Stochastic Generalized Porous Media and Fast Diffusion Equations *

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Abstract

We present a generalization of Krylov-Rozovskii's result on the existence and uniqueness of solutions to monotone stochastic differential equations. As an application, the stochastic generalized porous media and fast diffusion equations are studied for σ-finite reference measures, where the drift term is given by a negative definite operator acting on a time-dependent function, which belongs to a large class of functions comparable with the so-called N-functions in the theory of Orlicz spaces.

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1 Introduction

The main aim of this paper is to solve stochastic partial differential equations (SPDE) of "porous media" type, i.e.

$$(1.1)\qquad \mathrm{d}X_t = \big(L\Psi(t,X_t) + \Phi(t,X_t)\big)\,\mathrm{d}t + B(t,X_t)\,\mathrm{d}W_t\ ,$$

where L is a partial (or pseudo) differential operator of order (less than or equal to) two, so e.g. $L = \Delta$ (or $L = -(-\Delta)^\alpha$, $\alpha \in (0,1]$) on $\mathbb{R}^d$ or an open subset thereof (see Example 3.3 below). The maps $\Psi, \Phi : [0,T] \times \mathbb{R} \to \mathbb{R}$ (possibly also random) are to fulfill certain monotonicity conditions (cf. **(A1)**, **(A2)** in Section 3 below). B is a Hilbert-Schmidt operator valued Lipschitz map and W_t a Brownian motion on a suitable Hilbert space of generalized functions (see below for details). A typical example for Ψ is

$$\Psi(s) = \sum_{i=1}^{m} \alpha_i \operatorname{sign}(s)\,|s|^{r_i}\ , \quad s \in \mathbb{R}\ ,$$

with pairwise distinct $r_1, \dots, r_m > 0$ and $\alpha_i \in (0,\infty)$. For $r_i > 1$ this corresponds to the classical porous medium equation and for $r_i \in (0,1)$ to the so-called fast diffusion equation. Their behaviour is quite different. E.g. in the deterministic case, when L is the Dirichlet Laplacian on an open bounded domain in $\mathbb{R}^d$, it is well-known that if $r_i > 1$ the solution decays algebraically fast in t, and if $r_i \in (0,1)$, it decays to zero in finite time. We refer to [1, 2, 11] and the references therein, also for historical remarks.

In recent years, the stochastic version of the porous medium equation has been studied intensively, see [7] for the existence, uniqueness and long-time behavior of some stochastic generalized porous media equations with finite reference measures, [13] for the stochastic porous media equation on $\mathbb{R}^d$ where the reference (Lebesgue) measure is infinite and [3, 16] for the analysis of the corresponding Kolmogorov equations. See also [19] for large deviations for a class of generalized porous media equations.

Our analysis of (1.1) is based on the so-called variational approach and requires monotonicity assumptions on the coefficients. More precisely, we extend classical (impressive) work by N.V. Krylov and B. Rozovskii [15] (see in particular Theorems II.2.1 and II.2.2 there and also the pioneering work by E. Pardoux [16, 17]) to make it applicable to SPDE (1.1) for very general non-linearities Ψ and Φ. The corresponding abstract result is proved in Section 2 below (cf. in particular Theorem 2.1) where we also describe the new framework. In addition, we include a detailed proof of the crucial Itô formula, proved in [15, Theorem I.3.1], adapted to our more general situation, in the Appendix.

One of the main points in applying the variational approach is to find a suitable Gelfand triple

$$V \subset H \subset V^*\ ,$$

where V is a separable reflexive Banach space and V^* its dual, and H is a Hilbert space. In the case of (1.1) it turns out that the appropriate Hilbert space is just the Green space of the operator L, i.e. the dual of the zero-order Dirichlet space determined by L (see (3.1) below and, in particular, Proposition 3.1). For this to be well-defined we need that the semigroup generated by L is transient. If $L = \Delta$ on a bounded open subset Λ of $\mathbb{R}^d$ with Dirichlet boundary conditions, then $H = H^{-1}(\Lambda)$ (i.e. the dual space of the Sobolev space $H_0^{1,2}(\Lambda)$). But also the case $L = -(-\Delta)^\alpha$, $\alpha \in (0, \frac{d}{2}) \cap (0, 1]$, $\Lambda = \mathbb{R}^d$ is included (hence the cases studied in [13] are all covered by our results). Apart from this, the main novelty of our applications is that (unlike in [7]) we can include the case where $r_i < 1$. Shortly speaking, the condition on Ψ is that the function

$$s \mapsto s\,\Psi(s)\ , \quad s \in \mathbb{R},$$

is (equal to or appropriately) comparable to a Young function N, and Φ should be such that we can treat it as a kind of perturbation (see conditions **(A1)**, **(A2)**, respectively, in Section 3 below). Then the appropriate choice of V is

$$V := L_N \cap H\ ,$$

where L_N is the Orlicz space determined by N (cf. e.g. [18]). All details are contained in Section 3. We refer in particular to Theorem 3.9 there, which summarizes the main results.

Finally, we mention that standardly by the variational approach one also obtains information about the qualitative behaviour of solutions. The corresponding results in our case are stated in Proposition 2.2 below.

2 Monotone stochastic equations

We refer to [15] for the extensive literature on the subject and in particular, to the pioneering work in the stochastic case due to Pardoux [16, 17]. Let G be a real separable Hilbert space, W_t a cylindrical Brownian motion on G, i.e. for an ONB $\{g_1, g_2, \cdots\}$ of G, $W_t := \big(B_t^i\, g_i\big)_{i\in\mathbb{N}}$ for a sequence of independent one-dimensional Brownian motions $\{B_t^i\}_{i\geq 1}$ on a complete filtered probability spaces $(\Omega, \mathscr{F}, \mathscr{F}_t; P)$. The filtration is assumed to be right continuous. Next, let $(H, \langle\ ,\ \rangle_H)$ be another real separable Hilbert space. Let V be a reflexive Banach space such that the embedding $V \subset H$ is dense and continuous, i.e. V is dense in H w.r.t. $\|\cdot\|_H$ and $\|v\|_H \leq c\|v\|_V$ for some constant $c > 0$ and all $v \in V$. Let V^* be the dual space of V and ${}_{V^*}\langle\ ,\ \rangle_V$ denotes the corresponding dualization. Identifying H with its dual H^* we have

$$V \subset H \equiv H^* \subset V^*$$

continuously and densely. Note that, in particular, V^* is then also separable, hence so is V. Furthermore, ${}_{V^*}\langle\ ,\ \rangle_V|_{H\times V} = \langle\ ,\ \rangle_H$. Let $\mathscr{L}_{HS}(G; H)$ denote the set of all real Hilbert-Schmidt linear operators from G to H, which is a real separable Hilbert space under the inner product

$$\langle B_1, B_2\rangle_{\mathscr{L}_{HS}} := \sum_{i\geq 1}\langle B_1 g_i, B_2 g_i\rangle_H.$$

We consider the following stochastic equation on H:

$$\mathrm{d}X_t = A(t, X_t)\mathrm{d}t + B(t, X_t)\mathrm{d}W_t, \tag{2.1}$$

where

$$A : [0,\infty)\times V\times\Omega \to V^*, \quad B : [0,\infty)\times V\times\Omega \to \mathscr{L}_{HS}(G; H)$$

are progressively measurable, i.e. for any $t \geq 0$, these mappings restricted to $[0, t]\times V\times\Omega$ are measurable w.r.t. $\mathscr{B}([0, t])\times\mathscr{B}(V)\times\mathscr{F}_t$, where $\mathscr{B}(\cdot)$ is the Borel σ-field for a topological space. Below, writing $A(t, v)$ we mean the mapping $\omega \mapsto A(t, v, \omega)$; analogously for $B(t, v)$.

To solve equation (2.1), we need some assumptions. Let $T > 0$ be fixed. Let $(K, \|\cdot\|_K)$ be a real reflexive Banach space such that $L^p([0, T]\times\Omega \to V; \mathrm{d}t\times P) \subset K \subset L^1([0, T]\times\Omega \to V; \mathrm{d}t\times P)$ densely and continuously for some $p \in (1,\infty)$. By Lemma 2.4 below, its dual space is isometric to $(K^*, \|\cdot\|_{K^*})$, the completion of $L^\infty([0, T]\times\Omega \to V^*; \mathrm{d}t\times P)$ w.r.t.

$$\|z^*\|_{K^*} := \sup_{\|z\|_K\leq 1} \mathbb{E}\int_0^T {}_{V^*}\langle z_t^*, z_t\rangle_V\,\mathrm{d}t. \tag{2.2}$$

So, K^* is reflexive too and $K^* \subset L^{p/(p-1)}([0, T]\times\Omega \to V^*; \mathrm{d}t\times P)$ continuously and densely. Furthermore, ${}_{K^*}\langle Y, z\rangle_K = {}_{L^{p/(p-1)}}\langle Y, z\rangle_{L^p}$ if $Y \in K^*$ and $z \in L^p([0, T]\times\Omega \to V; \mathrm{d}t\times P)$.

Generalizing the framework in [15, Chapter 2], we make the following assumptions, where the first is for the reference spaces K and K^* and the remaining ones are for A and B.

(K) There exist a continuous function $R : V \to [0,\infty)$ with $R(x) = R(-x), x \in V$, and $R(0) = 0$, and two locally bounded real functions W_1, W_2 on $[0,\infty)$ with $W_1(r), W_2(r) \to \infty$ as $r \to \infty$ such that:
(i) For any sequence $z^{(n)} \in K, n \geq 1$, $\|z^{(n)}\|_K \to 0$ if and only if $\mathbb{E}\int_0^T R(z_t^{(n)})\mathrm{d}t \to 0$.
(ii) For any $z \in K$, $W_1\big(\mathbb{E}\int_0^T R(z_t)\mathrm{d}t\big) \leq \|z\|_K \leq W_2\big(\mathbb{E}\int_0^T R(z_t)\mathrm{d}t\big)$, where $W_1(\infty) := W_2(\infty) := \infty$.
(iii) There exists a constant $C > 0$ such that $R(x + y) \leq C\big(R(2x) + R(2y)\big)$ for all $x, y \in V$.
(iv) For any $h \in L^\infty([0, T]\times\Omega; \mathrm{d}t\times P), z \in K$, we have $hz \in K$ and $\|hz\|_K \leq \|h\|_\infty\|z\|_K$.

(H1) Hermicontinuity: for any $u, v, x \in V, \omega \in \Omega$ and any $t \in [0, T]$, the mapping

$$\mathbb{R} \ni \lambda \mapsto {}_{V^*}\langle A(t, u + \lambda v, \omega), x\rangle_V$$

is continuous.

(H2) Weak monotonicity: there exists a constant $c \in \mathbb{R}$ such that

$$2\ {}_{V^*}\langle A(\cdot, u) - A(\cdot, v), u - v\rangle_V + \|B(\cdot, u) - B(\cdot, v)\|^2_{\mathscr{L}_{HS}} \le c\|u - v\|^2_H \text{ for all } u, v \in V$$

holds on $[0, T] \times \Omega$.

(H3) Coercivity: there exist constants $c_1, c_2 > 0$ and an $\mathscr{F}_t$-adapted process $f \in L^1([0, T] \times \Omega; \mathrm{d}t \times P)$ such that

$$2\,{}_{V^*}\langle A(t, v), v\rangle_V + \|B(t, v)\|^2_{\mathscr{L}_{HS}} \le c_1\|v\|^2_H - c_2 R(v) + f_t$$

holds on Ω for all $v \in V$ and $t \in [0, T]$, where R is as in **(K)**.

(H4) There exists $c_3 > 0$ and an $\mathscr{F}_t$-adapted process $g \in L^1([0, T] \times \Omega, \mathrm{d}t \times P)$ such that

$$|\,{}_{V^*}\langle A(t, v), u\rangle_V| \le g_t + c_3(R(v) + R(u)) \text{ on } \Omega \text{ for all } t \in [0, T], u, v \in V,$$

where R is as in **(K)**.

Remark 2.1. (1) **(H4)** together with **(K)**(ii) implies that for all $z \in K$ the map

$$\tilde{z} \mapsto \mathbb{E}\int_0^T {}_{V^*}\langle A(t, z_t), \tilde{z}_t\rangle_V \mathrm{d}t, \quad \tilde{z} \in K,$$

is in K^* such that $K \ni z \mapsto \|A(\cdot, z)\|_{K^*}$ is bounded on bounded sets in K.

(2) It easily follows from **(H3)** and **(H4)** that for all $v \in V$ on Ω

$$\|B(t, v)\|^2_{\mathscr{L}_{HS}} \le c_1\|v\|^2_H + f_t + 2g_t + 4c_3 R(v), \quad t \in [0, T].$$

In particular, the function $K \in z \mapsto \mathbb{E}\int_0^T \|B(t, z_t)\|^2_{\mathscr{L}_{HS}}\mathrm{d}t$ is by **(K)**(ii) bounded on bounded sets in K, which are also bounded in $L^2([0, T] \times \Omega \to H; \mathrm{d}t \times P)$.

(3) We observe that $z \in K$ if and only if $z \in L^1([0, T] \times \Omega \to V; \mathrm{d}t \times P)$ with $\mathbb{E}\int_0^T R(z_t)\mathrm{d}t < \infty$. The the necessity is trivial by **(K)**(ii). On the other hand, for $z \in L^1([0, T] \times \Omega \to V; \mathrm{d}t \times P)$ with $\mathbb{E}\int_0^T R(z_t)\mathrm{d}t < \infty$ we have

$$K \ni z^{(n)} := z1_{\{|z|_V \le n\}} \to z \ \text{ in } L^1([0, T] \times \Omega \to V; \mathrm{d}t \times P).$$

We claim that $\{z^{(n)}\}$ is a Cauchy sequence in K so that it also converges to z in K. Otherwise, there exist $\varepsilon > 0$ and a subsequence $n_k \to \infty$ such that

$$\sup_{m > n_k} \|z^{(n_k)} - z^{(m)}\|_K \ge 2\varepsilon, \quad k \ge 1.$$

Then for any $k \ge 1$

$$m_k := \inf\{m > n_k : \|z^{(n_k)} - z^{(m)}\|_K \ge \varepsilon\} < \infty.$$

Letting $\tilde{z}^{(k)} = z^{(n_k)} - z^{(m_k)}$ we have $\|\tilde{z}^{(k)}\|_K \geq \varepsilon$ for all $k \geq 1$, which is contradictive to **(K)**(i) since $R(0) = 0$ and the dominated convergence theorem imply that

$$\limsup_{k\to\infty} \mathbb{E} \int_0^T R(\tilde{z}_t^{(k)}) \mathrm{d}t \leq \lim_{k\to\infty} \mathbb{E} \int_0^T R(z_t) 1_{\{n_k < |z_t|_V\}} \mathrm{d}t = 0.$$

(4) Let $Y \in K^*$, $z \in K$ and $h \in L^\infty\big([0,T]\times\Omega; \mathrm{d}t\times P\big)$. Let $z^{(n)} \in L^p\big([0,T]\times\Omega \to V; \mathrm{d}t\times P\big)$, $n \in \mathbb{N}$, such that $z^{(n)} \to z$ in K, hence $hz^{(n)} \to hz$ in K by **(K)**(iv). Assume that hY (which a priori is only in $L^{p/(p-1)}\big([0,T]\times\Omega \to V^*; \mathrm{d}t\times P\big)$) is in $L^\infty\big([0,T]\times\Omega \to V^*; \mathrm{d}t\times P\big)$. Then

$$ {}_{K^*}\langle Y, hz\rangle_K = \mathbb{E} \int_0^T {}_{V^*}\langle h_t Y_t, z_t\rangle_V \; \mathrm{d}t \;, \tag{2.3}$$

since

$$\begin{aligned} {}_{K^*}\langle Y, hz\rangle_K &= \lim_{n\to\infty} {}_{K^*}\langle Y, hz^{(n)}\rangle_K = \lim_{n\to\infty} \mathbb{E} \int_0^T {}_{V^*}\langle Y_t, h_t z_t^{(n)}\rangle_V \; \mathrm{d}t \\ &= \lim_{n\to\infty} \mathbb{E} \int_0^T {}_{V^*}\langle h_t Y_t, z_t^{(n)}\rangle_V \; \mathrm{d}t = \mathbb{E} \int_0^T {}_{V^*}\langle h_t Y_t, z_t\rangle_V \; \mathrm{d}t \;, \end{aligned}$$

since $z^{(n)} \to z$ in K, hence in $L^1\big([0,T]\times\Omega \to V; \mathrm{d}t\times P\big)$.

