ASYMPTOTIC NORMALITY OF OCCUPATION MEASURE OF MARKOV AND SEMI-MARKOV PROCESSES

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Abstract

The problem of asymptotic normality of occupation measure for Markov continuous time processes with the finite phase space is investigated in the book of G.Yin and Q.Zhang [1]. We propose some other approach based on asymptotic analysis of random evolution process by using a solution of the singular perturbation problem for reducible-invertible operator [2]. For simplicity the homogeneous in time Markov and semi-Markov processes are considered.

1 Introduction

The semi-Markov process $\alpha(t),\ t \geq 0$ in the finite phase space $E = \{1, \dots, N\}$ is given by the semi-Markov matrix

$$Q_{kr}(t) = p_{kr} F_k(t), \quad k,\, r \in E. \tag{1}$$

The stochastic matrix $P = [p_{kr}\,;\, k,\, r \in E]$ determines transition probabilities of the embedded Markov chain $\alpha_n = \alpha(\tau_n)$, $n \geq 0$:

$$p_{kr} = P\left\{\alpha_{n+1} = r \,|\, \alpha_n = k\right\} \tag{2}$$

The renewal moments of jump

$$\tau_{n+1} = \tau_n + \theta_{n+1}, \quad n \geq 0, \;\; \tau_0 = 0, \tag{3}$$

is determined by the distribution functions of sojourn times $\theta_{n+1},\ n \geq 0$:

$$F_k(t) = P\left\{\theta_{n+1} \leq t \,|\, \alpha_n = k\right\} \tag{4}$$

In particular case of exponential distributions $F_k(t) = 1 - \exp\left(-q_k t\right),\ k \in E$, the corresponding process $\alpha(t),\ t \geq 0$, is Markovian and can be defined by the generating matrix

$$Q = \left[q_{kr};\ k,\, r \in E\right], \quad q_{kr} = q_k p_{kr}, \quad k \neq r \tag{5}$$

$$q_{kk} = -q_k, \quad k \in E.$$

The **main assumption** is that the semi-Markov process $\alpha(t),\ t \geq 0$, is **ergodic** with the stationary distribution $\pi = (\pi_k,\, k \in E)$ which satisfies the relation

$$\pi_k q_k = q \rho_k, \quad k \in E, \quad q = \sum_{k \in E} \pi_k q_k \tag{6}$$

The vector $\rho = (\rho_k,\, k \in E)$ defines the stationary distribution of the embedded Markov chain α_n, $n \geq 0$. In semi-Markov case

$$q_k := 1/m_k, \quad m_k := \int_0^\infty \bar{F}_k(t)dt, \;\; \bar{F}_k(t) := 1 - F_k(t). \tag{7}$$

2 Occupation measure

Definition 2.1 *[1] The normalized occupation measure process for the semi-Markov process $\alpha(t)$, $t \geq 0$, in the series scheme with the small parameter series $\varepsilon \to 0\,(\varepsilon > 0)$ is defined by the integral functional*

$$\zeta_k^\varepsilon(t) = \varepsilon^{-1} \int_0^t \left[I\left(\alpha(s/\varepsilon^2) = k\right) - \pi_k \right] \beta_k ds, \tag{8}$$

where $I(A)$ is the indicator function of event A, $\beta = (\beta_k,\, k \in E)$ is the scaling vector.

It is worth noticing that under the main assumption the following convergence

$$EI\left(\alpha(s/\varepsilon^2) = k\right) \to \pi_k, \quad \varepsilon \to 0, \;\; k \in E$$

takes place. So, the normalizing factor ε^{-1} is used to obtain some non-trivial limit for the vector

$$\zeta^\varepsilon(t) = (\zeta_k^\varepsilon(t),\, k \in E)$$

as $\varepsilon \to 0$.

