^2 \end{aligned}$$

where $\|Z\|_{\mathcal{H}^2} < \infty$ and $\displaystyle\int_0^{\frac{(\alpha-1)T}{\alpha}} K(s)\, ds \to 0$ as $\alpha \downarrow 1$ since $K \in L^1([0,T])$. Altogether, we have an estimation of the second summand which is independent of $t \in [0,T]$ and we get the desired convergence in $\mathcal{H}^2(T,H)$. **Step 2:**

We show that the directional derivatives

$$\begin{aligned} \partial_1 \bar{\mathcal{F}} &: L_0^2 \times H^2(T,H) \times L_0^2 \to H^2(T,H) \\ \partial_2 \bar{\mathcal{F}} &: L_0^2 \times H^2(T,H) \times H^2(T,H) \to H^2(T,H) \end{aligned}$$

are continuous.

(a) The continuity of $\partial_1 \bar{\mathcal{F}}$ is obvious.
(b) To analyze the continuity of $\partial_2 \bar{\mathcal{F}}$ let $Y, Y_n, Z, Z_n \in \mathcal{H}^2(T,H)$, $n \in \mathbb{N}$, and $\xi, \xi_n \in \mathcal{L}_0^2$, $n \in \mathbb{N}$, such that $Y_n \to Y$ and $Z_n \to Z$ in $\mathcal{H}^2(T,H)$ and $\xi_n \to \xi$ in L_0^2 as $n \to \infty$. Then we have for all $t \in [0,T]$ that

$$\begin{aligned} & \|\partial_2 \mathcal{F}(\xi_n, Y_n; Z_n) - \partial_2 \mathcal{F}(\xi, Y; Z)\|_{\mathcal{H}^2} \\ \leq & \sup_{t\in[0,T]} \| \int_0^t S(t-s)(\partial F(Y_n(s))Z_n(s) - \partial F(Y(s))Z(s))\, ds\|_{L^2} \\ & + \sup_{t\in[0,T]} \| \int_0^{t+} \int_U S(t-s)\big(\partial_1 B(Y_n(s),y)Z_n(s) \\ & \qquad - \partial_1 B(Y(s),y)Z(s)\big)\, q(ds,dy)\|_{L^2}. \end{aligned}$$

First, we estimate the second summand. For this purpose we fix $\alpha > 1$ and use the isometric formula (1) to get that

$$\begin{aligned} & \|\int_0^{t+} \int_U S(t-s)\big(\partial_1 B(Y_n(s),y)Z_n(s) - \partial_1 B(Y(s),y)Z(s)\big)\, q(ds,dy)\|_{L^2} \\ \leq & \Big(E\Big[\int_0^t \int_U \|S(t-s)\partial_1 B(Y_n(s),y)(Z_n(s)-Z(s))\|^2 \, \nu(dy)\, ds\Big]\Big)^{\frac{1}{2}} \\ & + \Big(E\Big[\int_0^{\frac{t}{\alpha}} \int_U \|S(t-s)(\partial_1 B(Y_n(s),y) - \partial_1 B(Y(s),y))Z(s)\|^2 \, \nu(dy)\, ds\Big] \end{aligned}$$

$$
\begin{aligned}
&+E\Big[\int_{\frac{t}{\alpha}}^{t}\int_U \|S(t-s)(\partial_1 B(Y_n(s),y)-\partial_1 B(Y(s),y))Z(s)\|^2\,\nu(dy)\,ds\Big]\Big)^{\frac{1}{2}}\\
&\le\Big(\int_0^t K(s)\,ds\Big)^{\frac{1}{2}}\|Z_n-Z\|_{\mathcal{H}^2},\ \text{by lemma 16(ii),}\\
&+\Big(M_T^2E\Big[\int_0^{\frac{t}{\alpha}}\int_U \|S((\alpha-1)s)(\partial_1 B(Y_n(s),y)-\partial_1 B(Y(s),y))Z(s)\|^2\\
&\qquad\nu(dy)\,ds\Big]\\
&+E\Big[\int_{\frac{t}{\alpha}}^{t}4K(t-s)\|Z(s)\|^2\,ds\Big]\Big)^{\frac{1}{2}}\\
&\le\Big(\int_0^T K(s)\,ds\Big)^{\frac{1}{2}}\|Z_n-Z\|_{\mathcal{H}^2}\\
&+\Big(M_T^2E\Big[\int_0^{T}\int_U \|S((\alpha-1)s)(\partial_1 B(Y_n(s),y)-\partial_1 B(Y(s),y))Z(s)\|^2\\
&\qquad\nu(dy)\,ds\Big]\\
&+4\int_0^{\frac{(\alpha-1)T}{\alpha}}K(s)\,ds\|Z\|_{\mathcal{H}^2}^2\Big)^{\frac{1}{2}}.
\end{aligned}
$$

$\|Z_n-Z\|_{\mathcal{H}^2}\to 0$ as $n\to\infty$ by assumption and $\displaystyle\int_0^{\frac{(\alpha-1)T}{\alpha}}K(s)\,ds\to 0$ as $\alpha\downarrow 1$ by Lebesgue's theorem since $K\in L^1([0,T])$.
To show the convergence of the third term to 0 as $n\to\infty$ we use lemma 18. For fixed $s\in]0,T]$ the sequence of random variables $(Y_n(s),Z(s))$, $n\in\mathbb{N}$, converges in probability to $(Y(s),Z(s))$. Moreover, the mapping

$$
\begin{aligned}
f: H\times H &\to L^2(U,\mathcal{B},\nu;H)\\
(x,z)&\mapsto S((\alpha-1)s)\partial_1 B(x,\cdot)z
\end{aligned}
$$

is continuous. Hence, by lemma 18 it follows that

$$\|S((\alpha-1)s)(\partial_1 B(Y_n(s),\cdot)-\partial_1 B(Y(s),\cdot))Z(s)\|_{L^2(U,\mathcal{B},\nu;H)}^2\underset{n\to\infty}{\longrightarrow}0$$

in probability. In addition, this sequence is bounded by $4K((\alpha-1)s)\|Z(s)\|^2\in L^1(\Omega,\mathcal{F},P)$ which implies the uniform integrability. Therefore we get that

$$E\big[\|S((\alpha-1)s)(\partial_1 B(Y_n(s),\cdot)-\partial_1 B(Y(s),\cdot))Z(s)\|_{L^2(U,\mathcal{B},\nu;H)}^2\big]\underset{n\to\infty}{\longrightarrow}0.$$

Since the above expectation is bounded by $4K((\alpha-1)s)\|Z\|_{\mathcal{H}^2}^2$ where $4K((\alpha-1)\cdot)\|Z\|_{\mathcal{H}^2}^2\in L^1([0,T])$ we finally obtain that

$$
\begin{aligned}
&\int_0^T E\Big[\int_U \|S((\alpha-1)s)(\partial_1 B(Y_n(s),y)-\partial_1 B(Y(s),y))Z(s)\|^2\,\nu(dy)\Big]\,ds\\
&\underset{n\to\infty}{\longrightarrow}0.
\end{aligned}
$$

It is easy to see that the first summand converges to 0 by similar arguments.
Proof of (ii): Let $\bar{\xi}$, $\bar{\zeta}\in L_0^2$. Then by theorem 27 (i) we have the following representation of the Gâteaux derivative of X:

$$\partial X(\bar{\xi})\bar{\zeta}=\big[I-\partial_2\bar{\mathcal{F}}(\bar{\xi},X(\bar{\xi}))\big]^{-1}\partial_1\bar{\mathcal{F}}(\bar{\xi},X(\bar{\xi}))\bar{\zeta}$$

and therefore we have that

$$\partial X(\bar{\xi})\bar{\zeta} = \partial_1\bar{\mathcal{F}}(\bar{\xi}, X(\bar{\xi}))\bar{\zeta} + \partial_2\mathcal{F}(\bar{\xi}, X(\bar{\xi}))\partial X(\bar{\xi})\bar{\zeta}.$$

By (i) the assertion follows.
Proof of (iii): By theorem 12 the mild solution $X : L_0^2 \to H^2(T, H)$ is Lipschitz continuous. We denote the Lipschitz constant of X by $K_{T,2}$. Hence, we get that

$$\|\partial X(\xi)\zeta\|_{\mathcal{H}^2} \leq K_{T,2}\|\zeta\|_{L^2} \quad \text{for all } \xi, \zeta \in L_0^2$$

□

4 Gradient Estimates for the Resolvent Corresponding with the Mild Solution

In the first part of this section we make the following assumptions on the coefficients A, F and B.

Hypothesis H.2

- $(A, D(A))$ is the generator of a quasi-contractive C_0-semigroup $S(t)$, $t \geq 0$, on H, i.e. there exists $\omega_0 \geq 0$ such that $\|S(t)\|_{L(H)} \leq e^{\omega_0 t}$ for all $t \geq 0$.
- F is Lipschitz continuous and Gâteaux differentiable such that

  $$\partial F : H \times H \to H$$

  is continuous.
- F is dissipativ, i.e. $\langle \partial F(x)y, y\rangle \leq 0$ for all $x, y \in H$.
- $B : H \times U \to H$ such that
  - for all $y \in U$ $B(\cdot, y) : H \to H$ is constant,
  - there exists an integrable mapping $K : [0, T] \to [0, \infty[$ such that for all $t \in]0, T]$ and $x \in H$ holds

    $$\int_U \|S(t)B(x, y)\|^2 \, \nu(dy) \leq K(t)\big(1 + \|x\|\big)^2.$$

It is easy to check that, on condition that the assumptions of hypothesis H.2 are fulfilled,the coefficients A, F and B satisfy H.0 and H.1.
Under the assumptions of hypothesis H.2 we already proved in theorem 12 the existence of a mild solution of the following stochastic differential equation

$$\begin{cases} dX(t) &= [AX(t) + F(X(t))]\, dt + B(X(t), y)\, q(dt, dy) \\ X(0) &= x \in H. \end{cases} \tag{6}$$

Moreover, the mild solution $X : H \to H^2(T, H)$ is Gâteaux differentiable by theorem 14(i).