(5) For $Y \in K^*$ we have

$$ {}_{K^*}\langle Y, z\rangle_K = \mathbb{E} \int_0^T {}_{V^*}\langle Y_s, z_s\rangle_V \mathrm{d}s \leq \mathbb{E} \int_0^T |\, {}_{V^*}\langle Y_s, z_s\rangle_V| \mathrm{d}s \leq \|Y\|_{K^*} \|z\|_K, \quad z \in K. \tag{2.4}$$

Indeed, let $\xi :=_{V^*} \langle Y, z\rangle_V$ and for $N \geq 1$

$$\begin{aligned} z^{(N)} &:= \mathrm{sgn}(\xi) 1_{\{\|z\|_V \leq N, \|Y\|_{V^*} \leq N\}} z \in L^p([0,T]\times\Omega \to V; \mathrm{d}t\times P), \\ Y^{(N)} &:= 1_{\{\|z\|_V \leq N, \|Y\|_{V^*} \leq N\}} Y \in L^{p/(p-1)}([0,T]\times\Omega \to V^*; \mathrm{d}t\times P). \end{aligned}$$

Then for all $N \geq 1$

$$\begin{aligned} &\mathbb{E} \int_0^T |\xi_t| 1_{\{\|z\|_V \leq N, \|Y\|_{V^*} \leq N\}}(t) \mathrm{d}t = \mathbb{E} \int_0^T {}_{V^*}\langle Y_t^{(N)}, z_t^{(N)}\rangle_V \mathrm{d}t = {}_{K^*}\langle Y, z^{(N)}\rangle_K \\ &\leq \|Y\|_{K^*} \|z^{(N)}\|_K \leq \|Y\|_{K^*} \|z\|_K \end{aligned}$$

where we used (2.3) for the second equality and **(K)**(iv) in the last step. This implies (2.4) by letting $N \to \infty$.

(6) By [21, Proposition 26.4], **(H1)** and **(H2)** imply that for all $(t,\omega) \in [0,T]\times\Omega$ the map $u \mapsto A(t,u,\omega)$ is demicontinuous (i.e. $u_n \to u$ strongly in V implies $A(t,u_n,\omega) \to A(t,u,\omega)$ weakly in V^*). In particular, A is continuous if V is finite dimensional.

We remark that the assumptions made in [15, Chapter II] imply the present ones by taking $K := L^p([0,T]\times\Omega \to V; \mathrm{d}t\times P)$ for some $p > 1$, so that one has $K^* = L^{p/(p-1)}([0,T]\times\Omega \to V^*; \mathrm{d}t\times P)$, and $R := \|\cdot\|_V^p$, $W_1 = W_2 := |\cdot|^{1/p}$.

Definition 2.1. A continuous adapted process $\{X_t\}_{t\in[0,T]}$ on H is called a solution to (2.1), if $X \in L^2([0,T]\times\Omega \to H, \mathrm{d}t \times P)$, there exists a $\mathrm{d}t \times P$-version $\bar{X}$ of an element in K such that $X = \bar{X}$ $\mathrm{d}t \times P$-a.e., and P-a.s.

$$X_t = X_0 + \int_0^t A(s, \bar{X}_s)\mathrm{d}s + \int_0^t B(s, \bar{X}_s)\mathrm{d}W_s, \text{ for all } t \in [0,T].$$

Since the Bochner integral on general Banach spaces is only meaningful for pointwise (rather than a.e.) Banach space-valued functions, in Definition 2.1 we have to choose a V-valued progressively measurable version of X such that the right-hand side of the above formula makes sense.

Theorem 2.1. *Under* $(\mathbf{K}), (\mathbf{H1}), (\mathbf{H2}), (\mathbf{H3})$ *and* $(\mathbf{H4})$*, for any* $X_0 \in L^2(\Omega \to H, \mathscr{F}_0; P)$, (2.1) *has a unique solution and the solution satisfies* $\mathbb{E}\sup_{t\in[0,T]}\|X_t\|_H^2 < \infty$*. Moreover; if* $A(t,\cdot)(\omega)$ *and* $B(t,\cdot)(\omega)$ *are independent of* $t \in [0,T]$ *and* $\omega \in \Omega$*, then the solution is a Markov process.*

The uniqueness of the solution can be easily proved using the Itô formula for $\|X_t\|_H^2$ presented in Theorem 4.2 in the Appendix (see [15, Theorem I.3.2] for a special case). In fact, we have an even stronger statement formulated in the following proposition. Furthermore, in the case where A and B are independent of $t \in [0,T]$ and $\omega \in \Omega$, the Markov property can be easily proved by using the uniqueness as in [15].

Proposition 2.2. *Assume* $(\mathbf{K}), (\mathbf{H1}), (\mathbf{H2}), (\mathbf{H3})$ *and* $(\mathbf{H4})$*. Let* X *and* Y *be two solutions of* (2.1) *with* $X_0, Y_0 \in L^2(\Omega \to H, \mathscr{F}_0; P)$*. Let* c *be the* (*not necessarily positive*) *constant such that* $(\mathbf{H2})$ *holds. Then*

$$\mathbb{E}\|X_t - Y_t\|_H^2 \le \mathrm{e}^{ct}\mathbb{E}\|X_0 - Y_0\|_H^2, \quad t \in [0,T]. \tag{2.5}$$

Consequently, if moreover A *and* B *are independent of* $t \in [0,T]$ *and* $\omega \in \Omega$*, the semigroup* $(P_t)_{t\in[0,T]}$ *of the corresponding Markov process is a Feller semigroup with*

$$|P_tF(x) - P_tF(y)| \le \mathrm{e}^{ct/2}Lip(F)\|x-y\|_H, \quad t \in [0,T], x, y \in H,$$

where F *is an* H*-Lipschitz function with* $Lip(F)$ *the Lipschitz constant. In particular, if our assumptions hold for each* $T > 0$ *with* $\lambda := -c > 0$ *independent of* T*, then* (P_t) *has a unique invariant probability measure* μ *with* $\mu(\|\cdot\|_H^2) < \infty$ *and* (P_t) *converges exponentially fast to* μ*; more precisely, for any* H*-Lipschitz function* F*,*

$$|P_tF(x) - \mu(F)|^2 \le Lip(F)^2\mathrm{e}^{-\lambda t}\mu(\|x - \cdot\|_H^2), \quad x \in H.$$

Proof. By **(H2)** and the Itô formula (4.3) in the Appendix applied to $X_t - Y_t$, we have

$$
\begin{aligned}
&\mathbb{E}\|X_t - Y_t\|_H^2 - \mathbb{E}\|X_0 - Y_0\|_H^2 \\
&= 2\mathbb{E}\int_0^t \big\{ {}_{V^*}\langle A(s,\bar{X}_s) - A(s,Y_s), \bar{X}_s - \bar{Y}_s\rangle_V + \|B(s,\bar{X}_s) - B(s,\bar{Y}_s)\|_{\mathscr{L}_{HS}}^2 \big\}\mathrm{d}s \\
&\le c\int_0^t \mathbb{E}\|X_s - Y_s\|_H^2 \mathrm{d}s.
\end{aligned}
$$

This implies the first result immediately by Gronwall's lemma. The remainder of the proof is the same as that of (3) and (4) in [7, Theorem 1.3]. □

To prove the existence, we will construct a solution by the classical Galerkin method of finite-dimensional approximations as made in [15] (see also [7] for a special case).

Let $\{e_1, \cdots, e_n, \cdots\} \subset V$ be an ONB of H. Let $H_n := \mathrm{span}\{e_1, \cdots, e_n\}, n \ge 1$, and $P_n : V^* \to H_n$ is defined by

$$
P_n y := \sum_{i=1}^n {}_{V^*}\langle y, e_i\rangle_V e_i, \quad y \in V^*. \tag{2.6}
$$

Clearly, $P_n|_H$ is just the orthogonal projection onto H_n in H. Set

$$
W_t^{(n)} := \sum_{i=1}^n \langle W_t, g_i\rangle_G g_i = \sum_{i=1}^n B_t^i\, g_i.
$$

For each finite $n \ge 1$, we consider the following stochastic equation on H_n :

$$
\mathrm{d}\langle X_t^{(n)}, e_j\rangle_H = {}_{V^*}\langle A(t, X_t^{(n)}), e_j\rangle_V \mathrm{d}t + \langle B(t, X_t^{(n)})\mathrm{d}W_t^{(n)}, \mathrm{e}_j\rangle_H, \quad 1 \le j \le n, \tag{2.7}
$$

where $X_0^{(n)} := P_n X_0$. It is easily seen that we are in the situation covered by [14, Theorem 1.2] which implies that (2.7) has a unique continuous strong solution. Let

$$
J := L^2([0,T] \times \Omega \to \mathscr{L}_{HS}(G; H); \mathrm{d}t \times P). \tag{2.8}
$$

To construct the solution to (2.1), we need the following lemma.

Lemma 2.3. *Under the assumptions in Theorem* 2.1*, we have*

$$
\|X^{(n)}\|_K + \|A(\cdot, X^{(n)})\|_{K^*} + \sup_{t\in[0,T]} \mathbb{E}\|X_t^{(n)}\|_H^2 \le C
$$

for some constant $C > 0$ *and all* $n \ge 1$.

Proof. By Itô's formula and **(H3)**, we have

$$\begin{aligned}\mathrm{d}\|X_t^{(n)}\|_H^2 &= 2\ {}_{V^*}\langle A(t,X_t^{(n)}),X_t^{(n)}\rangle_V\mathrm{d}t + \|P_nB(t,X_t^{(n)})\|^2_{\mathscr{L}_{HS}(G;H)}\mathrm{d}t + \mathrm{d}M_t^{(n)}\\ &\le [-c_2R(X_t^{(n)}) + c_1\|X_t^{(n)}\|_H^2 + f_t]\mathrm{d}t + \mathrm{d}M_t^{(n)}.\end{aligned}$$

for a local martingale $M_t^{(n)}$. This implies

$$\begin{aligned}&-\mathbb{E}\|X_0\|_H^2 \le \mathbb{E}\mathrm{e}^{-c_1t}\|X_t^{(n)}\|_H^2 - \mathbb{E}\|X_0^{(n)}\|_H^2\\ &\le c_3 - c_4\mathbb{E}\int_0^t R(X_s^{(n)})\mathrm{d}s, \quad t\in[0,T]\end{aligned} \tag{2.9}$$

for some constants $c_3, c_4 > 0$. Then the proof is completed by **(K)** and **(H4)**. □

Lemma 2.4. *K^* is isometric to the dual space K' of K.*

Proof. Since $L^\infty([0,T]\times\Omega\to V^*;\mathrm{d}t\times P)$ is the dual space of $L^1([0,T]\times\Omega\to V;\mathrm{d}t\times P)$, and since the embedding $K\subset L^1([0,T]\times\Omega\to V;\mathrm{d}t\times P)$ is dense and continuous, the embedding

$$L^\infty([0,T]\times\Omega\to V^*;\mathrm{d}t\times P)\subset K'$$

is continuous too, but also dense by the Hahn-Banach Theorem because K is reflexive. Therefore, $K' = K^*$ since K' is complete and contains $L^\infty([0,T]\times\Omega\to V^*;\mathrm{d}t\times P)$. □

Proof of Theorem 2.1. We first recall that the uniqueness of the solution is included in Proposition 2.2. Hence, when A and B are independent of the time t and $\omega\in\Omega$, the Markov property follows immediately as in the proof of [15, Theorem II.2.4]. So, we only need to prove the existence, $\mathbb{E}\sup_{t\in[0,T]}\|X_t\|_H^2<\infty$ and the continuity of X_t in H.

(a) By the reflexivity of K and Lemmas 2.3, 2.4, and Remark 2.1, we have, for a subsequence $n_k\to\infty$:

(i) $X^{(n_k)}\to\bar{X}$ weakly in K and weakly in $L^2([0,T]\times\Omega\to H;\mathrm{d}t\times P)$;

(ii) $A(\cdot,X^{(n_k)})\to Y$ weakly in K^* (hence weakly in $L^{p/(p-1)}([0,T]\times\Omega\to V^*;\mathrm{d}t\times P)$);

(iii) $B(\cdot,X^{(n_k)})\to Z$ weakly in J and hence $\int_0^\cdot B(s,X_s^{(n_k)})\mathrm{d}W_s^{(n_k)}\to\int_0^\cdot Z_s\mathrm{d}W_s$ weakly in $L^\infty([0,T]\to L^2(\Omega\to H;P);\mathrm{d}t)$ (equipped with the supremum norm).

Here the second part in (iii) follows since also $B(\cdot,X^{(n_k)})\tilde{P}_{n_k}\to Z$ weakly in J, where $\tilde{P}_n$ is the orthogonal projection onto $\mathrm{span}\{g_1,\cdots,g_n\}$ in G, since

$$\int_0^\cdot B(s, X_s^{(n_k)})\mathrm{d}W_s^{(n_k)} = \int_0^\cdot B(s, X_s^{(n_k)})\tilde{P}_{n_k}\mathrm{d}W_s$$

and since a bounded linear operator between two Banach spaces is trivially weakly continuous. Since the approximants are progressively measurable, so are $\bar{X}, Y$ and Z.

Thus, by the definition of $X^{(n)}$ we have (since V is separable) that

$$(2.10) \quad {}_{V^*}\langle \bar{X}_t, e\rangle_V = \langle X_0, e\rangle_H + \int_0^t {}_{V^*}\langle Y_s, e\rangle_V \mathrm{d}s + \int_0^t \langle Z_s \mathrm{d}W_s, e\rangle_H, \text{ for all } \ e \in V \text{ a.e.-d}t \times P.$$

Defining

$$(2.11) \qquad X_t := X_0 + \int_0^t Y_s \mathrm{d}s + \int_0^t Z_s \mathrm{d}W_s, \quad t \in [0, T],$$

we have $X = \bar{X}$ $\mathrm{d}t \times P$-a.e. We note here, that since $Y \in K^* \subset L^{p/(p-1)}([0,T] \times \Omega \to V^*; \mathrm{d}t \times P)$, its integral in (2.11) always exists as a V^*-valued Bochner integral and is continuous in t. Therefore, X is a V^*-valued continuous adapted process.

Hence Theorem 4.2 applies to X in (2.11), so X is continuous in H and $\mathbb{E}\sup_{t\le T}\|X_t\|_H^2 < \infty$.