The main results are formulated below:

Theorem 2.1 *Under the main assumption of ergodicity of the semi-Markov process $\alpha(t)$, $t \geq 0$, on the finite phase space $E = \{1, 2, \ldots, N\}$ the normalized occupation measure process (8) weakly converges*

$$\zeta^\varepsilon(t) \Rightarrow W_\sigma(t), \quad \varepsilon \to 0. \tag{9}$$

The limit Wiener process $W_\sigma(t)$, $t \geq 0$, is defined by zero mean value and the variance matrix

$$B = \sigma\sigma^\star = \left[B_{kr};\, k,\, r \in E\right],$$

$$B_{kr} = \pi_k \beta_k R_{kr} \beta_r + \pi_r \beta_r R_{rk} \beta_k + B_{kr}^\mu \tag{10}$$

where

$$\mu_k = \left[m_k^{(2)} - m_k^2\right]/m_k, \quad m_k^{(2)} := \int_0^\infty t\bar{F}_k(t)dt \quad k \in E,$$

$$B_{kr}^\mu := \left[\pi_k \mu_k \delta_{kr} + \pi_k \pi_r (\hat{\mu} - \mu_k - \mu_r)\right] \beta_k \beta_r,$$

$$\hat{\mu} := \sum_{k\in E} \pi_k \mu_k,$$

and the potential matrix $R_0 = [R_{kr};\ k,\, r \in E]$ *is defined by the relations:*

$$QR_0 = R_0 Q = \Pi - I,$$

or by the form

$$R_0 = \int_0^\infty [P_t - \Pi]\, dt,$$

where P_t, $t \geq 0$, *is the semigroup defined by the generator* Q *(see (5)).*

The projector Π *acts as follows:*

$$\Pi\varphi(k) = \hat{\varphi}1\!\!1, \quad \hat{\varphi} := \sum_{k\in E} \pi_k \varphi(k), \quad 1\!\!1 := 1(k) \equiv 1, \quad k \in E.$$

Remark 2.1 *In particular case of Markov process* $\alpha(t)$, $t \geq 0$, *the variance matrix is defined as follows:*

$$B^0_{kr} = \pi_k \beta_k R_{kr} \beta_r + \pi_r \beta_r R_{rk} \beta_k, \quad k.\, r \in E \tag{11}$$

Remark 2.2 *The values* $\mu_k = \left[m_k^{(2)} - m_k^2\right]/m_k$, $k \in E$, *can be positive as well as negative [3]. For the exponential distribution* $\mu_k = 0$.

3 Algorithm of asymptotic normality

3.1 Random evolution approach

The random evolution approach, described in the book [4], means that the occupation measure process (8) is considered as a random evolution in the following form:

$$\zeta^\varepsilon(t) = u + \varepsilon^{-1} \int_0^t b(\alpha^\varepsilon(s))ds, \quad t \geq 0 \tag{12}$$

where

$$\alpha^\varepsilon(s) := \alpha(s/\varepsilon^2)$$

and the vector function

$$b(r) = (b_k(r),\, k \in E)$$

is defined by the relation

$$b_k(r) = [\delta_k(r) - \pi_k]\,\beta_k, \quad k \in E,\ r \in E \tag{13}$$

here

$$\delta_k(r) = \begin{cases} 1, & \text{if } r = k, \\ 0, & \text{otherwise} \end{cases}$$

is the Kronecker symbol.

Introduce the deterministic evolution in $\mathbb{R}^N$

$$u(t;\, r) := u + b(r)t, \quad t \geq 0,\ \ r \in E, \tag{14}$$

and the corresponding evolution in the Banach space $B(\mathbb{R}^N)$ of bounded real-valued test-functions $\varphi(u)$, $u \in \mathbb{R}^N$, with the sup-norm: $||\varphi|| := \sup_{u\in\mathbb{R}^N} |\varphi(u)|$, by the family of semigroups

$$B_t(r)\varphi(u) := \varphi(u(t;\, r)),\ \ t \geq 0,\ \ u(0;\, r) = u. \tag{15}$$

the generators of semigroups (15) are defined by the relation

$$B(r)\varphi(u) = b(r)\varphi'(u) := \sum_{k\in E} b_k(r)\varphi'_k(u), \tag{16}$$

where, by definition, vector

$$\varphi'(u) = \left(\varphi'_k(u) := \partial\varphi(u)/\partial u_k,\ k \in E\right).$$

