WEAK CONVERGENCE OF DIFFUSION PROCESSES ON WIENER SPACE

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ABSTRACT. Let γ be a Gaussian measure on a Suslin space X, H be the corresponding Cameron-Martin space and $\{e_i\} \subset H$ be an orthonormal basis of H. Suppose that $\mu_n = \rho_n \cdot \gamma$ is a sequence of probability measures which weakly converges to a probability measure $\mu = \rho \cdot \gamma$. Consider a sequence of Dirichlet forms $\{\mathcal{E}_n\}$, where $\mathcal{E}_n(f) = \int_X \|\nabla_H f\|^2_H \rho_n \, d\gamma$ and $\sqrt{\rho_n} \in W^{2,1}(\gamma)$. We prove some sufficient conditions for Mosco convergence

$$\mathcal{E}_n \to \mathcal{E},$$

where $\mathcal{E}(f) = \int_X \|\nabla_H f\|^2_H \rho \, d\gamma$. In particular, if X is a Hilbert space,

$$\sup_n \|\sqrt{\rho_n}\|_{W^{2,1}(\gamma)} < \infty$$

and $\frac{\partial_{e_i}\rho_n}{\rho_n}$ can be uniformly approximated by finite dimensional conditional expectations $\mathbb{E}^{\mathcal{F}_N}_{\mu_n}\left(\frac{\partial_{e_i}\rho_n}{\rho_n}\right)$ for every fixed e_i, then under broad assumptions $\mathcal{E}_n \to \mathcal{E}$ Mosco and the distributions of the associated stochastic processes converge weakly.

Keywords: Dirichlet forms; Mosco convergence; convergence of stochastic processes; Gaussian measures.

1. INTRODUCTION

In this paper we consider a Gaussian measure γ on a Suslin space X. Denote by H the corresponding Cameron-Martin space. It is assumed throughout the paper that $\mathrm{supp}(\gamma) = X$. Let us choose some orthonormal basis $\{e_i\}$ in H. Since H can be considered as a completion of X^* in $L^2(\gamma)$ (see [3] for details), we may assume without loss of generality that $e_i \in X^*$. Denote by $x_i := \hat{e}_i(x)$ the corresponding coordinate function. We denote by $\mathcal{F}C^\infty$ the linear span of smooth cylindrical functions, i.e. the functions of the type

$$f(x) = \varphi(x_1, \cdots, x_n),$$

where φ is a smooth compactly supported function on $\mathbb{R}^n$.

The reader may think for simplicity of $X = \mathbb{R}^\infty$ and the product measure $\gamma = (\gamma_1)^\infty$, where γ_1 is the standard Gaussian measure on $\mathbb{R}^1$. Another example is a separable Hilbert space, where γ is given by the formal expression $e^{-\left\langle Q^{-1}x, x\right\rangle_X} dx$ (see [3] for details). Here Q is a symmetric positive trace operator. In the case of $\mathbb{R}^\infty$ we take for x_i the i-th coordinate function.

Let $\{\rho_n\}$ be a sequence of probability densities (with respect to γ) on X such that $\sqrt{\rho_n} \in W^{2,1}(\gamma)$ for every n and, in addition, for every i

$$\sup_n \int_X \frac{\left(\partial_{e_i}\rho_n\right)^2}{\rho_n} \, d\gamma < \infty. \tag{1}$$

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Recall that $W^{2,1}(\gamma)$ consists of all functions which are weakly differentiable along H and admit the finite norm

$$\|f\|^2_{W^{2,1}(\gamma)} = \int_X f^2 \, d\gamma + \int_X \|\nabla_H f\|^2_H \, d\gamma.$$

Here $\nabla_H f$ denotes the standard Malliavin gradient of f.

Consider the following sequence of Dirichlet forms:

$$\mathcal{E}^0_n(\varphi) = \int_X \|\nabla_H \varphi\|^2_H \rho_n \, d\gamma, \quad \varphi \in \mathcal{F}C^\infty_0.$$

It is known, that $(\mathcal{E}^0_n, \mathcal{F}C^\infty_0)$ is closable in $L^2(\mu_n)$ (see [18]) under assumption $\sqrt{\rho_n} \in W^{2,1}(\gamma)$. In addition, the Markov uniqueness for $(\mathcal{E}^0_n, \mathcal{F}C^\infty_0)$ holds. This means that the operator

$$\mathcal{L} = \sum_{i=1}^{\infty} \partial^2_{e_i} - \Big[x_i - \partial_{e_i} \ln(\rho_n)\Big] \partial_{e_i}$$

is essentially self-adjoint in $L^2(\mu_n)$ and the corresponding strong and weak Sobolev spaces coincide (see [23], [24], [5] and [7], Theorem 5.1, Corollary 5.1). The closure of $(\mathcal{E}^0_n, \mathcal{F}C^\infty_0)$ will be denoted by $(\mathcal{E}_n, \mathcal{D}(\mathcal{E}_n))$.

