KUSUOKA-STROOCK FORMULA ON CONFIGURATION SPACE AND REGULARITIES OF LOCAL TIMES WITH JUMPS

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Abstract. In this paper, we first extend the classical Itô stochastic integral to the case of measurable fields of Hilbert spaces. Then, a Kusuoka-Stroock formula on configuration space is proved. Using this formula, we study the fractional regularities of local times with jumps in the sense of the Malliavin calculus.

Contents

1. Introduction

The purpose of the present paper is to establish some regularity results for local times of semimartingales with jumps. By regularities we mean here smoothness in the sense of the

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Malliavin calculus, i.e., its membership in Sobolev spaces over the basic probability space on the one side, and its quasi sure existence in the sense of capacities of the associated Dirichlet forms on the other side. The study of such regularities have been the subject of several works, see e.g. [21, 17, 24] for the Brownian case and [1, 11] for continuous semimartingales.

Originally, these two kinds of regularities were studied separately and no connection between them were realized. A connection was later established in [11, 19] which says that smoothness in Sobolev spaces can lead to quasi sure existence. In the present paper we propose to develop the ideas of these latter papers on Wiener-Poisson spaces.

To this end, as is already seen on the Wiener space, one has to differentiate stochastic integrals in the sense of the Malliavin calculus, and thus a Kusuoka-Stroock formula(cf. [13]) on the commutation relation between the gradient operator and the stochastic integral is needed, and this in turn involves stochastic integration of gradients of differentiable functionals. On Wiener space one can use the well developed theory of stochastic integration in Hilbert space since the common tangent space at all points is the Cameron-Martin subspace; this is essentially the case even for the continuous path space of a Riemannian manifold since one can use stochastic parallel transport to transport vector fields into the same Hilbert space (see [16]). But on the configuration space the situation is different since no such stochastic parallel transport yet exists.

So, as preparation our first task is the construction of a new kind of stochastic integrals. Compared with the standard ones, the main feature of such integrals consists in that the integrand takes its values at any sample point of the underlying probability space, say $\omega \in \Omega$, in a Hilbert space which *depends* on ω, and it will be considered as a function of time taking values in a direct integral of measurable field of Hilbert spaces. This construction will be done in Section 2 and we hope that this construction will be of independent interest.

In Section 3 we will introduce all necessary terminology concerning configuration spaces.

The said Kusuoka-Stroock formula will be stated and proved in Section 4.

Then after briefly recalling the theory of quasi-regular Dirichlet forms in Section 5 and defining fractional Sobolev spaces in Section 6, we state and prove our main results in the final section.

Convention: The letter C with or without subscripts will denote a positive constant, which is unimportant and may change from one line to another line.

2. Extension of stochastic integrals

The aim of this section is to develop a theory of stochastic integration of processes taking values in the direct integral of measurable fields of Hilbert spaces with respect to martingales and random measures.

2.1. **Preliminaries on measurable fields of Hilbert spaces.** We begin by introducing notions of the theory of measurable fields of $\mathbb{K}$-Hilbert spaces in a language familiar to probabilists. Here $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$. For a systematic study see [22].

Definition 2.1. *(* [22, Ch. IV, Definition 8.9]*) Let $(\Omega, \mathcal{F})$ be a measurable space. An $\mathcal{F}$-measurable field of Hilbert spaces on $(\Omega, \mathcal{F})$ is a family $\big(H(\omega)\big)_{\omega\in\Omega}$ of separable Hilbert spaces indexed by $\omega \in \Omega$ together with a linear subspace $\mathcal{M} = \mathcal{M}(\mathcal{F})$ of the product vector space $\prod_{\omega\in\Omega} H(\omega)$ with the following properties:*

*(**A**1) For any $\zeta \in \mathcal{M}$, the function $\Omega \ni \omega \mapsto \|\zeta(\omega)\|_{H(\omega)}$ is $\mathcal{F}$-measurable.*

*(**A**2) For any $\eta \in \prod_{\omega\in\Omega} H(\omega)$, if the function $\omega \in \Omega \mapsto \big(\zeta(\omega), \eta(\omega)\big)_{H(\omega)} \in \mathbb{K}$ is $\mathcal{F}$-measurable for every $\zeta \in \mathcal{M}$, then η belongs to $\mathcal{M}$.*

*(**A**3) There exists a countable subset $(h_n)_{n\in\mathbb{N}}$ of $\mathcal{M}$ such that for every $\omega \in \Omega$, $(h_n(\omega))_{n\in\mathbb{N}}$ is total in $H(\omega)$.*

*An element of $\mathcal{M}$ is called an $\mathcal{F}$-measurable vector field. The family in (**A**3) is called a fundamental sequence of $\mathcal{F}$-measurable vector fields.*