Notation: In the following we denote by $X(x)$ and $\partial X(x)h$ predictable representatives in $\mathcal{H}^2(T,H)$ of the respective equivalence classes in $H^2(T,H)$.

The Gâteaux derivative $\partial X(x)h$ of X in $x \in H$ in direction $h \in H$ fulfills the following equation:

$$\partial X(x)h(t) = S(t)h + \int_0^t S(t-s)\partial F\big(X(x)(s)\big)\partial X(x)h(s)\,ds \quad P\text{-a.s.}$$

for all $t \in [0,T]$ (see theorem 14(ii)).

Proposition 19. *There exists a continuous version $Y \in \mathcal{H}^2(T,H)$ of $\partial X(x)h$, $x, h \in H$, such that*

$$Y(t) = S(t)h + \int_0^t S(t-s)\partial F\big(X(x)(s)\big)Y(s)\,ds \text{ for all } t \in [0,T]$$

P-a.s.

Proof. Let $h \in H$ and $Y \in \mathcal{H}^2(T,H)$. Then Y has at least one predictable version which we denote again by Y. Define

$$\mathcal{G}(h,Y) := \Big(S(t)h + \int_0^t S(t-s)\partial F(X(x)(s))Y(s)\;ds\Big)_{t\in[0,T]}. \tag{7}$$

Then the appearing integral is well defined and an element of $\mathcal{H}^2(T,H)$. Moreover, $\mathcal{G}$ is well defined in the sense of version, i.e. taking another predictable version $\tilde{Y}$ of Y, then $\mathcal{G}(h,Y)$ is a version of $\mathcal{G}(h,\tilde{Y})$.
Define for $h \in H$ and $Y \in H^2(T,H)$, $\bar{\mathcal{G}}(h,Y)$ as the equivalence class of $\mathcal{G}(h,Z)$ w.r.t. $\|\,\|_{\mathcal{H}^2}$ for an arbitrary predictable representative $Z \in Y$. By the above considerations, in $\mathcal{H}^2(T,H)$, $\mathcal{G}(h,Z)$ is independent of the representative Z, i.e. $\bar{\mathcal{G}}$ is well defined. Moreover, there exists $\lambda_T > 0$ such that $\bar{\mathcal{G}} : H \times H^2_{\lambda_T}(T,H) \to H^2_{\lambda_T}(T,H)$ is a contraction in the second variable. By Banach's fixed point theorem we get the existence and uniqueness of an equivalence class $\bar{Z} \in \mathcal{H}^2_{\lambda_T}(T,H)$ such that for all $Y \in \bar{Z}$

$$Y(t) = S(t)h + \int_0^t S(t-s)\partial F\big(X(x)(s)\big)Y(s)\,ds \quad P\text{-a.s.}$$

for all $t \in [0,T]$. In particular, $\partial X(x)h \in \bar{Z}$.
Define now

$$Y(t) := S(t)h + \int_0^t S(t-s)\partial F\big(X(x)(s)\big)\partial X(x)h(s)\,ds,\ t \in [0,T].$$

Obviously, Y is a version of $\partial X(x)h$ and by the previous considerations we know that

$$Y(t) = S(t)h + \int_0^t S(t-s)\partial F\big(X(x)(s)\big)Y(s)\,ds\ P\text{-a.s.}$$

for all $t \in [0,T]$.
Moreover, both Y and the process $\Big(S(t)h+\int_0^t S(t-s)\partial F\big(X(x)(s)\big)Y(s)\,ds\Big)_{t\in[0,T]}$ are continuous. To show this let $Z \in \mathcal{H}^2(T,H)$.
Since

$$E[\int_0^T \|Z(s)\|\,ds] \leq T\|Z\|_{\mathcal{H}^2} < \infty$$

we get that

$$\int_0^t \|Z(s)\|\,ds < \infty \text{ for all } t \in [0,T] \ P\text{-a.s.}$$

Let now $u,t \in [0,T]$ with $u \leq t$ then

$$\begin{aligned}
&\|S(t)h + \int_0^t S(t-s)\partial F\big(X(x)(s)\big)Z(s)\,ds - S(u)h \\
&\qquad - \int_0^u S(u-s)\partial F\big(X(x)(s)\big)Z(s)\,ds\| \\
&\leq \|S(t)h - S(u)h\| \\
&\quad + \|\int_0^u \big(S(t-s) - S(s-u)\big)\partial F(X(x)(s))Z(s)\,ds\| \\
&\quad + \|\int_u^t S(t-s)\partial F(X(x)(s))Z(s)\,ds\|.
\end{aligned}$$

The first summand converges to 0 as $u \uparrow t$ or $t \downarrow u$ by the strong continuity of the semigroup.
As $\|Z(\cdot)\| \in L^1([0,T])$ P-a.s. the second and third summand converge to 0 as $u \uparrow t$ or $t \downarrow u$ by Lebesgue's dominated convergence theorem where the P-nullset does not depend on t and u.
Thus, we proved the existence of a continuous version Y of $\partial X(x)h$ such that

$$Y(t) = S(t)h + \int_0^t S(t-s)\partial F\big(X(x)(s)\big)Y(s)\,ds \ P\text{-a.s.}$$

for all $t \in [0,T]$ where by the above considerations also the right-hand side is continuous. By the continuity of both sides we get that

$$Y(t) = S(t)h + \int_0^t S(t-s)\partial F\big(X(x)(s)\big)Y(s)\,ds$$

for all $t \in [0,T]$ P-a.s. $\square$

In the following we have to distinguish between the case $A \in L(H)$ and the case of an arbitrary, possibly unbounded generator $(A, D(A))$.

4.1 First Case: $A \in L(H)$

Proposition 20. *Let $Y \in \mathcal{H}^2(T,H)$ be a continuous version of $\partial X(x)h$ such that*

$$Y(t) = S(t)h + \int_0^t S(t-s)\partial F\big(X(x)(s)\big)Y(s)\,ds \text{ for all } t \in [0,T]$$

P-a.s. Then

$$Y(t) = h + \int_0^t AY(s)\,ds + \int_0^t \partial F(X(x)(s))Y(s)\,ds \textit{ for all } t \in [0,T]$$

P-a.s.

Proof. Since

$$E[\int_0^T \|Y(s)\|\,ds] \le T\|Y\|_{\mathcal{H}^2} < \infty$$

we get that

$$\int_0^t \|Y(s)\|\,ds < \infty \text{ for all } t \in [0,T] \; P\text{-a.s.}$$

and therefore we have that P-a.s.

$$S(t-\cdot)\partial F\big(X(x)(\cdot)\big)Y(\cdot) \in L^1([0,t]) \text{ for all } t \in [0,T]. \tag{8}$$

Then we obtain that P-a.s. for all $t \in [0,T]$ that

$$\begin{aligned}
&\int_0^t AY(s)\,ds \\
=&\int_0^t AS(s)h\,ds + \int_0^t A\Big(\int_0^s S(s-u)\partial F\big(X(x)(u)\big)Y(u)\,du\Big)\,ds \\
=&\int_0^t AS(s)h\,ds + \int_0^t \int_0^s AS(s-u)\partial F\big(X(x)(u)\big)Y(u)\,du\,ds, \\
&\text{by proposition 30, the fact that } A \in L(H) \text{ and (8)}, \\
=&\int_0^t \frac{d}{ds}S(s)h\,ds + \int_0^t \int_u^t \frac{d}{ds}S(s-u)\partial F\big(X(x)(u)\big)Y(u)\,ds\,du, \\
&\text{by proposition 37}, \\
=&\,S(t)h - h + \int_0^t S(t-u)\partial F\big(X(x)(u)\big)Y(u)\,du \\
&-\int_0^t \partial F\big(X(x)(u)\big)Y(u)\,du, \text{ by proposition 33}, \\
=&\,Y(t) - h - \int_0^t \partial F\big(X(x)(u)\big)Y(u)\,du.
\end{aligned}$$

Finally, we get that

$$Y(t) = h + \int_0^t AY(s)\,ds + \int_0^t \partial F(X(x)(s))Y(s)\,ds \text{ for all } t \in [0,T]$$

P-a.s. □

Let now $Y \in \mathcal{H}^2(T,H)$ be a version of $\partial X(x)h$ such that there exists a P-nullset $N \in \mathcal{F}$ such that for all $\omega \in N^c$ and $t \in [0,T]$

(i) $Y(\cdot,\omega)$ is continuous and $Y(0,\omega) = h$

(ii) $\int_0^t \|Y(s,\omega)\|\, ds < \infty$ and

(iii) $Y(t,\omega) = h + \int_0^t AY(s,\omega)\, ds + \int_0^t \partial F(X(x)(s,\omega))Y(s,\omega)\, ds \qquad (9)$

Then, using proposition 33 and differentiating both sides of (9) we obtain that for all $\omega \in N^c$:

$$\begin{aligned} &Y'(t,\omega) = AY(t,\omega) + \partial F(X(x)(t,\omega))Y(t,\omega) \text{ for } \lambda\text{-a.e. } t \in [0,T] \\ \Rightarrow\ &\frac{1}{2}\frac{d}{dt}\|Y(t,\omega)\|^2 = \langle Y'(t,\omega), Y(t,\omega)\rangle \qquad (10) \\ &\qquad = \langle AY(t,\omega) + \partial F(X(x)(t,\omega))Y(t,\omega), Y(t,\omega)\rangle \\ &\text{for } \lambda\text{-a.e. } t \in [0,T]. \qquad (11) \end{aligned}$$

Proposition 21. *For all $\omega \in N^c$ and $t \in [0,T]$*

$$\|Y(t,\omega)\|^2 - \|Y(0,\omega)\|^2 = \int_0^t \frac{d}{ds}\|Y(s,\omega)\|^2\, ds.$$