Let $\mathcal{F}^N$ be the sigma-algebra generated by functions depending only on $(x_1, \cdots, x_N)$. We write

$$\mathbb{E}^N_m f := \mathbb{E}^{\mathcal{F}_N}_m f$$

for the conditional expectation of $f \in L^1(m)$ with respect to some Borel probability measure m and sigma-algebra $\mathcal{F}_N$. Let P_N be projection on the first N coordinates: $P_N x = \sum_{i=1}^N x_i e_i$. Set $S_N(x) = x - P_N(x)$. Then γ can be represented as a product measure $\gamma = \big(\gamma \circ P_N^{-1}\big) \times \big(\gamma \circ S_N^{-1}\big)$. It is known that

$$\mathbb{E}^N_\gamma f = \int f(P_N x + S_N x) \, d\gamma \circ S_N^{-1}$$

(see [3]). In particular, this formula implies that for every $f \in W^{1,1}(\gamma)$

$$\nabla_H \mathbb{E}^N_\gamma f = \mathbb{E}^N_\gamma \big(\nabla^N_H f\big).$$

Here

$$\nabla^N_H f = \big(\partial_{e_1} f, \cdots, \partial_{e_N} f, 0, \cdots, 0, \cdots\big).$$

Let $\Omega = C([0,\infty) \to X)$ be the space of all continuous mappings from $[0,\infty)$ into X and X_t be the coordinate function on Ω such that $X_t(\omega) = \omega(t)$. We denote by

$$\{\Omega, X_t, \mathcal{F}_t, \mathcal{P}^n_x, x \in X\}$$

the diffusion process associated with $(\mathcal{E}_n, \mathcal{D}(\mathcal{E}_n))$. This process exists by a result from [2]. Define

$$\mathcal{P}_{\mu_n} = \int_X \mathcal{P}^n_x \rho_n \, d\gamma.$$

By a recent result of Posilicano and Zhang [22], if $\sqrt{\rho}_n \to \sqrt{\rho}$ in $W^{2,1}(\gamma)$ and $\sqrt{\rho_n}, \sqrt{\rho} \in W^{2+\varepsilon,1}$ for some $\varepsilon > 0$, then

$$\mathcal{P}_{\mu_n} \to \mathcal{P}_\mu$$

in total variation norm on $\mathcal{F}_t$ for any $t > 0$.

In this paper we study some sufficient conditions for weak convergence $\mathcal{P}_{\mu_n} \to \mathcal{P}_\mu$. It turns out that the weak convergence of the distributions of the processes takes place under much weaker assumptions. For example, if X is a Hilbert space, the weak convergence holds under assumption $\sup_n \|\nabla_H \sqrt{\rho_n}\|_{W^{2,1}(\gamma)} < \infty$ and some uniform approximation property of $\frac{\partial_{e_i}\rho_n}{\rho_n}$ by the finite dimensional conditional expectations for every fixed e_i. We stress that we *don't assume* any *strong convergence of* $\frac{\partial_{e_i}\rho_n}{\rho_n}$. We emphasize that the problem of the weak convergence of the diffusion processes appears in many physical models without L^2-strong convergence of the corresponding logarithmic partial derivatives (see, for instance, [9]). Note that unlike [22] we require γ-a.e. positivity of the densities ρ_n, ρ.

In this paper we apply the techniques of Mosco convergence of quadratic forms defined on different L^2-spaces, initiated in [27], [13] and developed in [11], [12]. We give below the basic definitions of this theory and refer the reader to the cited works for detailed proofs.

Some other results on Mosco convergence and convergence of stochastic processes can be found in [1], [10], [14], [15], [16], [20], [21], [25], [26].

2. Main results

We assume throughout the paper that

$$\mu_n \to \mu$$

weakly, where $\mu_n = \rho_n \cdot \gamma$, $\mu = \rho \cdot \gamma$ and $\rho_n > 0$, $\rho > 0$ γ-a.e. Moreover, we assume that $\rho_n \to \rho$ weakly in $L^1(\gamma)$. Every Dirichlet form $\mathcal{E}$ of the type

$$\mathcal{E}(f) = \int \|\nabla f\|^2 \, dm \quad \text{or} \quad \mathcal{E}(f) = \int \Big(\frac{\partial f}{\partial x_i}\Big)^2 \, dm$$

is considered on the space $L^2(m)$. Here m means some Borel non-negative measure and ∇ denotes either the Malliavin gradient on the Wiener space or the standard gradient on $\mathbb{R}^d$.

Following [27], [13] we consider the union $\mathcal{H} = \cup_n L^2(\mu_n)$ and introduce a convergence on $\mathcal{H}$. We say that a sequence of functions $\{f_n\}$, where $f_n \in L^2(\mu_n)$, converges weakly to $f \in L^2(\mu)$ if $\int_X f_n \varphi \, d\mu_n \to \int_X f \varphi \, d\mu$ for every $\varphi \in \mathcal{F}C_0^\infty$. If, in addition, $\|f_n\|_{L^2(\mu_n)} \to \|f\|_{L^2(\mu)}$, we say that $f_n \to f$ strongly. The sequence of bounded operators $\{B_n\}$, where $B_n \in L(L^2(\mu_n))$, strongly converge to a bounded operator $B \in L(L^2(\mu))$ if $B_n f_n \to Bf$ strongly for every strongly convergent $f_n \to f$.

We set $\mathcal{E}(f) := \infty$ for every $f \notin \mathcal{D}(\mathcal{E})$.

Definition 2.1. *We say that a sequence $\{\mathcal{E}_n : L^2(\mu_n) \to \overline{\mathbb{R}}\}$ of quadratic symmetric forms Mosco converges to a quadratic form $\mathcal{E}$ on $L^2(\mu)$ if the following conditions are fulfilled:*

(M1) *If $f_n \to f$ weakly, then*

$$\mathcal{E}(f) \le \underline{\lim}_n \mathcal{E}_n(f_n).$$

(M2) *For every $f \in L^2(\mu)$ there exists a strongly convergent sequence $f_n \to f$ such that*

$$\mathcal{E}(f) = \lim_n \mathcal{E}_n(f_n).$$

It is known that the following statements are equivalent:

1. $\{\mathcal{E}_n\}$ Mosco converges to $\mathcal{E}$
2. the corresponding associated semigroups $\{T_{n,t}\}$ strongly converges to T_t for every $t > 0$.