Thus, it remains to verify that

$$(2.12) \qquad B(\cdot, \bar{X}) = Z, \quad A(\cdot, \bar{X}) = Y, \quad \text{a.e.-d}t \times P.$$

To this end, we first note that for any nonnegative $\psi \in L^\infty([0,T], \mathrm{d}t)$ it follows from (i) that

$$\begin{aligned}\mathbb{E}\int_0^T \psi(t)\|\bar{X}_t\|_H^2 \mathrm{d}t &= \lim_{k\to\infty}\mathbb{E}\int_0^T \langle \psi(t)\bar{X}_t, X_t^{(n_k)}\rangle_H \mathrm{d}t \\ &\le \left(\mathbb{E}\int_0^T \psi(t)\|\bar{X}_t\|_H^2 \mathrm{d}t\right)^{1/2} \liminf_{k\to\infty}\left(\mathbb{E}\int_0^T \psi(t)\|X_t^{(n_k)}\|_H^2 \mathrm{d}t\right)^{1/2} < \infty.\end{aligned}$$

Since $X = \bar{X}$ $\mathrm{d}t \times P$-a.e., this implies

$$(2.13) \qquad \liminf_{k\to\infty}\mathbb{E}\int_0^T \psi(t)\|X_t^{(n_k)}\|_H^2 \mathrm{d}t \ge \mathbb{E}\int_0^T \psi(t)\|X_t\|_H^2 \mathrm{d}t.$$

By (2.11) and the Itô formula (4.3) in the Appendix, using the elementary product rule we obtain that

$$(2.14) \qquad \mathbb{E}\mathrm{e}^{-ct}\|X_t\|_H^2 - \mathbb{E}\|X_0\|_H^2 = \mathbb{E}\int_0^t \mathrm{e}^{-cs}\big\{2\ {}_{V^*}\langle Y_s, \bar{X}_s\rangle_V + \|Z_s\|_{\mathscr{L}_{HS}}^2 - c\|X_s\|_H^2\big\}\mathrm{d}s.$$

Furthermore, for any $\phi \in K \cap L^2([0,T]\times\Omega \to H; \mathrm{d}t \times P)$,

$$
\begin{aligned}
&\mathbb{E}\mathrm{e}^{-ct}\|X_t^{(n_k)}\|_H^2 - \mathbb{E}\|X_0^{(n_k)}\|_H^2 \\
&= \mathbb{E}\int_0^t \mathrm{e}^{-cs}\big\{2\ {}_{V^*}\langle A(s,X_s^{(n_k)}), X_s^{(n_k)}\rangle_V + \|P_{n_k}B(s,X_s^{(n_k)})\tilde{P}_{n_k}\|_{\mathscr{L}_{HS}}^2 - c\|X_s^{(n_k)}\|_H^2\big\}\mathrm{d}s \\
&\le \mathbb{E}\int_0^t \mathrm{e}^{-cs}\big\{2\ {}_{V^*}\langle A(s,X_s^{(n_k)}), X_s^{(n_k)}\rangle_V + \|B(s,X_s^{(n_k)})\|_{\mathscr{L}_{HS}}^2 - c\|X_s^{(n_k)}\|_H^2\big\}\mathrm{d}s. \\
&= \mathbb{E}\int_0^t \mathrm{e}^{-cs}\Big\{2\ {}_{V^*}\langle A(s,X_s^{(n_k)}) - A(s,\phi_s), X_s^{(n_k)} - \phi_s\rangle_V \\
&\quad + \|B(s,X_s^{(n_k)}) - B(s,\phi_s)\|_{\mathscr{L}_{HS}}^2 - c\|X_s^{(n_k)} - \phi_s\|_H^2\Big\}\mathrm{d}s \\
&\quad + \mathbb{E}\int_0^t \mathrm{e}^{-cs}\Big\{2\ {}_{V^*}\langle A(s,\phi_s), X_s^{(n_k)}\rangle_V + 2\ {}_{V^*}\langle A(s,X_s^{(n_k)}) - A(s,\phi_s), \phi_s\rangle_V \\
&\quad - \|B(s,\phi_s)\|_{\mathscr{L}_{HS}}^2 + 2\langle B(s,X_s^{(n_k)}), B(s,\phi_s)\rangle_{\mathscr{L}_{HS}} - 2c\langle X_s^{(n_k)}, \phi_s\rangle_H + c\|\phi_s\|_H^2\Big\}\mathrm{d}s.
\end{aligned}
\tag{2.15}
$$

Note that by **(H2)** the first of the two summands above is negative. Hence by letting $k \to \infty$ we conclude by (i)–(iii), Fubini's theorem, Remark 2.1, (2.4) and (2.13) that for every nonnegative $\psi \in L^\infty([0,T];\mathrm{d}t)$

$$
\begin{aligned}
&\mathbb{E}\int_0^T \psi(t)(\mathrm{e}^{-ct}\|X_t\|_H^2 - \|X_0\|_H^2)\mathrm{d}t \\
&\le \mathbb{E}\int_0^T \psi(t)\Big(\int_0^t \mathrm{e}^{-cs}\Big\{2\ {}_{V^*}\langle A(s,\phi_s), \bar{X}_s\rangle_V + 2\ {}_{V^*}\langle Y_s - A(s,\phi_s), \phi_s\rangle_V \\
&\quad - \|B(s,\phi_s)\|_{\mathscr{L}_{HS}}^2 + 2\langle Z_s, B(s,\phi_s)\rangle_{\mathscr{L}_{HS}} - 2c\langle X_s, \phi_s\rangle_H + c\|\phi_s\|_H^2\Big\}\mathrm{d}s\Big)\mathrm{d}t.
\end{aligned}
$$

Inserting (2.14) for the left hand side and rearranging as above we arrive at

$$
\begin{aligned}
0 \ge &\mathbb{E}\int_0^T \psi(t)\Big(\int_0^t \mathrm{e}^{-cs}\big\{2\ {}_{V^*}\langle Y_s - A(s,\phi_s), \bar{X}_s - \phi_s\rangle_V \\
&+ \|B(s,\phi_s) - Z_s\|_{\mathscr{L}_{HS}}^2 - c\|X_s - \phi_s\|_H^2\big\}\mathrm{d}s\Big)\mathrm{d}t.
\end{aligned}
\tag{2.16}
$$

Taking $\phi = \bar{X}$ we obtain from (2.16) that $Z = B(\cdot,\bar{X})$. Finally, first applying (2.16) to $\phi = \bar{X} - \varepsilon\tilde{\phi}\, e$ for $\varepsilon > 0$ and $\tilde{\phi} \in L^\infty([0,T]\times\Omega;\mathrm{d}t \times P), e \in V$, then dividing both sides by ε and letting $\varepsilon \to 0$, by the dominated convergence theorem, the hemicontinuity of A, **(K)** and **(H4)**, we obtain

$$0 \geq \mathbb{E}\int_0^T \psi(t)\bigg(\int_0^t \mathrm{e}^{-cs}\tilde{\phi}_s\ {}_{V^*}\langle Y_s - A(s,\bar{X}_s), e\rangle_V \mathrm{d}s\bigg)\mathrm{d}t.$$

By the arbitrariness of ψ and $\tilde{\phi}$, we conclude that $Y = A(\cdot,\bar{X})$. This completes the proof. □

3 Stochastic generalized porous medium and fast diffusion equations

In this section we shall discuss applications. Let $(E,\mathscr{B},\mathbf{m})$ be a σ-finite measure space with countably generated σ-algebra $\mathscr{B}$. Let $(L,\mathscr{D}(L))$ be a negative definite self-adjoint operator on $L^2(\mathbf{m})$ with $\mathrm{Ker}(L)=\{0\}$. We shall use $\langle\ ,\ \rangle$ and $\|\cdot\|$ to stand for the inner product and the norm in $L^2(\mathbf{m})$ respectively, we also denote $\langle f,g\rangle := \mathbf{m}(fg) := \int fg\mathrm{d}\mathbf{m}$ for any two functions f,g such that $fg\in L^1(\mathbf{m})$. Consider the quadratic form $(\mathscr{E},D(\mathscr{E}))$ on $L^2(\mathbf{m})$ associated with $(L,\mathscr{D}(L))$, i.e.

$$D(\mathscr{E}) := \mathscr{D}\big(\sqrt{-L}\big)$$

and

$$\mathscr{E}(u,v) := \mathbf{m}\big(\sqrt{-L}u\,\sqrt{-L}v\big)\ ;\quad u,v\in D(\mathscr{E}).$$

Let $\mathscr{F}_e$ be the abstract completion of $D(\mathscr{E})$ with respect to the norm

$$\|\cdot\|_{\mathscr{F}_e} := \mathscr{E}(\cdot,\cdot)^{\frac{1}{2}}\ ,$$

and let $\mathscr{F}_e^*$ be its dual space. Note that both $(\mathscr{F}_e,\mathscr{E})$ and $\mathscr{F}_e^*$, with the inner product induced by the Riesz isomorphism, are Hilbert spaces. Now we define

$$H := \mathscr{F}_e^*\ ,\quad \langle\cdot,\cdot\rangle_H := \langle\cdot,\cdot\rangle_{\mathscr{F}_e^*}\ , \tag{3.1}$$

i.e. $\mathscr{F}_e^*$ will be the state space of our SDE (2.1). In order to make our general results from the previous section work, we shall use the further assumption, that $(\mathscr{E},D(\mathscr{E}))$ is a transient Dirichlet space in the sense of [12, Section 1.5]. $(\mathscr{E},\mathscr{F}_e)$, where $\mathscr{E}$ also denotes the extension of $\mathscr{E}$ to $\mathscr{F}_e$, is called *extended Dirichlet space* in [12], from which we also adopt the notation. In this case, $\mathscr{F}_e^*$ is also called the corresponding *Green space*. Since we do not assume the reader to be familiar with all these notions, we shall first formulate some abstract conditions on $(\mathscr{E},D(\mathscr{E}))$ (i.e. on $(L,D(L))$) such that our proofs below work. Then, partly on the basis of results from [12], we shall prove that these abstract assumptions hold if $(\mathscr{E},D(\mathscr{E}))$ is a transient Dirichlet space. Furthermore, we shall briefly describe whole classes of concrete examples.

From now on, we are going to assume that the following condition holds:

(L1) There exists a strictly positive $g\in L^1(\mathbf{m})\cap L^\infty(\mathbf{m})$ such that $\mathscr{F}_e\subset L^1(g\cdot\mathbf{m})$ continuously.

Our space V will be heuristically given as $V := H \cap L_N$, where N is a nice Young function and L_N the corresponding Orlicz space. More precisely, let $N \in C(\mathbb{R})$ be a Young function, i.e. a nonnegative, continuous, convex and even function such that $N(s) = 0$ if and only if $s = 0$, and

$$\lim_{s\to 0} \frac{N(s)}{s} = 0, \quad \lim_{s\to\infty} \frac{N(s)}{s} = \infty.$$

For any function f on E with $\mathbf{m}(N(\alpha f)) < \infty$ for some $\alpha > 0$, define

$$\|f\|_{L_N} := \inf\{\lambda \geq 0 : \mathbf{m}(N(f/\lambda)) \leq 1\}.$$

Then the space $(L_N, \|\cdot\|_{L_N})$, where

$$L_N(\mathbf{m}) := \{f : \|f\|_N < \infty\},$$

is a real separable Banach space, which is called the Orlicz space induced by the Young-function N (cf. [18, Proposition 1.2.4]). There is an equivalent norm defined by using the dual function:

$$N^*(s) := \sup\{r|s| - N(r) : r \geq 0\}, \quad s \in \mathbb{R},$$

which is once again a Young function. More precisely, letting

$$\|f\|_{(N)} := \sup\{\langle f, g\rangle : \mathbf{m}(N^*(g)) \leq 1\},$$

one has (see [18, Theorem 1.2.8 (ii)])

$$\|\cdot\|_{L_N} \leq \|\cdot\|_{(N)} \leq 2\|\cdot\|_{L_N}. \tag{3.2}$$

The function N is called Δ_2-regular, if there exists constant $c > 0$ such that

$$N(2s) \leq c\big(N(s) + 1_{\{\mathbf{m}(E)<\infty\}}\big), \quad s \in \mathbb{R}.$$

In this paper we assume that N and N^* are Δ_2-regular. By [18, Proposition 1.2.11(iii) and Theorem 1.2.13], $L_N(\mathbf{m})$ and $L_{N^*}(\mathbf{m})$ are dual spaces of each other with dualization given by $\langle f, g\rangle = \mathbf{m}(fg)$, $f \in L_N$, $g \in L_{N^*}$. Hence they are reflexive. By the Δ_2-regularity, $f \in L_N(\mathbf{m})$ if and only if $\mathbf{m}(N(f)) < \infty$.

Now let us give a precise definition of V. Define $V := H \cap L_N(\mathbf{m})$ in the following sense:

$$V := \big\{u \in L_N(\mathbf{m}) \;\big|\; \exists\, c \in (0,\infty) \text{ such that } \mathbf{m}(uv) \leq c\,\|v\|_{\mathscr{F}_e}\ \forall\, v \in \mathscr{F}_e \cap L_{N^*}\big\} \tag{3.3}$$

equipped with the norm

$$\|u\|_V := \|u\|_{L_N} + \|u\|_{\mathscr{F}_e^*} = \|u\|_{L_N} + \|u\|_H\ .$$

In order that V becomes a subset of H and $(V, \|\cdot\|_V)$ a Banach space, we need to assume that $\mathscr{F}_e \cap L_{N^*}$ is a dense subset of $\mathscr{F}_e$. For later use, we even make the following stronger assumption:

(N1) $\mathscr{F}_e \cap L_{N^*}$ is a dense subset of both $\mathscr{F}_e$ and L_{N^*}.

By **(N1)**, V can be considered as a subset of $H = \mathscr{F}_e^*$ by identifying $u \in V$ with the map $\bar{u} : \mathscr{F}_e \cap L_{N^*} \to \mathbb{R}$ defined by

$$\bar{u}(v) := \mathbf{m}(uv)\ , \quad v \in \mathscr{F}_e \cap L_{N^*}\ .$$

Then obviously, $V \subset H$ continuously, and it is easy to see that V is complete with respect to $\|\cdot\|_V$. The density of V in H is, however, not clear. Therefore, we assume:

(N2) V is a dense subset of both $H = \mathscr{F}_e^*$ and L_N.

The second part of **(N2)** we shall only need later. As mentioned above, we shall later prove that **(L1)**, **(N1)** and **(N2)** always hold if $(\mathscr{E}, D(\mathscr{E}))$ is a transient Dirichlet space. Let V^* be the dual space of V, so using $H \equiv H^*$ we have

$$V \subset H \subset V^* \quad \text{continuously and densely.}$$

Note that, since V is complete, the map

$$V \ni u \ \mapsto\ (u, \mathbf{m}(u\,\cdot\,)) \in L_N \times \mathscr{F}_e^*$$

is an isomorphism from V to a closed subspace of $L_N \times \mathscr{F}_e^*$ which is reflexive. So, V itself is reflexive.

Proposition 3.1. *Assume that $(\mathscr{E}, D(\mathscr{E}))$ is a transient Dirichlet space. Then conditions* **(L1)***,* **(N1)** *and* **(N2)** *hold.*

With respect to the length of this paper we do not recall all necessary definitions and notions here, but refer to [12, Section 1.5]. We shall stick exactly to the terminology and notation introduced there.

Before we prove Proposition 3.1, we need the following fact on Δ_2-regular Young functions.