Proof. Let $\omega \in N^c$ and $t \in [0,T]$. By proposition 35 we have the show that the mapping $f : [0,t] \to \mathbb{R}, s \mapsto \|Y(s,\omega)\|^2$ is absolutely continuous.
As first step we prove that $g : [0,t] \to \mathbb{R}, s \mapsto \|Y(s,\omega)\|$ is absolutely continuous, i.e. we show that given $\varepsilon > 0$ there exists $\delta > 0$ such that $\sum_{i=1}^n |g(t_i) - g(s_i)| < \varepsilon$ whenever $\sum_{i=1}^n |t_i - s_i| < \delta$ for any finite set of disjoint intervals such that $]s_i, t_i[\subset [0,t]$ for each i.
Let $\varepsilon > 0$. For any set of disjoint intervals such that $]s_i, t_i[\subset [0,t]$ for each i we have

$$\begin{aligned} \sum_{i=1}^n |g(t_i) - g(s_i)| &= \sum_{i=1}^n \big|\|Y(t_i,\omega)\| - \|Y(s_i,\omega)\|\big| \\ &\le \sum_{i=1}^n \|Y(t_i,\omega) - Y(s_i,\omega)\| \\ &\le \sum_{i=1}^n \int_{s_i}^{t_i} \|AY(s,\omega) + \partial F(X(x)(s,\omega))Y(s,\omega)\|\, ds \\ &= \int_{\bigcup_{i=1}^n]s_i,t_i[} \|AY(s,\omega) + \partial F(X(x)(s,\omega))Y(s,\omega)\|\, ds. \end{aligned}$$

Since $\|AY(\cdot,\omega) + \partial F(X(x)(\cdot,\omega))Y(\cdot,\omega)\| \in L^1([0,T], d\lambda)$ there exists $\delta > 0$ such that

$$\int_{\bigcup_{i=1}^n]s_i,t_i[} \|AY(\omega,s) + \partial F(X(x)(\omega,s))Y(\omega,s)\|\, ds < \varepsilon$$

provided $\sum_{i=1}^n |t_i - s_i| = \lambda(\bigcup_{i=1}^n]s_i, t_i[) < \delta$.
Now we use the fact that the product of two functions which are absolutely continuous on a finite interval $[a,b]$ is again absolutely continuous (see [11, 9.3 Example 7, p.161]) and obtain that $\|Y(\cdot,\omega)\|^2 = \|Y(\cdot,\omega)\|\|Y(\cdot,\omega)\|$ is absolutely continuous on $[0,t]$. Now, the assertion follows by proposition 35. $\square$

Integrating both sides of equation (10), using the previous proposition and taking into account the dissipativity of F we obtain for all $\omega \in N^c$ and $t \in [0,T]$ that

$$\begin{aligned}
\|Y(t,\omega)\|^2 - \|Y(0,\omega)\|^2 &= \int_0^t \frac{d}{ds}\|Y(s,\omega)\|^2\,ds \\
= 2\int_0^t \langle AY(s,\omega) &+ \partial F(X(x)(s,\omega))Y(s,\omega), Y(s,\omega)\rangle\,ds \\
\leq 2\int_0^t \langle AY(s,\omega), &Y(s,\omega)\rangle\,ds.
\end{aligned}$$

Since A is the generator of the quasi-contractive C_0-semigroup $S(t)$, $t \geq 0$, we get by the following calculation that $\langle Ax, x\rangle \leq \omega_0\|x\|^2$ for all $x \in H$:

$$\begin{aligned}
\langle Ax, x\rangle &= \lim_{t\downarrow 0}\frac{1}{t}\langle S(t)x - x, x\rangle \leq \lim_{t\downarrow 0}\frac{1}{t}(\|S(t)x\|\|x\| - \|x\|^2) \\
&\leq \lim_{t\downarrow 0}\frac{1}{t}(e^{\omega_0 t} - 1)\|x\|^2 = (\frac{d}{dt}e^{\omega_0 t})_{|\,t=0}\,\|x\|^2 = \omega_0\|x\|^2.
\end{aligned}$$

Consequently,

$$\|Y(t,\omega)\|^2 - \|h\|^2 = \|Y(t,\omega)\|^2 - \|Y(0,\omega)\|^2 \leq 2\int_0^t \omega_0\|Y(s,\omega)\|^2\,ds.$$

Using Gronwall's lemma (see [17, Lemma 6.12]) we can conclude that $\|Y(t)\|^2 \leq e^{2\omega_0 t}\|h\|^2$ for all $t \in [0,T]$ P-a.s. Since Y is a version of $\partial X(x)h$, finally, we have an exponentially estimation for $\|\partial X(x)h(t)\|$, $t \in [0,T]$:

$$\|\partial X(x)h(t)\| \leq e^{\omega_0 t}\|h\| \quad P\text{-a.s. for all } t \in [0,T].$$

4.2 Second case: $(A, D(A))$ is a (possibly) unbounded operator

In this section we need stronger assumptions on the measure ν and the coefficient B.
For the second part of this chapter we make the following assumptions on the coefficients A, F and B and the measure ν.

Hypothesis H.2'

- $(A, D(A))$ is the generator of a quasi-contractive C_0-semigroup $S(t)$, $t \geq 0$, on H, i.e. there exists $\omega_0 \geq 0$ such that $\|S(t)\|_{L(H)} \leq e^{\omega_0 t}$ for all $t \geq 0$.
- F is Lipschitz continuous and Gâteaux differentiable such that

  $$\partial F : H \times H \to H$$

  is continuous.
- F is dissipativ, i.e. $\langle \partial F(x)y, y\rangle \leq 0$ for all $x, y \in H$.

- $\nu(U) < \infty$.
- $B : H \times U \to H$, $(x, y) \mapsto z$ is constant.

If ν and B satisfy hypothesis H.2' then we obtain for every C_0-semigroup $T(t)$, $t \geq 0$, on H that

$$\int_U \|T(t)B(x,y)\|^2 \, \nu(dy) \leq \sup_{t \in [0,T]} \|T(t)\|^2_{L(H)} \|z\|^2 \nu(U)(1 + \|x\|)^2$$

for all $t \in [0,T]$ and $x \in H$, i.e $T(t)B$, $t \in [0,T]$, satisfies hypothesis H.2.

Since $(A, D(A))$ is the generator of a quasi-contractive C_0-semigroup $S(t)$, $t \geq 0$, there is a constant $\omega_0 \geq 0$ such that $\|S(t)\|_{L(H)} \leq e^{\omega_0 t}$ for all $t \geq 0$. By 39 A can be approximated by the Yosida-approximation A_n, $n \in \mathbb{N}$, $n > \omega_0$. Each A_n, $n > \omega_0$, is an element of $L(H)$ and, by proposition 40, again the infinitesimal generator of a quasi-contractive C_0-semigroup $S_n(t)$, $t \geq 0$, $n \in \mathbb{N}$, $n > \omega_0$, such that

$$\|S_n(t)\|_{L(H)} \leq exp(\frac{\omega_0 n t}{n - \omega_0}) \text{ for all } t \geq 0,\ n > \omega_0.$$

Thus, we get that the coefficients A_n, F and B, $n \in \mathbb{N}$, $n > \omega_0$, fulfill the assumptions of H.2. and so those of H.0 and H.1.
Now, we can derive for $n > \omega_0$ the existence of a unique mild solution $X_n(x)$ of the following stochastic differential equation

$$\begin{cases} dX(t) &= [A_n X(t) + F(X(t))] \, dt + z \, q(dt, dy) \\ X(0) &= x \in H \end{cases} \tag{12}$$

which is Gâteaux differentiable as a mapping from H to $H^2(T, H)$.

We define $\mathcal{F}_n$ and $\bar{\mathcal{F}}_n : H \times H^{2,\lambda}(T,H) \to H^{2,\lambda}(T,H)$, $n > \omega_0$, as in chapter 5, section 1 for the coefficients A_n, $n > \omega_0$, F and B. Since A_n, $n > \omega_0$, F and B fulfill H.0 and H.1 we get by theorem 12 the existence of a unique mild solution $X_n : H \to H^2(T, H)$ of (12) as the implicit function of $\bar{\mathcal{F}}_n$, i.e. $\bar{\mathcal{F}}_n(x, X_n(X)) = X_n(x)$ in $H^2(T, H)$. By theorem 14 $X_n : H \to H^2(T, H)$, $n > \omega_0$, is Gâteaux differentiable.

Notation: In the following we denote by $X_n(x)$ and $\partial X_n(x)H$, $n > \omega_0$, $x, h \in H$, predictable representatives in $\mathcal{H}^2(T, H)$ of the respective equivalence classes in $H^2(T, H)$.

Since $A_n \in L(H)$ for all $n \in \mathbb{N}$, $n > \omega_0$, we already know by section 4.1 that for all $x, h \in H$, $t \in [0,T]$ and $n > \omega_0$ holds

$$\|\partial X_n(x)h(t)\| \leq e^{\omega_n t}\|h\| \quad P\text{-a.s.} \tag{13}$$

where $\omega_n := \dfrac{\omega_0 n}{n - \omega_0}$.
Our next aim is to show that $X(x)$ and $\partial X(x)h$ are the limits in $\mathcal{H}^2(T, H)$ of $(X_n(x))_{n \in \mathbb{N}, n > \omega_0}$ and $(\partial X_n(x)h)_{n \in \mathbb{N}, n > \omega_0}$, respectively. For this purpose we use theorem 28.
We have to check that the mappings $\mathcal{F}, \mathcal{F}_n$, $n \in \mathbb{N}$, fulfill the conditions of theorem 28 if we set $\Lambda := H$ and $E := H^2_{\lambda_0}(T, H)$ for an appropriate $\lambda_0 \geq 0$.