3.2 The characterization of the random evolution process (12)

The characterization of the random evolution process (12) is considered in two cases:

i) The switching process $\alpha(t),\ t \geq 0$, is Markovian given by the generator (5).

ii) The switching process $\alpha(t),\ t \geq 0$, is semi-Markovian, given by the semi-Markov kernel (1).

Lemma 3.1 *i) The coupled Markov process*

$$\zeta^\varepsilon(t), \quad \alpha^\varepsilon(t) := \alpha(t/\varepsilon^2),\ t \geq 0,$$

can be characterized by the generator

$$\mathbb{L}^\varepsilon\varphi(u,\, r) = \left[\varepsilon^{-2}Q + \varepsilon^{-1}B(r)\right]\varphi(u,\, r). \tag{17}$$

ii) The extended Markov renewal process

$$\zeta_n^\varepsilon = \zeta^\varepsilon(\tau_n^\varepsilon),\ \alpha_n = \alpha(\tau_n),\ \tau_n^\varepsilon := \varepsilon^2\tau_n,\, n \geq 0,$$

can be characterized by the compensative operator [4].

$$\mathbb{L}^\varepsilon\varphi(u,\, r) = \varepsilon^{-2}q_r\left[\int_0^\infty F_r(dt)B_{\varepsilon t}(r)P\varphi(u,\, r) - \varphi(u,\, r)\right] \tag{18}$$

or, in other equivalent form

$$\mathbb{L}^\varepsilon\varphi(u,\, r) = \left[\varepsilon^{-2}Q + \varepsilon^{-1}B^\varepsilon(r)\right]\varphi(u,\, r), \tag{19}$$

where Q is the generator of the associated Markov process $\alpha^0(t),\ t \geq 0$, given by the generating matrix (5) with intensities (7). The operators

$$B^\varepsilon(r)\varphi(u,\, r) = q_r B(r)\int_0^\infty \bar{F}_r(t)B_{\varepsilon t}(r)dt\, P\varphi(u,\, r) \tag{20}$$

$$F_0^\varepsilon(r) = \int_0^\infty F_r(dt)B_{\varepsilon t}(r) = I + B(r)F_1^\varepsilon(r),$$

$$F_1^\varepsilon(r) = \int_0^\infty \bar{F}_r(t)B_{\varepsilon t}(r)dt.$$

Corollary 3.1 *The compensative operator (19) admit the following asymptotic extension on the test-functions $\varphi(u) \in C^k(\mathbb{R}^N),\, k \geq 3$:*

$$\mathbb{L}^\varepsilon\varphi(u,\, r) = \left[\varepsilon^{-2}Q + \varepsilon^{-1}B(r)P + B_0(r)P + \theta^\varepsilon(r)\right]\varphi(u,\, r), \tag{21}$$

where

$$B_0(r)\varphi(u) = \bar{m}_r^{(2)}B^2(r)\varphi(u), \quad \bar{m}_r^{(2)} := m_r^{(2)}/m_r \tag{22}$$

and the residual term $\theta^\varepsilon(r)$ is negligible:

$$||\theta^\varepsilon(r)\varphi(u)|| \to 0, \quad \varepsilon \to 0, \quad \varphi(u) \in C^k(\mathbb{R}^N), \ \ k \geq 3$$

also, here by the definition

$$B^2(r)\varphi(u) := \sum_{k,\, k' \in E} b_k(r)b_{k'}(r)\varphi''_{kk'}(u),$$

and

$$\varphi''_{kk'}(u) := \partial^2\varphi(u)/\partial u_k\partial u_{k'}.$$

Proof of Lemma 3.1. The first representation (17) follows from the asymptotic analysis of conditional expectation