Lemma 3.2. *Let N be Δ_2-regular. Then:*

(i) There exists $q \in (2, \infty)$ such that

$$N(rs) \le r^q\big(N(s) + 2 \cdot 1_{\{\mathbf{m}(E)<\infty\}}\big)\ , \quad r \ge 2,\ s \ge 0. \tag{3.4}$$

(ii) $L^1(\mathbf{m}) \cap L^q(\mathbf{m}) \subset L_N(\mathbf{m})$ continuously, where the intersection is equipped with the norm $\|\ \|_1 + \|\ \|_q$ and q is as in (i).

Proof. (i) Let $n \geq 1$ be such that $r \in [2^n, 2^{n+1})$. If N is Δ_2-regular with constant $C > 2$, then for $p_1 := \log C / \log 2$

$$\begin{aligned} N(rs) &\leq N(2^{n+1}s) \leq C^{n+1}\big(N(s) + \sum_{i=0}^{n} C^{-i} \cdot 1_{\{\mathbf{m}(E)<\infty\}}\big) \leq 2^{p_1(n+1)}\big(N(s) + 2 \cdot 1_{\{\mathbf{m}(E)<\infty\}}\big) \\ &\leq r^{p_1(n+1)/n}\big(N(s) + 2 \cdot 1_{\{\mathbf{m}(E)<\infty\}}\big) . \end{aligned}$$

Thus, (3.4) holds by taking $q := 2p_1 > 2$.

(ii) We have for all $\lambda > 0$ and $f \in L^1(\mathbf{m}) \cap L^q(\mathbf{m})$

$$\begin{aligned} \mathbf{m}\left(N\left(\frac{|f|}{\lambda}\right)\right) &= \mathbf{m}\left(1_{\{\frac{|f|}{\lambda}\geq 2\}} N\left(\frac{|f|}{\lambda}\right)\right) + \mathbf{m}\left(1_{\{\frac{|f|}{\lambda}<2\}} N\left(\frac{|f|}{\lambda}\right)\right) \\ &\leq \frac{N(1)+2}{\lambda^q}\mathbf{m}(|f|^q) + \sup_{0\leq s\leq 2}\frac{N(s)}{s} \cdot \frac{1}{\lambda}\mathbf{m}(|f|) \end{aligned}$$

where we used (i) in the last step. For

$$\lambda := \big(2\mathbf{m}(|f|^q)(N(1)+2)\big)^{\frac{1}{q}} + 2\mathbf{m}(|f|) \sup_{0\leq s\leq 2}\frac{N(s)}{s} ,$$

the right hand side is less than 1. Hence, the assertion follows. □

Proof of Proposition 3.1. We first note that **(L1)** holds by [12, Theorem 1.5.3 (α) and (β)] (see also [12, Lemma 1.5.5]).

Now let us prove **(N1)**. We first note that $L^1(\mathbf{m}) \cap L^\infty(\mathbf{m})$ is a dense subset of $L_{N^*}(\mathbf{m})$. This follows from Lemma 3.2(ii) applied to L_{N^*} and since for any $f \in L_N$ such that

$$\mathbf{m}(vf) = 0 \quad \text{for all } v \in L^1(\mathbf{m}) \cap L^\infty(\mathbf{m})$$

it follows that $f = 0$. So, to show that $\mathscr{F}_e \cap L_{N^*}$ is dense in L_{N^*}, by Lemma 3.2(ii) it suffices to show that

$$D(\mathscr{E}) \cap L^1(\mathbf{m}) \cap L^\infty(\mathbf{m}) \quad \text{is dense in } L^1(\mathbf{m}) \cap L^q(\mathbf{m}), \tag{3.5}$$

where q is as in Lemma 3.2(ii) with N^* replacing N. But (3.5) is a well-known fact about Dirichlet spaces. To show that $\mathscr{F}_e \cap L_{N^*}$ is dense in $\mathscr{F}_e$ it suffices to show that

$$L^1(\mathbf{m}) \cap L^\infty(\mathbf{m}) \cap D(\mathscr{E}) \quad \text{is dense in } D(\mathscr{E}) \tag{3.6}$$

with respect to the norm $\mathscr{E}_1(\,\cdot\,,\,\cdot\,)^{\frac{1}{2}} := \big(\mathscr{E}(\,\cdot\,,\,\cdot\,) + \|\,\cdot\,\|_2\big)^{1/2}$. So, let $u \in D(\mathscr{E})$ such that

$$\mathscr{E}_1(v, u) = 0 \quad \text{for all } v \in L^1(\mathbf{m}) \cap L^\infty(\mathbf{m}) \cap D(\mathscr{E}).$$

Then for $f \in L^1(\mathbf{m}) \cap L^2(\mathbf{m})$, since $G_1 f \in L^1(\mathbf{m}) \cap L^\infty(\mathbf{m}) \cap D(\mathscr{E})$, where $(G_\lambda)_{\lambda>0}$ is the resolvent associated to $(\mathscr{E}, D(\mathscr{E}))$,

$$\mathbf{m}(fu) = \mathscr{E}_1(G_1 f, u) = 0 \ .$$

Hence $u = 0$, because $L^1(\mathbf{m}) \cap L^\infty(\mathbf{m})$ is dense in $L^2(\mathbf{m})$.

Now let us show **(N2)**. Let g be as in **(L1)** and consider the set

$$\mathscr{G} := \left\{ h \cdot g \;\middle|\; h \in L^\infty(\mathbf{m}) \right\} .$$

Then $\mathscr{G} \subset L^1(\mathbf{m}) \cap L^\infty(\mathbf{m}) \subset L_N(\mathbf{m})$. Since $L^1(\mathbf{m}) \cap L^\infty(\mathbf{m})$ is dense in $L_N(\mathbf{m})$, it follows that $\mathscr{G}$ is dense in $L_N(\mathbf{m})$. Furthermore, by [12, Theorem 1.5.4] for every $f \in \mathscr{G}$ there exists $Gf \in \mathscr{F}_e$ such that

$$\mathbf{m}(fv) = \mathscr{E}(Gf, v) \quad \text{for all } v \in \mathscr{F}_e.$$

Hence $\mathscr{G} \subset V$, so V is dense in $L_N(\mathbf{m})$.

Now we show that $\mathscr{G}$ is dense in $H = \mathscr{F}_e^*$. So, let $v \in \mathscr{F}_e$ such that

$$\mathbf{m}(fv) = 0 \quad \text{for all } f \in \mathscr{G}.$$

Then

$$\mathbf{m}(hvg) = 0 \quad \text{for all } h \in L^\infty(\mathbf{m}),$$

and hence, since $vg \in L^1(\mathbf{m})$, $vg = 0$ $\mathbf{m}$-a.e. Since g is strictly positive, it follows that $v = 0$ $\mathbf{m}$-a.e. and **(N2)** is proved. □

Example 3.3. *There are plenty of examples of transient Dirichlet spaces described in the literature with E being e.g. a manifold or a fractal. Here, we shall briefly refer to cases where $E := D \subset \mathbb{R}^d$ and $\mathbf{m}$ is the Lebesgue measure and which are presented in detail in [12] (see Examples 1.5.1–1.5.3). For example, if L is the Friedrichs extension of a symmetric uniformly elliptic operator of second order with Dirichlet boundary conditions on an open domain $D \subset \mathbb{R}^d$, then all the above applies. But there are also examples with Neumann boundary conditions, as e.g. the Laplacian on $D :=$ half space in $\mathbb{R}^d$. Furthermore, for $D := \mathbb{R}^d$ we can take $L = (-\Delta)^\alpha$ with its standard domain, if $\alpha \in (0, \frac{d}{2}) \cap (0, 1]$. For details we refer to [12].*

Now let us return to our general situation described at the beginning of this section, i.e. the only conditions on L and the Δ_2-regular (dual) Young functions N and N^* are **(L1)**, **(N1)** and **(N2)**. The following is then standard and more a question of notation than contents. Nevertheless, we include a short proof.

Lemma 3.4. *(i) The map $\bar{L} : \mathscr{F}_e \to \mathscr{F}_e^*$ defined by*

$$\bar{L}v := -\mathscr{E}(v, \cdot\,) , \quad v \in \mathscr{F}_e \tag{3.7}$$

(i.e. the Riesz isomorphism of $\mathscr{F}_e$ and $\mathscr{F}_e^*$ multiplied by (-1)) is the unique continuous linear extension of the map

$$D(L) \ni v \;\mapsto\; \mathbf{m}(Lv \,\cdot\,) \in \mathscr{F}_e^* \ . \tag{3.8}$$

(ii) Let $v \in \mathscr{F}_e \cap L_{N^*}$, $u \in \mathscr{F}_e^* \cap L_N = V$. Then

$$\langle \bar{L}v, \bar{u} \rangle_{\mathscr{F}_e^*} = -\mathbf{m}(vu) .$$

(iii) The map $\bar{L} : \mathscr{F}_e \cap L_{N^*} \to \mathscr{F}_e^* \subset V^*$ has a unique continuous linear extension (again denoted by) $\bar{L} : L_{N^*} \to V^*$ and this extension satisfies

$$ {}_{V^*}\langle \bar{L}v, u \rangle_V = -\mathbf{m}(vu) \quad \text{for all } v \in L_{N^*},\ u \in V. \tag{3.9}$$

Proof. (i) For all $v \in D(L)$, $w \in D(\mathscr{E})$ we have

$$\mathbf{m}(Lv\, w) = -\mathscr{E}(v, w) .$$

By the density of $D(\mathscr{E})$ in its completion $\mathscr{F}_e$ it follows that the linear map in (3.8) really takes values in $\mathscr{F}_e^*$ and that it is continuous as a map from $\mathscr{F}_e$ with domain $D(L)$ with values in $\mathscr{F}_e^*$. Since $D(L)$ is dense in $(D(\mathscr{E}), \mathscr{E}_1)$, hence in $\mathscr{F}_e$, the assertion follows.

(ii) By Riesz's representation theorem there exists $Gu \in \mathscr{F}_e$ such that $\bar{u}$ $(:= \mathbf{m}(u\,\cdot))$ $=$ $\mathscr{E}(Gu, \cdot\,)$. Hence,

$$\langle \bar{L}v, \bar{u} \rangle_{\mathscr{F}_e^*} = -\big\langle \mathscr{E}(v, \cdot\,), \mathscr{E}(Gu, \cdot\,) \big\rangle_{\mathscr{F}_e^*} = -\mathscr{E}(v, Gu) = -\mathbf{m}(vu) .$$

(iii) Let $v \in \mathscr{F}_e \cap L_{N^*}$, $u \in \mathscr{F}_e^* \cap L_N = V$. Then by (ii)

$$ {}_{V^*}\langle \bar{L}v, u \rangle_V = \langle \bar{L}v, \bar{u} \rangle_{\mathscr{F}_e^*} = -\mathbf{m}(vu) , \tag{3.10}$$

hence,

$$\left| {}_{V^*}\langle \bar{L}v, u \rangle_V \right| \le c\, \|v\|_{L_{N^*}} \|u\|_{L_N} \le c\, \|v\|_{L_{N^*}} \|u\|_V$$

for some positive constant c independent of u and v. By **(N2)** it follows that

$$\|\bar{L}v\|_{V^*} \le c\, \|v\|_{L_{N^*}} \quad \text{for all } v \in \mathscr{F}_e \cap L_{N^*}.$$

Hence, by the second part ot **(N1)**, the desired extension $\bar{L} : L_{N^*} \to V^*$ exists and (3.10) extends to (3.9). □

For simplicity, we write L instead of $\bar{L}$ and u instead of $\bar{u} \in V$ below, hence, consider V as a subset of $H = \mathscr{F}_e^*$, hence of V^* in particular.

To define the nonlinear operators A and B in (2.1), let

$$\Psi, \Phi : [0, \infty) \times \mathbb{R} \times \Omega \to \mathbb{R} \tag{3.11}$$

be progressively measurable, i.e. for any $t \ge 0$, restricted to $[0,t] \times \mathbb{R} \times \Omega$ they are measurable w.r.t. $\mathscr{B}([0,t]) \times \mathscr{B}(\mathbb{R}) \times \mathscr{F}_t$. We assume that for fixed $(t, \omega) \in [0, \infty) \times \Omega$, $\Psi(t, \cdot)(\omega)$ and $\Phi(t, \cdot)(\omega)$ are continuous.

Finally, let

(3.12) $B : [0, T] \times V \times \Omega \to \mathscr{L}_{HS}(G; H)$ be progressively measurable satisfying **(H2)** with $A \equiv 0$ (i.e., B is Lipschitz with respect to H-norm, uniformly in $(t, \omega) \in [0, T] \times \Omega$).

We distinguish two sets of conditions on Ψ and Φ:

(A1) $\Phi(t,s) = h_t s$, $t \in [0,T]$, $s \in \mathbb{R}$, for some $\mathscr{F}_t$-adapted $h \in L^\infty\big([0,T] \times \Omega; \mathrm{d}t \times P\big)$ and there exist Δ_2-regular dual Young functions N and N^*, a nonnegative $\mathscr{F}_t$-adapted process $f \in L^1\big([0,T] \times \Omega; \mathrm{d}t \times P\big)$ and a constant $c \geq 1$, such that for all $s, s_1, s_2 \in \mathbb{R}$ on $[0,T] \times \Omega$

$$(\Psi 1) \qquad (s_2 - s_1)\big(\Psi(\,\cdot\,, s_2) - \Psi(\,\cdot\,, s_1)\big) \geq 0\,.$$

$$(\Psi 2) \qquad s\Psi(\,\cdot\,, s) \geq N(s) - 1_{\{\mathbf{m}(E)<\infty\}} \cdot f\,.$$

$$(\Psi 3) \qquad s\Psi(\,\cdot\,, s) \leq c\big(N(s) + 1_{\{\mathbf{m}(E)<\infty\}} \cdot f\big)\,.$$

$$(\Psi 4) \qquad N^*\big(\Psi(\,\cdot\,, 0)\big)\, 1_{\{\mathbf{m}(E)<\infty\}} \in L^1\big([0,T] \times \Omega; \mathrm{d}t \times P\big)\,.$$

If Φ is not just the identity times h as above, we need to restrict to the Young function considered in Example 3.5(i), i.e.

$$N(s) := \sum_{i=1}^m \varepsilon_i |s|^{r_i+1}\,, \quad s \in \mathbb{R}, \tag{3.13}$$

for some pairwise distinct $r_1, \dots, r_m > 0$ and $\varepsilon_1, \dots, \varepsilon_m > 0$, and consider the following condition:

(A2) There exist $r_1, \dots, r_m > 0$ such that for N as in (3.13) the set $L\big(D(L) \cap L_N(\mathbf{m})\big)$ is dense in $L^{r_i+1}(\mathbf{m})$ and $L^{-1} : L^{r_i+1}(\mathbf{m}) \to L^{r_i+1}(\mathbf{m})$, $1 \leq i \leq m$, is bounded. Furthermore, there exist a nonnegative $\mathscr{F}_t$-adapted process $f \in L^1\big([0,T] \times \Omega, \mathrm{d}t \times P\big)$ and constants $c \geq 0$, $\delta_1, \dots, \delta_m > 0$ such that for all $s, s_1, s_2 \in \mathbb{R}$ we have on $[0,T] \times \Omega$

$(\Psi 1)'$ $\displaystyle (s_2 - s_1)\big(\Psi(\,\cdot\,, s_2) - \Psi(\,\cdot\,, s_1)\big) \geq \sum_{i=1}^m \delta_i\, |s_2 - s_1|^{r_i+1}\,.$

$(\Psi 2)'$ $N(s) - f \cdot 1_{\{\mathbf{m}(E)<\infty\}} \leq s\,\Psi(\,\cdot\,, s) \leq c\big(N(s) + f \cdot 1_{\{\mathbf{m}(E)<\infty\}}\big)$
with N as in (3.13).

