

Uniqueness of solutions to weak parabolic equations for measures

V.I. Bogachev¹, G. Da Prato², M. Röckner³, W. Stannat⁴

Abstract

We study uniqueness of parabolic equations for measures $\mu(dt dx) = \mu_t(dx)dt$ of the type $L^*\mu = 0$, satisfying $\mu_t \rightarrow \nu$ as $t \rightarrow 0$, where each μ_t is a probability measure on \mathbb{R}^d , $L = \partial_t + a^{ij}(t, x)\partial_{x_i}\partial_{x_j} + b^i(t, x)\partial_{x_j}$ is a differential operator on $(0, T) \times \mathbb{R}^d$ and ν is a given initial measure. One main result is that uniqueness holds under uniform ellipticity and Lipschitz conditions on a^{ij} but for b^i merely local integrability and coercivity conditions are sufficient.

2000 Mathematics Subject Classification AMS: 35K10, 60J60, 60J35, 47D07.

Key words : Parabolic equations for measures, Stochastic differential equations, Time dependent coefficients.

1 Introduction and main result

Let $T > 0$, let $A = (a^{ij})$ be a Borel measurable mapping on $[0, T] \times \mathbb{R}^d$ with values in the space of non-negative symmetric $d \times d$ matrices, and let $b = (b^i): [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ be Borel measurable. Consider the differential operators

$$L_{A,b}u(t, x) = \sum_{i,j=1}^d a^{ij}(t, x)\partial_i\partial_j u(t, x) + \sum_{i=1}^d b^i(t, x)\partial_i u(t, x), \quad (1.1)$$

and

$$Lu(t, x) := (L_{A,b} + \partial_t)u(t, x). \quad (1.2)$$

for $u \in C_0^\infty((0, T) \times \mathbb{R}^d)$. Here $\partial_i := \frac{\partial}{\partial x_i}$ and $\partial_t := \frac{\partial}{\partial t}$. Let μ be a locally finite (not necessarily non-negative) Borel measure on $(0, T) \times \mathbb{R}^d$ such that $a^{ij}, b^i \in L_{loc}^1(|\mu|)$ and

$$\int_{(0,T) \times \mathbb{R}^d} Lu \, d\mu = 0 \quad \text{for all } u \in C_0^\infty((0, T) \times \mathbb{R}^d), \quad (1.3)$$

abbreviated

$$L^*\mu = 0. \quad (1.4)$$

¹Department of Mechanics and Mathematics, Moscow State University, 119899 Moscow, Russia

²Scuola Normale Superiore di Pisa, Piazza dei Cavalieri 7, I-56125 Pisa, Italy

³Department of Mathematics, Purdue University, W. Lafayette, IN 47907, USA

⁴Fakultät für Mathematik, Universität Bielefeld, D-33501 Bielefeld, Germany

Under reasonable regularity assumptions on the coefficients (see below) it follows that

$$\mu(dt dx) = \mu_t(dx) dt$$

for some family of locally finite Borel measures μ_t , $t \in (0, T)$, on \mathbb{R}^d . Hence one can consider the initial value problem

$$\left\{ \begin{array}{l} \int_{(0,T) \times \mathbb{R}^d} Lu \, d\mu = 0 \quad \text{for all } u \in C_0^\infty((0, T) \times \mathbb{R}^d), \\ \lim_{t \rightarrow 0} \int_{\mathbb{R}^d} \zeta \, d\mu_t = \int_{\mathbb{R}^d} \zeta \, d\nu \quad \text{for all } \zeta \in C_0^\infty(\mathbb{R}^d), \end{array} \right. \quad (1.5)$$

for a given locally finite Borel measure ν on \mathbb{R}^d . We are mainly interested in the case when the initial condition ν and each measure μ_t in the solution family $(\mu_t)_{t \in (0, T)}$ are probability measures. It was shown in [4] (see also [3]) that under reasonable assumptions on the coefficients, there exists a solution $(\mu_t)_{t \in (0, T)}$ to the weak parabolic initial value problem (1.5) consisting of probability measures, for any probability measure ν on \mathbb{R}^d as initial condition. The aim of this paper is to specify conditions under which this solution is unique.

In the elliptic case weak equations for measures of type (1.4) have been studied quite extensively in recent years on domains in \mathbb{R}^d and in infinite dimensions (see the recent paper [7] and references therein). For the so far most general existence results we refer to [8] in the finite dimensional and to [9] (see also [13]) in the infinite dimensional case. Regarding uniqueness, there are only two papers [11] and [12] (the latter one a slightly more general and more self-contained than the first) which contain general results on uniqueness and these are only in finite dimensions.

Of course, in the elliptic (time independent) case, weak equations for measures of type (1.4) are closely connected to the question whether the solution μ is invariant for a semigroup generated by the operator L in some sense. We warn the reader that if this semigroup exists, the measure μ might not be invariant with respect to it, but maybe only subinvariant. We refer to [19] (and also to [11] and [12]) for a detailed discussion of this question with the essence that the invariance under the semigroup is strongly related to the uniqueness of the weak elliptic problem for measures defined analogously to (1.4).