Remark 2.2. *Consider $(\Omega, \mathcal{F})$ and $(H(\omega))_{\omega\in\Omega}$ as in Definition 2.1.*

(i) (See [22, Ch. IV, Lemma 8.10]*) Let $h_n \in \prod_{\omega\in\Omega} H(\omega)$, $n \in \mathbb{N}$, be such that $\omega \mapsto \big(h_n(\omega), h_m(\omega)\big)_{H(\omega)}$ is $\mathcal{F}$-measurable for all $n, m \in \mathbb{N}$, and $(h_n(\omega))_{n\in\mathbb{N}}$ is total in $H(\omega)$ for all $\omega \in \Omega$. Then*

$$\begin{aligned} \mathcal{M} &:= \mathcal{M}_h \qquad\qquad (2.1)\\ &:= \Big\{\zeta \in \prod_{\omega\in\Omega} H(\omega) \,\Big|\, \omega \mapsto \big(\zeta(\omega), h_n(\omega)\big)_{H(\omega)} \text{ is } \mathcal{F}\text{-measurable } \forall\, n \in \mathbb{N}\Big\} \end{aligned}$$

*satisfies (**A**1)–(**A**3) in Definition 2.1.*

(ii) (See [22, Ch. IV, Lemma 8.12]*) Let $\mathcal{M} = \mathcal{M}(\mathcal{F})$ be a linear subspace of $\prod_{\omega\in\Omega} H(\omega)$ satisfying (**A**1)–(**A**3) in Definition 2.1. Then $\omega \mapsto d(\omega) := \dim H(\omega)$ ($\in \mathbb{N} \cup \{0, +\infty\}$) is $\mathcal{F}$-measurable and there exists a fundamental sequence (w.r.t. $\mathcal{M}$) $(e_n)_{n\in\mathbb{N}}$ of $\mathcal{F}$-measurable vector fields such that for all $\omega \in \Omega$, $(e_n(\omega))_{n\leqslant d(\omega)}$ is an orthonormal basis of $H(\omega)$ and $e_n(\omega) = 0$ for all $n > d(\omega)$.*

*(iii) It follows from (**A**2) (and the proof of* [22, Ch. IV, Lemma 8.12]*) that if $\mathcal{M}$ is as in Definition 2.1 (satisfying (**A**1)–(**A**3)) and $(h_n)_{n\in\mathbb{N}}$ is any fundamental sequence, then $\mathcal{M} = \mathcal{M}_h$ with $\mathcal{M}_h$ defined in* (2.1).

(iv) Suppose

$$\omega \mapsto d(\omega) := \dim H(\omega) \quad \text{is } \mathcal{F}\text{-measurable.} \qquad (2.2)$$

Then picking an orthonormal basis $(e_n(\omega))_{n\leqslant d(\omega)}$ of $H(\omega)$ and setting $e_n(\omega) = 0$ for $n > d(\omega)$, $\omega \in \Omega$, we are in the situation of (i), so we obtain $\mathcal{M} := \mathcal{M}_e$ defined by (2.1) *satisfying (**A**1)–(**A**3) in Definition 2.1. Thus by (ii), the existence of $\mathcal{M}$ as in Definition 2.1 is equivalent to* (2.2). *Furthermore, let $\sigma(d)$ denote the σ-algebra generated by $\omega \mapsto d(\omega)$. Then*

$$\sigma(d) = \sigma(\{\omega \mapsto (e_n(\omega), e_m(\omega))_{H(\omega)}, n, m \in \mathbb{N}\}).$$

(v) Let $\mathcal{M}$ be as in Definition 2.1 and $(h_n)_{n\in\mathbb{N}}$ any fundamental sequence. It follows by (iv) and the construction of $(e_n)_{n\in\mathbb{N}}$ in [22, Ch. IV, Lemma 8.12]*) from $(h_n)_{n\in\mathbb{N}}$, that*

$$\sigma(d) \subset \sigma(\{\omega \mapsto (h_n(\omega), h_m(\omega))_{H(\omega)}, n, m \in \mathbb{N}\}).$$

(vi) If $H(\omega) = H$ for all ω, then $\mathcal{M}$ coincides with the set of all $\mathcal{F}/\mathcal{B}(H)$-measurable mappings from Ω to H.

The following is obvious.

Lemma 2.3. *If ζ is an $\mathcal{F}$-measurable vector field and $f : \Omega \to \mathbb{R}$ is an $\mathcal{F}$-measurable function, then $h := f\zeta$ is an $\mathcal{F}$-measurable vector field.*

Below let $\big(H(\omega)\big)_{\omega\in\Omega}$ (with $\mathcal{M}$) be an $\mathcal{F}$-measurable field of Hilbert spaces.

