Concentration of Invariant Measures for Stochastic Generalized Porous Media Equations

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July 4, 2006

Abstract

By using Bernstein functions, existence and concentration properties are studied for invariant measures of the infinitesimal generators associated to a large class of stochastic generalized porous media equations. In particular, results derived in [4] are extended to equations with non-constant and stronger noises. Analogous results are also proved for invariant probability measures for strong solutions.

AMS subject Classification: 60H15, 60A10
Keywords: Particle system, reaction-diffusion model, Dirichlet form, functional inequality.

1 Introduction

Let $(E, \mathscr{M}, \mathbf{m})$ be a probability space and $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ a Dirichlet form on $L^2(\mathbf{m})$, whose generator $(L, \mathscr{D}(L))$ has discrete spectrum. Let

$$0 > -\lambda_1 \geq -\lambda_2 \geq \cdots \to -\infty$$

be all eigenvalues of L counting multiplicity, and let $\{e_i\}$ be the corresponding unit eigenfunctions. Throughout the paper, let $r > 1$ be a fixed number and assume that $e_i \in L^{r+1}(\mathbf{m})$ for all $i \geq 1$. Let

$$H^1 := \Big\{x \in L^2(\mathbf{m}):\ \sum_{i=1}^{\infty} \lambda_i\, \mathbf{m}(e_i x)^2 < \infty\Big\}$$

and let $H := H^{-1}$ be the dual space of H^1 w.r.t. $L^2(\mathbf{m})$. Thus, H is the completion of $L^2(\mathbf{m})$ under the inner product

$$\langle x, y\rangle_H := \sum_{i=1}^{\infty} \frac{1}{\lambda_i}\langle e_i, x\rangle\langle e_i, y\rangle,$$

where $\langle\, ,\, \rangle$ is the inner product in $L^2(\mathbf{m})$.

Let Ψ and Φ be two continuous functions on $\mathbb{R}$ satisfying the following assumptions:

(H1) *either* $\Psi(0) = 0$ *or* $1 \in \mathscr{D}(L)$;

(H2) *there exist constants* $C, \eta > 0$ *and* $\sigma \geq 0$ *such that* $|\Psi(s)| + |\Phi(s)| \leq C(1 + |s|^r)$ *and*

$$(s-t)(\Psi(s) - \Psi(t)) \geq \eta|\xi_r(s) - \xi_r(t)|^2 + \sigma(s-t)^2, \quad s, t \in \mathbb{R},$$

where $\xi_r(s) := |s|^{(r+1)/2} sgn(s)$.

Next, let $\mathscr{L}_{HS}(L^2(\mathbf{m}); H)$ be the set of all Hilbert-Schmidt operators from $L^2(\mathbf{m})$ to H, and let $Q : L^{r+1}(\mathbf{m}) \to \mathscr{L}_{HS}(L^2(\mathbf{m}); H)$ be a measurable and bounded mapping. Let W_t be the cylindrical Brownian motion on $L^2(\mathbf{m})$, that is, $W_t = \{B_t^i e_i\}_{i\geq 1}$ for a $\{B_t^i\}$ a sequence of independent one-dimensional Brownian motions on a complete filtered probability space $(\Omega, \mathscr{F}, \mathscr{F}_t, P)$.

The main purpose of this paper is to study the invariant measures, in particular, their support (concentration) properties, associated to the following stochastic generalized porous medium equation (see [1, 2] and references within for the study of porous media equations):

$$\mathrm{d}X_t = (L\Psi(X_t) + \Phi(X_t))\mathrm{d}t + Q(X_t)\mathrm{d}W_t. \tag{1.1}$$

When $L = \Delta$ on a regular domain in $\mathbb{R}^d$, this equation has been studied intensively in [3, 4, 6, 7], where both weak solutions and invariant measures of the infinitesimal generator of (1.1) are investigated. Recently, under the a rather general framework, the existence, uniqueness and ergodicity of strong solutions to (1.1) have been proved in [8] and [12].

To introduce the infinitesimal generator L on the space of cylindrical functions, let

$$\begin{aligned}
&\mathscr{F}C_b^\infty := \big\{f(\langle\cdot, e_1\rangle, \cdots, \langle\cdot, e_N\rangle):\ N \geq 1, f \in C_b^\infty(\mathbb{R}^N)\big\},\\
&b_i(x) := \int_E \big(\Psi(x)Le_i + \Phi(x)e_i\big)\mathrm{d}\,\mathbf{m} =:\ \mathbf{m}\big(\Psi(x)Le_i + \Phi(x)e_i\big),\\
&q_{ij}(x) := \langle Q(x)e_i, e_j\rangle, \quad x \in L^{r+1}(\mathbf{m}), i, j \geq 1.
\end{aligned}$$

Then the infinitesimal generator L associated to (1.1) is expressed as

$$Lf := \sum_{i,j,k=1}^{\infty} q_{ki}q_{kj}\partial_i\partial_j f + \sum_{i=1}^{\infty} b_i\partial_i f, \quad f \in \mathscr{F}C_b^\infty,$$

where for $f(x) := f(\langle x, e_1\rangle, \cdots, \langle x, e_N\rangle)$,

$$\partial_i f(x) := (\partial_i f)(\langle x, e_1\rangle, \cdots, \langle x, e_N\rangle).$$

Recall that a probability measure μ on H is called an (infinitesimally) invariant measure of L, denoted by $L^*\mu = 0$, if $\mu(L^{r+1}(\mathbf{m})) = 1$ and

$$\int Lf \mathrm{d}\mu = 0, \quad f \in \mathscr{F}C_b^\infty.$$

We first study the existence and the concentration of μ using Bernstein functions. Recall that a positive function $f \in C[0,\infty)$ is called a Bernstein function if $f \in C_b^\infty(0,\infty)$ with $(-1)^n f^{(n)} \le 0$ for all $n \ge 1$. It is well-known that for any Bernstein function f, the operator $-f(-L)$ is still a sub-Markovian generator (cf. [11, Chapter 5]. Some typical examples of f are s^ε ($\varepsilon \in [0,1]$) and $\log(1+s)$.