A similar phenomenon is central also in the parabolic case studied in this paper. The main difference is to invoke the boundary condition for $t = 0$ in a proper way.

In order to formulate the main results of this paper and recall the existence result from [4] we first fix conditions on the coefficients a^{ij}, b^i which will be in force throughout the paper. Note that these conditions are purely *local in space*. Let $H^{p,1}(B)$ denote the set of all $f \in L^p(U) := L^p(U, dx)$, where dx is Lebesgue measure, with generalized partial derivatives $\partial_i f \in L^p(U)$ for all $1 \leq i \leq d$.

- (H1) *There exists $p > d + 2$ such that for every open ball $B \subset \mathbb{R}^d$ one has*
- (a) $\inf_{(t,x) \in [0,T] \times B} \det A(t, x) > 0$ and $\sup_{t \in [0,T], 1 \leq i, j \leq d} \|a^{ij}(t, \cdot)\|_{H^{p,1}(B)} < \infty$,
 - (b) $\int_0^T \int_B |b(t, x)|^p \, dx dt < \infty$.

Let $\mathcal{P}(\mathbb{R}^d)$ denote the set of all Borel probability measures on \mathbb{R}^d . We introduce

the following set of measures on $(0, T) \times \mathbb{R}^d$:

$$\mathcal{M}_{par}^{A,b,\nu} := \{\mu \mid \mu(dt, dx) = \mu_t(dx)dt, \mu_t \in \mathcal{P}(\mathbb{R}^d) \forall t \in (0, T) \quad (1.6)$$

and μ solves (1.5), where $|b| \in L^1((0, T) \times B, \mu)$ for every ball $B \subset \mathbb{R}^d\}$.

Here the subscript “par” refers to “parabolic”.

Theorem 1.1. *Assume (H1). Suppose, in addition, that the following condition holds:*

(H2) *Each a^{ij} is Hölder continuous in $t \in [0, T]$, locally uniformly with respect to $x \in \mathbb{R}^d$.*

Let $\mathcal{K} \subset \mathcal{M}_{par}^{A,b,\nu}$ be such that \mathcal{K} is convex and for all $\mu \in \mathcal{K}$ one has

$$(1 - L)(C_0^\infty([0, T) \times \mathbb{R}^d)) \text{ is dense in } L^1((0, T) \times \mathbb{R}^d, \mu). \quad (1.7)$$

Then $\#\mathcal{K} \leq 1$.

In the last section of this paper we shall specify examples of subsets \mathcal{K} as above. For completeness we recall the main existence result from [4]. Note that (H2) is not needed for this.

Theorem 1.2 (cf. Theorem 3.1 in [4]). *Assume that there exists $p > d + 2$ such that for every ball $B \subset \mathbb{R}^d$ (H1)–(a) holds and*

$$\sup_{t \in [0, T], 1 \leq i \leq d} \|b^i(t, \cdot)\|_{L^p(B)} < \infty.$$

Assume furthermore that there exists a nonnegative function $\Psi \in C^2(\mathbb{R}^d)$ with compact level sets and a constant $C \in [0, +\infty)$ such that

$$L\Psi \leq C(1 + \Psi) \quad \text{a.e. in } (0, T) \times \mathbb{R}^d.$$

Then for every probability measure ν on \mathbb{R}^d there exists a family $\mu = (\mu_t)_{t \in (0, T)}$ of probability measures on \mathbb{R}^d satisfying (1.5). In addition, setting $\mu_0 := \nu$ we have that the function $t \rightarrow \int_{\mathbb{R}^d} \zeta d\mu_t$ is continuous on $[0, T)$ for every $\zeta \in C_0^\infty(\mathbb{R}^d)$.

The organization of the rest of this paper is as follows.

In §2 we first fix some notation and recall results from [6] and [1] that we shall use below. Then we prove that any $\mu \in \mathcal{M}_{par}^{A,b,\nu}$ satisfying (1.7) is an extreme point of the convex set $\mathcal{M}_{par}^{A,b,\nu}$. This obviously implies the assertion of Theorem 1.1. The final §3 is devoted to applications.

Let us fix some notation. If μ is a (not necessarily nonnegative) locally finite Borel measure on an open subset $\Omega \subset \mathbb{R}^d$ and $p \in [1, \infty)$ we denote by $L_{loc}^p(\Omega, \mu)$ the class of all functions f such that $\chi f \in L^p(\Omega, \mu)$ for every $\chi \in C_0^\infty(\mathbb{R}^d)$. Here $L^p(\Omega, \mu) = L^p(\Omega, |\mu|)$ where $|\mu|$ is the variation of μ . Lebesgue measure is denoted by dx , and as usual $L_{loc}^p(\Omega) := L_{loc}^p(\Omega, dx)$. The Borel σ -algebra of a topological space X is denoted by $\mathcal{B}(X)$ and, for a space of real or complex valued functions $\mathcal{F}(X)$ on X , we denote by $\mathcal{F}_0(X)$ the subset of functions $f \in \mathcal{F}(X)$ with compact support, i.e., the closure of $\{f \neq 0\}$ is compact. So, $C_0^\infty(\Omega)$ is the class of all infinitely differentiable functions with compact support in Ω . Let $H^{p,r}(\Omega)$, $p \geq 1$, $r \geq 0$, be the standard Sobolev space of functions on Ω whose generalized derivatives up to order r are in $L^p(\Omega)$, equipped with its natural norm. Furthermore $H_{loc}^{p,r}(\Omega)$ denotes the class of functions f on Ω such that $\chi f \in H^{p,r}(\mathbb{R}^d)$ for every $\chi \in C_0^\infty(\Omega)$.