Proposition 22. *There exists* $\lambda_0 \geq 0$ *and* $\alpha \in [0,1[$ *such that for all* $n > \omega_0$ *and predictable* $Y, Z \in \mathcal{H}^2(T,H)$

$$\|\mathcal{F}_n(x,Y) - \mathcal{F}_n(x,Z)\|_{2,\lambda_0,T} \leq \alpha\|Y - Z\|_{2,\lambda_0,T} \quad \textit{and}$$
$$\|\mathcal{F}(x,Y) - \mathcal{F}(x,Z)\|_{2,\lambda_0,T} \leq \alpha\|Y - Z\|_{2,\lambda_0,T}.$$

Proof. By the proof of theorem 12 we know that for all $x \in H$ and predictable $Y, Z \in \mathcal{H}^2(T,H)$,

$$\|\mathcal{F}(x,Y) - \mathcal{F}(x,Z)\|_{2,\lambda,T} \leq M_T C\big(\frac{T}{2\lambda}\big)^{\frac{1}{2}}\|Y - Z\|_{2,\lambda,T} \quad \text{and}$$
$$\|\mathcal{F}_n(x,Y) - \mathcal{F}_n(x,Z)\|_{2,\lambda,T} \leq M_{T,n} C\big(\frac{T}{2\lambda}\big)^{\frac{1}{2}}\|Y - Z\|_{2,\lambda,T},\ n \in \mathbb{N},$$

where

$$M_T := \sup_{t\in[0,T]}\|S(t)\|_{L(H)} \leq e^{\omega_0 T} \quad \text{and}$$
$$M_{T,n} := \sup_{t\in[0,T]}\|S_n(t)\|_{L(H)} \leq exp\big(\frac{\omega_0 n T}{n - \omega_0}\big),\ n \in \mathbb{N},\ n > \omega_0.$$

As the sequence $exp(\frac{\omega_0 n T}{n-\omega_0})$, $n \in \mathbb{N}$, $n > \omega_0$, is convergent with limit $e^{\omega_0 T}$ it is bounded from above by a constant $K > 0$. If we choose $\lambda_0 \geq 0$ such that

$$\alpha := (K \vee M_T) C\big(\frac{T}{2\lambda_0}\big)^{\frac{1}{2}} \in [0,1[$$

then the assertion follows. □

Proposition 23. *For all* $x, y \in H$*,* $Z \in \mathcal{H}^2(T,H)$*, predictable, and* $\lambda \geq 0$ *the mappings*

$$\partial_1\mathcal{F}_n(x,\cdot)y : \mathcal{H}^2(T,H) \to \mathcal{H}^2(T,H)$$
$$\partial_2\mathcal{F}_n(x,\cdot)Z : \mathcal{H}^2(T,H) \to \mathcal{H}^2(T,H)$$

are continuous uniformly in $n \in \mathbb{N}$*,* $n > \omega_0$*.*

Proof. Since for $x, y \in H$ and $Z \in \mathcal{H}^2(T,H)$, predictable, $\partial_1\mathcal{F}_n(x,Z)y = (S_n(t)y)_{t\in[0,T]}$ the continuity of $\partial_1\mathcal{F}_n(x,\cdot)y$ uniformly in $n \in \mathbb{N}$, $n > \omega_0$, is obvious.
We have to show the continuity of

$$\partial_2\mathcal{F}_n(x,\cdot)Z\mathcal{H}^2(T,H) \to \mathcal{H}^2(T,H)$$
$$Y \mapsto \Big(\int_0^t S_n(t-s)\partial F(Y(s))Z(s)\,ds\Big)_{t\in[0,T]}.$$

Let $x \in H$ and $Y, Y_k, Z \in \mathcal{H}^2(T,H)$, predictable, $k \in \mathbb{N}$, such that $Y_k \underset{k\to\infty}{\longrightarrow} Y$ in $\mathcal{H}^2(T,H)$. Then we get for all $n > \omega_0$ that

$$\|\partial_2\mathcal{F}_n(x,Y)Z - \partial_2\mathcal{F}_n(x,Y_k)Z\|_{\mathcal{H}^2}$$

$$\leq M_{T,n} T^{\frac{1}{2}} E[\int_0^T \|\partial F(Y(s))Z(s) - \partial F(Y_k(s))Z(s)\|^2\, ds]^{\frac{1}{2}}$$
$$\leq K T^{\frac{1}{2}} E[\int_0^T \|\partial F(Y(s))Z(s) - \partial F(Y_k(s))Z(s)\|^2\, ds]^{\frac{1}{2}}.$$

(For the definition of $M_{T,n}$ and K see the proof of proposition 22.)
Since $\partial F : H\times H \to H$ is continuous we obtain by lemma 18 that $\|\partial F(Y)Z - \partial F(Y_k)Z\| \underset{k\to\infty}{\longrightarrow} 0$ in $\lambda_{|[0,T]} \otimes P$-measure.
Moreover,

$$\|\partial F(Y)Z - \partial F(Y_k)Z\|^2 \leq 4C^2 \|Z\|^2 \in L^1([0,T]\times\Omega, \lambda_{|[0,T]} \otimes P).$$

Hence we obtain that

$$E[\int_0^T \|\partial F(Y(s))Z(s) - \partial F(Y_k(s))Z(s)\|^2\, ds] \to 0 \text{ as } k \to \infty.$$

□

Proposition 24. *For all $x, y \in H$ and predictbale $Y, Z \in \mathcal{H}^2(T,H)$*

(i) $\mathcal{F}_n(x,Y) \to \mathcal{F}(x,Y)$ *as* $n \to \infty$, $n > \omega_0$,

(ii) $\partial_1 \mathcal{F}_n(x,Y)y \to \partial_1 \mathcal{F}(x,Y)y$ *as* $n \to \infty$, $n > \omega_0$,

(iii) $\partial_2 \mathcal{F}_n(x,Y)Z \to \partial_2 \mathcal{F}(x,Y)Z$ *as* $n \to \infty$, $n > \omega_0$,

in $\mathcal{H}^2(T,H)$.

Proof.
(i) Let $x \in H$ and $Y \in \mathcal{H}^2(T,H)$, predictable, then

$$\begin{aligned}
&\big(E[\|\mathcal{F}_n(x,Y)(t) - \mathcal{F}(x,Y)(t)\|^2]\big)^{\frac{1}{2}} \\
\leq &\big(E[\|S_n(t)x - S(t)x\|^2]\big)^{\frac{1}{2}} \\
&+ \big(E[\|\int_0^t S_n(t-s)F(Y(s)) - S(t-s)F(Y(s))\, ds\|^2]\big)^{\frac{1}{2}} \\
&+ \big(E[\|\int_0^{t+}\int_U S_n(t-s)z - S(t-s)z\, q(ds,dy)\|^2]\big)^{\frac{1}{2}} \\
\leq &\sup_{t\in[0,T]} \|S_n(t)x - S(t)x\| \\
&+ \big(E[T\int_0^T \sup_{t\in[0,T]} 1_{[0,t]}(s)\|S_n(t-s)F(Y(s)) - S(t-s)F(Y(s))\|^2\, ds]\big)^{\frac{1}{2}} \\
&+ \big(E[\int_0^t\int_U \|S_n(t-s)z - S(t-s)z\|^2\, \nu(dy)\, ds]\big)^{\frac{1}{2}} \\
\leq &\sup_{t\in[0,T]} \|S_n(t)x - S(t)x\| \\
&+ T^{\frac{1}{2}}\big(E[\int_0^T \sup_{t\in[0,T]} 1_{[0,t]}(s)\|S_n(t-s)F(Y(s)) - S(t-s)F(Y(s))\|^2\, ds]\big)^{\frac{1}{2}}
\end{aligned}$$

$$+\nu(U)^{\frac{1}{2}}\Big(\int_0^T \sup_{t\in[0,T]} 1_{[0,t]}(s)\|S_n(t-s)z - S(t-s)z\|^2\,ds\Big)^{\frac{1}{2}}.$$

$\sup_{t\in[0,T]}\|S_n(t)x - S(t)x\| \to 0$ as $n\to\infty$ for all $x \in H$ by proposition 40.
Again by proposition 40, for fixed $s \in [0,T]$

$$\sup_{t\in[0,T]} 1_{[0,t]}(s)\|S_n(t-s)F(Y(s)) - S(t-s)F(Y(s))\| \underset{n\to\infty}{\longrightarrow} 0 \qquad (1)$$

$$\text{and} \quad \sup_{t\in[0,T]} 1_{[0,t]}(s)\|S_n(t-s)z - S(t-s)z\| \underset{n\to\infty}{\longrightarrow} 0 \qquad (2).$$

Moreover, the first sequence (1) of mappings from $[0,T]\times\Omega$ to $\mathbb{R}$ is bounded by $(K+M_T)C(1+\|Y\|) \in L^2([0,T]\times\Omega, \lambda_{|[0,T]}\otimes P)$.
Hence, by Lebesgue's dominated convergence theorem we get that

$$E[\int_0^T \sup_{t\in[0,T]} 1_{[0,t]}(s)\|S_n(t-s)F(Y(s)) - S(t-s)F(Y(s))\|^2\,ds] \to 0$$

as $n\to\infty$, $n>\omega_0$.
The second sequence (2): $\sup_{t\in[0,T]} 1_{[0,t]}(\cdot)\|S_n(t-\cdot)z - S(t-\cdot)z\|$, $n\in\mathbb{N}$, $n > \omega_0$, is bounded by $(K+M_T)\|z\| \in L^2([0,T])$, thus, we obtain again by Lebesgue's theorem that $\int_0^T \sup_{t\in[0,T]} 1_{[0,t]}(s)\|S_n(t-s)z - S(t-s)z\|^2\,ds \to 0$ as $n\to\infty$, $n>\omega_0$.