Theorem 1.1. *Assume* **(H1), (H2)**. *Let $Q(x)e_i := q_i(x)e_i$ for some $q_i \in C(L^{r+1}(\mathbf{m}))$ such that $\sum_{i=1}^\infty \frac{q_i^2}{\lambda_i}$ is bounded.*

(1) *Let f be a Bernstein function with $f(\infty) = \infty$ and $\tilde{f}(s) := s/f(s)$ satisfying*

$$\mathbf{m}\big(\Phi(x)(\tilde{f}(-L))^{-1}x\big) \le \theta\,\mathbf{m}(|x|^{r+1}) + c, \quad x \in \mathrm{sgn}\{e_i : i \ge 1\} \tag{1.2}$$

for some constants $\theta < \eta f(\lambda_1)$ and $c > 0$. If

$$\sup_{L^{r+1}(\mathbf{m})} \sum_{i=1}^\infty \frac{f(\lambda_i)q_i^2}{\lambda_i} < \infty \tag{1.3}$$

then L has an invariant measure μ such that

$$\sum_{i=1}^\infty f(\lambda_i) \int \Big(\mathbf{m}(\xi_r(x)e_i)^2 + \sigma\,\mathbf{m}(xe_i)^2\Big)\mu(\mathrm{d}x) < \infty. \tag{1.4}$$

(2) *Assume that $|\Psi'(s)| \le C(1+|s|^r)$ and*

$$|\Phi(s)| \le \varepsilon|\Psi(s)| + C \text{ for some } C > 0, \varepsilon < \lambda_1 \text{ and all } s \in \mathbb{R}. \tag{1.5}$$

If

$$(1.6)\qquad \sup_{L^{r+1}(\mathbf{m})}\sum_{i}^{\infty} q_i^2\,\mathbf{m}(|e_i|^{r+1})^{2/(r+1)}<\infty,$$

then L has an invariant measure μ such that

$$(1.7)\qquad \sum_{i=1}^{\infty}\lambda_i\int_H \mathbf{m}(\Psi(x)e_i)^2\mu(\mathrm{d}x)<\infty.$$

If in particular $\tilde f$ is a Bernstein function, then (1.2) and (1.5) are implied by (see Lemma 2.1 below)

$$(1.8)\qquad \text{there exist } c>0 \text{ and } \varepsilon'<2\eta\lambda_1/(r+1) \text{ such that } |\Phi(s)|\le\varepsilon'|s|^r+c,\quad s\in\mathbb{R}.$$

Thus, we have the following consequence of Theorem 1.1 recovering the main results in [4]. In particular, Theorem 1.1(2) (or Corollary 1.2(1)) improves [4, Theorem 1.1 (ii)], where (1.6) is replaced by the stronger condition that $q_i's$ are constant such that $\sum_{i=1}^{\infty} q_i^2\|e_i\|_\infty^2<\infty$.

Corollary 1.2. *Consider the situation of Theorem 1.1 and let* (1.8) *hold.*

(1) *If* (1.6) *holds then $L^*\mu=0$ has a solution satisfying* (1.7).

(2) *Let f be a Bernstein function such that $f(\infty)=\infty$ and $\tilde f(s):=s/f(s)$ is also a Bernstein function* (*which is the case for e.g. $f(s):=s^\varepsilon,\varepsilon\in(0,1]$*). *Then* (1.3) *implies* (1.4) *for some invariant measure μ of L.*

Next, we consider the invariant measure for strong solutions to (1.1). According to [8] and more generally [12], an H-valued continuous adapted process X is called a strong solution to (1.1), if $X\in L^{r+1}([0,T]\times\Omega\times E;\mathrm{d}t\times P\times\mathbf{m})$ for any $T>0$ and

$$\langle X_t,e_i\rangle=\langle X_0,e_i\rangle+\int_0^t \mathbf{m}\big(\Psi(X_s)Le_i+\Phi(X_s)e_i\big)\mathrm{d}s+\sum_{j=1}^{\infty}\int_0^t q_{ji}(X_s)\mathrm{d}B_s^j,\quad t\ge0, i\ge1.$$

As observed in [8] and [12], this implies that

$$X_t=X_0+\int_0^t (L\Psi(X_s)+\Phi(X_s))\mathrm{d}s,\quad t>0$$

exists and is continuous w.r.t. t in H.

By [8, Theorem 1.1] (more general, by [12, Theorem 3.1]), if **(H2)** and

$$(1.9)\qquad \begin{aligned}&\frac{1}{2}\|Q(x)-Q(y)\|^2_{\mathscr{L}_{HS}(L^2(\mathbf{m});H)}+\langle\Phi(x)-\Phi(y),x-y)\rangle_H\\ &\le\theta\,\mathbf{m}(|x-y|^{r+1})+\delta\,\mathbf{m}(|x-y|^2),\quad x,y\in L^{r+1}(\mathbf{m}),\end{aligned}$$

holds for some $\theta < 2^{1-r}\eta$ and $\delta \le \sigma$, then (1.1) has a unique strong solution which is ergodic, and the unique invariant probability measure μ is concentrated on $L^{r+1}(\mathbf{m})$.