2 L^1 -uniqueness and extremality

In this section we fix $L_{A,b}$ and L as in (1.1), (1.2) respectively. We consider L with domain $C^\infty([0, T] \times \mathbb{R}^d)$. We always assume (H1) to hold, but recall this in all theorems.

Lemma 2.1. *Let $\mu \in \mathcal{M}_{par}^{A,b,\nu}$ (cf. (1.6)). Then $(L, C_0^\infty([0, T] \times \mathbb{R}^d))$ is dissipative on $L^1((0, T) \times \mathbb{R}^d, \mu)$ and therefore, in particular, closable.*

Proof. The last statement is standard. The first one follows by [15, Lemma 1.8, Appendix A] and Lemma 2.7 below. \square

As a consequence, we have for the closure $(\bar{L}^\mu, D(\bar{L}^\mu))$ of $(L, C_0^\infty([0, T] \times \mathbb{R}^d))$ on $L^1((0, T) \times \mathbb{R}^d, \mu)$ the following result, cf. [15, Appendix A].

Proposition 2.2. *For $\mu \in \mathcal{M}_{par}^{A,b,\nu}$ the following assertions are equivalent:*

(i) $(\bar{L}^\mu, D(\bar{L}^\mu))$ generates a C_0 -semigroup $(T_t^\mu)_{t \geq 0}$ (i. e. a strongly continuous semigroup of bounded operators $(T_t^\mu)_{t \geq 0}$) on $L^1((0, T) \times \mathbb{R}^d, \mu)$.

(ii) For one (hence all) $\lambda \in (0, +\infty)$ the set $(\lambda - L)(C_0^\infty([0, T] \times \mathbb{R}^d))$ is dense in $L^1((0, T) \times \mathbb{R}^d, \mu)$.

(iii) $(T_t^\mu)_{t \geq 0}$ is the only C_0 -semigroup on $L^1((0, T) \times \mathbb{R}^d, \mu)$ which has a generator extending $(L, C_0^\infty([0, T] \times \mathbb{R}^d))$ (i. e., “ L^1 -uniqueness” holds for $(L, C_0^\infty([0, T] \times \mathbb{R}^d))$).

In case any (hence all) of the assertions (i)–(iii) hold, $(T_t^\mu)_{t \geq 0}$ is a contraction semigroup, i. e., each T_t^μ is a contraction on $L^1((0, T) \times \mathbb{R}^d, \mu)$ and it is sub-Markovian, i. e., $f \in L^1((0, T) \times \mathbb{R}^d, \mu)$, $0 \leq f \leq 1$ implies $0 \leq T_t^\mu f \leq 1$ for all $t \geq 0$.

Proof. The equivalence of (i) and (ii) is a consequence of Lemma 2.1 and the well known Lumer–Phillips Theorem (see e.g. [17, Chapter I, Theorem 4.3]). The implication “(i) \Rightarrow (iii)” is trivial, and “(iii) \Rightarrow (i)” is due to W. Arendt [2, 4–II, Theorem 1.33].

For the last part, we note that $(T_t^\mu)_{t \geq 0}$ must consist of contractions by the dissipativity of $(L, C^\infty([0, T] \times \mathbb{R}^d))$, and the sub-Markov property was proved in [15, Lemma 1.9]. \square

Now we introduce the subset of all $M_{par}^{A,b,\nu}$ for which the assertions in Proposition 2.2 hold, the subscript “cg” refers to “closure generates” with Proposition 2.2–(i) in mind. So, define

$$M_{par,cg}^{A,b,\nu} = \{\mu \in M_{par}^{A,b,\nu} \mid \text{Proposition 2.2–(i) holds for } \mu\}.$$

We note that $M_{par}^{A,b,\nu}$ is a convex set and denote by $ext M_{par}^{A,b,\nu}$ the set of its extreme points.

Now we can formulate the main result of this section which obviously implies the assertion of Theorem 1.1.

Theorem 2.3. *Assume (H1). Then $M_{par,cg}^{A,b,\nu} \subset ext M_{par}^{A,b,\nu}$.*

Remark 2.4. (i) See [11, Lemma 4.2] for the corresponding result in the elliptic case.

(ii) In the proof of Theorem 2.3 below we do not really need Hypothesis (H1). As the reader will see, the proof goes through without any changes under the much weaker conditions (1.1)–(1.3) in [20] if, in addition, we know that all measures satisfying the first identity in (1.5) are equivalent (i. e., have the same zero sets). Under Hypothesis (H1) the latter follows from a result in [6] which we recall below.

Before we can prove Theorem 2.3 we need some preparations. We first recall the following two results from [6] and [1].