The proof of (ii) and (iii) can be done analoguously. □

By proposition 23 and proposition 24 we justified that the mappings

$$\bar{\mathcal{F}}_n : H\times H^2_{\lambda_0}(T,H) \to H^2_{\lambda_0}(T,H),\ n\in\mathbb{N},\ n>\omega_0, \text{ and}$$
$$\bar{\mathcal{F}} : H\times H^2_{\lambda_0}(T,H) \to H^2_{\lambda_0}(T,H)$$

fulfill the conditions of theorem 28 and, finally, we obtain that for all $x,h\in H$

$$X_n(x)\to X(x) \ \text{ and } \ \partial X_n(x)h \to \partial X(x)h \text{ in } \mathcal{H}^2_{\lambda_0}(T,H) \text{ as } n\to\infty.$$

In particular, we get for each $t\in[0,T]$ the existence of a subsequence $(n_k(t))_{k\in\mathbb{N}}$ such that

$$\partial X_{n_k(t)}(x)h(t) \underset{\substack{k\to\infty\\ n_k(t)>\omega_0}}{\longrightarrow} \partial X(x)h(t)\ P\text{-a.s.}$$

Thus, by (13), it follows that for all $t\in[0,T]$

$$\begin{aligned}\|\partial X(x)h(t)\| &= \lim_{\substack{k\to\infty\\ n_k(t)>\omega_0}} \|\partial X_{n_k(t)}(x)h(t)\| \le \lim_{\substack{k\to\infty\\ n_k(t)>\omega_0}} exp(\frac{\omega_0 n_k(t)}{n_k(t)-\omega_0}t)\|h\| \qquad (14)\\ &= e^{\omega_0 t}\|h\| \quad P\text{-a.s.}\end{aligned}$$

4.3 Gradient estimates for the resolvent

We define the transition kernels and the "resolvent" corresponding with the mild solution $X(x)$, $x\in H$, in the following way.
Let $f:(H,\mathcal{B}(H))\to(\mathbb{R},\mathcal{B}(\mathbb{R}))$, bounded. Define

$$p_t f(x) := E[f(X(x)(t)],\ t\in[0,T],\ x\in H, \text{ and}$$

$$R_\alpha f(x) := \int_0^\infty e^{-\alpha t} p_t f(x)\, dt,\ \alpha \geq 0.$$

Proposition 25. *If $f \in C_b^1(H,\mathbb{R})$ where*

$$C_b^1 := \{g : H \to \mathbb{R} \mid g \text{ is continuously Fréchet differentiable such that } \sup_{x\in H} \|Dg(x)\|_{L(H,\mathbb{R})} < \infty\}$$

then $R_\alpha f : H \to \mathbb{R}$ is Gâteaux differentiable for all $\alpha \geq 0$ and for all $x, h \in H$ and $\alpha \geq 0$

$$\partial R_\alpha f(x)h = \int_0^\infty e^{-\alpha t} E[Df(X(x)(t))\partial X(x)h(t)]\, dt.$$

Proof. Let $\alpha \geq 0$, $x, h \in H$ and $\varepsilon > 0$ then we get that

$$\begin{aligned}
&|\frac{R_\alpha f(x+\varepsilon h) - R_\alpha f(x)}{\varepsilon} - \int_0^\infty e^{-\alpha t} E[Df(X(x)(t))\partial X(x)h(t)]\, dt| \\
\leq &\int_0^\infty e^{-\alpha t} E\big[|\frac{f(X(x+\varepsilon h)(t)) - f(X(x)(t))}{\varepsilon} - Df(X(x)(t))\partial X(x)h(t)|\big]\, dt,
\end{aligned}$$

where by proposition 31

$$\begin{aligned}
&E\big[|\frac{f(X(x+\varepsilon h)(t)) - f(X(x)(t))}{\varepsilon} - Df(X(x)(t))\partial X(x)h(t)|\big] \\
=&E\big[|\int_0^1 Df\big(X(x)(t) - \sigma(X(x+\varepsilon h)(t) - X(x)(t))\big) \\
&\qquad\big(\frac{X(x+\varepsilon h)(t) - X(x)(t)}{\varepsilon}\big) - Df(X(x)(t))\partial X(x)h(t)\, d\sigma|\big] \\
\leq &E\big[\int_0^1 \|Df\big(X(x)(t) - \sigma(X(x+\varepsilon h)(t) - X(x)(t))\big)\|_{L(H,\mathbb{R})} \\
&\qquad\|\frac{X(x+\varepsilon h)(t) - X(x)(t)}{\varepsilon} - \partial X(x)h(t)\|\, d\sigma\big] \\
&+E\big[\int_0^1 \|Df\big(X(x)(t) - \sigma(X(x+\varepsilon h)(t) - X(x)(t))\big) \\
&\qquad - Df(X(x)(t))\|_{L(H,\mathbb{R})}\|\partial X(x)h(t)\|\, d\sigma\big] \\
\leq &\sup_{x\in H}\|Df(x)\|_{L(H,\mathbb{R})}\|\frac{X(x+\varepsilon h) - X(x)}{\varepsilon} - \partial X(x)h\|_{\mathcal{H}^2} \\
&+ \big(E\big[\int_0^1 \|Df\big(X(x)(t) - \sigma(X(x+\varepsilon h)(t) - X(x)(t))\big) \\
&\qquad - Df(X(x)(t))\|^2_{L(H,\mathbb{R})}\, d\sigma\big]\big)^{\frac{1}{2}}\|\partial X(x)h\|_{\mathcal{H}^2}.
\end{aligned}$$

Thus, we get that

$$\begin{aligned}
&|\frac{R_\alpha f(x+\varepsilon h) - R_\alpha f(x)}{\varepsilon} - \int_0^\infty e^{-\alpha t} E[Df(X(x)(t))\partial X(x)h(t)]\, dt| \\
\leq &\int_0^\infty e^{-\alpha t}\, dt \sup_{x\in H}\|Df(x)\|_{L(H,\mathbb{R})}\|\frac{X(x+\varepsilon h) - X(x)}{\varepsilon} - \partial X(x)h\|_{\mathcal{H}^2} \\
&+ \int_0^\infty e^{-\alpha t}\big(E\big[\int_0^1 \|Df\big(X(x)(t) - \sigma(X(x+\varepsilon h)(t) - X(x)(t))\big) \\
&\qquad - Df(X(x)(t))\|^2_{L(H,\mathbb{R})}\, d\sigma\big]\big)^{\frac{1}{2}}\, dt\, \|\partial X(x)h\|_{\mathcal{H}^2}.
\end{aligned}$$

The first summand converges to 0 as $\varepsilon \to 0$ as $X : H \to \mathcal{H}^2(T, H)$ is Gâteaux-differentiable.
To prove the convergence to 0 of the second summand we use lemma 18.
Since $X(t) : H \to L^2(\Omega, \mathcal{F}_t, P; H)$ is continuous we can conclude that for fixed $\sigma \in [0, 1]$

$$X(x)(t) - \sigma(X(x + \varepsilon h)(t) - X(x)(t)) \underset{\varepsilon \to 0}{\to} X(x)(t) \text{ in } P\text{-measure.}$$

Moreover, $Df : H \to L(H, \mathbb{R})$ is continuous and we obtain by lemma 18 that

$$\|Df(X(x)(t) - \sigma(X(x + \varepsilon h)(t) - X(x)(t))) - Df(X(x)(t))\|^2_{L(H,\mathbb{R})} \underset{\varepsilon \to 0}{\to} 0$$

in P-measure. As this sequence is bounded by $4 \sup_{x \in H} \|Df(x)\|^2_{L(H,\mathbb{R})} < \infty$ it follows that

$$E\big[\|Df(X(x)(t) - \sigma(X(x + \varepsilon h)(t) - X(x)(t))) - Df(X(x)(t))\|^2_{L(H,\mathbb{R})}\big] \underset{\varepsilon \to 0}{\to} 0.$$

Since this expectation is bounded by $4 \sup_{x \in H} \|Df(x)\|^2_{L(H,\mathbb{R})} < \infty$ we get by Lebesgue's dominated convergence theorem that

$$\int_0^1 E\big[\|Df(X(x)(t) - \sigma(X(x + \varepsilon h)(t) - X(x)(t))) - Df(X(x)(t))\|^2_{L(H,\mathbb{R})}\big]\, d\sigma \underset{\varepsilon \to 0}{\to} 0.$$

Finally, again by Lebesgue's theorem, we obtain that

$$\int_0^\infty e^{-\alpha t} E\Big[\int_0^1 \|Df(X(x)(t) - \sigma(X(x + \varepsilon h)(t) - X(x)(t))) - Df(X(x)(t))\|^2_{L(H,\mathbb{R})}\, d\sigma\Big]^{\frac{1}{2}}\, dt\, \|\partial X(x)h\|_{\mathcal{H}^2} \to 0 \text{ as } \varepsilon \to 0.$$

We proved the existence of the directional derivatives $\partial R_\alpha f(x, ; h)$, $x, h \in H$. Obviously, $\partial R_\alpha f(x, ; h) \in L(H, \mathbb{R})$ and therefore the assertion of the proposition follows. $\square$

Using the gradient estimate (14) for the mild solution and the representation of $\partial R_\alpha f(x)h$ we get, if $f \in C_b^1(H, \mathbb{R})$ and $\alpha > \omega_0$, that

$$\begin{aligned}
\|\partial R_\alpha f(x)h\| &= \|\int_0^\infty e^{-\alpha t} E[Df(X(x)(t))\partial X(x)h(t)]\, dt\| \\
&\le \int_0^\infty e^{-\alpha t} E[\sup_{x \in H} \|Df(x)\|_{L(H,\mathbb{R})} \|\partial X(x)h(t)\|]\, dt \\
&\le \int_0^\infty e^{-\alpha t} \sup_{x \in H} \|Df(x)\|_{L(H,\mathbb{R})} e^{\omega_0 t} \|h\|\, dt \\
&= \frac{1}{\alpha - \omega_0} \sup_{x \in H} \|Df(x)\|_{L(H,\mathbb{R})} \|h\|
\end{aligned}$$