Definition 2.4. *A map $\lambda : [0,1] \to \mathcal{M}$, $t \mapsto (\lambda(t,\omega))_{\omega\in\Omega}$ is called a process and it is called measurable if, in addition for some(hence every by Proposition 2.5 below) fundamental sequence $(h_n)_{n\in\mathbb{N}}$ of $\mathcal{M}$, $(t,\omega) \mapsto (\lambda(t,\omega), h_n(\omega))_{H(\omega)}$ is $\mathcal{B}([0,1]) \times \mathcal{F}$ measurable for all $n \in \mathbb{N}$.*

Proposition 2.5. *Let λ be a measurable process. Then for any $\zeta \in \mathcal{M}$, $(t,\omega) \mapsto (\lambda(t,\omega),\zeta(\omega))_{H(\omega)}$ is $\mathcal{B}([0,1])\times\mathcal{F}$ measurable. Moreover, the process $(t,\omega) \mapsto \|\lambda(t,\omega)\|_{H(\omega)}$ is $\mathcal{B}([0,1])\times\mathcal{F}$ measurable.*

Proof. Let $(e_n)_{n\in\mathbb{N}}$ be the Gram-Schmidt orthogonalization of $(h_n)_{n\in\mathbb{N}}$ as in the proof of [22, Ch. IV, Lemma 8.12]. Then from the procedure of the orthogonalization, we know that $(t,\omega) \mapsto (\lambda(t,\omega), e_n(\omega))_{H(\omega)}$ is measurable for each $n \in \mathbb{N}$. The results now follows from

$$(\lambda(t,\omega),\zeta(\omega))_{H(\omega)} = \sum_{n\in\mathbb{N}} (\lambda(t,\omega), e_n(\omega))_{H(\omega)} (\zeta(\omega), e_n(\omega))_{H(\omega)},$$

and

$$\|\lambda(t,\omega)\|^2_{H(\omega)} = \sum_{n\in\mathbb{N}} (\lambda(t,\omega), e_n(\omega))^2_{H(\omega)}.$$

□

For $\omega \in \Omega$, let $J(\omega)$ be an (orthogonal) projection on $H(\omega)$ and define for $\zeta \in \mathcal{M}$

$$(J\zeta(\omega))_{\omega\in\Omega} := (J(\omega)\zeta(\omega))_{\omega\in\Omega}. \tag{2.3}$$

Assume that

$$J\mathcal{M} \subset \mathcal{M}. \tag{2.4}$$

Then $\big(J(\omega)H(\omega)\big)_{\omega\in\Omega}$ with $J\mathcal{M}$ is obviously again an $\mathcal{F}$-measurable field of Hilbert spaces.

Now we want to see what happens when we replace $\mathcal{F}$ by a sub σ-algebra $\tilde{\mathcal{F}}$ on Ω. Let $\sigma(d)$ denote the σ-algebra generated by $\omega \mapsto d(\omega) := \dim H(\omega)$. Suppose

$$\sigma(d) \subset \tilde{\mathcal{F}}. \tag{2.5}$$

Note that if there exists $\mathcal{M}(\tilde{\mathcal{F}})$ satisfying (**A**1)–(**A**3) in Definition 2.1 with $\tilde{\mathcal{F}}$ replacing $\mathcal{F}$ then (2.5) must hold by Remark 2.2 (iv).

Now let $(h_n)_{n\in\mathbb{N}}$ be a fundamental sequence in $\mathcal{M}(\mathcal{F})$ such that $\omega \mapsto \big(h_n(\omega), h_m(\omega)\big)_{H(\omega)}$ is $\tilde{\mathcal{F}}$-measurable for all $n, m \in \mathbb{N}$. Let $(e_n)_{n\in\mathbb{N}}$ be constructed as in the proof of [22, Ch. IV, Lemma 8.12] starting with $(h_n)_{n\in\mathbb{N}}$. Define

$$\mathcal{M}_h(\tilde{\mathcal{F}}) := \left\{ \zeta \in \prod_{\omega\in\Omega} H(\omega) \;\middle|\; \omega \mapsto \big(\zeta(\omega), h_n(\omega)\big)_{H(\omega)} \text{ is } \tilde{\mathcal{F}}\text{-measurable } \forall n \in \mathbb{N} \right\}. \tag{2.6}$$

Then $\mathcal{M}_h(\tilde{\mathcal{F}})$ satisfies (**A**1)–(**A**3) in Definition 2.1 with $\tilde{\mathcal{F}}$ replacing $\mathcal{F}$ and (as above)

$$\mathcal{M}_h(\tilde{\mathcal{F}}) = \mathcal{M}_e(\tilde{\mathcal{F}}).$$

But, nevertheless, the space in (2.6) might depend on the choice of $(h_n)_{n\in\mathbb{N}}$.