Theorem 2.5. *Assume that (H1) holds. Let $\mu \neq 0$ be a locally finite Borel measure on $(0, T) \times \mathbb{R}^d$ satisfying the first identity in (1.5) (i.e., no boundary condition at $t = 0$ required). Then, there exists a strictly positive function $\varrho: (0, T) \times \mathbb{R}^d \rightarrow \mathbb{R}_+$ such that*

$$\mu(dt dx) = \varrho(t, x) dt dx,$$

ϱ is locally Hölder continuous on $(0, T) \times \mathbb{R}^d$ and for any $(t_1, t_2) \subset (0, T)$ one has

$$\int_{t_1}^{t_2} \|\varrho(t, \cdot)\|_{H^{p,1}(B)}^p dt < \infty$$

for all open balls $B \in \mathbb{R}^d$.

Proof. We apply [6, Corollary 3.9] and the remarks following it. \square

Lemma 2.6. *Let m be a positive measure on a measurable space (E, \mathcal{B}) and S a sub-Markovian bounded linear operator on $L^1(m) := L^1(E, \mathcal{B}, m)$ such that m is S -invariant, that is*

$$\int_E S f dm = \int_E f dm \quad \text{for all } f \in L^1(m) \cap L^\infty(m).$$

Suppose $\rho_1, \rho_2 \in L^1(m)$ (not necessarily nonnegative) such that $\rho_1 \cdot m$ and $\rho_2 \cdot m$ are S -invariant. Then $(\rho_1 \wedge \rho_2) \cdot m$ is S -invariant as well (where $\rho_1 \wedge \rho_2 := \min\{\rho_1, \rho_2\}$).

Proof. Since S is sub-Markovian, by Jensen's inequality S considered with domain $L^1(m) \cap L^\infty(m)$ extends to a linear contraction operator S_2 on $L^2(m)$. Let S_2^* denote its adjoint. Then S_2^* is again sub-Markovian (cf. [16, Chapter II, Proposition 4.1]) and extends to a linear contraction operator S^* on $L^1(m)$.

Let $\rho \in L^1(m)$ and $\rho_n := (\rho \wedge n) \vee (-n)$, $n \in \mathbb{N}$. Then for all $f \in L^1(m) \cap L^\infty(m)$

$$\int_E S f \rho dm = \lim_{n \rightarrow \infty} \int_E S f \rho_n dm = \lim_{n \rightarrow \infty} \int_E f S_2^* \rho_n dm = \int_E f S^* \rho dm.$$

So, $\rho \cdot m$ is S -invariant if and only if ρ belongs to the space of fix points of S^* , that is to the space

$$V := \{\rho \in L^1(m) \mid S^* \rho = \rho\}.$$

So, to prove the assertion we have to prove that V is a lattice. But if $\rho \in V$, then by the positivity of S^* we have $S^* \rho^+ \geq S^* \rho = \rho$ and $S^* \rho^+ \geq 0$, so $S^* \rho^+ \geq \rho^+$. Hence, since S^* is a contraction on $L^1(m)$,

$$0 \leq \int_E (S^* \rho^+ - \rho^+) dm \leq 0.$$

So, $\rho^+ \in V$. \square

We need one more lemma.

Lemma 2.7. *Let $\mu \in M_{par}^{A,b,\nu}$ (so $\mu(dt dx) = \mu_t(dx) dt$). Then, for every function $u \in C_0^\infty([0, T) \times \mathbb{R}^d)$, one has*

$$\int_{(0,T) \times \mathbb{R}^d} Lu d\mu = - \int_{\mathbb{R}^d} u(0, x) \nu(dx).$$

Proof. Let $u \in C_0^\infty([0, T] \times \mathbb{R}^d)$. By a limiting argument we may assume that $u(t, x) = \varphi(t)\zeta(x)$ for $\varphi \in C_0^\infty([0, T])$ and $\zeta \in C_0^\infty(\mathbb{R}^d)$. Since $\mu \in M_{par}^{A, b, \nu}$ we know that for dt -a.e. $\varepsilon, t \in [0, T]$ with $t > \varepsilon$, one has

$$\int_{\mathbb{R}^d} \zeta(x) \mu_t(dx) = \int_\varepsilon^t \int_{\mathbb{R}^d} L\zeta(s, x) \mu_s(dx) ds + \int_{\mathbb{R}^d} \zeta(x) \mu_\varepsilon(dx).$$

Hence

$$\begin{aligned} & - \lim_{\varepsilon \rightarrow 0} \int_\varepsilon^T \frac{d}{dt} \varphi(t) \int_{\mathbb{R}^d} \zeta(x) \mu_t(dx) dt \\ &= \varphi(0) \int_{\mathbb{R}^d} \zeta(x) \nu(dx) + \lim_{\varepsilon \rightarrow 0} \int_\varepsilon^T \varphi(t) \frac{d}{dt} \int_{\mathbb{R}^d} \zeta(x) \mu_t(dx) dt \\ &= \int_{\mathbb{R}^d} u(0, x) \nu(dx) + \lim_{\varepsilon \rightarrow 0} \int_\varepsilon^T \varphi(t) \int_{\mathbb{R}^d} L\zeta(t, x) \mu_t(dx) dt. \end{aligned}$$