$$E\left[\varphi(u+\triangle\zeta^{\varepsilon}(t),\,\alpha^{\varepsilon}_{t+\triangle})-\varphi(u,\,r)\,|\,\alpha^{\varepsilon}_{t}=r\right]=$$

$$E\left[\varphi(u+\varepsilon^{-1}b(r)\triangle,\,\alpha^{\varepsilon}_{t+\triangle})-\varphi(u,\,r)\,|\,\alpha^{\varepsilon}_{t}=r\right]=$$
$$\triangle\left[\varepsilon^{-2}Q+\varepsilon^{-1}B(r)\right]\varphi(u,\,r)+o(\triangle).$$

The second representation (18) follows from calculation of conditional expectation $\left(\triangle\zeta^{\varepsilon}_{n}:=\zeta^{\varepsilon}_{n+1}-\zeta^{\varepsilon}_{n}\right)$

$$E\left[\varphi(u+\triangle\zeta^{\varepsilon}_{n}(t),\,\alpha^{\varepsilon}_{n+1})-\varphi(u,\,r)\,|\,\zeta^{\varepsilon}_{n}=u,\;\alpha^{\varepsilon}_{n}=r\right]=$$

$$E\left[\varphi(u+\varepsilon\theta_{n+1}b(r),\,\alpha_{n+1})-\varphi(u,\,r)\,|\,\alpha^{\varepsilon}_{n}=r\right]=$$
$$\int_0^\infty F_r(dt)B_{\varepsilon t}(r)P\varphi(u,\,r)-\varphi(u,\,r).$$

The normalizing factor $\varepsilon^{-2}q_r$ leads to (18). The corollary 3.1 can be obtained by using the following transformation

$$F^{\varepsilon}_0(r):=\int_0^\infty F_r(dt)B_{\varepsilon t}(r)=I+\varepsilon B(r)F^{\varepsilon}_1(r),$$

$$F^{\varepsilon}_1(r):=\int_0^\infty F_r(dt)\int_0^{\varepsilon t}B_s(r)ds=\int_0^\infty \bar{F}_r(t)B_{\varepsilon t}(r)dt=$$
$$=m_rI+\varepsilon B(r)F^{\varepsilon}_2(r),$$
$$F^{\varepsilon}_2(r):=\int_0^\infty \bar{F}^{(2)}_r(t)B_{\varepsilon t}(r)dt=m^{(2)}_rI+\varepsilon B(r)F^{\varepsilon}_3(r),$$
$$F^{\varepsilon}_3(r):=\int_0^\infty \bar{F}^{(3)}_r(t)B_{\varepsilon t}(r)dt.$$

Here, by the definition,

$$\bar{F}^{(k+1)}_r(t)=\int_t^\infty \bar{F}^{(k)}_r(s)ds,\quad k\geq 1,\quad \bar{F}^{(1)}_r(t):=\bar{F}_r(t).$$

In above transformation the relations for the semigroup

$$B_{\varepsilon t}(r)=I+\varepsilon B(r)\int_0^{\varepsilon t}B_s(r)ds,$$

and

$$dB_{\varepsilon t}(r) = \varepsilon B(r) B_{\varepsilon t}(r) dt,$$

are used.