We employ the following fact from the theory of convergent Hilbert spaces:

Every sequence $\{f_n\}$ *such that* $\sup_n \|f_n\|_{L^2(\mu_n)} < \infty$ *admits a weakly convergent subsequence* $f_n \to f$. *In addition,* $\|f\|_{L^2(\mu)} \le \underline{\lim}_n \|f_n\|_{L^2(\mu_n)}$.

More on Mosco convergence see in [19], [27], [13], [11], [12].

Throughout the paper we consider the weighted Sobolev spaces $W^{2,1}(\mu)$ with respect to some measure μ (finite of infinite dimensional) which admits the logarithmic derivative β_{x_i} along every x_i. This means that for every test function φ

$$\int_X \varphi_{x_i}\, d\mu = -\int_X \varphi \beta_{x_i}\, d\mu.$$

The weak (Sobolev) derivative f_{x_i} of some function f is defined by the integration by parts in the following way:

$$\int_X \varphi_{x_i} f\, d\mu = -\int_X \varphi f \beta_{x_i}\, d\mu - \int_X \varphi f_{x_i}\, d\mu.$$

Then

$$W^{2,1}(\mu) = \Bigl\{ f \text{ is weakly differentiable and } f, \|\nabla_H f\|_H \in L^2(\mu) \Bigr\}.$$

By Theorem 5.1 from [7] $W^{2,1}(\mu)$ can be also defined as a completion of $\mathcal{F}C_0^\infty$ in the corresponding norm.

The following lemma can be proved by the compactness imbedding theorem for the sequences $\sqrt{\varrho_n} \subset W^{2,1}_{loc}(\mathbb{R}^d)$ and $\{f_n \varrho_n\} \subset W^{1,1}_{loc}(\mathbb{R}^d)$ (see [12] for details).

Lemma 2.2. *Let* $X = \mathbb{R}^d$. *Consider sequences of functions* $\{\varrho_n\}$, $\{f_n\}$ *such that* $\varrho_n > 0$. *Suppose that for every ball* $B \subset \mathbb{R}^d$:

$$\sup_n \Bigl(\int_B \varrho_n\, dx + \frac{1}{4} \int_B \frac{\|\nabla \varrho_n\|^2}{\varrho_n}\, dx \Bigr) = \sup_n \|\sqrt{\varrho_n}\|_{W^{2,1}(B)} < \infty.$$

and

$$\sup_n \Bigl[\int_B f_n^2 \varrho_n\, dx + \int_B (\nabla f_n)^2 \varrho_n\, dx \Bigr] < \infty.$$

Then there is a subsequence $\{n_m\}$ *such that* $\{\sqrt{\varrho_{n_m}}\}$ *converges in* $L^2_{loc}(\mathbb{R}^d)$ *and* $\{f_{n_m} \varrho_{n_m}\}$ *converges in* $L^1_{loc}(\mathbb{R}^d)$.

The proof of the following theorem can be found in [12] (Theorem 4.2). The idea is to prove first that $\mathcal{E}_n \to \mathcal{E}$ in the sense of Γ-convergence (which is weaker than Mosco convergence, see [6] for details). The next step is to prove that an uniformly bounded sequence of functions $\{f_n\}$ satisfying $\sup_n \mathcal{E}_n(f_n) < \infty$ admits a $\mathcal{H}$-strongly convergent subsequence. This can be verified with the help of Lemma 2.2. Then using the contraction properties of the Dirichlet forms and the cutoff approximations one easily completes the proof.

Theorem 2.3. *Let* $\{\varrho_n\}$ *be a sequence of positive functions such that every* $\varrho_n\, dx$ *is a probability measure and* $\varrho_n\, dx \to \varrho\, dx$ *weakly. In addition, we assume that* $\varrho > 0$ *almost everywhere and*

$$\sup_n \int_{\mathbb{R}^d} \frac{\bigl(\nabla \varrho_n\bigr)^2}{\varrho_n}\, dx < \infty.$$

Then $\mathcal{E}_n \to \mathcal{E}$ *Mosco, where* $\mathcal{E}_n(f) = \int_{\mathbb{R}^d} \|\nabla f\|^2 \varrho_n\, dx$, $\mathcal{E}(f) = \int_{\mathbb{R}^d} \|\nabla f\|^2 \varrho\, dx$.

In addition, (M1) holds for every partial form: if $f_n \to f$ weakly in the sense on $\mathcal{H}$-convergence, then for every $1 \le i \le N$

$$\int_X \Big(\frac{\partial f}{\partial x_i}\Big)^2 \varrho\, dx \le \underline{\lim}_n \int_X \Big(\frac{\partial f_n}{\partial x_i}\Big)^2 \varrho_n\, dx.$$

Now we are ready to prove the main result of this paper. We consider a centered Gaussian measure γ on X. Everywhere below $\|\cdot\|$ means the H-norm.

Theorem 2.4. *Let $\{\rho_n\}$ be a sequence of probability densities. Denote $\mu_n := \rho_n \cdot \gamma$, $\mu := \rho \cdot \gamma$. Suppose that $\{\mu_n\}$ satisfies the following assumptions*

1) *$\rho_n(x) > 0$, $\rho(x) > 0$ for γ-almost all x and $\rho_n \to \rho$ weakly in $L^1(\gamma)$.*
2) *for every n $\sqrt{\rho_n} \in W^{2,1}(\gamma)$ and for every i*

$$\sup_n \int_X \frac{\big(\partial_{e_i}\rho_n\big)^2}{\rho_n}\, d\gamma < \infty$$

3) *for every $i \in \mathbb{N}$*

$$\lim_N \Big[\sup_n \int_X \Big(\frac{\partial_{e_i}\rho_n}{\rho_n} - \mathbb{E}^N_{\mu_n}\Big[\frac{\partial_{e_i}\rho_n}{\rho_n}\Big]\Big)^2 d\mu_n\Big] = 0. \tag{2}$$

Then $\mathcal{E}_n \to \mathcal{E}$ Mosco, where

$$\mathcal{E}_n(f) = \int_X \|\nabla_H f\|^2 \rho_n\, d\gamma, \quad \mathcal{E}(f) = \int_X \|\nabla_H f\|^2 \rho\, d\gamma.$$

Proof. Let $f_n \to f$ be a $\mathcal{H}$-weakly convergent sequence of functions $f_n \in L^2(\mu_n)$ such that $\sup_n \mathcal{E}_n(f_n) < \infty$. Suppose first that $\sup_X |f_n| \le K$. Let us show that (M1) is fulfilled.