Note that by assumption each a^{ij} is continuous on $[0, T] \times \mathbb{R}^d$ and $b \in L^1((0, T) \times B, \mu)$ for all balls $B \subset \mathbb{R}^d$. So, $\varphi L\zeta = \varphi L_{A, b}\zeta \in L^1((0, T) \times \mathbb{R}^d, \mu)$. Hence the assertion follows. \square

Proof of Theorem 2.3. Let $\mu \in M_{par, cg}^{A, b, \nu}$ and let $\mu_i \in M_{par}^{A, b, \nu}$, $\alpha_i \in (0, 1)$, $i = 1, 2$ be such that $\alpha_1 + \alpha_2 = 1$ and

$$\mu = \alpha_1 \mu_1 + \alpha_2 \mu_2.$$

We have to show that $\mu_1 = \mu_2 = \mu$. We first note that by the Radon–Nikodym theorem we have

$$\mu_i = \sigma_i \cdot \mu, \quad i = 1, 2,$$

with some measurable functions $\sigma_i: (0, T) \times \mathbb{R}^d \rightarrow [0, \infty)$ and

$$\sigma_i \leq \frac{1}{\alpha_i}.$$

Now we can complete the proof using Lemmas 2.6 and 2.7. Indeed, by assumption there exists a sub-Markovian C_0 -semigroup $(T_t^\mu)_{t \geq 0}$ generated by $(\bar{L}^\mu, D(\bar{L}^\mu))$ on $L^1((0, T) \times \mathbb{R}^d, \mu)$. By Lemma 2.7 we have

$$\int_{(0, T) \times \mathbb{R}^d} Lu \, d\mu_1 = \int_{(0, T) \times \mathbb{R}^d} Lu \, d\mu_2 \quad \text{for all } u \in C_0^\infty([0, T] \times \mathbb{R}^d),$$

hence, since σ_1, σ_2 are bounded, we obtain

$$\int_{(0, T) \times \mathbb{R}^d} \bar{L}^\mu u \, \sigma_1 \, d\mu = \int_{(0, T) \times \mathbb{R}^d} \bar{L}^\mu u \, \sigma_2 \, d\mu \quad \text{for all } u \in D(\bar{L}^\mu).$$

Consequently, for all $u \in C_0^\infty([0, T] \times \mathbb{R}^d)$ and $t > 0$ we have

$$\int_{(0, T) \times \mathbb{R}^d} \bar{L}^\mu T_t^\mu u \, \sigma_1 \, d\mu = \int_{(0, T) \times \mathbb{R}^d} \bar{L}^\mu T_t^\mu u \, \sigma_2 \, d\mu.$$

Integrating over $(0, t)$ yields the equality

$$\int_{(0, T) \times \mathbb{R}^d} T_t^\mu u \, \sigma_1 \, d\mu - \int_{(0, T) \times \mathbb{R}^d} u \, \sigma_1 \, d\mu = \int_{(0, T) \times \mathbb{R}^d} T_t^\mu u \, \sigma_2 \, d\mu - \int_{(0, T) \times \mathbb{R}^d} u \, \sigma_2 \, d\mu,$$

which by a density argument then holds for all bounded measurable functions $u: (0, T) \times \mathbb{R}^d \rightarrow \mathbb{R}$. Applying Lemma 2.6 with $\varrho_1 := \sigma_1 - \sigma_2$, $\varrho_2 = 0$, and $\varrho_2 := \sigma_2 - \sigma_1$, $\varrho_1 = 0$, we obtain that for

$$\nu_1 := (\sigma_1 - \sigma_2)^+ \mu$$

and

$$\nu_2 := (\sigma_1 - \sigma_2)^- \mu$$

we have for $i = 1, 2$

$$\int_{(0, T) \times \mathbb{R}^d} T_t^\mu u \, d\nu_i = \int_{(0, T) \times \mathbb{R}^d} u \, d\nu_i \quad \text{for all } u \in C_0^\infty([0, T] \times \mathbb{R}^d).$$

Differentiating at $t = 0$ yields

$$\int_{(0, T) \times \mathbb{R}^d} Lu \, d\nu_i = 0 \quad \text{for all } u \in C_0^\infty([0, T] \times \mathbb{R}^d),$$

i.e., both measures ν_1 and ν_2 satisfy the first identity in (1.5). Hence by Theorem 2.5, since these two measures cannot be equivalent, either $(\sigma_1 - \sigma_2)^+ = 0$ or $(\sigma_1 - \sigma_2)^- = 0$ holds μ -a.e. Since both $\sigma_1(t, \cdot)$ and $\sigma_2(t, \cdot)$ are probability densities for each t , in either case it follows that $\sigma_1 = \sigma_2$, hence $\mu_1 = \mu_2$. \square

Remark 2.8. Let us replace in all of the above $[0, T] \times \mathbb{R}^d$ by $\mathbb{R} \times \mathbb{R}^d$ and require no boundary condition at $-\infty$. Let us define $\mathcal{M}_{par}^{A, b}$ correspondingly as in (1.6). Then all arguments go through, so the corresponding versions (that is, with $C_0^\infty(\mathbb{R} \times \mathbb{R}^d)$ in replace of $C_0^\infty([0, T] \times \mathbb{R}^d)$) of our main Theorems 2.3 and 1.2 also hold in this case. Since Lemma 2.7 is not necessary in this case we can even drop the condition on b in the definition of $\mathcal{M}_{par}^{A, b}$. Furthermore, we have to replace (H1)(b) by the assumption that $\int_{-T}^T \int_B |b(t, x)|^p \, dx \, dt < +\infty$ for all $T > 0$.