$(\Phi 1)$ $\Phi(\,\cdot\,, s) = hs + \Phi_0(\,\cdot\,, s)$, $s \in \mathbb{R}$, for some $\mathscr{F}_t$-adapted $h \in L^\infty\big([0,T] \times \Omega; \mathrm{d}t \times P\big)$ such that on $[0,T] \times \Omega$ we have

$$\big|\Phi_0(\,\cdot\,, s_2) - \Phi_0(\,\cdot\,, s_1)\big| \leq \sum_{i=1}^m \delta_i\, \|L^{-1}\|^{-1}_{\mathscr{L}(L^{r_i+1}(\mathbf{m}))}\, |s_2 - s_1|^{r_i}\,.$$

$(\Phi 2)$ $\displaystyle \big|\Phi_0(\,\cdot\,, s)\big| \leq \sum_{i=1}^m \tilde{\varepsilon}_i\, |s|^{r_i}\,,$
where $\tilde{\varepsilon}_i := \varepsilon\, \|L^{-1}\|^{-1}_{\mathscr{L}(L^{r_i+1}(\mathbf{m}))}\, \varepsilon_i$ for some $\varepsilon \in (0,1)$ (independent of $s \in \mathbb{R}$).

Remark 3.1. (i) If **(A1)** holds for Ψ, then it holds for $s \mapsto -\Psi(-s)$ with the same Young function N.

(ii) If $\mathbf{m}(E) = \infty$, then (Ψ2) and (Ψ3) imply that $\Psi(\,\cdot\,, 0) \equiv 0$.

(iii) We note that if $(\mathscr{E}, D(\mathscr{E}))$ is a Dirichlet form and for some $c \in (0, \infty)$

$$\mathscr{E}(u,u) \geq c \int u^2 \mathrm{d}\mathbf{m} \quad \text{for all } u \in D(\mathscr{E}),$$

then as is well known, $(L, D(L))$ has bounded inverses as in condition **(A2)** for all $r_i \geq 1$.

Example 3.5. *(i) Let $r_1, \ldots, r_m > 0$, $\delta_1, \ldots, \delta_m > 0$ and*

$$\Psi(s) := \operatorname{sign}(s) \sum_{i=1}^{m} \delta_i \, |s|^{r_i} \,.$$

*Define $N(s) := s\Psi(s)$. Then **(A1)** is fulfilled, provided we can show that N, which is obviously a Young function, and also N^* are Δ_2-regular. This is clear for N. To see the Δ_2-regularity of N^*, let $r := \min r_i$ and $\theta := 2^{1/r} > 1$. We have, for $s > 0$,*

$$\begin{aligned} N^*(2s) &:= \sup_{t>0} \Big\{ 2st - \sum_{i=1}^{m} \delta_i t^{r_i+1} \Big\} = \sup_{t>0} \Big\{ 2\theta st - \sum_{i=1}^{m} \theta^{r_i+1} \delta_i t^{r_i+1} \Big\} \\ &\leq \theta^{r+1} \sup_{t>0} \Big\{ 2\theta^{-r} st - \sum_{i=1}^{m} \delta_i t^{r_i+1} \Big\} = \theta^{r+1} N^*(s). \end{aligned} \tag{3.14}$$

(ii) Let Ψ, N be as in (i), so that $(\Psi 2)'$ trivially holds, but with $r_1, \ldots, r_m \geq 1$. Then an elementary calculation (see e.g. [21, p. 503]) shows that $(\Psi 1)'$ holds.

*(iii) Let $\Psi(s) := \operatorname{sign}(s)\, |s|^{\theta-1} \big(\log(1+|s|)\big)^r$, $s \in \mathbb{R}$, where $\theta \in (1, \infty)$, $r \in [1, \infty)$. Then Ψ satisfies **(A1)** with $N(s) := s\,\Psi(s) = |s|^{\theta} \big(\log(1+|s|)\big)^r$, $s \in \mathbb{R}$. Obviously, N is a Young function which together with its dual N^* is easily checked to be Δ_2-regular.*

(iv) Time dependent examples are easily obtained by e.g. multiplying Ψ or Φ by a bounded adapted process which is bounded below by a strictly positive constant.

To define A as in (2.1), we need one more lemma.

Lemma 3.6. *(i) Let **(A1)** hold. Then for all $s \in \mathbb{R}$*

$$N^*(c^{-1}\Psi(\,\cdot\,, s)) \leq N(s) + \big[3f + N^*\big(c^{-1}\Psi(\,\cdot\,, 0)\big) \big] \cdot 1_{\{\mathbf{m}(E) < \infty\}} \quad \text{on } [0,T] \times \Omega.$$

*(ii) Let **(A2)** hold and let N be as in (3.13). Then there exists $\tilde{c} \in (0, \infty)$ such that for all $s \in \mathbb{R}$*

$$N^*\big(\Phi_0(\,\cdot\,, s)\big) \leq \tilde{c}\, N(s) \quad \text{on } [0,T] \times \Omega.$$

Proof. (i) Let $s \in \mathbb{R}$. By Remark 3.1(i) we may assume that $s \geq 0$. Fix $t \in [0,T]$ and suppose first that $\Psi(t,s) \geq 0$, which by Remark 3.1(ii) and (Ψ1) is always the case if $\mathbf{m}(E) = \infty$. Then by (Ψ3)

$$\begin{aligned} N^*(c^{-1}\Psi(t,s)) &= \sup_{r\geq 0}\big[rc^{-1}\Psi(t,s) - N(r)\big] \\ &\leq \sup_{r\geq 0}\big[rc^{-1}(\Psi(t,s) - \Psi(\,\cdot\,,r))\big] + f\cdot 1_{\{\mathbf{m}(E)<\infty\}} \\ &= c^{-1}\sup_{s\geq r\geq 0}\big[r(\Psi(t,s) - \Psi(\,\cdot\,,r))\big] + f\cdot 1_{\{\mathbf{m}(E)<\infty\}}\ , \end{aligned}$$

where we used (Ψ1) in the last step. But since $N \geq 0$ the last term due to (Ψ2) is dominated by

$$c^{-1}s\Psi(\,\cdot\,,s) + 2f\,1_{\{\mathbf{m}(E)<\infty\}}\ .$$

Now (i) follows by (Ψ3), for such $s \geq 0$, respectively is completely proved if $\mathbf{m}(E) = \infty$.

If $\mathbf{m}(E) < \infty$ and $\Psi(t,s) < 0$, then $|\Psi(t,s)| \leq |\Psi(t,0)|$ by (Ψ1). Hence,

$$N^*\big(c^{-1}\,\Psi(t,s)\big) = N^*\big(c^{-1}|\Psi(t,s)|\big) \leq N^*\big(c^{-1}|\Psi(t,0)|\big)$$

and (i) also follows in this case.

(ii) Fix $1 \leq i \leq m$ and define $N_i(s) := |s|^{r_i+1}$, $s \in \mathbb{R}$. Then

$$N_i^*(s) = \left(\frac{1}{r_i+1}\right)^{\frac{1}{r_i}} \frac{r_i}{r_i+1}\,|s|^{\frac{r_i+1}{r_i}} =: c_i\,|s|^{\frac{r_i+1}{r_i}}$$

(as an elementary calculation shows). Hence, by (Φ2) and (3.14) for all $s \in \mathbb{R}$ we have on $[0,T]\times\Omega$

$$\begin{aligned} N^*\big(\Phi_0(\,\cdot\,,s)\big) &= N^*\big(|\Phi_0(\,\cdot\,,s)|\big) \leq N^*\Big(\sum_{i=1}^m \tilde{\varepsilon}_i\,|s|^{r_i}\Big) \leq N^*\Big(\sum_{i=1}^m \tilde{\varepsilon}_i\,|s|^{r_i}\Big) \\ &\leq \sup_{\xi\geq 0}\Big[\xi\sum_{i=1}^m \tilde{\varepsilon}_i\cdot\big(|s|^{r_i} - \xi^{r_i}\big)\Big] \leq \sum_{i=1}^m \tilde{\varepsilon}_i\,N_i^*\big(|s|^{r_i}\big) = \sum_{i=1}^m \tilde{\varepsilon}_i c_i\,|s|^{r_i+1}\ . \end{aligned}$$

Hence, the assertion follows with $\tilde{c} := \varepsilon\max_{1\leq i\leq m}\big(c_i\,\|L^{-1}\|^{-1}_{\mathscr{L}(L^{r_i+1}(\mathbf{m}))}\big)$. □

Fix Ψ, Φ as above satisfying **(A1)** or **(A2)**. We could define $A : [0,T]\times V\times\Omega \to V^*$ by

$$A(t,v,\omega) := L\Psi(t,v,\omega) + \Phi(t,v,\omega)\ , \quad t\in[0,T],\ v\in V,\ \omega\in\Omega. \tag{3.15}$$

By Lemma 3.4(iii) and Lemma 3.6(i) the first summand is a well-defined element in V^*. By Lemma 3.6(ii) this is also true for the second summand since $V \subset V^*$ and $L_{N^*} = (L_N)^* \subset V^*$ since V is dense in L_N by **(N2)**. But in order to show that A satisfies our assumptions **(H1)**–**(H4)**, we need estimates on Φ in (3.15) and we have to compare it with $L\Psi$. Therefore, we need to define the second summand in (3.15) in a more convenient way which is, however, only equivalent under additional assumptions. We refer to [7] for such a case and to the calculation in Remark 3.2 below. So, we define for $t \in [0,T]$, $v \in V$, $\omega \in \Omega$

$$\bar{\Phi}(t,v,\omega) := h_t \cdot v - \mathbf{m}\big(\Phi_0(t,v,\omega)\, L^{-1} \cdot\big) \tag{3.16}$$

and

$$A(t,v,\omega) := L\Psi(t,v,\omega) + \bar{\Phi}(t,v,\omega)\ . \tag{3.17}$$

Since by assumption Φ_0 is only nonzero if **(A2)** holds, and then $L^{-1} : L_N \to L_N$ is continuous, it follows by Lemma 3.6(ii) that $V \ni u \mapsto \mathbf{m}\big(\Phi_0(t,v,\omega)\, L^{-1}u\big)$ is a continuous linear functional on L_N, hence on $V = \mathscr{F}_e^* \cap L_N$, so belongs indeed to V^*. With this definition of A we shall then be able to verify our conditions **(H1)**–**(H4)** on the basis of our assumptions **(A1)** and **(A2)**.

Remark 3.2. To avoid confusion below, we denote the continuous extension of the inverse L^{-1} in **(A2)** to all of L_N by $\tilde{L}^{-1}$. Now let $v \in V$ and $u \in V$ such that $\tilde{L}^{-1}u \in D(L) \cap L_{N^*}$. Then by Lemma 3.4(iii)

$$-\mathbf{m}(v\, \tilde{L}^{-1}u) = {}_{V^*}\langle \bar{L}\tilde{L}^{-1}u, v\rangle_V = \langle \bar{L}\tilde{L}^{-1}u, v\rangle_{\mathscr{F}_e^*} = \langle u, v\rangle_{\mathscr{F}_e^*} = {}_{V^*}\langle v, u\rangle_V\ .$$

So, if the set of such u is dense in V, the above definitions of $\Phi(t,v,\omega)$ and $\bar{\Phi}(t,v,\omega)$ are equivalent (cf. [7] for an example), since V is dense in L_N.

Now we define

$$R(v) := \mathbf{m}(N(v)) + \|v\|_H^2\ , \quad v \in V, \tag{3.18}$$

and

$$K := L_N\big([0,T] \times E \times \Omega; \mathrm{d}t \times \mathbf{m} \times P\big) \cap L^2\big([0,T] \times \Omega \to H; \mathrm{d}t \times P\big) \tag{3.19}$$

with norm

$$\|\cdot\|_K := \|\cdot\|_{L_N([0,T]\times E\times\Omega;\mathrm{d}t\times\mathbf{m}\times P)} + \|\cdot\|_{L^2([0,T]\times\Omega\to H;\mathrm{d}t\times P)}\ . \tag{3.20}$$

That the intersection in (3.19) is meaningful follows from the last inclusion in the following lemma and the definition of V $(= L_N(\mathbf{m}) \cap \mathscr{F}_e^*)$. It also follows that K is complete (hence a Banach space) and reflexive.

Lemma 3.7. *Let N be a Δ_2-regular Young function and $q \in (2,\infty)$ as in Lemma 3.2. Then the following embeddings are dense and continuous*

$$\begin{aligned} L^q\big([0,T] \times \Omega \to V; \mathrm{d}t \times P\big) &\subset L^q\big([0,T] \times \Omega \to L_N(\mathbf{m}); \mathrm{d}t \times P\big) \\ &\subset L_N\big([0,T] \times E \times \Omega; \mathrm{d}t \times \mathbf{m} \times P\big) \subset L^1\big([0,T] \times \Omega \to L_N(\mathbf{m}); \mathrm{d}t \times P\big)\ . \end{aligned}$$

Proof. The assertion with respect to the first inclusion is clear, by (the second half of) condition **(N2)**. To prove the assertion for the second inclusion, let $g \in L_{N^*}\big([0,T] \times E \times \Omega; \mathrm{d}t \times \mathbf{m} \times P\big)$ such that

$$\bar{\mathbf{m}}(N(g)) \quad \left(:= \int N(g)\, \mathrm{d}\bar{\mathbf{m}}\right) \quad \leq 1 \,,$$

where $\bar{\mathbf{m}} := \mathrm{d}t \times \mathbf{m} \times P$. Then for all $f \in L_N(\bar{\mathbf{m}})$, since $s_1 \cdot s_2 \leq N(s_1) + N^*(s_2)$ for all $s_1, s_2 \in \mathbb{R}$,

$$\begin{aligned}
\bar{\mathbf{m}}(f \cdot g) &\leq \bar{\mathbf{m}}\big(N(f)\big) + 1 \\
&\leq \int\int_0^T \mathbf{m}\left(N\left(\big(\|f(t,\,\cdot\,,\omega)\|_{L_N(\mathbf{m})} + 2\big)\, \frac{f(t,\,\cdot\,,\omega)}{\|f(t,\,\cdot\,,\omega)\|_{L_N(\mathbf{m})} + 2}\right)\right) \mathrm{d}t\; P(\mathrm{d}\omega) + 1 \\
&\leq \int\int_0^T \big(\|f(t,\,\cdot\,,\omega)\|_{L_N(\mathbf{m})} + 2\big)^q\, \big(1 + 2\mathbf{m}(E)\, 1_{\{\mathbf{m}(E)<\infty\}}\big)\; \mathrm{d}t\; P(\mathrm{d}\omega) + 1 \,,
\end{aligned}$$

where we used (3.4) in the last step. Now by (3.2) we obtain that for some constants $a, b > 0$ (independent of f)

$$\|f\|_{L_N(\bar{\mathbf{m}})} \leq a \int\int_0^T \big\|f(t,\,\cdot\,,\omega)\big\|^q_{L_N(\mathbf{m})}\; \mathrm{d}t\; P(\mathrm{d}\omega) + b \,.$$