Set:

$$\rho^N_n := \mathbb{E}^N_\gamma\big(\rho_n\big), \quad f^N_n := \mathbb{E}^N_{\mu_n}\big(f_n\big), \quad \mu^N_n := \big(\rho^N_n\big)\cdot\gamma.$$

We note that $\mu^N_n = \big(\mu_n \circ P_N^{-1}\big) \times \big(\gamma \circ S_N^{-1}\big)$. Obviously, $\{\mu^N_n\}$ is a sequence of probability measures and $\mu^N_n \to \mu^N$ weakly. In what follows we consider some integrals involving the measures $\{\mu^N_n\}$ and some functions depending only on $(x_1, \cdots, x_N)$. Since these integrals can be reduced to the finite dimensional ones, we may use some finite dimensional results (like compactness embedding of Sobolev spaces etc.).

First we show that L^2-norms of logarithmic derivatives of $\{\mu^N_n\}$ are uniformly bounded for every fixed N. Then we show that the values of the corresponding Dirichlet forms on $\{f^N_n\}$ are bounded by some constant.

Since $\rho_n > 0$, one has

$$\mathbb{E}^N_{\mu_n}(f_n) = \frac{\mathbb{E}^N_\gamma(f_n\rho_n)}{\mathbb{E}^N_\gamma(\rho_n)}.$$

Hence

$$\int_X \frac{\big\|\nabla_H \rho^N_n\big\|^2}{\rho^N_n}\, d\gamma = \int_X \frac{\big\|\mathbb{E}^N_\gamma\big[\nabla^N_H\rho_n\big]\big\|^2}{\mathbb{E}^N_\gamma \rho_n}\, d\gamma = \int_X \Big\|\frac{\mathbb{E}^N_\gamma\big[\nabla^N_H\rho_n\big]}{\mathbb{E}^N_\gamma \rho_n}\Big\|^2 \mathbb{E}^N_\gamma \rho_n\, d\gamma$$

$$= \int_X \Big\|\frac{\mathbb{E}^N_\gamma\big[\nabla^N_H\rho_n\big]}{\mathbb{E}^N_\gamma \rho_n}\Big\|^2 \rho_n\, d\gamma = \int_X \Big\|\mathbb{E}^N_{\mu_n}\Big[\frac{\nabla^N_H\rho_n}{\rho_n}\Big]\Big\|^2 \rho_n\, d\gamma \le \int_X \mathbb{E}^N_{\mu_n}\Big[\Big\|\frac{\nabla^N_H\rho_n}{\rho_n}\Big\|^2\Big] \rho_n\, d\gamma$$

$$= \int_X \frac{\|\nabla^N_H\rho_n\|^2}{\rho_n}\, d\gamma \le \sum_{i=1}^N \int_X \frac{\big(\partial_{x_i}\rho_n\big)^2}{\rho_n}\, d\gamma.$$

It follows from (1) that the right-hand side of this estimate is uniformly bounded for every fixed N. Further we get

$$\int_X \big(f_n^N\big)^2 \rho_n^N \, d\gamma = \int_X \big(f_n^N\big)^2 \mathbb{E}_\gamma^N(\rho_n) \, d\gamma = \int_X \big(f_n^N\big)^2 \rho_n \, d\gamma \le \int_X \mathbb{E}_{\mu_n}^N \big(f_n\big)^2 \rho_n \, d\gamma = \|f_n\|^2_{L^2(\mu_n)}.$$

Now let us analyze $\int_X \|\nabla_H f_n^N\|_H^2 \, d\mu_n^N$. One has

$$\begin{aligned}
\nabla_H \mathbb{E}_{\mu_n}^N(f_n) &= \frac{\mathbb{E}_\gamma^N(\nabla_H^N f_n \rho_n) + \mathbb{E}_\gamma^N(f_n \nabla_H^N \rho_n)}{\mathbb{E}_\gamma^N(\rho_n)} - \frac{\mathbb{E}_\gamma^N(f_n\rho_n)\mathbb{E}_\gamma^N(\nabla_H^N \rho_n)}{\Big(\mathbb{E}_\gamma^N(\rho_n)\Big)^2} \\
&= \mathbb{E}_{\mu_n}^N(\nabla_H^N f_n) + \mathbb{E}_{\mu_n}^N\Big[f_n \frac{\nabla_H^N \rho_n}{\rho_n}\Big] - \mathbb{E}_{\mu_n}^N(f_n)\mathbb{E}_{\mu_n}^N\Big[\frac{\nabla_H^N \rho_n}{\rho_n}\Big] \\
&= \mathbb{E}_{\mu_n}^N(\nabla_H^N f_n) + \mathbb{E}_{\mu_n}^N\Big[f_n\Big(\frac{\nabla_H^N \rho_n}{\rho_n} - \mathbb{E}_{\mu_n}^N\Big[\frac{\nabla_H^N \rho_n}{\rho_n}\Big]\Big)\Big] = I + II.
\end{aligned}$$