In this case, i.e. with $\mathbb{R} \times \mathbb{R}^d$ in place of $[0, T] \times \mathbb{R}^d$, there are easy examples where $\mathcal{M}_{par}^{A, b, \nu}$ contains more than one element.

Consider e.g. the case $a_{ij} = \frac{1}{2} \delta_{ij}$ and $b^i(t, x) = x^i$, $t \in \mathbb{R}, x \in \mathbb{R}^d$. Then obviously the standard normal distribution μ_0 belongs to $\mathcal{M}_{par}^{A, b}$. Define for all $z \in \mathbb{R}^d$, $t \in \mathbb{R}$, the map $\tau_{e^{-t}z}: \mathbb{R}^d \rightarrow \mathbb{R}^d$ by

$$\tau_{e^{-t}z}(y) := y + e^{-t}z, \quad y \in \mathbb{R}^d,$$

and the measure

$$\mu_t^z := \mu_0 \circ (\tau_{e^{-t}z})^{-1}.$$

Set

$$\mu^z(dt \, dx) := \mu_t^z(dx) \, dt.$$

Then we easily check that $\mu^z \in M_{par}^{A, b}$ for all $z \in \mathbb{R}^d$. Moreover, in this case we have

$$\text{ext } M_{par}^{A, b} = \{\mu^z \mid z \in \mathbb{R}^d\}.$$

We refer to [18] (see also [14]) for a detailed proof, even in the case \mathbb{R}^d is replaced by a Hilbert space.

So, e.g. if

$$\mu(dt \, dx) := \frac{1}{2} \mu^z(dt \, dx) + \frac{1}{2} dt \, \mu_0(dx), \quad z \in \mathbb{R}^d, z \neq 0,$$

then by Theorem 2.3, $\mu \notin M_{par, cg}^{A, b}$, so $(L^\mu, D(L^\mu))$ generates no C_0 -semigroup on $L^1(\mathbb{R} \times \mathbb{R}^d, \mu)$.

3 Applications

The first two results in this section are easy consequences of [20].

Assume (H1) and (H2) hold and let $\mu \in \mathcal{M}_{par}^{A,b,\nu}$. Then by Theorem 2.5 we have

$$\mu(dt dx) = \varrho(t, x) dt dx,$$

ϱ is locally Hölder continuous on $(0, T) \times \mathbb{R}^d$ and $\partial_j \varrho \in L_{loc}^p((0, T) \times \mathbb{R}^d)$, $1 \leq j \leq d$. Define the logarithmic derivative $\beta_\mu = (\beta_\mu^1, \dots, \beta_\mu^d)$ of μ with respect to the metric given by A as follows:

$$\beta_\mu^i := \sum_{j=1}^d (\partial_j a^{ij} + a^{ij} \varrho^{-1} \partial_j \varrho), \quad i = 1, \dots, d. \quad (3.8)$$

Proposition 3.1. *Assume (H1) and (H2) and define \mathcal{K} to be the set of all measures $\mu \in \mathcal{M}_{par}^{A,b,\nu}$ satisfying the following three conditions for all $1 \leq i, j \leq d$:*

(i) $\partial_j a^{ij} \in L^1((0, T) \times B, \mu)$ for all open balls $B \subset \mathbb{R}^d$.

(ii) $a^{ij} \in L^1((0, T) \times \mathbb{R}^d, \mu)$.

(iii) $b^i - \beta_\mu^i \in L^1((0, T) \times \mathbb{R}^d, \mu)$.

Then $\#\mathcal{K} \leq 1$.

Proof. By [20, Corollary 1.14 (a)] we have that each $\mu \in \mathcal{K}$ satisfies (1.7). By Theorem 1.1, it remains to show that \mathcal{K} is convex. Let $\mu_1, \mu_2 \in \mathcal{K}$ and $\lambda \in (0, 1)$. Then obviously

$$\mu := \lambda \mu_1 + (1 - \lambda) \mu_2$$

satisfies (i) and (ii). To see that it also satisfies (iii) we first recall that

$$\mu_i(dt dx) = \varrho_i(t, x) dt dx, \quad i = 1, 2,$$

with ϱ_i as in Theorem 2.5. Then an easy calculation shows that for all $1 \leq i \leq d$ we have

$$b^i - \beta_\mu^i = (\lambda \varrho_1 + (1 - \lambda) \varrho_2)^{-1} [\lambda \varrho_1 (b^i - \beta_{\mu_1}^i) + (1 - \lambda) \varrho_2 (b^i - \beta_{\mu_2}^i)], \quad (3.9)$$

which is obviously in $L^1((0, T) \times \mathbb{R}^d, \mu)$, since

$$\mu(dt dx) = (\lambda \varrho_1(t, x) + (1 - \lambda) \varrho_2(t, x)) dt dx.$$