In the same spirit of Theorem 1.1, the following theorem provides stronger concentration properties of μ.

Theorem 1.3. *Assume* **(H2)** *and* (1.9) *for some $\theta < 2^{1-r}\eta$ and $\delta \le \sigma$. Let μ be the unique invariant probability measure of the strong solution to* (1.1).

(1) *Let f be a Bernstein function such that $f(\infty) = \infty$ and* (1.2) *holds. If $\sigma > 0$ then* (1.3) *implies*

$$\sum_{i=1}^{\infty} f(\lambda_i) \int \mathbf{m}(xe_i)^2 \mathrm{d}\mu < \infty. \tag{1.10}$$

(2) *If $\Phi = 0$ and $|\Psi'(s)| \le C(1+|s|^r)$ for some $C > 0$ and all $s \in \mathbb{R}$, then* (1.6) *implies* (1.7).

2 Proofs of Theorem 1.1 and Corollary 1.2

To prove Theorem 1.1, we follow the line of arguments in [4] to make use of the general result [5, Theorem 5.1]. To this end, let

$$E_n := \mathrm{span}\{e_1, \cdots, e_n\},$$
$$L_n := \sum_{i=1}^{n} q_i^2 \partial_i^2 + \sum_{i=1}^{n} \partial_i.$$

Proof of Theorem 1.1 (1). Take $V(x) := \sum_{i=1}^{\infty} \frac{f(\lambda_i)}{\lambda_i} \langle x, e_i \rangle^2,\ x \in H$ and

$$\Theta(x) := \begin{cases} \sum_{i=1}^{\infty} f(\lambda_i)\big(\mathbf{m}(\xi_r(x)e_i)^2 + \sigma\, \mathbf{m}(xe_i)^2\big), & \text{if } x \in L^{r+1}(\mathbf{m}), \\ \infty, & \text{otherwise.} \end{cases}$$

We have $E_n \subset \{\Theta < \infty\}$ for any $n \ge 1$. To apply [5, Theorem 5.1], it suffices to verify the following:

(i) $V|_{E_n}$ is smooth and is a compact function, that is, $\{x \in E_n : V(x) \le r\}$ is a relatively compact set in $E_n,\ n \ge 1$.

(ii) Θ is a compact function in H.

(iii) q_i and b_i are continuous on E_n and $\{\Theta \le r\}$ in the topology of $H,\ n \ge 1, r > 0$.

(iv) For any $i \ge 1$ there exist a constant $c_i > 0$ and a positive function δ_i with $\delta_i(s) \to 0$ as $s \to \infty$ such that $|b_i(x)| \le c_i + (\delta_i \circ \Theta(x))\Theta(x), x \in E_n, n \ge 1$.

(v) There exist two constant $c, \kappa > 0$ such that $L_n V(x) \le c - \kappa\Theta(x),\ x \in E_n, n \ge 1$.

Once these conditions are satisfied, L_n has an invariant measure on E_n (hence on H by setting $\mu_n(H \setminus E_n) = 0$) such that $\mu_n(\Theta) \le c$ for some constant $c > 0$ and all $n \ge 1$, so that $\{\mu_n\}$ is tight and, up to subsequence, converges weakly to some probability measure μ solving $L^*\mu = 0$ with $\mu(\Theta) \le c$.

(i) is obvious for the above specific function V, while (ii) follows immediately from the Sobolev embedding theorem since $f(\lambda_i) \to \infty$ as $i \to \infty$. So, below we verify (iii), (iv) and (v) respectively.

Proof of (iii). Since Ψ and Φ are continuous, the continuity of q_i and b_i on E_n is trivial. So, we only prove their continuity on $A_r := \{\Theta \le r\}$. Let $x_n \in A_r$ with $x_n \to x \in A_r$ in the topology of H. We intend to show that $q_i(x_n) \to q_i(x)$ and

$$\lim_{n\to\infty} \big(|\,\mathbf{m}\big((\Psi(x_n) - \Psi(x))e_i\big)| + |\,\mathbf{m}\big((\Phi(x_n) - \Phi(x))e_i\big)|\big) = 0. \tag{2.1}$$

By Sobolev's embedding theorem, A_r is relatively compact in $L^2(\mathbf{m})$. Thus, $x_n \to x$ in H implies the convergence in $L^2(\mathbf{m})$ and hence, $q_i(x_n) \to q_i(x)$ according to the continuity of q_i in $L^{r+1}(\mathbf{m})$. If (2.1) does not hold, let, for instance,

$$|\,\mathbf{m}\big((\Psi(x_{n_k}) - \Psi(x))e_i\big)| \ge \varepsilon_0$$

for some $\varepsilon_0 > 0$ and a subsequence $n_k \to \infty$. Since $x_n \to x$ in $L^{r+1}(\mathbf{m})$, there exists a subsequence n_k' of n_k such that $x_{n_k'} \to x$ a.e.- $\mathbf{m}$. Moreover, since $|\Psi(x_n)| \le c(1+|x_n|^r)$ and $e_i \in L^{r+1}(\mathbf{m})$, $x_n \to x$ in $L^{r+1}(\mathbf{m})$ implies the uniform integrability of $(\Psi(x_n) - \Psi(x))e_i$ in $L^1(\mathbf{m})$. Therefore, by the dominated convergence theorem we arrive at

$$\lim_{k\to\infty} \mathbf{m}\big((\Psi(x_{n_k'}) - \Psi(x))e_i\big) = 0$$

which is a contradiction.