Finally, we have

$$\|\partial R_\alpha f(x)\|_{L(H,\mathbb{R})} \le \frac{1}{\alpha - \omega_0} \sup_{x \in H} \|Df(x)\|_{L(H,\mathbb{R})} \text{ for all } \alpha > \omega_0 \text{ and } f \in C_b^1(H, \mathbb{R}).$$

A Existence, continuity and differentiability of implicit functions

Let $(E, \| \|)$ and $(\Lambda, \| \|_\Lambda)$ be two Banach spaces. In the whole chapter we consider a mapping $G : \Lambda \times E \to E$ which is a contraction in the second variable, i.e. there exists an $\alpha \in [0,1[$ such that

$$\|G(\lambda, x) - G(\lambda, y)\| \leq \alpha \|x - y\| \text{ for all } \lambda \in \Lambda,\ x, y \in E. \tag{15}$$

Then, by Banach's fixed point theorem, we get the existence of a unique implicit function $\varphi : \Lambda \to E$, i.e.

$$\varphi(\lambda) = G(\lambda, \varphi(\lambda)) \text{ for all } \lambda \in \Lambda.$$

Theorem 26 (Continuity of the implicit function). *(i) If for all $x \in E$ the mapping $G(\cdot, x) : \Lambda \to E$ is continuous then $\varphi : \Lambda \to E$ is continuous.*

(ii) If there exists a constant $L \geq 0$ such that

$$\|G(\lambda, x) - G(\tilde{\lambda}, x)\|_E \leq L \|\lambda - \tilde{\lambda}\|_\Lambda \textit{ for all } x \in E$$

then $\varphi : \Lambda \to E$ is Lipschitz continuous.

Proof. [15, Theorem D.1, p.164] □ □

To analyze the differentiability of the implicit function we adapt an idea first proposed in [33]. We introduce two further Banach spaces $(\Lambda_0, \| \|_{\Lambda_0})$ and $(E_0, \| \|_{E_0})$ continuously embedded in $(\Lambda, \| \|_\Lambda)$ and $(E, \| \|_E)$, respectively. We assume that $G : \Lambda \times E \to E$ and $G : \Lambda_0 \times E_0 \to E_0$ fulfill condition (15) with the same $\alpha \in [0,1[$.

Theorem 27 (First order differentiability). *We assume that the mapping $G : \Lambda \times E \to E$ fulfills the following conditions.*

1. *$G(\cdot, x) : \Lambda \to E$ is continuous for all $x \in E$,*
2. *for all $\lambda, \mu \in \Lambda$ and all $x, y \in E$ there exist the directional derivatives*

$$\partial_1 G(\lambda, x; \mu) = E - \lim_{h \to \infty} \frac{G(\lambda + h\mu, x) - G(\lambda, x)}{h}$$
$$\partial_2 G(\lambda, x; y) = E - \lim_{h \to \infty} \frac{G(\lambda, x + hy) - G(\lambda, x)}{h}$$

and $\partial_1 G : \Lambda \times E \times \Lambda \to E$ and $\partial_2 G : \Lambda \times E \times E \to E$ are continuous.

Then the implicit function $\varphi : \Lambda \to E$ is Gâteaux differentiable such that the mapping $\Lambda \times \Lambda \to E$, $(\lambda, \mu) \mapsto \partial\varphi(\lambda)\mu$ is continuous and

$$\partial\varphi(\lambda)\mu = [I - \partial_2 G(\lambda, \varphi(\lambda))]^{-1} \partial_1 G(\lambda, \varphi(\lambda))\mu \tag{16}$$

for all $\lambda, \mu \in \Lambda$

Proof. [15, Theorem D.8, p.168] □ □

Theorem 28. *Let $G_n : \Lambda \times E \to E$, $n \in \mathbb{N}$, such that*

$$\|G_n(\lambda, x) - G_n(\lambda, y)\| \le \alpha\|x - y\| \qquad \text{for all } \lambda \in \Lambda \text{ and all } x, y \in E \text{ and } n \in \mathbb{N}.$$

Moreover, assume that the mappings G and G_n, $n \in \mathbb{N}$, fulfill the following conditions.

1. *$G(\cdot, x)$ and $G_n(\cdot, x)$, $n \in \mathbb{N}$, are continuous for all $x \in E$,*
2. *G, G_n, $n \in \mathbb{N}$, are Gâteaux differentiable such that*

$$\begin{aligned} \partial_1 G : \Lambda \times E \times \Lambda \to E \text{ and } \partial_2 G : \Lambda \times E \times E \to E \\ \partial_1 G_n : \Lambda \times E \times \Lambda \to E \text{ and } \partial_2 G_n : \Lambda \times E \times E \to E,\ n \in \mathbb{N}, \end{aligned}$$

   are continuous,
3. *$\partial_1 G_n(\lambda, \cdot)\mu$ and $\partial_2 G_n(\lambda, \cdot)x$, $\lambda, \mu \in \Lambda$, $x \in E$, are continuous uniformly in $n \in \mathbb{N}$,*
4. *$G_n \to G$, $\partial_1 G_n \to \partial_1 G$ and $\partial_2 G_n \to \partial_1 G$ pointwisely as $n \to \infty$.*

Then there exist unique implicit functions $\varphi, \varphi_n : \Lambda \to E$, $n \in \mathbb{N}$, such that $G(\lambda, \varphi(\lambda)) = \varphi(\lambda)$ and $G_n(\lambda, \varphi_n(\lambda)) = \varphi_n(\lambda)$, $n \in \mathbb{N}$, for all $\lambda \in \Lambda$.
φ and φ_n, $n \in \mathbb{N}$, are Gâteaux differentiable.

Moreover, $\varphi_n(\lambda) \to \varphi(\lambda)$ and $\partial\varphi_n(\lambda)\mu \to \partial\varphi(\lambda)\mu$ as $n \to \infty$ for all $\lambda, \mu \in \Lambda$.

Proof. For all $\lambda \in \Lambda$ we have that

$$\begin{aligned} &\|\varphi_n(\lambda) - \varphi(\lambda)\| = \|G_n(\lambda, \varphi_n(\lambda)) - G(\lambda, \varphi(\lambda))\| \\ &\le \|G_n(\lambda, \varphi_n(\lambda)) - G_n(\lambda, \varphi(\lambda))\| + \|G_n(\lambda, \varphi(\lambda)) - G(\lambda, \varphi(\lambda))\| \\ &\le \alpha\|\varphi_n(\lambda) - \varphi(\lambda)\| + \|G_n(\lambda, \varphi(\lambda)) - G(\lambda, \varphi(\lambda))\|. \end{aligned}$$

Subtracting on both sides of the above equation $\alpha\|\varphi_n(\lambda) - \varphi(\lambda)\|$ and dividing by $(1 - \alpha)$ we get that

$$\|\varphi_n(\lambda) - \varphi(\lambda)\| \le \frac{1}{1 - \alpha}\|G_n(\lambda, \varphi(\lambda)) - G(\lambda, \varphi(\lambda))\| \underset{n \to \infty}{\longrightarrow} 0$$

by assumption.
By theorem 27 (i) φ and φ_n, $n \in \mathbb{N}$, are Gâteaux differentiable. Using the representation (16) of the Gâteaux derivatives of φ_n, $n \in \mathbb{N}$, and φ we can estimate $\|\partial_n\varphi(\lambda)\mu - \partial\varphi(\lambda)\mu\|$, $\lambda, \mu \in \Lambda$, in the following way:

$$\begin{aligned} &\|\partial_n\varphi(\lambda)\mu - \partial\varphi(\lambda)\mu\| \\ &\le \|\partial_2 G_n(\lambda, \varphi_n(\lambda))\partial\varphi_n(\lambda)\mu - \partial_2 G(\lambda, \varphi(\lambda))\partial\varphi(\lambda)\mu\| \\ &\quad + \|\partial_1 G_n(\lambda, \varphi_n(\lambda))\mu - \partial_1 G(\lambda, \varphi(\lambda))\mu\| \\ &\le \|\partial_2 G_n(\lambda, \varphi_n(\lambda))\partial\varphi_n(\lambda)\mu - \partial_2 G_n(\lambda, \varphi_n(\lambda))\partial\varphi(\lambda)\mu\| \end{aligned}$$

$$
\begin{aligned}
&+ \sup_{m\in\mathbb{N}} \|\partial_2 G_m(\lambda, \varphi_n(\lambda))\partial\varphi(\lambda)\mu - \partial_2 G_m(\lambda, \varphi(\lambda))\partial\varphi(\lambda)\mu\| \\
&+ \|\partial_2 G_n(\lambda, \varphi(\lambda))\partial\varphi(\lambda)\mu - \partial_2 G(\lambda, \varphi(\lambda))\partial\varphi(\lambda)\mu\| \\
&+ \sup_{m\in\mathbb{N}} \|\partial_1 G_m(\lambda, \varphi_n(\lambda))\mu - \partial_1 G_m(\lambda, \varphi(\lambda))\mu\| \\
&+ \|\partial_1 G_n(\lambda, \varphi(\lambda))\mu - \partial_1 G(\lambda, \varphi(\lambda))\mu\|
\end{aligned}
$$

Since $\|\partial_2 G_n(\lambda, \varphi_n(\lambda))\partial\varphi_n(\lambda)\mu - \partial_2 G_n(\lambda, \varphi_n(\lambda))\partial\varphi(\lambda)\mu\| \leq \alpha\|\partial_n\varphi(\lambda)\mu - \partial\varphi(\lambda)\mu\|$ we obtain that

$$
\begin{aligned}
&\|\partial_n\varphi(\lambda)\mu - \partial\varphi(\lambda)\mu\| \\
\leq &\frac{1}{1-\alpha}\Big(\sup_{m\in\mathbb{N}} \|\partial_2 G_m(\lambda, \varphi_n(\lambda))\partial\varphi(\lambda)\mu - \partial_2 G_m(\lambda, \varphi(\lambda))\partial\varphi(\lambda)\mu\| \\
&\quad + \|\partial_2 G_n(\lambda, \varphi(\lambda))\partial\varphi(\lambda)\mu - \partial_2 G(\lambda, \varphi(\lambda))\partial\varphi(\lambda)\mu\| \\
&\quad + \sup_{m\in\mathbb{N}} \|\partial_1 G_m(\lambda, \varphi_n(\lambda))\mu - \partial_1 G_m(\lambda, \varphi(\lambda))\mu\| \\
&\quad + \|\partial_1 G_n(\lambda, \varphi(\lambda))\mu - \partial_1 G(\lambda, \varphi(\lambda))\mu\|\Big) \\
&\xrightarrow[n\to\infty]{} 0,
\end{aligned}
$$

since $\varphi_n(\lambda) \to \varphi(\lambda)$ as $n \to \infty$ and by the assumptions on the mappings G_n, $n \in \mathbb{N}$, and G. □

B Properties of the Bochner integral

Let $(X, \| \|)$ be a Banach space, $\mathcal{B}(X)$ the Borel σ-field of X and $(\Omega, \mathcal{F}, \mu)$ a measure space with finite measure μ.