The proof is complete. \square

Proposition 3.2. *Assume (H1) and (H2). Let $V \in C^{1,2}([0, T] \times \mathbb{R}^d)$ be such that $\lim_{|x| \rightarrow \infty} V(t, x) = +\infty$ uniformly in $t \in [0, T]$. Let \mathcal{K} be the set of all $\mu \in \mathcal{M}_{par}^{A,b,\nu}$ satisfying condition (i) in Proposition 3.1 and such that for some $\alpha_0 = \alpha_0(\mu) \in (0, \infty)$ one has*

$$L_{A,2\beta_\mu-b} V - \partial_t V \leq \alpha_0 V.$$

Then $\#\mathcal{K} = 1$.

Proof. By [20, Corollary 1.14 (b)] we have that each $\mu \in \mathcal{K}$ satisfies (1.7). By Theorem 1.1 it remains to show that \mathcal{K} is convex. Let $\mu_1, \mu_2 \in \mathcal{K}$ and $\lambda \in (0, 1)$. Letting

$$\mu := \lambda \mu_1 + (1 - \lambda) \mu_2,$$

we obtain from (3.9) that

$$\begin{aligned} & L_{A,2\beta_\mu-b}V - \partial_t V \\ &= (\lambda \varrho_1 + (1-\lambda)\varrho_2)^{-1} [\lambda \varrho_1 (L_{A,2\beta_{\mu_1}-b}V - \partial_t V) + (1-\lambda)\varrho_2 (L_{A,2\beta_{\mu_2}-b}V - \partial_t V)] \\ &\leq \alpha_0 V. \end{aligned}$$

Hence \mathcal{K} is convex. \square

Now we are going to give concrete global conditions on A, b and ν so that problem (1.5) has a unique solution. The proof of the corresponding theorem relies on a combination of recent results of [4], [10] and Proposition 3.1 above.

Theorem 3.3. *Assume (H1) and (H2). Suppose that the following global assumptions on A and b hold:*

(iv) *the measure ν has finite entropy, i.e., $\nu(dx) = \varrho_0(x)dx$ for some $\varrho_0 \in L^1(\mathbb{R}^d)$ and*

$$\int_{\mathbb{R}^d} \varrho_0(x) \log \varrho_0(x) dx < +\infty.$$

(v) *There exists $\varepsilon \in (0, \infty)$ such that*

$$\varepsilon Id \leq A(t, x) \leq \varepsilon^{-1} Id$$

for all $(t, x) \in [0, T] \times \mathbb{R}^d$.

(vi) *There exists $\Lambda \in (0, \infty)$ such that for all $x, y \in \mathbb{R}^d$ one has*

$$\sup\{|a^{ij}(t, x) - a^{ij}(t, y)| : t \in [0, T], 1 \leq i \leq j \leq d\} \leq \Lambda|x - y|.$$

(vii) *There exists $c \in (0, \infty)$ such that for all $(t, x) \in [0, T] \times \mathbb{R}^d$ one has*

$$\langle b(t, x), x \rangle \leq c(1 + |x|^2),$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product on \mathbb{R}^d , and for some $k \in \mathbb{N}$ one has

$$|b(t, x)| \leq c(1 + |x|^{2k}) \quad \text{as well as} \quad \int_{\mathbb{R}^d} |x|^{2k} \nu(dx) < \infty,$$

or there exist numbers $\alpha, \gamma, \delta, c, k \in (0, \infty)$ such that for all $(t, x) \in [0, T] \times \mathbb{R}^d$

$$\langle b(t, x), x \rangle \leq \gamma - (2\varepsilon^{-1}ck + \delta)|x|^{2k},$$

with ε as in (v), and

$$|b(t, x)| \leq \alpha \exp\left(\frac{c}{2}|x|^{2k}\right) \quad \text{as well as} \quad \int_{\mathbb{R}^d} \exp\left(\frac{c}{2}|x|^{2k}\right) \nu(dx) < \infty.$$

Then there exists a unique family $\{\mu_t, t \in (0, T]\}$ of probability measures on \mathbb{R}^d solving (1.5).

Proof. By [4, Theorem 3.1 and Examples 2.5 (i) and (iii)] the existence of the desired family $\mu_t, t \in (0, T]$, follows and we have that

$$|b| \in L^2((0, T) \times \mathbb{R}^d, \mu). \tag{3.10}$$

By our assumptions (v) and (vi) condition (i) and (ii) in Proposition 3.1 hold. Because of (v), (3.8) and (3.10) it suffices to show that

$$\frac{\nabla \varrho}{\varrho} \in L^1((0, T) \times \mathbb{R}^d, \mu)$$

in order to verify condition (iii). But by [10, Theorem 2.1] we even have that

$$\frac{\nabla \varrho}{\varrho} \in L^2((0, T) \times \mathbb{R}^d, \mu).$$

Hence the above family μ_t , $t \in (0, T]$, is unique. \square

Remark 3.4. (i) We would like to emphasize that in the above theorem no dissipativity type conditions are assumed on the operator $L_{A,b}$. Nevertheless, one can prove uniqueness for problem (1.5).