For our applications we have to take into account the time evolution. So, in the sequel we assume that for each $\omega \in \Omega$ there is a family of projections $(J_t(\omega))_{t\in[0,1]}$ in $H(\omega)$ such that each J_t satisfies (2.4) and for $s, t \in [0,1]$

$$J_t(\omega)J_s(\omega) = J_{s\wedge t}(\omega).$$

Set

$$H_t(\omega) := J_t(\omega)H(\omega), \quad h_t(\omega) := J_t(\omega)h(\omega).$$

We also fix a filtration $(\mathcal{F}_t)_{t\in[0,1]}$ of sub-σ-algebras of $\mathcal{F}$ and assume the following compatibility condition with $(J_t(\omega))_{t\in[0,1]}$

(***A**4*) *There exists a fundamental sequence $(h_n)_{n\in\mathbb{N}}$ of $\mathcal{M}$ such that for every n, m and every $t \in [0,1]$, the function:*

$$\Omega \ni \omega \mapsto (h_{n,t}(\omega), h_{m,t}(\omega))_{H(\omega)} = (h_n(\omega), h_{m,t}(\omega))_{H(\omega)} \in \mathbb{K}$$

is $\mathcal{F}_t$-measurable.

In case (**A**4) holds, set

$$\mathcal{M}_t := \left\{ \zeta \in \prod_{\omega\in\Omega} H_t(\omega) \;\middle|\; \omega \mapsto (\zeta(\omega), h_n(\omega))_{H(\omega)} \in \mathcal{F}_t \text{ for every } n \in \mathbb{N} \right\}.$$

Then obviously(cf. (2.6))

$$\mathcal{M}_t = (J_t\mathcal{M})_{J_t h}(\mathcal{F}_t). \tag{2.7}$$

So, by Remark 2.2 (i), $\mathcal{M}_t$ satisfies (**A**1)–(**A**3) in Definition 2.1 with $\mathcal{F}_t$ replacing $\mathcal{F}$. Elements of $\mathcal{M}_t$ are called $\mathcal{F}_t$-measurable.

Remark 2.6. *(i) Not every fundamental sequence of $\mathcal{M}$ satisfies (**A**4) above.*
(ii) In the same way as in Remark 2.2 (vi), when $H(\omega)$ is independent of ω and $J_t = I$ for all $t \in [0,1]$, $\mathcal{M}_t$ is just the set of all $\mathcal{F}_t$-measurable mappings from Ω to H.

The following lemma is important for the definition of stochastic integrals below.

Lemma 2.7. *(i) For $s < t$, $\mathcal{M}_s \subset \mathcal{M}_t$.*
(ii) For $\xi \in \mathcal{M}_t$ and $\zeta \in \mathcal{M}_s$,

$$\omega \mapsto (\xi(\omega), \zeta(\omega))_{H(\omega)} \in \mathcal{F}_{t\vee s}.$$

Proof. (i) is obvious by definition.
(ii) is a special case of Proposition 2.5. □

Definition 2.8. *A process λ is called $\mathcal{F}_t$-adapted if $\lambda(t) \in \mathcal{M}_t$ for all $t \in [0,1]$. All measurable and adapted processes are denoted by $\mathcal{A}$. A process λ is continuous(resp. left and right continuous) if for all $\omega \in \Omega$, the path $t \mapsto \lambda(t,\omega)$ is continuous in $H(\omega)$(resp. left and right continuous).*

The following lemma is immediate from Proposition 2.5.

Lemma 2.9. *For any $\lambda \in \mathcal{A}$, the process $(t,\omega) \mapsto \|\lambda(t,\omega)\|_{H(\omega)}$ is a real measurable and adapted process. Moreover, any left or right continuous process λ is measurable.*

2.2. **Example: Tangent bundle on configuration space.** Before proceeding, we first give a concrete example which shows the necessity of considering measurable fields of Hilbert spaces, and which is indeed our main motivation to extend the Itô stochastic integral to the case of measurable fields of Hilbert spaces.

Let X be a connected C^∞ complete Riemannian manifold. The tangent space of X at point x will be denoted by T_xX. $TX := \cup_{x\in X} T_xX$ denotes the tangent bundle. The Riemannian metric on X associates to each $x \in X$ an inner product $\langle\cdot,\cdot\rangle_{T_xX}$ on T_xX. The associated norm will be denoted by $|\cdot|_{T_xX}$.

Let us consider the product space $\mathbb{X} := [0,1] \times X$. The configuration space $\Gamma_{\mathbb{X}}$ over $\mathbb{X}$ is defined as the set of all locally finite subsets(configurations) in $\mathbb{X}$:

$$\Gamma_{\mathbb{X}} := \left\{ \gamma \subset \mathbb{X} : \#(\gamma \cap [0,1] \times K) < \infty \text{ for any compact } K \subset X \right\}.$$

Here $\#(\Lambda)$ denotes the cardinality of a set Λ.