Obviously,

$$\int_X I^2 \, d\mu_n^N = \int_X I^2 \, d\mu_n = \int_X I^2 \rho_n \, d\gamma \le \int_X \|\nabla_H f_n\|^2 \rho_n \, d\gamma$$

and

$$\begin{aligned}
&\int_X II^2 \, d\mu_n^N = \int_X II^2 \, d\mu_n = \int_X II^2 \rho_n \, d\gamma \le K^2 \int_X \Big(\frac{\nabla_H^N \rho_n}{\rho_n} - \mathbb{E}_{\mu_n}^N\Big[\frac{\nabla_H^N \rho_n}{\rho_n}\Big]\Big)^2 \rho_n \, d\gamma \\
&= K^2 \Big[\int_X \frac{\|\nabla_H^N \rho_n\|^2}{\rho_n} \, d\gamma - \int_X \Big(\mathbb{E}_{\mu_n}^N\Big[\frac{\nabla_H^N \rho_n}{\rho_n}\Big]\Big)^2 \rho_n \, d\gamma\Big].
\end{aligned}$$

In the same way we show the similar estimates for the partial logarithmic derivatives along e_i, where $N > i$

$$\sup_n \int_X \frac{\big(\partial_{e_i}\rho_n^N\big)^2}{\rho_n^N} \, d\gamma \le \sup_n \int_X \frac{\big(\partial_{e_i}\rho_n\big)^2}{\rho_n} \, d\gamma < \infty. \tag{3}$$

In particular we get that $\sup_n \|\beta_{x_i}^{N,n}\|_{L^2(\mu_n^N)} < \infty$, where $\beta_{x_i}^{N,n}$ is the logarithmic partial derivative of μ_n^N along e_i. Indeed

$$\beta_{x_i}^{N,n} = \frac{\partial_{e_i}\rho_n^N}{\rho_n^N} - x_i.$$

Hence by the Young inequality and the log-Sobolev inequality

$$\begin{aligned}
\frac{1}{2}\|\beta_{x_i}^{N,n}\|_{L^2(\mu_n^N)} &\le \int_X \frac{\big(\partial_{e_i}\rho_n^N\big)^2}{\rho_n^N} \, d\gamma + \int_X x_i^2 \rho_n^N \, d\gamma \\
&\le \int_X \frac{\big(\partial_{e_i}\rho_n^N\big)^2}{\rho_n^N} \, d\gamma + 3\Big(\int_X e^{\frac{x_i^2}{3}-1} \, d\gamma + \int_X \rho_n^N \log \rho_n^N \, d\gamma\Big) \\
&\le \int_X \frac{\big(\partial_{e_i}\rho_n^N\big)^2}{\rho_n^N} \, d\gamma + 3\int_X e^{\frac{x_i^2}{3}-1} \, d\gamma + 3\int_X \frac{\|\nabla \rho_n^N\|^2}{2\rho_n^N} \, d\gamma.
\end{aligned}$$

Hence $\sup_n \|\beta_{x_i}^{N,n}\|_{L^2(\mu_n^N)} < \infty$ and one can apply Theorem 2.3 to the sequence $\{\mu_n^N\}$. Further we get analogously to the above estimates

$$\partial_{e_i}\mathbb{E}_{\mu_n}^N(f_n) = \mathbb{E}_{\mu_n}^N(\partial_{e_i} f_n) + II_{i,n,N} \tag{4}$$

and

$$\int_X \left(\mathbb{E}^N_{\mu_n} \partial_{e_i} f_n\right)^2 d\mu_n \le \int_X \left(\partial_{e_i} f_n\right)^2 \rho_n \, d\gamma \tag{5}$$

$$\int_X II^2_{i,n,N} \, d\mu^N_n \le K^2 \Big[\int_X \frac{(\partial_{e_i}\rho_n)^2}{\rho_n} \, d\gamma - \int_X \Big(\mathbb{E}^N_{\mu_n} \Big[\frac{\partial_{e_i}\rho_n}{\rho_n} \Big] \Big)^2 \rho_n \, d\gamma \Big]. \tag{6}$$

We know that $\mu^N_n \to \mu^N$ weakly. Moreover, by (3) $\sqrt{\rho^N_n}$ is uniformly bounded in $W^{2,1}(\gamma)$ for every fixed N. Using the compactness embedding $W^{2,1}(B) \to L^2(B)$ for every ball $B \subset \mathbb{R}^N$, and applying the standard diagonal procedure, one can extract a subsequence $\{\rho^N_{n_m}\}$ which converges in L^1_{loc}. By the weak convergence $\mu^N_n \to \mu^N$, the limit of $\{\rho^N_{n_m}\}$ coincides with $\rho^N := \mathbb{E}^N_\gamma(\rho)$, hence the whole sequence $\{\rho^N_n\}$ converges in L^1_{loc} to the same limit. In addition, by the log-Sobolev inequality and (3)

$$\sup_n \int_X \rho^N_n \log \rho^N_n \, d\gamma \le \sup_n \int_X \frac{\|\nabla^N_H \rho^N_n\|^2}{2\rho^N_n} \, d\gamma < \infty.$$