Proposition 29. *Let $f \in L^1(\Omega, \mathcal{F}, \mu; X)$. Then*

$$\int \varphi \circ f \, d\mu = \varphi\Big(\int f \, d\mu\Big)$$

holds for all $\varphi \in X^* = L(X, \mathbb{R})$.

Proof. [8, Proposition E.11, p.356] □

Proposition 30. *Let Y be a further Banach space, $\varphi \in L(X, Y)$ and $f \in L^1(\Omega, \mathcal{F}, \mu; X)$ such that $\varphi \circ f$ is strongly measurable. Then*

$$\int \varphi \circ f \, d\mu = \varphi\Big(\int f \, d\mu\Big).$$

Proof. [9, Proposition 1.6, p.21] □

Proposition 31 (Fundamental theorem). *Let $-\infty < a < b < \infty$ and $f \in C^1([a, b]; X)$. Then*

$$f(t) - f(s) = \int_s^t f'(u) \, du := \begin{cases} \int 1_{[s,t]}(u) f'(u) \, du & \text{if } s \leq t \\ -\int 1_{[t,s]}(u) f'(u) \, du & \text{otherwise} \end{cases}$$

for all $s, t \in [a, b]$ where du denotes the Lebesgue measure on $\mathcal{B}(\mathbb{R})$.

Proof. [15, Proposition A.7, p.152] □

Proposition 32. *Let $[a,b]$ be a finite interval and $f \in L^1([a,b],\mathcal{B}([a,b]),\lambda;\mathbb{R})$, where λ denotes the Lebesgue measure. Then the mapping $F:[a,b]\to\mathbb{R}$, $s\mapsto\int_a^s f(t)\,dt$, is differentiable λ-a.e. on $[a,b[$ and $F'(s)=f(s)$ for λ-a.e. $s\in[a,b[$.*

Proof. [11, Chapter 4, Theorem 12, p.89] □

Proposition 33. *Let $[a,b]$ be a finite interval and let $f \in L^1([a,b],\mathcal{B}([a,b]),\lambda;X)$, where λ denotes the Lebesgue measure. Then the mapping $F:[a,b]\to X$, $s\mapsto\int_a^s f(t)\,dt$, is differentiable λ-a.e. on $[a,b[$ and $F'(s)=f(s)$ for λ-a.e. $s\in[a,b[$.*

Proof. Since $f([a,b])$ is separable there exist x_n, $n\in\mathbb{N}$, such that $\{x_n\,|\,n\in\mathbb{N}\}$ is a dense subset of $f([a,b])$. Then $\|f-x_n\|\in L^1([a,b],\lambda)$ for all $n\in\mathbb{N}$. Consequently, by proposition 32 the mappings $F_n:[a,b]\to\mathbb{R}$, $s\mapsto\int_a^s\|f(t)-x_n\|\,dt$, $n\in\mathbb{N}$, are differentiable λ-a.e. on $[a,b[$ and $F_n(s)=\|f(s)-x_n\|$ for all $n\in\mathbb{N}$ and for λ-a.e. $s\in[a,b[$.
Then we get for λ-a.e. $s\in[a,b[$ that

$$\begin{aligned}
&\limsup_{h\to 0}\|\frac{1}{h}\big(\int_a^{s+h}f(t)\,dt-\int_a^s f(t)\,dt\big)-f(s)\|\\
=&\limsup_{h\to 0}\|\frac{1}{h}\int_s^{s+h}(f(t)-f(s)\,dt\|\\
\le&\limsup_{h\to 0}\frac{1}{h}\int_s^{s+h}\|f(t)-f(s)\|\,dt\\
\le&\limsup_{h\to 0}\frac{1}{h}\int_s^{s+h}\|f(t)-x_n\|\,dt-\|f(s)-x_n\|\\
=&2\|f(s)-x_n\|.
\end{aligned}$$

Choosing a subsequence x_{n_k}, $k\in\mathbb{N}$, such that $\|f(s)-x_{n_k}\|\to 0$ as $k\to\infty$ we obtain that for λ-a.e. $s\in[a,b[$ holds

$$\|\frac{1}{h}\big(\int_a^{s+h}f(t)\,dt-\int_a^s f(t)\,dt\big)-f(s)\|\to 0\ \text{ as } h\to 0.$$

□

Definition 34 (Absolut continuity). Let $-\infty\le a<b\le\infty$. A function $f:[a,b]\to\mathbb{R}$ is absolutely continuous (on $[a,b]$) if for every $\varepsilon>0$ there exists $\delta>0$ such that $\sum_{i=1}^n|f(x_i)-f(y_i)|<\varepsilon$ whenever $\sum_{i=1}^n|x_i-y_i|<\delta$ for any set of disjoint intervals such that $(x_i,y_i)\subset[a,b]$ for each $i\in\{1,\dots,n\}$.

Proposition 35. *Let $[a,b]$ be a finite interval and $f:[a,b]\to\mathbb{R}$ absolutely continuous, then if $x\in[a,b]$*

$$f(x)-f(a)=\int_a^x f'(t)\,dt$$

Proof. [11, Chapter 9, Corollary 3, p.162] □

C Complements

In this chapter we present some results, needed in the theorem 14, for the drift part $\int_0^t S(t-s)F(X(s))\,ds$, $t \in [0,T]$, of equation (3). They can also be found in [FrKn 2002].

Theorem 36. *Assume that F fulfills hypotheses H.0 and H.1.*

(i) Let $Y, Z \in \mathcal{H}^2(T,H)$, predictable. Then $1_{[0,t]}(\cdot)S(t-\cdot)\partial F(Y(\cdot))Z(\cdot)$ is P-a.s. Bochner integrable on $[0,T]$.

(ii) Let $Y, Z \in \mathcal{H}^2(T,H)$, predictable. Then

$$\sup_{t\in[0,T]} \| \int_0^t S(t-s)\big(\frac{F(Y(s)+hZ(s))-F(Y(s))}{h} - \partial F(Y(s))Z(s)\big)\,ds\|_{L^2}$$
$$\leq M_T T^{\frac{1}{2}} E\big[\int_0^T \|\frac{F(Y(s)+hZ(s))-F(Y(s))}{h} - \partial F(Y(s))Z(s)\|^2\,ds\big]^{\frac{1}{2}}$$
$$\underset{h\to 0}{\longrightarrow} 0.$$

(iii) Let $Y, Y_n, Z, Z_n \in \mathcal{H}^2(T,H)$, predictable, $n \in \mathbb{N}$, such that $Y_n \to Y$ and $Z_n \to Z$ in $\mathcal{H}^2(T,H)$. Then

$$\sup_{t\in[0,T]} \| \int_0^t S(t-s)(\partial F(Y_n(s))Z_n(s) - \partial F(Y(s))Z(s))\,ds\|_{L^2}$$
$$\underset{n\to\infty}{\longrightarrow} 0.$$

Proof. (i) Since Y is predictable and F is $\mathcal{B}(H)/\mathcal{B}(H)$-measurable the process $\partial F(Y(\cdot))Z(\cdot)$ is predictable. Moreover, $\|\partial F(Y)Z\| \leq C\|Z\| \in L^1(\Omega\times[0,T], P\otimes\lambda)$. Hence, $\partial F(Y(\cdot))Z(\cdot)$ is P-a.s. Bochner integrable.

(ii) The estimate is an easy calculation. Then by Lebesgue's dominated convergence theorem the convergence to 0 follows (see also [15, Proof of Theorem 4.3.(i), Step 1, (b), (1.), p.97]).

(iii)

$$\sup_{t\in[0,T]} \| \int_0^t S(t-s)(\partial F(Y_n(s))Z_n(s) - \partial F(Y(s))Z(s))\,ds\|_{L^p}$$

can be estimated by

$$M_T T^{\frac{p-1}{p}} \Big[C T^{\frac{1}{p}} \|Z_n - Z\|_{\mathcal{H}^p}$$
$$+ \big(E\big[\int_0^T \|\partial F(Y_n(s))Z(s) - \partial F(Y(s))Z(s)\|^p\,ds\big]\big)^{\frac{1}{p}}\Big].$$

$\|Z_n - Z\|_{\mathcal{H}^p} \to 0$ as $n \to \infty$ by assumption. The second summand converges to 0 as $n \to \infty$, by the continuity of ∂F, lemma 18 and the fact that

$$\|\partial F(Y_n(s))Z(s) - \partial F(Y(s))Z(s)\|^p \leq 2^p C^p \|Z\|^p \in L^1(\Omega \times [0,T], \mathcal{P}_T, P \times \lambda)$$

(see also [15, Proof of Theorem 4.3.(i), Step 2, (b), (1.), p.100/101]). □

D The Theorem of Hille-Yosida

Let $(E, \| \|)$ be a separable Banach space.