We can identify $\gamma \in \Gamma_{\mathbb{X}}$ with a positive $\mathbb{Z}_+ \cup \{+\infty\}$-valued Radon measure on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$ by

$$\gamma = \sum_{(t,x)\in\gamma} \delta_{(t,x)},$$

where $\sum_{(t,x)\in\emptyset} \delta_{(t,x)}$ = zero measure by convention and $\delta_{(t,x)}$ denotes the Dirac measure at point (t, x). The space $\Gamma_{\mathbb{X}}$ can hence be endowed with the vague topology, i.e., the weakest topology on $\Gamma_{\mathbb{X}}$ such that the maps

$$\Gamma_{\mathbb{X}} \ni \gamma \mapsto \gamma(f) := \int_0^1 \int_X f(t,x)\gamma(\mathrm{d}t, \mathrm{d}x) = \sum_{(t,x)\in\gamma} f(t,x) \in \mathbb{R}$$

are continuous for all $f \in C_0(\mathbb{X})$ (the set of all continuous functions on $\mathbb{X}$ with compact support). Let $\mathcal{B}(\Gamma_{\mathbb{X}})$ denote the corresponding Borel σ-algebra on $\Gamma_{\mathbb{X}}$.

To define the filtration on the space $\Gamma_{\mathbb{X}}$ we let $\mathcal{F}_t^\Gamma$ be the smallest σ-algebra on $\Gamma_{\mathbb{X}}$ such that all the mappings $\Gamma_{\mathbb{X}} \ni \gamma \mapsto \gamma(B) := \#(\gamma \cap B) \in \mathbb{Z}^+, B \in \mathcal{B}([0,t] \times X)$ are measurable. In particular, $\mathcal{F}^\Gamma := \mathcal{F}_1^\Gamma = \mathcal{B}(\Gamma_{\mathbb{X}})$.

Definition 2.10. *The tangent space $T_\gamma(\Gamma_{\mathbb{X}})$ to the configuration space $\Gamma_{\mathbb{X}}$ at the point $\gamma \in \Gamma_{\mathbb{X}}$ is defined as the Hilbert space of γ-square-integrable time dependent sections(measurable vector fields) $v : \mathbb{X} \mapsto TX$ with the scalar product*

$$(v_1, v_2)_{T_\gamma(\Gamma_{\mathbb{X}})} := \int_0^1 \int_X \langle v_1(t,x), v_2(t,x)\rangle_{T_xX}\gamma(\mathrm{d}t, \mathrm{d}x),$$

for $v_1, v_2 \in T_\gamma(\Gamma_{\mathbb{X}})$. The norm in $T_\gamma(\Gamma_{\mathbb{X}})$ is denoted by $|\cdot|_{T_\gamma(\Gamma_{\mathbb{X}})}$.

Let $\mathcal{V}_0(X)$ denote the set of continuous vector fields on X with compact support, endowed with the topology of compact uniform convergence, $\{V_1, V_2, \cdots\}$ a dense subset of $\mathcal{V}_0(X)$ and $\{f_1, f_2, \cdots\}$ the totality of all polynomials with rational coefficients on $[0, 1]$. Then for every $\gamma \in \Gamma_{\mathbb{X}}$, $\mathcal{U} := \{V_i(x)f_j(t), i, j \in \mathbb{N}\}$ is a dense subset of $H(\gamma) := L^2([0,1] \times X \mapsto TX, \gamma) = T_\gamma(\Gamma_{\mathbb{X}})$. Hence it follows from Remark 2.2 (i) that

$$\mathcal{M} := \left\{ V \in \prod_{\gamma\in\Gamma_{\mathbb{X}}} H(\gamma) \;\middle|\; \gamma \mapsto (V(\gamma), Z(\gamma))_{H(\gamma)} \text{ is } \mathcal{F}^\Gamma\text{-measurable for all } Z \in \mathcal{U} \right\}$$

satisfies (**A**1)-(**A**3) of Definition 2.1. Furthermore, we define

$$J_t h := h1_{[0,t]}.$$

We clearly have that $\mathcal{U}$ together with J_t satisfies (**A**4). Then of course we have the corresponding $\mathcal{M}_t$.

2.3. **Martingale with values in Hilbert measurable fields.** Apart from $(H(\omega))_{\omega\in\Omega}$, $\mathcal{M}$, $(J_t)_{t\in[0,1]}$ and $(\mathcal{F}_t)_{t\in[0,1]}$ in subsection 2.1, we now also fix a probability measure P on $(\Omega, \mathcal{F})$ so that $(\Omega, \mathcal{F}, P)$ is complete and $(\mathcal{F}_t)_{t\in[0,1]}$ satisfies the usual conditions. A fundamental sequence $(h_n)_{n\in\mathbb{N}}$ of $\mathcal{M}$ satisfying (**A**4) is fixed.