Hence $\{\rho^N_n\}$ is uniformly γ-integrable and

$$\rho^N_n \to \rho^N \quad \text{in } L^1(\gamma). \tag{7}$$

Using the above estimates for $\|f^N_n\|_{L^2(\mu^N_n)}$, $\|\nabla_H f^N_n\|_{L^2(\mu^N_n)}$ and positivity of $\{\rho^N_n\}$, we get by Lemma 2.2 that there exist a subsequence (denoted again by f^N_n) and a function $\tilde{f}^N$ such that $f^N_n \to \tilde{f}^N$ γ-a.e. By the properties of conditional expectations $|f^N_n| \le K$. Let us show that $\tilde{f}^N = \mathbb{E}^N_\mu f$. Indeed, fix a bounded continuous $\varphi(x_1, \cdots, x_N)$:

$$\lim_{n\to\infty} \int_X f^N_n \varphi \rho_n \, d\gamma = \lim_{n\to\infty} \int_X f^N_n \varphi \rho^N_n \, d\gamma = \lim_{n\to\infty} \int_X f^N_n \varphi \rho^N \, d\gamma + \lim_{n\to\infty} \int_X f^N_n \varphi (\rho^N_n - \rho^N) \, d\gamma.$$

The first term tends to $\int_X \tilde{f}^N \varphi \rho^N \, d\gamma = \int_X \tilde{f}^N \varphi \rho \, d\gamma$ by the Lebesque domination convergence theorem. The second one can be estimated by

$$\sup(|\varphi|) K \|\rho^N_n - \rho^N\|_{L^1(\gamma)}.$$

This obviously tends to zero by (7). Hence

$$\lim_{n\to\infty} \int_X f^N_n \varphi \rho_n \, d\gamma = \int_X \tilde{f}^N \varphi \rho \, d\gamma.$$

In the other hand, we get by the weak $\mathcal{H}$-convergence $f_n \to f$

$$\lim_n \int_X f^N_n \varphi \rho_n \, d\gamma = \lim_n \int_X \left(\mathbb{E}^N_{\mu_n} f_n\right) \varphi \rho_n \, d\gamma = \lim_n \int_X f_n \varphi \rho_n \, d\gamma = \int_X f \varphi \rho \, d\gamma.$$

Hence

$$\int_X f \varphi \rho \, d\gamma = \int_X \tilde{f}^N \varphi \rho \, d\gamma$$

and $\mathbb{E}^N_\mu f = \tilde{f}^N$. Using the $L^1(\gamma)$-convergence of ρ^N_n to ρ^N and the uniform boundedness of $\{f^N_n\}$, we obtain that $f^N_n \to \mathbb{E}^N_\mu f$ $\mathcal{H}$-weakly (even $\mathcal{H}$-strongly) in the sense of convergent $L^2(\mu^N_n)$-spaces.

Then using Theorem 2.3 (which is applicable since all the function and measures are essentially finite dimensional), (4), (5), (6) and assumption 3) of the theorem, we get for every $i \in \mathbb{N}$

$$\int_X \Big(\frac{\partial \mathbb{E}^N_\mu f}{\partial x_i}\Big)^2 \, d\mu \le \underline{\lim}_n \int_X \Big(\frac{\partial \mathbb{E}^N_\mu f_n}{\partial x_i}\Big)^2 \, d\mu_n \tag{8}$$

and
(9)

$$\underline{\lim}_n \int_X \Big(\frac{\partial \mathbb{E}^N_\mu f_n}{\partial x_i}\Big)^2 \, d\mu_n \le \underline{\lim}_n \int_X \Big(\frac{\partial f_n}{\partial x_i}\Big)^2 \, d\mu_n + M_{i,N} + 2M^{1/2}_{i,N} \Big[\underline{\lim}_n \int_X \Big(\frac{\partial f_n}{\partial x_i}\Big)^2 \, d\mu_n\Big]^{1/2},$$

where

$$M_{i,N} = \sup_n \int_X II^2_{i,n,N} \, d\mu_n.$$

In addition, we get that $\mathbb{E}^N_\mu f$ belongs to the maximal domain of definition (we note that since $\|\sqrt{\rho^N}\|_{W^{2,1}(\gamma)} < \infty$, it coincides with the minimal one) of the finite dimensional gradient Dirichlet form associated with the measure μ^N.

Now let us show that $f \in W^{2,1}(\mu)$. The sequence $\frac{\partial \mathbb{E}^N_\mu f}{\partial x_i}$ is bounded in $L^2(\mu)$ according to (9). Hence one can extract a subsequence (denoted again by $\frac{\partial \mathbb{E}^N_\mu f}{\partial x_i}$), which converges weakly to $F \in L^2(\mu)$. Take a smooth cylindrical test function $\varphi = \varphi(x_1, \cdots, x_m)$. One obtains for $i \le m$

$$\lim_N \int_X \varphi \frac{\partial \mathbb{E}^N_\mu f}{\partial x_i} \rho \, d\gamma = \int_X \varphi F \rho \, d\gamma.$$

Integrating by parts we get

$$\int_X \varphi \frac{\partial \mathbb{E}^N_\mu f}{\partial x_i} \rho \, d\gamma = -\int_X \mathbb{E}^N_\mu(f)\big(\varphi_{x_i} - x_i\varphi\big)\rho \, d\gamma - \int_X \mathbb{E}^N_\mu(f)\varphi\rho_{x_i} \, d\gamma.$$

Note that the functions $\mathbb{E}^N_\mu f$ are uniformly bounded and tends γ-a.e. to f. Hence, we get in the limit

$$\int_X \varphi F \rho \, d\gamma = -\int_X f\big(\varphi_{x_i} - x_i\varphi\big)\rho \, d\gamma - \int_X f\varphi\rho_{x_i} \, d\gamma.$$

This implies that $f \in W^{2,1}(\mu)$ and $\frac{\partial f}{\partial x_i} = F$. By the property of weak convergence we get from (8)

$$\int_X \Big(\frac{\partial f}{\partial x_i}\Big)^2 \rho \, d\gamma = \lim_N \int_X \Big(\frac{\partial \mathbb{E}^N_\mu f}{\partial x_i}\Big)^2 \rho \, d\gamma \le \lim_N \underline{\lim}_n \int_X \Big(\frac{\partial \mathbb{E}^N_\mu f_n}{\partial x_i}\Big)^2 \, d\mu_n$$