Proof of (iv): Let $\|\cdot\|_p$ denote the norm in $L^p(\mathbf{m})$. Since $e_i \in L^{r+1}(\mathbf{m})$ and

$$\Theta(x) \ge f(\lambda_1)\,\mathbf{m}(|x|^{r+1}), \quad x \in E_n, n \ge 1,$$

(H2) implies

$$\begin{aligned} |b_i(x)| &\le c_i\big(\|\Psi(x)\|_{(r+1)/r} + \|\Phi(x)\|_{(r+1)/r}\big) \\ &\le c_i'(1 + \mathbf{m}(|x|^{r+1})^{r/(r+1)}) \le c_i' + c_i''\Theta(x)^{r/(r+1)}, \quad x \in E_n, n \ge 1 \end{aligned}$$

for some constants $c_i, c_i', c_i'' > 0$. Thus, (iv) holds for $\delta_i(s) := c_i'' s^{-1/(r+1)}$.

Proof of (v): By the definition of V, we have

$$
\begin{aligned}
L_nV(x) &= 2\sum_{i=1}^{n}\frac{q_i(x)^2 f(\lambda_i)}{\lambda_i} + 2\sum_{i=1}^{n}\frac{b_i(x)f(\lambda_i)}{\lambda_i}\,\mathbf{m}(xe_i)\\
&\le 2\sum_{i=1}^{\infty}\frac{q_i(x)^2 f(\lambda_i)}{\lambda_i} + 2\sum_{i=1}^{n}\frac{b_i(x)f(\lambda_i)}{\lambda_i}\,\mathbf{m}(xe_i) =: C + I,
\end{aligned}
\tag{2.2}
$$

where C is bounded. By the definition of b_i and noting that $\tilde{f}(s) := s/f(s)$, we obtain from (1.2) that

$$
\begin{aligned}
I &= \int_E \Big(\Psi(x)(-f(-L)x) + \Phi(x)(\tilde{f}(-L))^{-1}x\Big)\mathrm{d}\,\mathbf{m}\\
&\le -\,\mathbf{m}\big(\Psi(x)f(-L)x\big) + \theta\,\mathbf{m}(|x|^{r+1}) + c, \quad x\in E_n, n\ge 1.
\end{aligned}
\tag{2.3}
$$

Moreover, since f is a Bernstein function, $T_t := \mathrm{e}^{-tf(-L)}$ is a sub-Markovian semigroup. Let $K_t := 1 - T_t 1 \ge 0$ and let J_t be the symmetric sub-probability measure on $E\times E$ determined by $J_t(A\times B) := \mathbf{m}(1_A T_t 1_B), A, B \in \mathscr{M}$. Since for any $x\in E_n$ one has $\Psi(x)\in L^{(r+1)/r}(\mathbf{m})$ and

$$
\frac{x - T_t x}{t} = \sum_{i=1}^{n}\frac{1-\mathrm{e}^{-f(\lambda_i)t}}{t}\mu(xe_i)e_i \to f(-L)x \text{ in } L^{r+1}(\mathbf{m}) \text{ as } t\to 0,
$$

by the symmetry of T_t we obtain,

$$
\begin{aligned}
&\int_E \Psi(x)(f(-L)x)\mathrm{d}\,\mathbf{m} = \lim_{t\to 0}\frac{1}{t}\int_E \Psi(x)(x - T_t x)\mathrm{d}\,\mathbf{m}\\
&= \lim_{t\to 0}\frac{1}{2t}\Big\{\int_{E\times E}\big(x(u)-x(v)\big)\big(\Psi\circ x(u) - \Psi\circ x(v)\big)J_t(\mathrm{d}u,\mathrm{d}v) + \mathbf{m}(K_t x\Psi(x))\Big\}.
\end{aligned}
\tag{2.4}
$$

We now consider the two situations in **(H1)** respectively.

(a) If **(H1)** holds with $\Psi(0)=0$, then **(H2)** with $t=0$ implies $s\Psi(s)\ge \eta|\xi_r(s)|^2+\sigma s^2$. Thus, (2.4) and **(H2)** lead to

$$
\begin{aligned}
&\int_E \Psi(x)(f(-L)x)\mathrm{d}\,\mathbf{m}\\
&\ge \limsup_{t\to 0}\frac{1}{2t}\Big\{\int_{E\times E}\Big[\eta\big(\xi_r\circ x(u) - \xi_r\circ x(v)\big)^2 + \sigma\big(x(u)-x(v)\big)^2\Big]J_t(\mathrm{d}u,\mathrm{d}v)\\
&\qquad\qquad + \mathbf{m}\Big(K_t\big[\eta\xi_r(x)^2 + \sigma x^2\big]\Big)\Big\}\\
&= \eta\sum_{i=1}^{n} f(\lambda_i)\,\mathbf{m}(\xi_r(x)e_i)^2 + \sigma\sum_{i=1}^{n} f(\lambda_i)\,\mathbf{m}(xe_i)^2, \qquad x\in E_n.
\end{aligned}
\tag{2.5}
$$

Since by the Poincaré inequality one has $\Theta(x) \geq f(\lambda_1)\,\mathbf{m}(|x|^{r+1})$, combining (2.2), (2.3), (2.5) and noting that $\theta < \eta f(\lambda_1)$, we prove (v) for some $\kappa, c > 0$.