In the following, we shall identify two vector fields $\xi, \zeta \in \mathcal{M}$ if $\xi(\omega) = \zeta(\omega)$ for P-almost all $\omega \in \Omega$. Let $\mathfrak{H}$ be the set of all $\zeta \in \mathcal{M}$ such that

$$\|\zeta\|_{\mathfrak{H}} := \left\{ \int_\Omega \|\zeta(\omega)\|^2_{H(\omega)} P(\mathrm{d}\omega) \right\}^{1/2} < +\infty. \tag{2.8}$$

Then $\mathfrak{H}$ is a Hilbert space which is called the direct integral of measurable fields of Hilbert spaces and is denoted by

$$\mathfrak{H} := \int_{\Omega}^{\oplus} H(\omega) P(\mathrm{d}\omega). \tag{2.9}$$

In the same way using $\mathcal{M}_t(\omega)$ instead of $\mathcal{M}(\omega)$ we get $\mathfrak{H}_t$. Clearly, $\mathfrak{H}_t$ is a closed subspace of $\mathfrak{H}$. Let Π_t denote the corresponding orthogonal projection. As usual, for $\xi \in \mathfrak{H}$ we set

$$\mathbb{E}(\xi|\mathcal{M}_t) := \Pi_t \xi,$$

and call it the conditional expectation of ξ with respect to $\mathcal{M}_t$.

Definition 2.11. *Let λ be a measurable adapted process. If, for all $0 \leqslant s < t \leqslant 1$, $\mathbb{E}(\lambda(t)|\mathcal{M}_s) = \lambda(s)$, we then call λ an $\mathcal{M}_t$-martingale.*

Lemma 2.12. *Let λ be an $\mathcal{M}_t$-martingale. For $0 \leqslant s < t \leqslant 1$ and $\zeta \in \mathcal{M}_s$, we have*

$$\mathbb{E}\left((\lambda(t), \zeta)^2_{H(\cdot)}\right) \geqslant \mathbb{E}\left((\lambda(s), \zeta)^2_{H(\cdot)}\right).$$

Proof. Noting that $\zeta(\cdot)(\lambda(s), \zeta)_{H(\cdot)} \in \mathcal{M}_s$, we have

$$\begin{aligned} 0 &\leqslant \mathbb{E}\left((\lambda(t) - \lambda(s), \zeta)^2_{H(\cdot)}\right) \\ &= \mathbb{E}\left((\lambda(t), \zeta)^2_{H(\cdot)}\right) - 2\mathbb{E}\left((\lambda(t), \zeta)_{H(\cdot)}(\lambda(s), \zeta)_{H(\cdot)}\right) + \mathbb{E}\left((\lambda(s), \zeta)^2_{H(\cdot)}\right) \\ &= \mathbb{E}\left((\lambda(t), \zeta)^2_{H(\cdot)}\right) - \mathbb{E}\left((\lambda(s), \zeta)^2_{H(\cdot)}\right), \end{aligned}$$

which gives the result. □

Proposition 2.13. *Let λ be an $\mathcal{M}_t$-martingale. Then $\{\|\lambda(t)\|^2_{H(\cdot)}, t \in [0,1]\}$ is a real submartingale with respect to $\mathcal{F}_t$.*

Proof. Let $0 \leqslant s < t \leqslant 1$. It suffices to prove that for any $F \in \mathcal{F}_s$

$$\mathbb{E}\left(\|\lambda(t)\|^2_{H(\cdot)} 1_F\right) \geqslant \mathbb{E}\left(\|\lambda(s)\|^2_{H(\cdot)} 1_F\right).$$

Let $(e_{n,s})_{n\in\mathbb{N}}$ be the Gram-Schmidt orthogonalization of $(h_{n,s})_{n\in\mathbb{N}}$ as in the proof of [22, Ch. IV, Lemma 8.12]. Then, $e_{n,s} \in \mathcal{M}_s$. Hence, by Lemmas 2.12 and 2.3

$$\begin{aligned} \mathbb{E}\left(\|\lambda(t)\|^2_{H(\cdot)} 1_F\right) &\geqslant \mathbb{E}\left(\|J_s\lambda(t)\|^2_{H(\cdot)} 1_F\right) \\ &= \sum_n \mathbb{E}\left((J_s\lambda(t), e_{n,s})^2_{H(\cdot)} 1_F\right) \\ &= \sum_n \mathbb{E}\left((\lambda(t), 1_F e_{n,s})^2_{H(\cdot)}\right) \\ &\geqslant \sum_n \mathbb{E}\left((\lambda(s), 1_F e_{n,s})^2_{H(\cdot)}\right) \\ &= \mathbb{E}\left(\|\lambda(s)\|^2_{H(\cdot)} 1_F\right). \end{aligned}$$

□

Let us also prove the following Kolmogorov's continuity criterion.