(ii) An application where we have uniqueness for problem (1.5) under dissipativity conditions is presented in [5].

Acknowledgement.

This work was supported by the research program “Equazioni di Kolmogorov” from the Italian “Ministero della Ricerca Scientifica e Tecnologica”. Support by the DFG-Research Group 399, the SFB-701, the BIBOS-Research Center, the INTAS project 99-559, the RFBR project 04-01-00748, the Russian–Japanese Grant 05-01-02941-JF, the DFG Grant 436 RUS 113/343/0(R), is gratefully acknowledged.

This work was initiated and completed during a very pleasant stay at Scuola Normale Superiore in Pisa of the first and third named authors who would like to thank for the warm hospitality and excellent working conditions.

References

- [1] S. Albeverio, V.I. Bogachev and M. Röckner, *On uniqueness of invariant measures for finite and infinite dimensional diffusions*, Comm.Pure Appl. Math, **52**, 325–362, 1999.
- [2] W. Arendt, *The abstract Cauchy problem, special semigroups and perturbation*. In: One-parameter semigroups of positive operators, Edited by R. Nagel, Berlin: Springer 1986.
- [3] V.I. Bogachev, G. Da Prato and M. Röckner, *Existence of solutions to weak parabolic equations for measures*, Proc. London Math. Soc., (3), **88**, 753–774, 2004.
- [4] V.I. Bogachev, G. Da Prato and M. Röckner, *On parabolic equations for measures*, Preprint SFB-701 N 06-010, Universität Bielefeld, 2006.
- [5] V.I. Bogachev, G. Da Prato and M. Röckner, *Maximal dissipativity for Kolmogorov operators with time dependent singular drifts on Hilbert spaces*, Preprint 2006.
- [6] V.I. Bogachev, N.V. Krylov and M. Röckner, *On regularity of transition probabilities and invariant measures of singular diffusions under minimal conditions*, Comm. Partial Diff. Equations, **26**, no. 11–12, 2037–2080, 2001.

- [7] V.I. Bogachev, N.V. Krylov and M. Röckner, *Elliptic equations for measures: regularity and global bounds of densities*. J. Math. Pures Appl., **85**, 743–757, 2006.
- [8] V.I. Bogachev and M. Röckner, *A generalization of Khasminskii's theorem on existence of invariant measures for locally integrable drifts*, Theory Probab. Appl., **45**, 417–436, 2000.
- [9] V.I. Bogachev and M. Röckner, *Elliptic equations for measures on infinite dimensional spaces and applications*, Probab. Theory Relat. Fields, **120**, 445–496, 2001.
- [10] V.I. Bogachev, M. Röckner and S.V. Shaposhnikov, *Global regularity and bounds for solutions of parabolic equations for probability measures*, Teor. Veroyatn. i Primen., **50**, no. 4, 952–974, 2005 (in Russian); English transl.: Theory Probab. Appl., **50**, 2006.
- [11] V.I. Bogachev, M. Röckner and W. Stannat *Uniqueness of invariant measures and essential m -dissipativity for diffusion operators in L^1* , In: Infinite Dimensional Stochastic Analysis (Ph. Clément, F. den Hollander, J. van Neerven and B. de Pagter Eds.). Royal Netherlands Academy of Arts and Sciences, Amsterdam, pp. 39–54, 2000.
- [12] V.I. Bogachev, M. Röckner and W. Stannat *Uniqueness of solutions of elliptic equations and uniqueness of invariant measures of diffusions*, Sbornik Math., **193**, 7, 3–36, 2002.
- [13] V.I. Bogachev, M. Röckner and F.-Y. Wang, *Elliptic equations for invariant measures on finite and infinite dimensional manifolds*, J. Math. Pures Appl., **80**, 177–221, 2001.
- [14] G. Da Prato and J. Zabczyk, *Stochastic equations in infinite dimensions*, Cambridge University Press, 1992.
- [15] A. Eberle, *Uniqueness and non-uniqueness of singular diffusion operators*, Lecture Notes in Mathematics, **1718**, Springer-Verlag, 1999.
- [16] Z. M. Ma and M. Röckner, *Introduction to the Theory of (Non Symmetric) Dirichlet Forms*, Springer-Verlag, 1992.
- [17] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Springer-Verlag, 1983.
- [18] M. Röckner, *On the parabolic Martin boundary of the Ornstein–Uhlenbeck process on Wiener space*. Ann. Prob., **20**, 1063–1085. 1992.
- [19] W. Stannat, *(Nonsymmetric) Dirichlet operators on L^1 : existence, uniqueness and associated Markov processes*, Annali Scuola Normale Super. di Pisa Cl. Sci. (4) **28**, no. 1, 99–140, 1999.
- [20] W. Stannat, *Time-dependent diffusion operators on L^1* , J. Evol. Equat., **4**, 463–495, 2004.