Hence by (9) and (2)

$$\int_X \Big(\frac{\partial f}{\partial x_i}\Big)^2 \rho \, d\gamma \le \underline{\lim}_n \int_X \Big(\frac{\partial f_n}{\partial x_i}\Big)^2 \rho_n \, d\gamma.$$

Since this inequality holds for every i, we get

$$\int_X |\nabla_H f|^2 \rho \, d\gamma = \sum_{i=1}^{\infty} \int_X \Big(\frac{\partial f}{\partial x_i}\Big)^2 \rho \, d\gamma \le \sum_{i=1}^{\infty} \underline{\lim}_n \int_X \Big(\frac{\partial f_n}{\partial x_i}\Big)^2 \rho_n \, d\gamma$$

$$\le \underline{\lim}_n \sum_{i=1}^{\infty} \int_X \Big(\frac{\partial f_n}{\partial x_i}\Big)^2 \rho_n \, d\gamma = \underline{\lim}_n \int_X |\nabla_H f_n|^2 \rho_n \, d\gamma < \infty.$$

The proof of (M1) for a sequence of uniformly bounded functions $\{f_n\}$ is complete.

Now let us consider the general case. Let $\{f_n\}$ be a sequence of functions such that $f_n \to f$ weakly in $\mathcal{H}$ and $\sup_n \mathcal{E}_n(f_n) < \infty$. Note that

$$\int_{|f_n|\rho_n > K} |f_n|\rho_n \, d\gamma \le \int_{|f_n|>\sqrt{K}} |f_n|\rho_n \, d\gamma + \int_{\rho_n > \sqrt{K}} |f_n|\rho_n \, d\gamma$$
$$\le \frac{\int_X |f_n|^2 \rho_n \, d\gamma}{\sqrt{K}} + \Bigl(\int_X |f_n|^2 \rho_n \, d\gamma\Bigr)^{1/2} \Bigl(\int_{\rho_n > \sqrt{K}} \rho_n \, d\gamma\Bigr)^{1/2}.$$

Since $\{\rho_n\}$ is weakly compact in $L^1(\gamma)$, we get that $|f_n|\rho_n$ is uniformly integrable, hence weakly compact in $L^1(\gamma)$. By the standard subsequence arguments $f_n\rho_n \to f\rho$ weakly in $L^1(\gamma)$.

Let us consider the cutoff functions $f_{n,N} = \bigl(f_n \wedge N\bigr) \vee (-N)$, where $N \in \mathbb{N}$. Extract some $\mathcal{H}$-weakly convergent subsequence $f_{n,N} \to f^N$ (denoted by the same index). By the same reason as above $f_{n,N}\rho_n \to f^N\rho$ weakly in $L^1(\gamma)$. Representing every $f_{n,N}$ as the difference of the positive and non-positive part $f_{n,N} = f^+_{n,N} - f^-_{n,N}$ one can assume that

$$f^+_{n,N} \to f_{1,N} \quad \text{and} \quad f^-_{n,N} \to f_{2,N}$$

$\mathcal{H}$-weakly to some non-negative functions $f_{1,N}$, $f_{2,N}$. Obviously,

$$0 \le f_{1,N} \le f^+ \wedge N \quad \text{and} \quad 0 \le f_{2,N} \le f^- \wedge N.$$

Hence $|f^N| \le |f| \wedge N$. By the above result and the contraction properties of $\mathcal{E}_n$ (see [18])

$$\mathcal{E}(f^N) \le \underline{\lim}_n \mathcal{E}_n(f_{n,N}) \le \underline{\lim}_n \mathcal{E}_n(f_n). \tag{10}$$

It remains to show that $\mathcal{E}(f) \le \underline{\lim}_N \mathcal{E}(f^N)$. If we prove that $f^N \to f$ weakly in $L^2(\mu)$ (at least for some subsequence), the proof will be complete. Since $|f^N| \le f$ and $\int_X f^2\rho \, d\gamma < \infty$, one can choose a $L^2(\mu)$-weakly convergent subsequence $f^N \to \tilde{f}$. By Komlós theorem (see [4], Theorem 4.7.23) for every $\varepsilon > 0$ there exists $X_\varepsilon \subset X$ such that $\gamma(X \setminus X_\varepsilon) \le \varepsilon$ and (for an appropriate subsequence which we denote by the same index)

$$\int_{X_\varepsilon} \Bigl| \frac{f_1\rho_1 + \cdots + f_n\rho_n}{n} - f\rho \Bigr| d\gamma \to 0$$

and

$$\int_{X_\varepsilon} \Bigl| \frac{f_{1,N}\rho_1 + \cdots + f_{n,N}\rho_n}{n} - f^N\rho \Bigr| d\gamma \to 0.$$

Note that

$$\int_X \Bigl| \frac{f_1\rho_1 + \cdots + f_n\rho_n}{n} - \frac{f_{1,N}\rho_1 + \cdots + f_{n,N}\rho_n}{n} \Bigr| d\gamma \le \frac{1}{n}\sum_{i=1}^n \int_X \Bigl| f_i - f_{i,N} \Bigr| \rho_i \, d\gamma$$
$$\le \frac{1}{n}\sum_{i=1}^n \int_{|f_i| \ge N} |f_i|\rho_i \, d\gamma \le \frac{1}{N} \sup_i \int_X f_i^2 \rho_i \, d\gamma.$$

Hence

$$\lim_N \int_{X_\varepsilon} |f - f^N|\rho \, d\gamma \le \lim_N \Bigl(\frac{1}{N} \sup_i \int_X f_i^2 \rho_i \, d\gamma \Bigr) = 0.$$

Hence $f^N \to f$ in measure. This implies that $\tilde{f} = f$. The proof of (M1) is complete.