Hence, $\|\cdot\|_{L_N(\bar{\mathbf{m}})}$ is bounded on bounded sets of $L^q\big([0,T] \times \Omega \to L_N(\mathbf{m}); \mathrm{d}t \times P\big)$, so the second embedding in the assertion is continuous. To show its density, it is enough to prove that

$$L_0^\infty(\bar{\mathbf{m}}) \subset L^q\big([0,T] \times \Omega \to L_N(\mathbf{m}); \mathrm{d}t \times P\big) \,,$$

where $L_0^\infty(\bar{\mathbf{m}})$ denotes the set of all $f \in L^\infty(\bar{\mathbf{m}})$ such that for some $E_0 \in \mathscr{B}$ with $\mathbf{m}(E_0) < \infty$, $\{f \neq 0\} \subset [0,T] \times E_0 \times \Omega$. (Obviously, $L_0^\infty(\bar{\mathbf{m}})$ is dense in $L_N(\bar{\mathbf{m}})$.) But by Lemma 3.2(ii) there exists $c \in (0,\infty)$ such that for all $f \in L_0^\infty(\bar{\mathbf{m}})$

$$\begin{aligned}
&\int\int \big\|f(t,\,\cdot\,,\omega)\big\|^q_{L_N(\mathbf{m})}\; \mathrm{d}t\; P(\mathrm{d}\omega) \\
&\leq c \int\int \big(\|f(t,\,\cdot\,,\omega)\|^q_{L^q(\mathbf{m})} + \|f(t,\,\cdot\,,\omega)\|^q_{L^1(\mathbf{m})}\big)\; \mathrm{d}t\; P(\mathrm{d}\omega) \\
&\leq c\, \|f\|^q_{L^\infty(\bar{\mathbf{m}})}\; T\big(\mathbf{m}(E_0) + \mathbf{m}(E_0)^q\big) < \infty \,.
\end{aligned}$$

Now let us prove the assertion for the last inclusion. Let $f \in L_N(\bar{\mathbf{m}}) \setminus \{0\}$. As above by (3.2) we obtain that for $\mathrm{d}t \times P$-a.e. $(t,\omega) \in [0,T] \times \Omega$

$$\big\|f(t,\,\cdot\,,\omega)\big\|_{L_N(\mathbf{m})} \leq \mathbf{m}\big(N(f(t,\,\cdot\,,\omega))\big) + 1 \,.$$

Hence,

$$\int\!\!\int \big\|f(t,\,\cdot\,,\omega)\big\|_{L_N(\mathbf{m})}\,\mathrm{d}t\;P(\mathrm{d}\omega) \le \bar{\mathbf{m}}(N(f)) + T$$
$$\le \bar{\mathbf{m}}\Bigg(N\bigg(\frac{f}{\|f\|_{L_N(\bar{\mathbf{m}})}}\,\big[\|f\|_{L_N(\bar{\mathbf{m}})}+2\big]\bigg)\Bigg) + T$$
$$\le \big(\|f\|_{L_N(\bar{\mathbf{m}})}+2\big)^q\,\big(1+2T\,\mathbf{m}(E)\,1_{\{\mathbf{m}(E)<\infty\}}\big) + T\ ,$$

where we used (3.4) in the last step. Since the last expression is bounded on bounded sets of $L_N(\bar{\mathbf{m}})$, the third continuous embedding in the assertion is proved. Its density is then obvious since clearly $L_0^\infty(\bar{\mathbf{m}}) \subset L^1\big([0,T]\times\Omega \to L_N(\mathbf{m}); \mathrm{d}t\times P\big)$. □

By definition and Lemma 3.7 it now follows that for q as in Lemma 3.2

$$L^q\big([0,T]\times\Omega\to V;\mathrm{d}t\times P\big) \subset K \subset L^1\big([0,T]\times\Omega\to V;\mathrm{d}t\times P\big)$$

continuously, and both embeddings are dense, since L^q is dense in L^1. So, it remains to check our general conditions **(K)**, **(H1)**–**(H4)**.

Proposition 3.8. *For R and K defined in* (3.18), (3.19), *respectively, conditions* **(K)**, **(H1)**–**(H4)** *from Section 2 hold for A defined in* (3.17) *and* (3.16).

Proof. To prove **(K)**(i) it suffices to show that for any sequence $z^{(n)} \in K$, $n\in\mathbb{N}$, one has $\|z^{(n)}\|_{L_N(\bar{\mathbf{m}})} \to 0$ if and only if $\bar{\mathbf{m}}(N(z^{(n)})) \to 0$, where as before $\bar{\mathbf{m}} := \mathrm{d}t\times\mathbf{m}\times P$. So, assume $\|z^{(n)}\|_{L_N(\bar{\mathbf{m}})} \to 0$. Then $z^{(n)} \to 0$ in $\bar{\mathbf{m}}$-measure, because for all $k\in\mathbb{N}$ there exists $n_k\in\mathbb{N}$ such that

$$\bar{\mathbf{m}}\Bigg(N\bigg(\frac{|z^{(n_k)}|}{2^{-k}}\bigg)\Bigg) \le 1\ ,$$

hence by the convexity and continuity of N

$$\bar{\mathbf{m}}\bigg(N\Big(\sum_{k=1}^\infty |z^{(n_k)}|\Big)\bigg) \le \sum_{k=1}^\infty 2^{-k}\,\bar{\mathbf{m}}\big(N(2^k\,|z^{(n_k)}|)\big) < \infty\ ,$$

so $\sum_{k=1}^\infty |z^{(n_k)}| < \infty$ $\bar{\mathbf{m}}$-a.e. Since this is true for any subsequence of $z^{(n)}$, $n\in\mathbb{N}$, this really implies that $z^{(n)}\to 0$ in $\bar{\mathbf{m}}$-measure.

On the other hand, by (3.4) for N^* (with possibly different $q>2$), for any $\varepsilon\in(0,2^{1-q}]$ we have

$$N(\varepsilon s) := \sup_{r>0}\{|s|\varepsilon r - N^*(r)\} = \varepsilon^{q/(q-1)}\sup_{r>0}\{|s|\varepsilon^{-1/(q-1)}r - N^*(r)\varepsilon^{-q/(q-1)}\}$$
$$\le \varepsilon^{q/(q-1)}\sup_{r>0}\{|s|\varepsilon^{-1/(q-1)}r - N^*(\varepsilon^{-1/(q-1)}r)\} + 2\cdot 1_{\{\mathbf{m}(E)<\infty\}}$$
$$= \varepsilon^{q/(q-1)}N(s) + 2\cdot 1_{\{\mathbf{m}(E)<\infty\}},\quad s\in\mathbb{R}.$$

This implies, for $(0 \neq)\ \|z^{(n)}\|_{L_N(\bar{\mathbf{m}})} \leq 2^{1-q}$,

$$N(z^{(n)}) = N\left(\frac{z^{(n)}\|z^{(n)}\|_{L_N(\bar{\mathbf{m}})}}{\|z^{(n)}\|_{L_N(\bar{\mathbf{m}})}}\right) \leq \|z^{(n)}\|^{q/(q-1)}_{L_N(\bar{\mathbf{m}})}\ N\left(\frac{z^{(n)}}{\|z^{(n)}\|_{L_N(\bar{\mathbf{m}})}}\right) + 2T\,\mathbf{m}(E)\,1_{\{\mathbf{m}(E)<\infty\}}\ .$$

But the right hand side converges in $L^1(\bar{\mathbf{m}})$, hence $N(z^{(n)}) \to 0$ in $L^1(\bar{\mathbf{m}})$.

Now assume that $N(z^{(n)}) \to 0$ in $L^1(\bar{\mathbf{m}})$, hence in $\bar{\mathbf{m}}$-measure. Then $z^{(n)} \to 0$ in $\bar{\mathbf{m}}$-measure. But for $\lambda \in \left(0, \frac{1}{2}\right)$ by (3.4) we have

$$N\left(\frac{z^{(n)}}{\lambda}\right) \leq \lambda^{-q}\big(N(z^{(n)}) + 2 \cdot 1_{\{\mathbf{m}(E)<\infty\}}\big)$$

and the right hand side converges in $L^1(\bar{\mathbf{m}})$. Hence, $N\big(\frac{z^{(n)}}{\lambda}\big) \to 0$ in $L^1(\mathbf{m})$, so for sufficiently large n

$$\|z^{(n)}\|_{L_N(\bar{\mathbf{m}})} \leq \lambda\ ,$$

thus $\|z^{(n)}\|_{L_N(\bar{\mathbf{m}})} \to 0$ since $\lambda \in \left(0, \frac{1}{2}\right)$ was arbitrary.

Now we verify **(K)**(ii). By (3.4) and since $N(s)$ is increasing in $|s|$, we have for $z \in K\setminus\{0\}$

$$\begin{aligned}
\bar{\mathbf{m}}\big(N(z)\big) &\leq \bar{\mathbf{m}}\left(N\left(\frac{z(\|z\|_{L_N(\bar{\mathbf{m}})} + 2)}{\|z\|_{L_N(\bar{\mathbf{m}})}}\right)\right) \\
&\leq \bar{\mathbf{m}}\left(\left[N\left(\frac{z}{\|z\|_{L_N(\bar{\mathbf{m}})}}\right) + 2 \cdot 1_{\{\mathbf{m}(E)<\infty\}}\right]\ \big(\|z\|_{L_N(\bar{\mathbf{m}})} + 2\big)^q\right) \\
&\leq \big(1 + 2T\,\mathbf{m}(E)\,1_{\{\mathbf{m}(E)<\infty\}}\big)\ \big(\|z\|_{L_N(\bar{\mathbf{m}})} + 2\big)^q\ .
\end{aligned}$$

This implies that for some $c \in (0, \infty)$

$$\begin{aligned}
\|z\|_K &\geq c\left(\big[\bar{\mathbf{m}}\big(N(z)\big)\big]^{1/q} + \left(\mathbb{E}\int_0^T \|z_t\|_H^2\ \mathrm{d}t\right)^{1/2}\right) - 2 \\
&\geq c\left(\mathbb{E}\int_0^T R(z_t)\ \mathrm{d}t\right)^{1/q} - c - 2\ ,
\end{aligned}$$

where we used the elementary estimate $(a+b)^{1/q} \leq a^{1/q} + b^{1/2} + 1$, $a, b \geq 0$, $q \geq 2$. On the other hand, since by (3.2) and because of $st \leq N(s) + N^*(t)$ we have

$$\|z\|_{L_N(\bar{\mathbf{m}})} \leq \bar{\mathbf{m}}\big(N(z)\big) + 1\ ,$$

it follows that

$$\|z\|_K \leq \mathbb{E}\int_0^T \mathbf{m}\big(N(z_t)\big)\ \mathrm{d}t + 1 + \left(\mathbb{E}\int_0^T \|z_t\|_H^2\ \mathrm{d}t\right)^{1/2} \leq \mathbb{E}\int_0^T R(z_t)\ \mathrm{d}t + 2\ .$$

Therefore, **(K)**(ii) holds for

$$W_1(r) := cr^{1/q} - c - 2\ , \quad W_2(r) := r + 2\ , \quad r \geq 0.$$

Since N is convex,

$$\begin{aligned} R(x+y) &:= \mathbf{m}(N(x+y)) + \|x+y\|_H^2 \\ &\leq \frac{1}{2}\mathbf{m}(N(2x) + N(2y)) + 2\|x\|_H^2 + 2\|y\|_H^2 = \frac{1}{2}(R(2x) + R(2y)). \end{aligned}$$

So, **(K)**(iii) holds for $C = \frac{1}{2}$. **(K)**(iv) is clear, since obviously for $z \in L_N(\bar{\mathbf{m}})$ and $h \in L^\infty\big([0,T] \times \Omega; \mathrm{d}t \times P\big)$

$$\|hz\|_{L_N(\bar{\mathbf{m}})} \leq \|h\|_{L^\infty(\mathrm{d}t \times P)}\, \|z\|_{L_N(\bar{\mathbf{m}})}\ .$$

Now we are going to prove **(H1)**–**(H4)**.

(H1): Let $u, v, x \in V = L_N(\mathbf{m}) \cap \mathscr{F}_e^*$ and $\lambda \in \mathbb{R}$. Then by (3.14), (3.13), Lemma 3.6 and (3.8) on $[0,T] \times \Omega$ we have

$$\begin{aligned} &{}_{V^*}\big\langle A(\,\cdot\,, u + \lambda v), x\big\rangle_V \\ &\quad = -\mathbf{m}\big(\Psi(\,\cdot\,, u+\lambda v)\, x\big) + h_t\ {}_{V^*}\langle u + \lambda v, x\rangle_V - \mathbf{m}\big(\Phi_0(\,\cdot\,, u+\lambda v)\, L^{-1}x\big)\ . \end{aligned}$$

But by Lemma 3.6 on $[0,T] \times \Omega$

$$\begin{aligned} \big|\Psi(\,\cdot\,, u+\lambda v) \cdot x\big| &\leq N^*\big(c^{-1}\,\Psi(\,\cdot\,, u+\lambda v)\big) + N(cx) \\ &\leq N(u+\lambda v) + \big[3f + N^*\big(c^{-1}\,\Psi(\,\cdot\,, 0)\big)\big] \cdot 1_{\{\mathbf{m}(E)<\infty\}} + N(cx) \end{aligned}$$

and

$$\big|\Phi_0(\,\cdot\,, u+\lambda v)\big|\, |L^{-1}x| \leq N^*\big(\Phi_0(\,\cdot\,, u+\lambda v)\big) + N(L^{-1}x) \leq \tilde{c}\, N(u+\lambda v) + N(L^{-1}x)\ ,$$

where for $\lambda \in [-1,1]$

$$N(u+\lambda v) = N\big(|u+\lambda v|\big) \leq N\big(|u| + |v|\big)\ .$$

So **(H1)** follows by the continuity of Ψ, Φ_0 in the spatial variable and Lebesgue's dominated convergence theorem.

(H2): Let $u, v \in V$. Then as above on $[0,T] \times \Omega$

$$\begin{aligned} &{}_{V^*}\big\langle A(\,\cdot\,, u) - A(\,\cdot\,, v), u - v\big\rangle_V \\ &\quad = -\mathbf{m}\Big(\big(\Psi(\,\cdot\,, u) - \Psi(\,\cdot\,, v)\big)\,(u-v)\Big) + h_t\ {}_{V^*}\langle u-v, u-v\rangle_V \\ &\qquad - \mathbf{m}\Big(\big(\Phi_0(\,\cdot\,, u) - \Phi_0(\,\cdot\,, v)\big)\, L^{-1}(u-v)\Big)\ . \end{aligned} \tag{3.21}$$

In case **(A1)** holds, the latter is dominated by $\|h\|_{L^\infty(\mathrm{d}t\times P)}\|u-v\|_H^2$. In case **(A2)** holds, the absolute value of the last summand is by (Φ1) dominated by

$$\begin{aligned}
&\sum_{i=1}^m \delta_i \|L^{-1}\|^{-1}_{\mathscr{L}(L^{r_i+1}(\mathbf{m}))}\, \mathbf{m}\big(|u-v|^{r_i}\,\big|L^{-1}(u-v)\big|\big)\\
&\quad\le \sum_{i=1}^m \delta_i \Big(\mathbf{m}(|u-v|^{r_i+1})\Big)^{r_i/(r_i+1)}\, \mathbf{m}(|u-v|^{r_i+1})^{1/(r_i+1)}\\
&\quad\le \mathbf{m}\Big(\big(\Psi(\,\cdot\,,u)-\Psi(\,\cdot\,,v)\big)\,(u-v)\Big)\ ,
\end{aligned}$$

where we first used Hölder's inequality and then (Ψ1)$'$. So, also in case **(A2)** holds, the right hand side of (3.21) is dominated by $\|h\|_{L^\infty(\mathrm{d}t\times P)}\|u-v\|_H^2$ on $[0,T]\times\Omega$, so **(H2)** is proved.