(b) If **(H1)** holds with $1 \in \mathscr{D}(L)$ then $1 \in \mathscr{D}(f(-L))$. Moreover, by **(H2)** with $t = 0$ one has $s\Psi(s) \geq \eta(\xi_r(s))^2 + \sigma s^2 + \Psi(0)s$. Hence, (2.4) implies

$$
\begin{aligned}
&\int_E \Psi(x)(-f(-L)x)\mathrm{d}\,\mathbf{m} \\
&\geq \limsup_{t\to 0} \frac{1}{2t}\bigg\{ \int_{E\times E} \Big[\eta\big(\xi_r\circ x(u) - \xi_r \circ x(v)\big)^2 + \sigma\big(x(u)-x(v)\big)^2\Big] J_t(\mathrm{d}u,\mathrm{d}v) \\
&\qquad\qquad + \mathbf{m}\big(K_t\big[\eta\xi_r(x)^2 + \sigma x^2\big]\big)\bigg\} - \frac{\Psi(0)}{2}\mathbf{m}(xf(-L)1) \\
&= \eta\sum_{i=1}^n f(\lambda_i)\,\mathbf{m}(\xi_r(x)e_i)^2 + \sigma\sum_{i=1}^n f(\lambda_i)\,\mathbf{m}(xe_i)^2 - c\,\mathbf{m}(|x|^2)^{1/2},\ \ x \in E_n
\end{aligned}
\tag{2.6}
$$

for some $c > 0$ and all $x \in E_n, n \geq 1$. Therefore, (v) holds by the same reason as in (a). □

Proof of Theorem 1.1(2). We modify the proof of [4, Theorem 1.1(ii)] by using our weaker assumptions. Since (1.6) is stronger than (1.3), Theorem 1.1(1) applies. It suffices to prove that if (1.6) holds instead of (1.3), then the invariant measure obtained in the proof of Theorem 1.1(1) also satisfies (1.7). Following the line of [4], we take $\Xi(s) := \int_0^s \Psi(t)\mathrm{d}t$ and

$$V_n(x) := \mathbf{m}(\Xi(x)), \qquad x \in E_n.$$

Then V_n is a compact function on E_n. By (1.6) we have

$$
\begin{aligned}
L_nV_n(x) =& \sum_{i=1}^n q_i(x)^2\,\mathbf{m}(\Psi'(x)e_i^2) + \sum_{i=1}^n b_i(x)\,\mathbf{m}(\Psi(x)e_i) \\
\leq& c_0\,\mathbf{m}(|x|^{r+1})^{(r-1)/(r+1)} - \sum_{i=1}^n \lambda_i\,\mathbf{m}(\Psi(x)e_i)^2 \\
&+ \frac{1}{2\lambda_1}\sum_{i=1}^n \mathbf{m}\big(\Phi(x)e_i\big)^2 + \frac{\lambda_1}{2}\sum_{i=1}^n \mathbf{m}(\Psi(x)e_i)^2, \quad x \in E_n
\end{aligned}
\tag{2.7}
$$

for some $c_0 > 0$. Noting that $|\Xi(s)| \leq c(1+|s|^{r+1})$ for some $c > 0$ and that the proof of Theorem 1.1(1) implies $\mu_n(|\cdot|^{r+1}) \leq C$ for some constant $C > 0$ and all $n \geq 1$, we obtain from (2.7) that

$$\int_H \sum_{i=1}^N \lambda_i\,\mathbf{m}(\Psi(x)e_i)^2\mathrm{d}\mu_n \leq c_1 + \frac{1}{\lambda_1}\int_H \mathbf{m}(\Phi(x)^2)\mathrm{d}\mu_n, \quad n \geq N \geq 1 \tag{2.8}$$

for some $c_1 > 0$.

(a) Assume that Φ is bounded. Then $\mathbf{m}(\Phi(\cdot)^2)$ is a bounded continuous function on $L^{1+r}(\mathbf{m})$. Since $|\Psi(x)| \le C(1+|x|^{r+1})$ for some $C > 0$ and $e_i \in L^{r+1}(\mathbf{m})$, there exists $c_i > 0$ such that

$$\mathbf{m}(\Phi(x)e_i)^2 \le c_i(1+\mathbf{m}(|x|^{r+1})^{2/(r+1)}), \quad x \in L^{r+1}(\mathbf{m}).$$

Noting that $r+1 > 2$ and $\mu_n(\mathbf{m}(|\cdot|^{r+1}) \le C$ for some $C > 0$ and all $n \ge 1$, we conclude that $\mathbf{m}(\Psi(x)e_i)^2$ is uniformly integrable w.r.t. μ_n, that is,

$$\lim_{N\to\infty} \sup_{n\ge 1} \int_H \mathbf{m}(\Psi(x)e_i)^2 1_{\{\mathbf{m}(\Psi(x)e_i)^2 > N\}} \mu_n(\mathrm{d}x) = 0. \tag{2.9}$$