The proof of condition (M2) is much easier and follows from the Markov uniqueness property of the limiting form and Lemma 2.8 of [12].

□

Remark 2.5. *Since* $\frac{\partial_{e_i}\rho_n}{\rho_n} \in L^2(\mu_n)$*, we get by the martingale property of conditional expectations that for every* n

$$\lim_N \int_X \Bigl(\frac{\partial_{e_i}\rho_n}{\rho_n} - \mathbb{E}^N_{\mu_n}\Bigl[\frac{\partial_{e_i}\rho_n}{\rho_n}\Bigr]\Bigr)^2 d\mu_n = 0.$$

Assumption 3) of Theorem 2.4 requires the uniform convergence in n*. Note that*

$$\int_X \Bigl(\mathbb{E}^N_{\mu_n}\Bigl[\frac{\partial_{e_i}\rho_n}{\rho_n}\Bigr]\Bigr)^2 \rho_n\, d\gamma = 4\int_X \Bigl(\partial_{e_i}\sqrt{\mathbb{E}^N_{\gamma}(\rho_n)}\Bigr)^2 d\gamma$$

if $N > i$*.*

Corollary 2.6. *Suppose that conditions 1)-2) of Theorem 2.4 hold. Suppose in addition that for every* $i \in \mathbb{N}$

$$\lim_{M\to\infty} \sup_{n\geq M} \int_X \frac{\bigl(\partial_{e_i}\rho_n\bigr)^2}{\rho_n}\, d\gamma = 0$$

(for example $\sqrt{\rho_n} \to \sqrt{\rho}$ *in* $W^{2,1}(\gamma)$*). Since*

$$\int_X \Bigl(\frac{\partial_{e_i}\rho_n}{\rho_n} - \mathbb{E}^N_{\mu_n}\Bigl[\frac{\partial_{e_i}\rho_n}{\rho_n}\Bigr]\Bigr)^2 d\mu_n \leq \int_X \frac{\bigl(\partial_{e_i}\rho_n\bigr)^2}{\rho_n}\, d\gamma,$$

we get by the previous remark that condition 3) of Theorem 2.4 holds and $\mathcal{E}_n \to \mathcal{E}$ *Mosco.*

In the following corollary we prove the convergence of the associated stochastic processes on a Hilbert space under additional assumption $\sup_n \|\sqrt{\rho_n}\|_{W^{2,1}(\gamma)} < \infty$, which is stronger that assumption 2) of Theorem 2.4.

Corollary 2.7. *Suppose that* X *is a Hilbert space and the assumptions of Theorem 2.4 are fulfilled. Suppose in addition, that* $\sup_n \|\sqrt{\rho_n}\|_{W^{2,1}(\gamma)} < \infty$*. Then* $\mathcal{P}_{\mu_n} \to \mathcal{P}_\mu$ *weakly.*

Proof. The tightness of $\{\mathcal{P}_{\mu_n}\}$ was shown in [17], Theorem 3.1. The assumptions of Theorem 3.1, [17] are obviously fulfilled except of

$$\sup_n \int_X \|x\|^2_X\, d\mu_n < \infty, \tag{11}$$

where $\|\cdot\|_X$ is the Hilbert norm on X. Let us show that (11) holds indeed. By the Young inequality

$$\varepsilon\|x\|^2_X\rho_n \leq e^{\varepsilon\|x\|^2_X - 1} + \rho_n\log\rho_n.$$

By the Fernique theorem (see [3]) $e^{\varepsilon\|x\|^2_X} \in L^1(\gamma)$ for a small enough ε and

$$\sup_n \int_X \rho_n \log\rho_n\, d\gamma < \sup_n \int_X \frac{\|\nabla_H\rho_n\|^2_H}{2\rho_n}\, d\gamma < \infty$$

by the log-Sobolev inequality. This gives (11).

Now let us show that every weak limiting point of $\{\mathcal{P}_{\mu_n}\}$ coincides with $\mathcal{P}_\mu$. To this end let us fix some $\mathcal{F}C_0^\infty$-functions $f_0, \cdots, f_m$. Then

$$\int f_0(X^n_0)f_1(X^n_{t_1})f_2(X^n_{t_1+t_2})\cdots f_m(X^n_{t_1+\cdots+t_m})\, dP_{\mu_n}$$
$$= \int f_0 T^n_{t_1}(f_1 T^n_{t_2}(f_2\cdots T^n_{t_m}(f_m))\cdots)\, d\mu_n,$$

where every T_t^n is associated with $\mathcal{E}_n$. By Theorem 2.4 and properties of Mosco convergence $T_t^n \to T_t$ strongly in $\mathcal{H}$. We note that $\varphi f_n \to \varphi f$ for every $\varphi \in \mathcal{F}C_0^\infty$ if $f_n \to f$ $\mathcal{H}$-strongly and $|f_n| \le C$. This follows easily from the weak $L^1(\gamma)$-convergence of ρ_n to ρ and basic properties of $\mathcal{H}$-convergence. Since every T_t^n is sub-Markovian, one gets

$$\int f_0(X_0^n) f_1(X_{t_1}^n) f_2(X_{t_1+t_2}^n) \cdots f_m(X_{t_1+\cdots+t_m}^n)\, dP_{\mu_n}$$
$$\to \int f_0(X_0) f_1(X_{t_1}) f_2(X_{t_1+t_2}) \cdots f_m(X_{t_1+\cdots+t_m})\, dP_{\mu}.$$

The proof is complete. □

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