(H3): Let $v\in V$. Then as above on $[0,T]\times\Omega$,

$$_{V^*}\big\langle A(\,\cdot\,,v),v\big\rangle_V = -\mathbf{m}\big(\Psi(\,\cdot\,,v)\,v\big) + h_t\ {}_{V^*}\langle v,v\rangle_V - \mathbf{m}\big(\Phi_0(\,\cdot\,,v)\,L^{-1}v\big)\ .$$

By (Ψ2), (Ψ2)$'$, respectively, and (Φ2), this is on $[0,T]\times\Omega$ dominated by

$$-(1-\varepsilon)\,\mathbf{m}\big(N(v)\big) + f\,\mathbf{m}(E)\,1_{\{\mathbf{m}(E)<\varepsilon\}} + \|h\|_{L^\infty(\mathrm{d}t\times P)}\,\|v\|_H^2\ .$$

By the definition of R (cf. (3.18)) condition **(H3)** now follows.

(H4): Let $u,v\in V$. Then as above on $[0,T\times\Omega]$

$$\begin{aligned}
&\big|\,{}_{V^*}\langle A(\,\cdot\,,v),u\rangle_V\big|\\
&\quad\le \mathbf{m}\big(|\Psi(\,\cdot\,,v)|\,|u|\big) + \|h\|_{L^\infty(\mathrm{d}t\times P)}\,\|v\|_H\,\|u\|_H + \mathbf{m}\big(|\Phi_0(\,\cdot\,,v)|\,|L^{-1}u|\big)\ .
\end{aligned}$$

But by Lemma 3.6 on $[0,T]\times\Omega$

$$\begin{aligned}
c^{-1}\,\big|\Psi(\,\cdot\,,v)\big|\,|u| &\le N^*\big(c^{-1}\,\Psi(\,\cdot\,,v)\big) + N(u)\\
&\le N(v) + \big[3f + N^*\big(c^{-1}\,\Psi(\,\cdot\,,0)\big)\big]\cdot 1_{\{\mathbf{m}(E)<\infty\}} + N(u)\ .
\end{aligned}$$

Furthermore, by (Φ2), Hölder's and Young's inequality

$$\begin{aligned}
\mathbf{m}\big(|\Phi_0(\,\cdot\,,v)|\,|L^{-1}u|\big) &\le \varepsilon\sum_{i=1}^m \varepsilon_i\big(\mathbf{m}(|v|^{r_i+1})\big)^{r_i/(r_i+1)}\,\big(\mathbf{m}(|u|^{r_i+1})\big)^{1/(r_i+1)}\\
&\le \varepsilon\sum_{i=1}^m \varepsilon_i\bigg(\frac{r_i}{r_i+1}\,\mathbf{m}\big(|v|^{r_i+1}\big) + \frac{1}{r_i+1}\,\mathbf{m}\big(|u|^{r_i+1}\big)\bigg)\\
&\le \varepsilon\big(N(v)+N(u)\big)\ ,
\end{aligned}$$

and **(H4)** follows. □

Now our general results from Section 2 apply to this case.

Theorem 3.9. *Assume that conditions* **(L1)**, **(N1)** *and* **(N2)** *hold and let* Φ, Ψ *be as in* (3.11) *satisfying* **(A1)** *or* **(A2)** *for some Young function* N *and let* B *be as in* (**??**). *Then for any* $X_0 \in L^2(\Omega \to H, \mathscr{F}_0; P)$ *the SPDE*

$$\mathrm{d}X_t = \left[L\Psi(t, X_t) + \bar{\Phi}(t, X_t)\right] \mathrm{d}t + B(t, X_t)\, \mathrm{d}W_t$$

has a unique continuous H*-valued solution (in the sense of Definition 2.1 with* H *being the Green space of* L *and* K *defined by* (3.19)*) with initial condition* X_0*. The solution satisfies*

$$\mathbb{E} \sup_{t\in[0,T]} \|X_t\|_H^2 < \infty$$

and all further assertions from Theorem 2.1 and Proposition 2.2 also hold.

4 Appendix: the Itô formula for the square of the norm

In this section we aim to prove the Itô formula for $\|X_t\|_H^2$ which has been used in the paper. This formula has been established in [15] for the case where $K := L^p([0,T] \times \Omega \to V; \mathrm{d}t \times P)$ for some $p > 1, R := \|\cdot\|_V^p, W_1 = W_2 := \|\cdot\|^{1/p}$.

As in [15], let us first consider the piecewise-constant approximation of a process in K by using an argument of Doob (see [10, Ch. IX, §5], or [20, page 75]).

Lemma 4.1. *Assume* **(K)** *and let* $X : [0,T] \times \Omega \to V^*$ *be* $\mathscr{B}([0,T]) \times \mathscr{F}/\mathscr{B}(V^*)$*-measurable such that* $X = \bar{X}$ $\mathrm{d}t \times P$*-a.e. for some* $\mathrm{d}t \times P$*-version* $\bar{X}$ *of an element in* K*. Then there exists a sequence of partitions* $I_l := \{0 = t_0^l < t_1^l < \cdots < t_{k_l}^l = T\}$ *such that* $I_l \subset I_{l+1}$ *and* $\delta(I_l) := \max_i(t_i^l - t_{i-1}^l) \to 0$ *as* $l \to \infty$ *and for*

$$\bar{X}^l := \sum_{i=2}^{k_l} 1_{[t_{i-1}^l, t_i^l)} X_{t_{i-1}^l}, \quad \tilde{X}^l := \sum_{i=1}^{k_l - 1} 1_{[t_{i-1}^l, t_i^l)} X_{t_i^l}, \quad l \geq 1$$

we have $\bar{X}^l, \tilde{X}^l$ *are* V*-valued* P*-a.s. and* ($\mathrm{d}t \times P$*-versions of*) *elements in* K*. Furthermore,*

$$\lim_{l\to\infty} \left\{\|\bar{X} - \bar{X}^l\|_K + \|\bar{X} - \tilde{X}^l\|_K\right\} = 0.$$

In particular, $X_{t_i^l} \in V$ *for all* $l \geq 1, 1 \leq i \leq k_l - 1$.

Proof. For simplicity we assume that $T = 1$ and let X be extended to $\mathbb{R} \times \Omega$ by setting $X|_{[0,1]^c} = 0$. Since $L^p([0,T] \times \Omega \to V; \mathrm{d}t \times P)$ is dense in K, it is easy to see from **(K)** that there exists $\Omega' \subset \Omega$ with full probability such that $\int_0^1 R(\bar{X}_t)\mathrm{d}t) < \infty$ holds on Ω' and for any $\omega \in \Omega'$, there exists a sequence $\{f_n\} \subset C(\mathbb{R}; V)$ with compact support such that

$$\int_{\mathbb{R}} R(4f_n(t) - 4\bar{X}_t(\omega))\mathrm{d}t \leq \frac{1}{n}, \qquad n \geq 1.$$

Thus for every $n \geq 1$ it follows from **(K)**(iii), since $R(v) \to 0$ as $v \to 0$, and $f_n \in C_0(\mathbb{R}; V)$, that

$$\begin{aligned}
&\limsup_{s\to\infty} \int_R R(\bar{X}_{t+1}(\omega) - \bar{X}_t(\omega))\mathrm{d}t \\
&\leq C \limsup_{s\to\infty} \int_R \big\{R(2\bar{X}_{s+t}(\omega) - 2f_n(t+s) + 2\bar{X}_t(\omega) - 2f_n(t)) + R(2f_n(t+s) - 2f_n(t))\big\}\mathrm{d}t \\
&\leq C^2 \limsup_{s\to\infty} \int_R \big\{R(4\bar{X}_{s+t}(\omega) - 4f_n(t+s)) + R(4\bar{X}_t(\omega) - 4f_n(t))\big\}\mathrm{d}t \leq \frac{2C^2}{n}.
\end{aligned}$$

Note here that since by continuity R is bounded on sufficiently small balls around 0 and since each f_n is uniformly continuous we really have by dominated convergence that for all $n \geq 1$

$$\lim_{s\to 0} \int_{\mathbb{R}} R(2f_n(s+t) - 2f_n(t))\mathrm{d}t = 0.$$

Letting $n \to \infty$ we arrive at

$$\lim_{\delta\to 0} \int_{\mathbb{R}} R(\bar{X}_{\delta+s}(\omega) - \bar{X}_s(\omega))\mathrm{d}s = 0, \quad \omega \in \Omega'. \tag{4.1}$$

Now, given $t \in \mathbb{R}$, let $[t]$ denote the biggest integer $\leq t$. Let $\gamma_n(t) := 2^{-n}[2^n t]$, $n \geq 1$. Shifting the integral in (4.1) by t and taking $\delta = \gamma_n(t) - t$ we obtain

$$\lim_{n\to\infty} \int_{\mathbb{R}} R(\bar{X}_{\gamma_n(t)+s} - \bar{X}_{t+s})\mathrm{d}s = 0 \ \text{ on } \Omega'.$$

Moreover, since $R(0) = 0$ and by **(K)**(iii) and Remark 2.1(3)

$$\begin{aligned}
\int_0^1 R(\bar{X}_{\gamma_n(t)+s} - \bar{X}_{t+s})\mathrm{d}s &\leq 1_{[-2,2]}(t) C \int_{\mathbb{R}} \big\{R(2\bar{X}_{\gamma_n(t)+s}) + R(2\bar{X}_{t+s})\big\}\mathrm{d}s \\
&= 2C 1_{[-2,2]}(t) \int_0^1 R(2\bar{X}_s)\mathrm{d}s < \infty.
\end{aligned}$$

So, by the dominated convergence theorem, we obtain that

(4.2) $$0=\lim_{n\to\infty}\mathbb{E}\int_{\mathbb{R}}\mathrm{d}t\int_0^1 R(\bar X_{\gamma_n(t)+s}-\bar X_{t+s})\mathrm{d}s\geq\lim_{n\to\infty}\mathbb{E}\int_0^1\mathrm{d}s\int_0^1 R(\bar X_{\gamma_n(t-s)+s}-\bar X_t)\mathrm{d}t\,.$$

Given $s\in[0,1)$ and $n\geq 1$, let the partition $I_n(s)$ be defined by

$$t_0^n(s):=0,\ t_i^n(s):=\Big(s-\frac{[2^ns]}{2^n}\Big)+\frac{i-1}{2^n},\ 1\leq i\leq 2^n,\ t_{2^n+1}^n(s):=1.$$

Then, for $t\in[t_{i-1}^n(s),t_i^n(s))$ one has $t-s\in[2^{-n}(i-[2^ns]-2),2^{-n}(i-[2^ns]-1))$ and hence,

$$\gamma_n(t-s)+s=\big\{2^{-n}(i-[2^ns]-2)+s\big\}^+=t_{i-1}^n(s),\quad 1\leq i\leq 2^n+2.$$

Therefore, (4.2) implies

$$\lim_{n\to\infty}\mathbb{E}\int_0^1\mathrm{d}s\int_0^1 R(\bar X_t-\bar X_t^{n,s})\mathrm{d}t=0,$$

where $\bar X^{n,s}$ is the process defined as $\bar X^l$ for the partition $I_n(s)$ but with $X_{t_i^l(s)}$ replaced by $\bar X_{t_i^l(s)}$. Similarly, the same holds for $\tilde X^{n,s}$ in place of $\bar X^{n,s}$ by using $\tilde\gamma_n:=\gamma_n+2^{-n}$ instead of γ_n, where $\tilde X^{n,s}$ is defined as $\tilde X^l$ for the partition $I_n(s)$ but with $X_{t_i^l(s)}$ replaced by $\bar X_{t_i^l(s)}$. Hence, there exist a subsequence $n_k\to\infty$ and a ds-zero set $N_1\in\mathscr{B}([0,1])$ such that

$$\lim_{k\to\infty}\mathbb{E}\int_0^1\big\{R(\bar X_t-\bar X_t^{n_k,s})+R(\bar X_t-\tilde X_t^{n_k,s})\big\}\mathrm{d}t=0,\quad s\in[0,1]\setminus N_1.$$

Since for $1\leq i\leq 2^n$ the maps $s\mapsto t_i^n(s)$ are piecewise C^1-diffeomorphisms, the image measures of ds under these maps are absolutely continuous with respect to ds. Therefore, since $\bar X=X$ d$s\times P$-a.e., there exists a ds-zero set $N_2\in\mathscr{B}([0,1])$ such that

$$\bar X_{t_i^n(s)}=X_{t_i^n(s)}\ \text{ a.s.},\quad s\in[0,1]\setminus N_2, 1\leq i\leq 2^n, n\geq 1.$$

Since for any $s\in[0,1]\setminus(N_1\cup N_2)$ one has $\mathbb{E}R(\bar X_{t_i^n(s)})<\infty$ and by Remark 2.1(3) $z\in K$ if and only if $z\in L^1([0,1]\times\Omega;\mathrm{d}t\times P)$ with $\mathbb{E}\int_0^1 R(z_t)\mathrm{d}t<\infty$, the map

$$[0,1]\times\Omega\ni(t,\omega)\mapsto z_t^{i,n}:=X_{t_i^n(s)}(\omega)\in V$$

is once again in K. Therefore, fixing $s\in[0,1]\setminus(N_1\cup N_2)$, the sequence of the corresponding partitions $I_{n_l}(s), l\geq 1$, has all properties of the assertion. □

Remark 4.1. As follows from the above proof all the partition points $t_i^l, l\geq 1, 1\leq i\leq k_l-1$, in the assertion of Lemma 4.1 can be chosen outside an a priori given Lebesgue zero set in $[0,T]$.

Theorem 4.2. *Assume* **(K)**. *Let* $X_0 \in L^2(\Omega \to H; \mathscr{F}_0; P)$, $Y \in K^*$, Y *progressively measurable, and* $Z \in J$. *Define the continuous* V^**-valued adapted process*

$$X_t := X_0 + \int_0^t Y_s \mathrm{d}s + \int_0^t Z_s \mathrm{d}W_s, \quad t \in [0,T].$$

If there exists a $\mathrm{d}t \times P$*-version* $\bar{X}$ *of an element in* K *such that* $X = \bar{X}$ $\mathrm{d}t \times P$*-a.e., then* X_t *is a continuous process on* H *such that* $\mathbb{E}\sup_{t\le T}\|X_t\|_H^2 < \infty$ *and* P*-a.s.*

$$\|X_t\|_H^2 = \|X_0\|_H^2 + \int_0^t (2\ {}_{V^*}\langle Y_s, \bar{X}_s\rangle_V + \|Z_s\|_{\mathscr{L}_{HS}}^2)\mathrm{d}s + 2\int_0^t \langle Z_s \mathrm{d}W_s, X_s\rangle_H, \quad t \in [0,T]. \tag{4.3}$$

Setting ${}_{V^*}\langle Y_s, X_s\rangle_V = 0$ *for* $X_s \notin V$ *we may replace in the right-hand side* $\bar{X}_s$ *by* X_s.

Proof. Since $M_t := \int_0^t Z_s \mathrm{d}W_s$ is already a continuous martingale on H and since $Y \in K^* \subset L^{p/(p-1)}([0,T]\times\Omega \to V^*; \mathrm{d}t \times P)$ is progressively measurable, $\int_0^t Y_s \mathrm{d}s$ is a continuous adapted process on V^*. Thus, X is a continuous adapted process on V^*, hence is $\mathscr{B}([0,T]) \times \mathscr{F}/\mathscr{B}(V^*)$-measurable. Then, due to Lemma 4.1 and Remark 2.1(5), the remainder of the proof is similar to that of [15, Theorem I.3.1]. We include a complete proof below for the readers' convenience.