Since $e_i \in L^{p_i}(\mathbf{m})$ and $p_i > r+1$, there exists $q_i < r+1$ such that $\mathbf{m}(\Psi(x)e_i)^2$ and $\mathbf{m}(\Phi(x)e_i)^2$ are continuous in x with respect to the topology of $L^{q_i}(\mathbf{m})$. Since, as observed in the proof of (iii), Θ is a compact function in $L^{r+1}(\mathbf{m})$, $\{\mu_n\}$ is tight in $L^{r+1}(\mathbf{m})$ due to $\mu_n(\Theta) \le C$ for some $c > 0$ and all $n \ge 1$. Hence, we may assume that $\mu_n \to \mu$ weakly in $L^{r+1}(\mathbf{m})$. Therefore, (2.9) implies

$$\begin{aligned}
&\lim_{n\to\infty} \int_H \mathbf{m}(\Psi(x)e_i)^2 \mu_n(\mathrm{d}x) \\
&= \lim_{N\to\infty} \lim_{n\to\infty} \int_H [\mathbf{m}(\Psi(x)e_i)^2 \wedge N] \mu_n(\mathrm{d}x) = \int_H \mathbf{m}(\Psi(x)e_i) \mu(\mathrm{d}x).
\end{aligned}$$

Combining this with (2.8) by first letting $n \to \infty$ then $N \to \infty$, we arrive at

$$\int_H \sum_{i=1}^\infty \lambda_i \, \mathbf{m}(\Psi(x)e_i)^2 \mathrm{d}\mu \le c_1 + \frac{1}{\lambda_1} \int_H \mathbf{m}(|\Phi(x)|^2) \mathrm{d}\mu < \infty.$$

Combining this with (1.5), there exists a constant $C > 0$ independent of the upper bound of Φ such that

$$\int_H \sum_{i=1}^\infty \lambda_i \, \mathbf{m}(\Psi(x)e_i)^2 \mathrm{d}\mu \le C. \tag{2.10}$$

(b) In general, we take, as in [4], $\Phi_n := (\Phi \wedge n) \vee (-n)$, $n \ge 1$ and let $\tilde{\mu}_n$ be the corresponding invariant measure of L with Φ_n in place of Φ such that as in (2.10)

$$\int_H \sum_{i=1}^\infty \lambda_i \, \mathbf{m}(\Psi(x)e_i)^2 \tilde{\mu}_n(\mathrm{d}x) \le C, \quad n \ge 1 \tag{2.11}$$

holds for some constant $C > 0$. Since $|\Psi(s) - \Psi(t)| \ge \eta|s-t|$ for all $s,t \in \mathbb{R}$, the same reasoning as in the proof of (iii) leads to the compactness of Θ in $L^{r+1}(\mathbf{m})$. Hence, $\{\tilde{\mu}_n\}$

is tight in $L^{2r}(\mathbf{m})$ so that we may assume that $\tilde{\mu}_n \to \mu$ weakly in $L^{2r}(\mathbf{m})$ as $n \to \infty$. Since $2r > r+1$, (2.11) implies (2.9) and hence, it is easy to check that $L^*\mu = 0$ and (1.7) holds. □

Finally, Corollary 1.2 follows immediately from Theorem 1.1 and the following lemma.

Lemma 2.1. *Consider the situation of Theorem 1.1.*

(1) (1.8) *implies* (1.5).

(2) *If* $\tilde{f}$ *is a Bernstein function then* (1.8) *implies* (1.2) *for some* $\theta < \eta f(\lambda_1)$ *and* $c > 0$.

Proof. The first assertion follows immediately since **(H2)** with $t = 0$ implies $|\Psi(s)| \geq \eta|s|^r - c$ for some constant $c > 0$.

To prove the second assertion, let $\|\cdot\|_p$ denote the norm in $L^p(\mathbf{m})$. Since $\tilde{f}$ is a Bernstein function, $\tilde{T}_s := \mathrm{e}^{-s\tilde{f}(-L)}$ is a sub-Markovian semigroup. In particular, $\|\tilde{T}_s\|_{\infty\to\infty} \leq 1$. On the other hand, by the spectral mapping theorem one has $\|\tilde{T}_s\|_{2\to 2} \leq \mathrm{e}^{-\tilde{f}(\lambda_1)s}$. Then it follows from Riesz-Thorin's interpolation theorem that

$$\|\tilde{T}_s\|_{r+1\to r+1} \leq \mathrm{e}^{-2\tilde{f}(\lambda_1)s/(r+1)}, \quad s \geq 0.$$

Hence,

$$\|\tilde{f}(-L)^{-1}x\|_{r+1} \leq \int_0^\infty \|\tilde{T}_s x\|_{r+1}\mathrm{d}s \leq \frac{\|x\|_{r+1}(r+1)}{2\tilde{f}(\lambda_1)}.$$

Combining this with (1.8) we obtain

$$\mathbf{m}\big(\Phi(x)(\tilde{f}(-L)^{-1}x)\big) \leq \|\Phi(x)\|_{(r+1)/r}\|\tilde{f}(-L)^{-1}x\|_{r+1} \leq \frac{(r+1)\varepsilon_0\,\mathbf{m}(|x|^{r+1})}{2\tilde{f}(\lambda_1)} + c_0$$

for some $\varepsilon_0 < 2\eta\lambda_1/(r+1)$ and $c_0 > 0$. This completes the proof. □

3 Proof of Theorem 1.3

We first briefly recall the construction of the strong solution in [8]. For any $n \geq 1$ and any $x \in H$, let $r_t^{(n)} := (r_{t,1}^{(n)}, \cdots, r_{t,n}^{(n)})$ solve the following SDE on $\mathbb{R}^n$:

(3.1)
$$\mathrm{d}r_{t,i}^{(n)} = \sum_{j=1}^n q_{ji}\Big(\sum_{k=1}^n r_{t,k}^{(n)}e_k\Big)\mathrm{d}B_t^j - \lambda_i\mathbf{m}\Big(e_i\Psi\Big(\sum_{k=1}^n r_{t,k}^{(n)}e_k\Big)\Big)\mathrm{d}t + \mathbf{m}\Big(e_i\Phi\Big(\sum_{k=1}^n r_{t,k}^{(n)}e_k\Big)\Big)\mathrm{d}t$$

with $r_{0,i}^{(n)} = \langle x, e_i\rangle,\ 1 \leq i \leq n$. Since the coefficients are continuous under our assumption, the solution to (2.1) exists uniquely (see [9] for a much stronger result). As shown in [8], under **(H2)** and (1.9) there exists a subsequence n_k such that $X^{n_k}(x) := \sum_{i=1}^n r_{t,i}^{(n)}e_i \to$

$X(x)$ weakly in $L^{r+1}([0,T]\times\Omega\times E; \mathrm{d}t\times P\times \mathbf{m})$ for any $T>0$, where $X(x)$ is the unique strong solution to (1.1) with $X_0(x)=x$.

Proof of Theorem 1.3(1). Let V and Θ be as in the proof of Theorem 1.1 (1). By Itô's formula and the calculations in the proof of (v) using $\sum_{j=1}^\infty q_{ji}^2$ in place of q_i^2, we have

$$\mathrm{d}V(X_t^n(x)) \le \mathrm{d}M_t + c\mathrm{d}t - \kappa\Theta(X_t^{(n)}(x))\mathrm{d}t$$

for some local martingale M_t and some constants $c,\kappa>0$. Since $V(x)\le c\,\mathbf{m}(|x|^2)$ for some $c>0$, this implies

$$\mathbb{E}\int_0^1\sum_{i=1}^n f(\lambda_i)\,\mathbf{m}(X_t^{(n)}(x)e_i)^2\mathrm{d}t \le \frac{1}{\sigma}\mathbb{E}\int_0^1\Theta(X_t^{(n)}(x))\mathrm{d}t \le C + C\,\mathbf{m}(|x|^2) \tag{3.2}$$

for some constant $C>0$. On the other hand, by the proof of [10, Theorem II.2.1] on page 1246 (see also [8, Section 3]) there exists a subsequence $n_k\to\infty$ such that $X^{(n_k)}(x)\to X(x)$ weakly in $L^{r+1}([0,1]\times E\times\Omega; \mathrm{d}t\times\,\mathbf{m}\times P)$. Then, for any $N\ge 1$ we have

$$\begin{aligned}
&\mathbb{E}\int_0^1\sum_{i=1}^N f(\lambda_i)\,\mathbf{m}(X_t(x)e_i)^2\,\mathrm{d}t = \mathbb{E}\int_0^1\sum_{i=1}^N f(\lambda_i)\lim_{k\to\infty}\,\mathbf{m}(X_te_i)\,\mathbf{m}(X_t^{(n_k)}(x)e_i)\,\mathrm{d}t\\
&\le \frac{1}{2}\mathbb{E}\int_0^1\sum_{i=1}^N f(\lambda_i)\,\mathbf{m}(X_t(x)e_i)^2\,\mathrm{d}t + \frac{1}{2}\liminf_{k\to\infty}\mathbb{E}\int_0^1\sum_{i=1}^N f(\lambda_i)\,\mathbf{m}(X_t^{(n_k)}(x)e_i)^2\,\mathrm{d}t.
\end{aligned}$$

Combining this with (3.2) we arrive at

$$\mathbb{E}\int_0^1\sum_{i=1}^N f(\lambda_i)\,\mathbf{m}(X_t(x)e_i)^2\,\mathrm{d}t \le C + C\,\mathbf{m}(|x|^2).$$

Since μ is the invariant probability measure of X and, similarly to [8, Theorem 1.1], $\int \mathbf{m}(|x|^{r+1})\mathrm{d}\mu<\infty$, we obtain

$$\begin{aligned}
\int_H\sum_{i=1}^N f(\lambda_i)[\,\mathbf{m}(xe_i)^2\wedge N]\mu(\mathrm{d}x) &= \int_H\bigg(\mathbb{E}\int_0^1\sum_{i=1}^N f(\lambda_i)\big(N\wedge\,\mathbf{m}(X_t(x)e_i)^2\big)\,\mathrm{d}t\bigg)\mu(\mathrm{d}x)\\
&\le C + C\int\,\mathbf{m}(|x|^2)\mu(\mathrm{d}x)<\infty,\quad N>1.
\end{aligned}$$

Then the proof is completed by letting $N\to\infty$. $\square$

Proof of Theorem 1.3(2). Since $\Phi=0$, according to [8, (3.6)] we have

$$\Psi(X^{(n_k)}(x))\to\Psi(X(x)) \text{ weakly in } L^{(r+1)/r}([0,T]\times\Omega\times E; \mathrm{d}t\times P\times\,\mathbf{m}),\quad T>0.$$