Theorem 2.14. *Let λ be a measurable process. Assume that there are positive constants p, α, C such that for all $s, t \in [0,1]$*

$$\mathbb{E}\|\lambda(t,\cdot) - \lambda(s,\cdot)\|^p_{H(\cdot)} \leqslant C|t-s|^{1+\alpha}. \tag{2.10}$$

Then there exists a continuous process $\tilde\lambda$ such that for every $t \in [0,1]$, $\tilde\lambda(t,\cdot) = \lambda(t,\cdot)$ a.e. and for any $\epsilon \in (0, \frac{\alpha}{p})$

$$\mathbb{E}\left(\sup_{s\neq t\in[0,1]} \frac{\|\tilde\lambda(t,\cdot)-\tilde\lambda(s,\cdot)\|^p_{H(\cdot)}}{|t-s|^{\epsilon p}}\right) < +\infty. \tag{2.11}$$

Proof. We use the Sobolev embedding theorem to prove the assertion (see Da Prato-Zabczyk [9]). By Proposition 2.5, the function

$$(t,s,\omega) \mapsto \frac{\|\lambda(t,\omega)-\lambda(s,\omega)\|^p_{H(\omega)}}{|t-s|^{2+\epsilon p}} =: f(t,s,\omega)$$

is $\mathcal{B}([0,1]^2)\times\mathcal{F}$-measurable. For any $\epsilon \in (0, \frac{1+\alpha}{p})$, by Fubini's theorem we have

$$\begin{aligned}\mathbb{E}\left(\int_{[0,1]^2} f(t,s,\cdot)\mathrm{d}s\mathrm{d}t\right) &= \int_{[0,1]^2} \mathbb{E}f(t,s,\cdot)\mathrm{d}s\mathrm{d}t \\ &\leqslant C\int_{[0,1]^2} |t-s|^{\alpha-\epsilon p-1}\mathrm{d}s\mathrm{d}t < +\infty. \end{aligned} \tag{2.12}$$

So, $P(\Omega_0)=1$, where

$$\Omega_0 := \left\{\omega\in\Omega : \int_{[0,1]^2} f(t,s,\omega)\mathrm{d}s\mathrm{d}t < +\infty\right\} \in \mathcal{F}.$$

For any $\omega_0 \in \Omega_0$, by the Sobolev embedding theorem there is a continuous map $t \mapsto \tilde\lambda(t,\omega_0) \in H(\omega_0)$ such that $\tilde\lambda(\cdot,\omega_0) = \lambda(\cdot,\omega_0)$ a.e. on $[0,1]$ and

$$\sup_{s\neq t\in[0,1]} \frac{\|\tilde\lambda(t,\omega_0)-\tilde\lambda(s,\omega_0)\|^p_{H(\cdot)}}{|t-s|^{p\epsilon}} \leqslant C\int_{[0,1]^2} f(t,s,\omega_0)\mathrm{d}s\mathrm{d}t. \tag{2.13}$$

Note that for any $t\in(0,1]$

$$\tilde\lambda(t,\omega_0) = \lim_{n\to\infty} n\int_{t-1/n}^t \tilde\lambda(s,\omega_0)\mathrm{d}s = \lim_{n\to\infty} n\int_{t-1/n}^t \lambda(s,\omega_0)\mathrm{d}s =: \lim_{n\to\infty}\lambda_n(t,\omega_0).$$

Clearly, λ_n is a continuous process for each $n\in\mathbb{N}$. For the limit, we obtain that $\tilde\lambda(t,\cdot)\in\mathcal{M}$ for each $t\in(0,1]$. Similarly, $\tilde\lambda(0,\cdot)\in\mathcal{M}$ follows from

$$\tilde\lambda(0,\omega_0) = \lim_{n\to\infty} n\int_0^{1/n} \tilde\lambda(s,\omega_0)\mathrm{d}s = \lim_{n\to\infty} n\int_0^{1/n} \lambda(s,\omega_0)\mathrm{d}s.$$

On the other hand, by Fubini's theorem again we have for almost all $t\in[0,1]$

$$P(\omega\in\Omega : \tilde\lambda(t,\omega) = \lambda(t,\omega)) = 1,$$

which together with (2.10) gives that $\tilde\lambda(t,\cdot)=\lambda(t,\cdot)$ a.e. for all $t\in[0,1]$. The estimate (2.11) follows from (2.12) and (2.13). ☐

2.4. **Stochastic integration with respect to real martingales.** Now suppose that we are given, on a complete filtered probability space, a direct integral of measurable fields of Hilbert spaces $\mathfrak{H}$ and a real continuous square integrable martingale M. For simplicity we suppose that the square variation process $\langle M\rangle_t$ of M is absolutely continuous with respect to the Lebesgue measure.