(a) Note that by [15, Lemma I.4.2],

$$\begin{aligned}\|X_t\|_H^2 =& \|X_s\|_H^2 + 2\int_s^t {}_{V^*}\langle Y_r, X_t\rangle_V \mathrm{d}r + 2\langle X_s, M_t - M_s\rangle_H \\ &+ \|M_t - M_s\|_H^2 - \|X_t - X_s - M_t + M_s\|_H^2\end{aligned} \tag{4.4}$$

holds for all $t > s$ such that $X_t, X_s \in V$. Indeed, this follows immediately by noting that

$$\begin{aligned}&\|M_t - M_s\|_H^2 - \|X_t - X_s - M_t + M_s\|_H^2 + 2\langle X_s, M_t - M_s\rangle_H \\ &= 2\langle X_t, M_t - M_s\rangle_H - \|X_t - X_s\|_H^2 \\ &= 2\langle X_t, X_t - X_s\rangle_H - 2\int_s^t {}_{V^*}\langle Y_r, X_t\rangle_V \mathrm{d}r - \|X_t\|_H^2 - \|X_s\|_H^2 + 2\langle X_t, X_s\rangle_H \\ &= \|X_t\|_H^2 - \|X_s\|_H^2 - 2\int_s^t {}_{V^*}\langle Y_r, X_t\rangle_V \mathrm{d}r.\end{aligned}$$

(b) As in [15, Lemma 4.3], we have

$$\mathbb{E}\sup_{t\in[0,T]} \|X_t\|_H^2 < \infty. \tag{4.5}$$

Indeed, by (4.4), for any $t = t_i^l \in I_l \setminus \{0, T\}$ given in Lemma 4.1,

$$
\begin{aligned}
\|X_t\|_H^2 - \|X_0\|_H^2 &= \sum_{j=0}^{i-1}(\|X_{t^l_{j+1}}\|_H^2 - \|X_{t^l_j}\|_H^2) \\
&= 2\int_0^t {}_{V^*}\langle Y_s, \tilde{X}^l_s\rangle_V \mathrm{d}s + 2\int_0^t \langle \bar{X}^l_s, Z_s \mathrm{d}W_s\rangle_H + 2\langle X_0, \int_0^{t^l_1} Z_s \mathrm{d}W_s\rangle_H \\
&\quad + \sum_{j=0}^{i-1}\big(\|M_{t^l_{j+1}} - M_{t^l_j}\|_H^2 - \|X_{t^l_{j+1}} - X_{t^l_j} - M_{t^l_{j+1}} + M_{t^l_j}\|_H^2\big).
\end{aligned} \tag{4.6}
$$

By Remark 2.1(5), Lemma 4.1 and **(K)**(ii),

$$
\mathbb{E}\int_0^T |\, {}_{V^*}\langle Y_s, \tilde{X}^l_s\rangle_V|\mathrm{d}s \le c\|\tilde{X}^l\|_K \le c_1 \tag{4.7}
$$

for some constant $c_1 > 0$ independent of l. Moreover, by the Burkholder-Davis inequality,

$$
\begin{aligned}
\mathbb{E}\sup_{t\le T}\bigg|\int_0^t \langle \bar{X}^l_s, Z_s\mathrm{d}W_s\rangle_H\bigg| &\le 3\mathbb{E}\bigg(\int_0^T \|\bar{X}^l_s\|_H^2 \mathrm{d}\langle M\rangle_s\bigg)^{1/2} \\
&\le \frac{1}{4}\mathbb{E}\sup_{k_l-1\ge j\ge 0}\|X_{t^l_j}\|_H^2 + 9\mathbb{E}\langle M\rangle_T,
\end{aligned} \tag{4.8}
$$

where $\langle M\rangle_t$ is the increasing process part of $\|M_t\|_H^2$. Finally, for all $l \ge 1$

$$
\limsup_{l\to\infty}\mathbb{E}\sum_{j=0}^{k_l-1}\|M_{t^l_{j+1}} - M_{t^l_j}\|_H^2 \le \mathbb{E}\langle M\rangle_T. \tag{4.9}
$$

Combining (4.6)–(4.9), we obtain

$$
\mathbb{E}\sup_{t\in I_l\setminus\{T\}}\|X_t\|_H^2 \le c_2
$$

for some constant $c_2 > 0$ independent of l. Therefore, letting $l \uparrow \infty$ and setting $I := \cup_{l\ge 1} I_l \setminus \{T\}$, with I_l as in Lemma 4.1, we obtain

$$
\mathbb{E}\sup_{t\in I}\|X_t\|_H^2 < \infty.
$$

Since for all $t \in [0, T]$

$$
\sum_{j=1}^N {}_{V^*}\langle X_t, e_j\rangle_V^2 \uparrow \|X_t\|_H^2 \text{ as } N \uparrow \infty,
$$

where as usual for $x \in V^* \setminus H$ we set $\|x\|_H := \infty$, it follows that $t \mapsto \|X_t\|_H$ is lower semicontinuous P-a.s. Since I is dense in $[0,T]$, we arrive at $\sup_{t\le T}\|X_t\|_H^2 = \sup_{t\in I}\|X_t\|_H^2$. Thus, (4.5) holds.

(c) Next, since $\sup_{t\in[0,T]}\|X_t\|_H < \infty$, the proof of [15, Lemma 4.5] implies

$$\lim_{l\to\infty}\sup_{t\le T}\left|\int_0^t \langle X_s - \bar{X}_s^l, Z_s \mathrm{d}W_s\rangle_H\right| = 0 \ \text{ in probability.} \tag{4.10}$$

We repeat the proof here for completeness. We first note that because of (b) and its continuity in V^* the process X is weakly continuous in H, and therefore, since $\mathscr{B}(H)$ is generated by H^*, progressively measurable as an H-valued process. Hence, for any $n \ge 1$ the process $P_n X_s$, where P_n is as defined in (2.6), is continuous in H so that

$$\lim_{l\to\infty}\int_0^T \|P_n(X_s - \bar{X}_s^l)\|_H^2 \mathrm{d}\langle M\rangle_s = 0.$$

Therefore, it suffices to show that for any $\varepsilon > 0$,

$$\begin{aligned} &\lim_{n\to\infty}\sup_{l\ge 1} P\left(\sup_{t\le T}\left|\int_0^t \langle (1-P_n)\bar{X}_s^l, Z_s\mathrm{d}W_s\rangle_H\right| > \varepsilon\right) = 0,\\ &\lim_{n\to\infty} P\left(\sup_{t\le T}\left|\int_0^t \langle (1-P_n)X_s, Z_s\mathrm{d}W_s\rangle_H\right| > \varepsilon\right) = 0\,. \end{aligned} \tag{4.11}$$

For any $\delta \in (0,1)$ and $N > 1$ by the Burkholder-Davis inequality

$$\begin{aligned} &P\left(\sup_{t\le T}\left|\int_0^t \langle (1-P_n)\bar{X}_s^l, Z_s\mathrm{d}W_s\rangle_H\right| > \varepsilon\right) \le \frac{3\delta}{\varepsilon} + P\left(\int_0^T \|\bar{X}_s^l\|_H^2 \mathrm{d}\langle (1-P_n)M\rangle_s > \delta\right)\\ &\le \frac{3\delta}{\varepsilon} + P\left(\sup_{t\le T}\|X_t\|_H > N\right) + \frac{N^2}{\delta}\mathbb{E}\langle (1-P_n)M\rangle_T. \end{aligned}$$

By letting first $n\to\infty$, then $N\to\infty$ and finally $\delta\to 0$, we prove the first equality in (4.11). Similarly, the second equality also holds.

(d) As in [15, Lemma I.4.6], we first prove (4.3) for $t \in I$. So, fix $t \in I$. We may assume that $t \ne 0$. In this case for each sufficiently large $l \ge 1$ there exists a unique $0 < i < k_l$ such that $t = t_i^l$. We have $X_{t_j^l} \in V$ a.s. for all j. By Lemma 4.1, Remark 2.1(5) and (4.10) the sum of the first three terms in the right-hand side of (4.6) converges in probability to $2\int_0^t {}_{V^*}\langle Y_s, \bar{X}_s\rangle_V \mathrm{d}s + 2\int_0^t \langle X_s, Z_s\mathrm{d}W_s\rangle_H$. Then

$$\|X_t\|_H^2 - \|X_0\|_H^2 = 2\int_0^t {}_{V^*}\langle Y_s, \bar{X}_s\rangle_V \mathrm{d}s + 2\int_0^t \langle X_s, Z_s\mathrm{d}W_s\rangle_H + \langle M\rangle_t - \varepsilon_0,$$

where

$$\varepsilon_0 := \lim_{l\to\infty} \sum_{t^l_{j+1}\le t} \|X_{t^l_{j+1}} - X_{t^l_j} - M_{t^l_{j+1}} + M_{t^l_j}\|_H^2$$

exists. So, to prove (4.3) for t as above, it suffices to show that $\varepsilon_0 = 0$. Since for any $\varphi \in V$,

$$\langle X_{t^l_{j+1}} - X_{t^l_j} - M_{t^l_{j+1}} + M_{t^l_j}, \varphi\rangle_H = \int_{t^l_j}^{t^l_{j+1}} {}_{V^*}\langle Y_s, \varphi\rangle_V \mathrm{d}s,$$

letting $\tilde{M}^l$ and $\bar{M}^l$ be defined as $\tilde{X}^l$ and $\bar{X}^l$ respectively, for M replacing X, we obtain for every $n \ge 1$

$$\begin{aligned}\varepsilon_0 = \lim_{l\to\infty}\Bigg(&\int_0^t {}_{V^*}\langle Y_s, \tilde{X}^l_s - \bar{X}^l_s - P_n(\tilde{M}^l_s - \bar{M}^l_s)\rangle_V \mathrm{d}s \\ &- \big\langle X(t^l_1) - X(0) - M(t^l_1) + M(0), P_n M(0) - X(0)\big\rangle_H \\ &- \sum_{t^l_{j+1}\le t} \langle X_{t^l_{j+1}} - X_{t^l_j} - M_{t^l_{j+1}} + M_{t^l_j}, (1-P_n)(M_{t^l_{j+1}} - M_{t^l_j})\rangle_H \Bigg).\end{aligned}$$

By the weak continuity of X in H the second term converges to zero as $l \to \infty$. Lemma 4.1 and Remark 2.1(5) imply that $\int_0^t {}_{V^*}\langle Y_s, \tilde{X}^l_s - \bar{X}^l_s\rangle_V \mathrm{d}s \to 0$ in probability. Moreover, since $P_n M_s$ is a continuous process in V, $\int_0^t {}_{V^*}\langle Y_s, P_n(\tilde{M}^l_s - \bar{M}^l_s)\rangle_V \mathrm{d}s \to 0$ as $l \to \infty$. Thus,

$$\begin{aligned}\varepsilon_0 \le &\lim_{l\to\infty}\Big(\sum_{t^l_{j+1}\le t} \|X_{t^l_{j+1}} - X_{t^l_j} - M_{t^l_{j+1}} + M_{t^l_j}\|_H^2\Big)^{1/2}\Big(\sum_{t^l_{j+1}\le t} \|(1-P_n)(M_{t^l_{j+1}} - M_{t^l_j})\|_H^2\Big)^{1/2} \\ =&\varepsilon_0^{1/2}\langle (1-P_n)M\rangle_t^{1/2},\end{aligned}$$

which goes to zero as $n \to \infty$ since M_t is a square-integrable martingale in H. Therefore, $\varepsilon_0 = 0$.

(e) Now, take $\Omega' \in \mathscr{F}$ with full probability such that the limit in (4.10) is a pointwise limit in Ω' for some subsequence (denoted again by $l \to \infty$) and (4.3) holds for all $t \in I$ on Ω'. If $t \notin I$, for any $l \ge 1$ there exists a unique $j(l) < k_l$ such that $t \in (t^l_{j(l)}, t^l_{j(l)+1}]$. Letting $t(l) := t^l_{j(l)}$, we have $t(l) \uparrow t$ as $l \uparrow \infty$. By (4.3) for $t \in I$, for any $l > m$ we have, on Ω' (since the above applies to $X - X_{t(m)}$ replacing X)

$$
\begin{aligned}
\|X_{t(l)} - X_{t(m)}\|_H^2 &= 2\int_{t(m)}^{t(l)} {}_{V^*}\langle Y_s, \bar{X}_s - X_{t(m)}\rangle_V \mathrm{d}s \\
&\qquad + 2\int_{t(m)}^{t(l)} \langle X_s - X_{t(m)}, Z_s \mathrm{d}W_s\rangle_H + \langle M\rangle_{t(l)} - \langle M\rangle_{t(m)} \\
&= 2\int_0^T 1_{[t(m),t(l)]}(s)\, {}_{V^*}\langle Y_s, \bar{X}_s - \bar{X}_s^m\rangle_V \mathrm{d}s \\
&\qquad + 2\int_0^T 1_{[t(m),t(l)]}(s) \langle X_s - \bar{X}_s^m, Z_s \mathrm{d}W_s\rangle_H + \langle M\rangle_{t(l)} - \langle M\rangle_{t(m)}.
\end{aligned} \tag{4.12}
$$

Thus, by the continuity of $\langle M\rangle_t$ and (4.10) (holding pointwise on Ω'), we have that

$$
\lim_{m\to\infty} \sup_{l>m} \left\{ 2\left| \int_0^T 1_{[t(m),t(l)]}(s)\langle\, X_s - \bar{X}_s^m, Z_s \mathrm{d}W_s\rangle_H \right| + |\langle M\rangle_{t(l)} - \langle M\rangle_{t(m)}| \right\} = 0 \tag{4.13}
$$

holding on Ω'. Furthermore, by Lemma 4.1 and by Remark 2.1(5), selecting another subsequence if necessary we have for some $\Omega'' \in \mathscr{F}$ with full probability and $\Omega'' \subset \Omega'$, that on Ω''

$$
\lim_{m\to\infty} \int_0^T |\, {}_{V^*}\langle Y_s, \bar{X}_s - \bar{X}_s^m\rangle_V| \mathrm{d}s = 0.
$$

Since for all $t \notin I$

$$
\sup_{l>m} \int_{t(m)}^{t(l)} |\, {}_{V^*}\langle Y_s, \bar{X}_s - \bar{X}_s^m\rangle_V| \mathrm{d}s \le \int_0^T |\, {}_{V^*}\langle Y_s, \bar{X}_s - \bar{X}_s^m\rangle_V| \mathrm{d}s,
$$

we have that

$$
\lim_{m\to\infty} \sup_{l>m} \int_{t(m)}^{t(l)} {}_{V^*}\langle Y_s, \bar{X}_s - \bar{X}_s^m\rangle_V \mathrm{d}s = 0
$$

holds on Ω''.

Combining this with (4.12) and (4.13), we conclude that

$$
\lim_{m\to\infty} \sup_{l\ge m} \|X_{t(l)} - X_{t(m)}\|_H^2 = 0
$$

holds on Ω''. Thus, $(X_{t(l)})_{l\in\mathbb{N}}$ converges in H on Ω''. Since we know that $X_{t(l)} \to X_t$ in V^*, it converges to X_t strongly in H on Ω''. Therefore, by the formula for $t(l)$ and letting $l \to \infty$, we obtain (4.3) on Ω'' also for all $t \notin I$.

Finally, since the right hand side of (4.3) is on Ω'' continuous in $t \in [0, T]$, so must be its left hand side, i.e. $t \mapsto \|X_t\|_H$ is continuous on $[0, T]$. Therefore, the weak continuity of X_t in H implies its strong continuity in H. □

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