Proposition 37. *Let $S(t)$, $t \geq 0$ be a C_0-semigroup on E and let $(A, D(A))$ be its infinitesimal generator. If $x \in D(A)$ then $S(t)x \in D(A)$ and*

$$\frac{d}{dt}S(t)x = AS(t)x = S(t)Ax \text{ for all } t \geq 0.$$

Proof. [25, I. Theorem 2.4, p.4/5] □

Proposition 38 (Hille-Yosida). *Let $(A, D(A))$ be a linear operator on E. Then the following statements are equivalent.*
(i) A is the infinitesimal generator of a C_0-semigroup $S(t)$, $t \geq 0$, such that there exist constants $M \geq 1$ and $\omega \geq 0$ such that $\|S(t)\|_{L(E)} \leq Me^{\omega t}$ for all $t \geq 0$.
(ii) A is closed and $D(A)$ is dense in E, the resolvent set $\rho(A)$ contains the interval $]\omega, \infty[$ and the following estimates for the resolvent $G_\alpha := (\alpha - A)^{-1}$, $\alpha \in \rho(A)$, associated to A hold

$$\|G_\alpha^k\|_{L(H)} \leq \frac{M}{(\alpha - \omega)^k}\ ,\ k \in \mathbb{N},\ \alpha > \omega.$$

Proof. [25, I. Theorem 5.3, p.20] □

Let $(A, D(A))$ be the infinitesimal generator of a C_0-semigroup $S(t)$, $t \geq 0$, such that there exist constants $M \geq 1$ and $\omega \geq 0$ such that $\|S(t)\|_{L(E)} \leq Me^{\omega t}$ for all $t \geq 0$. We define now the Yosida-approximation of A. For $n \in \mathbb{N}$, $n > \omega$, define

$$A_n := nAG_n = nG_nA.$$

Proposition 39. *Let $(A, D(A))$ be the infinitesimal generator of a C_0-semigroup $S(t)$, $t \geq 0$, such that there exist constants $M \geq 1$ and $\omega \geq 0$ such that $\|S(t)\|_{L(E)} \leq Me^{\omega t}$ for all $t \geq 0$. Then*

$$\lim_{n \to \infty} A_n x = Ax \text{ for all } x \in D(A).$$

Proof. Let $x \in D(A)$ and $n > \omega$, then

$$\|nG_nx - x\|_E = \|G_n(nx - Ax) + G_nAx - x\|_E$$

$$= \|G_n Ax\|_E \leq \frac{M}{n-\omega}\|Ax\|_E \underset{n\to\infty}{\longrightarrow} 0.$$

But, by proposition 38, $D(A)$ is dense in E and $\|nG_n x\|_{L(E)} \leq \frac{Mn}{n-\omega}$, where the sequence $\frac{Mn}{n-\omega}$, $n > \omega$, is convergent and therefore bounded. Hence we get for arbitrary $x \in E$ that $\|nG_n x - x\|_E \to 0$.
In particular, we obtain for all $x \in D(A)$ that

$$A_n x = nG_n Ax \underset{n\to\infty}{\longrightarrow} Ax.$$

□

Proposition 40. *Let $(A, D(A))$ be the infinitesimal generator of a strongly continuous semigroup $S(t)$, $t \geq 0$, such that there exist constants $M \geq 1$ and $\omega \geq 0$ such that $\|S(t)\|_{L(E)} \leq Me^{\omega t}$ for all $t \geq 0$. Moreover, let A_n, $n \in \mathbb{N}$, $n > \omega$, be the Yosida-approximation of A. Then*

$$S(t)x = \lim_{n\to\infty} S_n(t)x \text{ locally uniformly in } t \geq 0 \text{ for all } x \in E$$

where $S_n(t) := e^{tA_n}$, $t \geq 0$, and the following estimate holds

$$\|S_n(t)\|_{L(E)} \leq Mexp(\frac{\omega nt}{n-\omega}) \text{ for all } t \geq 0,\ n > \omega.$$

Proof. [25, I. Theorem 5.5, p.21] □

References

[1] Albeverio, S., Wu, J.-L., Zhang, T.-S.: Parabolic SPDEs driven by Poisson white noise. Stochastic Processes and their Applications 74 (1998), p.21-36

[2] Applebaum, D., Wu, J.-L.: Stochastic partial differential equations driven by Lévy space-time white noise. Random Operators and Stochastic Equations 8 (2000), p.245-261

[3] Applebaum, D., Tang, F.: Stochastic flows of diffeomorphisms on manifolds driven by infinite-dimensional semimartingales with jumps. Stochastic Processes and their Applications 92 (2001), p.219-236

[4] Applebaum, D.: Lévy processes - from probability to finance and quantum groups. Notices of the AMS 51 (2004) number 11, p.1336-1342

[5] Applebaum, D.: Martingale-valued measures, Ornstein-Uhlenbeck processes with jumps and operator self-decomposability in Hilbert space, to appear in Seminaire de Probabilites

[6] Blanchard, P., Brüning, E.: Variational Methods in Mathematical Physics. Berlin-Heidelberg-New York: Springer Verlga 1982

[7] Chow, Y., Teicher, H.: Probability Theory. New York-Heidelberg-Berlin: Springer Verlag 1978

[8] Cohn, D.L.: Measure Theory. Boston: Birkhäuser 1980

[9] Da Prato, G., Zabczyk, J.: Stochastic Equations in Infinite Dimensions. Cambridge: Cambridge University Press 1992

[10] Da Prato, G., Zabczyk, J.: Ergodicity for Infinite Dimensional Systems. Cambridge: Cambridge University Press 1996

[11] de Barra,G.: Measure Theory and Integration. Chichester: Ellis Horwood Limited . Distributed by: New York-Chichester-Brisbane-Toronto: John Wiley & Sons 1977

[12] Dellacherie, C., Meyer, P.-A.: Probability and Potential. Theory of Martingales. Amsterdam-New York-Oxford: North-Holland Publishing Company 1982

[13] Eberlein, E., Raible, S.: Term structure models driven by general Lévy processes. Math. Finance, 9 (1) (1999), p.31-53

[14] Ethier, S.N., Kurtz, T.G.: Markov Processes: Characterization and Convergence. New York-Chichester-Brisbane-Toronto-Singapore: John Wiley & Sons 1986

[15] Frieler, K., Knoche, C.: Solutions of stochastic differential equations in infinite dimensional Hilbert spaces and their dependence on initial data. Diplomathesis, preprint no.03-06-119 of the additional series of the research center BiBoS (Bielefeld-Bonn-Stochastics), Faculty of Mathematics, University of Bielefeld, Bielefeld, Germany, 2001

[16] Hausenblas, E.: SPDEs driven by Poisson Random Measure: Existence and Uniqueness. Electr. J. Prob. 10 (2005), p. 1496-1546,

[17] Hackenbroch, W., Thalmaier, A.: Stochastische Analysis: eine Einführung in die Theorie der stetigen Semimartingale. Stuttgart : Teubner 1994

[18] Ikeda, N., Watanabe, S.: Stochastic Differential Equations and Diffusion Processes. Amsterdam-Oxford-New York: North-Holland Publishing Company 1981

[19] Imkeller, P., Pavlyukevich, I.: First exit times of solutions of non-linear stochastic differential equations driven by symmetric Lévy processes with α-stable components. arXiv: math.PR/0409246 v1, 15. September 2004

[20] Kallianpur, G., Wolpert, R.: Infinite dimensional stochastic differential equation models for spatially distributed neurons. Applied Mathematics and Optimization 12 (1984), p.125-172

[21] Knoche, C.: Mild Solutions of SPDE's Driven by Poisson Noise in Infinite Dimensions and their Dependence on Initial Conditions. Preprint no.05-003 of the Collaborative Research Centre 701, Faculty of Mathematics, University of Bielefeld, Bielefeld, Germany, 2005

[22] Mandrekar, V., Rüdiger, B.: Existence and uniqueness of path wise solutions for stochastic integral equations driven by non Gaussian noise on separable Banach spaces. Preprint no. 61 of the Collaborative Research Centre 611, Faculty of Mathematics, University Bonn, Bonn, Germany (2003)

[23] Metivier, M.: Semimartingales. A course on Stochastic Processes.Berlin-New York: W. de Gruyter 1982

[24] Müller, C.: The heat equation with Lévy noise. Stochastic Processes and their Applications 74 (1998), p.67-82

[25] Pazy, A.: Semigroups of Linear Operators. Berlin-Heidelberg-New York-Tokyo: Springer Verlag 1983

[26] Peszat, S.: Existence and uniqueness of the solution for stochastic equations on Banach spaces. Stochastics and Stochastics Reports 55 (1995), p.167-193

[27] Protter, P.: Stochastic Integration and Differential Equations : a New Approach. Berlin [u.a.] : Springer Verlag 1990

[28] Raible, S.: Lévy processes in finance: Theory, numerics and empirical facts. PhD Thesis. Faculty of Mathematics, University Freiburg, Freiburg, Germany, 2000

[29] Rall, W.: Branching dendritic trees and motoneuron resistivity. Experimental Neurology 1 (1959) , p.491-527

[30] Rüdiger, B.: Stochastic integration with respect to compensated Poisson random measures on separable Banach spaces. Stochastics and Stochastics Reports 76 (2004), no. 3, p.213-242

[31] Walsh, J.B.: A stochastic model of neural response. Adv. Appl. Prob. 13 (1981), p.231-281

[32] Walsh, J.B.: An introduction to stochastic partial differential equations. In: Ecole dÉté de Probabilité de St. Flour XIV, Lecture Notes in Math. 1180. Berlin: Springer Verlag 1986, p.266-439

[33] Zabczyk, J.: Parabolic equations on Hilbert spaces. In: Da Prato, G (Editor), Stochastic PDE's and Kolmogorov Equations in Infinite Dimensions, Lecture Notes in Mathematics, Cetraro, Italy 1998, Berlin-Heidelberg-New York: Springer Verlag 1998, p.117-213