As result we get the expansion

$$F_0^\varepsilon(r) = \int_0^\infty F_r(dt) B_{\varepsilon t}(r) = I + \varepsilon m_r B(r) + \varepsilon^2 m_r^{(2)} B^2(r) + \varepsilon^2 \theta^\varepsilon(r),$$

with the negligibly term $\theta^\varepsilon(r)$. Here, by the definition

$$m_r^{(2)} := \int_0^\infty \bar{F}_r^{(2)}(t) dt = \int_0^\infty t \bar{F}(t) dt.$$

3.3 Problem of singular perturbation

The asymptotic representations (17) and (21) can be used to construct the limit operator by using a solution of singular perturbation problem in the following form (see [2], Lemma 3.2, p.52).

$$\mathbb{L}^\varepsilon \varphi^\varepsilon := \left[\varepsilon^{-2} Q + \varepsilon^{-1} B(r)\right] \left[\varphi(u) + \varepsilon \varphi_1(u, r) + \varepsilon^2 \varphi_2(u, r)\right] = \mathbb{L}_0 \varphi(u) + \theta_l^\varepsilon(r) \varphi(u), \tag{23}$$

with the negligibly term

$$||\theta_l^\varepsilon(r)\varphi|| \Rightarrow 0, \quad \varepsilon \to 0, \quad \varphi(u) \in C^k(\mathbb{R}^N),\, k \geq 3.$$

The limit operator is given by the relation

$$\mathbb{L}_0 = \Pi B(r) R_0 B(r) \Pi. \tag{24}$$

The limit operator in the case of semi-Markov switching is constructed by using a solution of singular perturbation problem for the truncated generator (21) in the following form (see [2], Lemma 3.2, p.52).

$$\mathbb{L}_0^\varepsilon \varphi^\varepsilon := \left[\varepsilon^{-2} Q + \varepsilon^{-1} B(r) P + B_0(r) P\right] \left[\varphi(u) + \varepsilon \varphi_1(u, r) + \varepsilon^2 \varphi_2(u, r)\right] = \mathbb{L} \varphi(u) + \theta_l^\varepsilon(r) \varphi(u), \tag{25}$$

with he negligibly term

$$||\theta_l^\varepsilon(r)\varphi|| \Rightarrow 0, \quad \varepsilon \to 0, \quad \varphi(u) \in C^k(\mathbb{R}^N),\, k \geq 3.$$

The limit operator $\mathbb{L}$ is calculated by the formula

$$\mathbb{L} = \Pi B(r) P R_0 B(r) \Pi + \Pi B_0(r) \Pi. \tag{26}$$

Corollary 3.2 *i) The limit operator (24) in case of Markov process is given by the relation*

$$\mathbb{L}_0\varphi(u) = \frac{1}{2}B_0\varphi''(u) := \frac{1}{2}\sum_{k,\,r=1}^{N} B^0_{kr}\varphi''_{kr}(u), \tag{27}$$

where the variance matrix $B_0 = [B^0_{kr};\ k,\, r \in E]$ *is given in (11).*

ii)The limit operator (26) in the case of semi-Markov process is given by the relation

$$\mathbb{L}_0\varphi(u) = \frac{1}{2}B\varphi''(u) := \frac{1}{2}\sum_{k,\,r=1}^{N} B_{kr}\varphi''_{kr}(u), \tag{28}$$

where the variance matrix $B = [B_{kr};\ k,\, r \in E]$ *is given in (10).*

To proof the Corollary 3.2 in the case of Markov process the limit operator $\mathbb{L}_0$ is calculated by the relation (24) using the definition of the operator $B(r)$ in (13) and (16).

The calculation of the limit operator $\mathbb{L}$ in the case of semi-Markov process is more complicated. At first we use the following relation (see [2])

$$PR_0 = R_0 + m\,[\Pi - I]\,.$$

So in (26) we set

$$\mathbb{L} = \Pi B(r) R_0 B(r)\Pi + \Pi\mu_r B^2(r)\Pi \tag{29}$$

with

$$\mu_r := m^{(2)}_r q_r - m_r = \left[m^{(2)}_r - m^2_r\right]/m_r \tag{30}$$

As result we obtain the second order operator $\mathbb{L}$ given by the variance matrix (10).

The representations (23) and (25) can be used to proof the convergence for the finite dimensional distributions (see, for example [5]). To proof the weak convergence in Theorem we have to use additional compact containment conditions (see [4], [6]).

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