Then for any $N \geq 1$,

$$\begin{aligned}
&\mathbb{E}\int_0^1 \mathbf{m}(\Psi(X_t(x))e_i)^2 1_{\{|\,\mathbf{m}(\Psi(X_t(x))e_i)|\leq N\}}\mathrm{d}t \\
&= \lim_{k\to\infty} \mathbb{E}\int_0^1 \mathbf{m}(\Psi(X_t(x))e_i) 1_{\{|\,\mathbf{m}(\Psi(X_t(x))e_i)|\leq N\}}\, \mathbf{m}(\Psi(X_t^{(n_k)}(x))e_i)\mathrm{d}t \\
&\leq \frac{1}{2}\mathbb{E}\int_0^1 \mathbf{m}(\Psi(X_t(x))e_i)^2 1_{\{|\,\mathbf{m}(\Psi(X_t(x))e_i)|\leq N\}}\mathrm{d}t + \frac{1}{2}\liminf_{k\to\infty}\mathbb{E}\int_0^1 \mathbf{m}(\Psi(X_t^{(n_k)}(x))e_i)^2\mathrm{d}t.
\end{aligned}$$

This implies

$$\begin{aligned}
\mathbb{E}\int_0^1 \mathbf{m}(\Psi(X_t(x))e_i)^2\mathrm{d}t &= \lim_{N\to\infty}\mathbb{E}\int_0^1 \mathbf{m}(\Psi(X_t(x))e_i)^2 1_{\{|\,\mathbf{m}(\Psi(X_t(x))e_i)|\leq N\}}\mathrm{d}t \\
&\leq \liminf_{k\to\infty}\mathbb{E}\int_0^1 \mathbf{m}(\Psi(X_t^{(n_k)}(x))e_i)^2\mathrm{d}t.
\end{aligned} \tag{3.3}$$

On the other hand, let $V(x) := \mathbf{m}(\Xi(x))$ be defined as in the proof of Theorem 1.1(2). Since $\Phi = 0$, by Itô's formula and our assumptions we obtain

$$\begin{aligned}
\mathrm{d}V(X_t^{(n)}(x)) &= \mathrm{d}M_t + \Big[\sum_{i,j=1}^n\sum_{k=1}^\infty q_{ki}(x)q_{kj}(x)\,\mathbf{m}(\Psi'(x)e_ie_j) - \sum_{i=1}^n \lambda_i\,\mathbf{m}(\Psi(X_t^{(n)}(x))e_i)^2\Big]\mathrm{d}t \\
&\leq \mathrm{d}M_t + \big[c\,\mathbf{m}(|X_t^{(n)}(x)|^{r+1})^{(r-1)/(r+1)} - \sum_{i=1}^n \lambda_i\,\mathbf{m}(\Psi(X_t^{(n)}(x))e_i)^2\big]\mathrm{d}t
\end{aligned}$$

for some $c > 0$ and some local martingale M_t. Since V is bounded below and $|V(x)| \leq c(1 + \mathbf{m}(|x|^{r+1}))$ for some $c > 0$ and all $x \in L^{r+1}(\mathbf{m})$, this implies

$$\begin{aligned}
&\mathbb{E}\int_0^1 \sum_{i=1}^n \lambda_i\,\mathbf{m}(\Psi(X_t^{(n)}(x))e_i)^2\mathrm{d}t \\
&\leq c + c\,\mathbf{m}(|x|^{r+1}) + c\mathbb{E}\int_0^1 \mathbf{m}(|X_t^{(n)}(x)|^{r+1})^{(r-1)/(r+1)}\mathrm{d}t
\end{aligned} \tag{3.4}$$

for some $c > 0$ and all $n \geq 1, x \in L^{r+1}(\mathbf{m})$. Moreover, since (1.6) is stronger than (1.3), we have (3.2) for some constant $C > 0$. Combining the second inequality in (3.2) with the fact that $\Theta(x) \geq f(\lambda_1)\,\mathbf{m}(|x|^{r+1})$, we obtain

$$\mathbb{E}\int_0^1 \mathbf{m}((X_t^{(n)}(x)|^{r+1})^{(r-1)/(r+1)}\mathrm{d}t \leq c(1 + \mathbf{m}(|x|^2))$$

for some $c > 0$ and all $n \geq 1, x \in L^{r+1}(\mathbf{m})$. Thus, it follows from (3.4) that

$$\mathbb{E}\int_0^1 \sum_{i=1}^n \lambda_i\,\mathbf{m}(\Psi(X_t^{(n)}(x))e_i)^2\mathrm{d}t \le c(1+\mathbf{m}(|x|^{r+1})),\quad x\in L^{r+1}(\mathbf{m})$$

for some constant $c>0$. Combining this with (3.3) and noting that μ is the invariant measure of X, we arrive at

$$\begin{aligned}\int_H \sum_{i=1}^N \lambda_i\big[\mathbf{m}(\Psi(x)e_i)^2\wedge N\big]\mu(\mathrm{d}x) &= \int_H \mathbb{E}\int_0^1 \sum_{i=1}^N \lambda_i\big[\mathbf{m}(\Psi(X_t(x))e_i)^2\wedge N\big]\mu(\mathrm{d}x)\\ &\le c\int_H (1+\mathbf{m}(|x|^{r+1}))\mu(\mathrm{d}x)<\infty.\end{aligned}$$

Then the proof is completed by letting $N\to\infty$. $\square$

Acknowledgements Feng-Yu Wang (the corresponding author) would like to thank the University of Bielefeld for a pleasant stay when this paper was initiated. This work is Supported in part by the DFG through the Forschergruppe "Spectral Analysis, Asymptotic Distributions and Stochastic Dynamics", CRC 701, the BiBoS Research Centre, NNSFC(10121101) and RFDP(20040027009).

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