Set for $p\geqslant 2$

$$L^M_p := \left\{\lambda\in\mathcal{A} : \|\lambda\|^p_{L^M_p} := \mathbb{E}\left(\int_0^1 \|\lambda(s,\cdot)\|^p_{H(\cdot)}\mathrm{d}\langle M\rangle_s\right) < +\infty\right\},$$

where $\mathcal{A}$ is the set of measurable and adapted processes in Definition 2.8.

We first define stochastic integrals for step functions.

Definition 2.15. *Let λ be a step function in L_2^M of the form:*

$$\lambda(t,\omega) = \sum_{k=0}^{n-1} \lambda_k(\omega) 1_{[t_k,t_{k+1})}(t),$$

where $0 = t_0 < t_1 < \cdots < t_n = 1$ and $\lambda_k \in \mathfrak{H}_{t_k} \subset \mathcal{M}_{t_k}$. Define for $t \in [0,1]$

$$I_t^M(\lambda) := \int_0^t \lambda(s)\mathrm{d}M(s) := \sum_{k=0}^{n-1} \lambda_k(M(t_{k+1} \wedge t) - M(t_k \wedge t)) \in \mathcal{M}_t$$

and call it the stochastic integral of λ with respect to M on $[0,t]$.

This integral enjoys many familiar properties of the ordinary stochastic integral. For example we have

Lemma 2.16. *Let λ, $\tilde{\lambda}$ be step functions in L_2^M and set $\xi(t) := I_t^M(\lambda)$ and $\tilde{\xi}(t) := I_t^M(\tilde{\lambda})$.*

(i) For any $a, b \in \mathbb{R}$ and $t \in [0,1]$

$$I_t^M(a \cdot \lambda + b \cdot \tilde{\lambda}) = a \cdot I_t^M(\lambda) + b \cdot I_t^M(\tilde{\lambda}).$$

(ii) $\{\xi(t), t \in [0,1]\}$ is a continuous $\mathcal{M}_t$-martingale. In particular, the real process $\{\|\xi(t)\|^2_{H(\cdot)}, t \in [0,1]\}$ is a continuous submartingale.

(iii) $\{(\xi(t), \tilde{\xi}(t))_{H(\cdot)}, t \in [0,1]\}$ is a real semimartingale and

$$\begin{aligned} \mathrm{d}(\xi(t), \tilde{\xi}(t))_{H(\cdot)} &= \Big((\lambda(t), \tilde{\xi}(t))_{H(\cdot)} + (\xi(t), \tilde{\lambda}(t))_{H(\cdot)}\Big)\mathrm{d}M(t) \\ &\quad + (\lambda(t), \tilde{\lambda}(t))_{H(\cdot)} \mathrm{d}\langle M\rangle_t. \end{aligned} \tag{2.14}$$

In particular, we have the following isometry property

$$\|I_1^M(\lambda)\|_{\mathfrak{H}} = \|\lambda\|_{L_2^M}. \tag{2.15}$$

(iv) If $A(t)$ is a real valued adapted process of bounded variation, then for any $t \in [0,1]$

$$\xi(t)A(t) = \int_0^t A(s)\lambda(s)\mathrm{d}M(s) + \int_0^t \xi(s)\mathrm{d}A(s). \tag{2.16}$$

Proof. (i) needs no proof.

(ii) From Definition 2.15, we know that $t \mapsto \xi(t)$ is a continuous and adapted process. Let $0 \leqslant t_i \leqslant s < t_{i+1} \leqslant t_j \leqslant t < t_{j+1} \leqslant 1$ and $\zeta \in \mathfrak{H}_s$, we have

$$\begin{aligned} (\xi(t), \zeta)_{\mathfrak{H}} &= \sum_{k=0}^{j} \mathbb{E}((\lambda_k, \zeta)_{H(\cdot)}(M(t_{k+1} \wedge t) - M(t_k \wedge t))) \\ &= \sum_{k=0}^{i} \mathbb{E}((\lambda_k, \zeta)_{H(\cdot)}(M(t_{k+1} \wedge s) - M(t_k))) \\ &\quad + \sum_{k=i+1}^{j} \mathbb{E}((\lambda_k, \zeta)_{H(\cdot)}(M(t_{k+1} \wedge t) - M(t_k))) \\ &\quad + \mathbb{E}((\lambda_i, \zeta)_{H(\cdot)}(M(t_{i+1}) - M(s))) \\ &= (\xi(s), \zeta)_{\mathfrak{H}}, \end{aligned}$$

where we used Lemma 2.7 (ii) and the martingale property in the last step. This implies that $\mathbb{E}(\xi(t)|\mathcal{M}_s) = \xi(s)$.

(iii) Without loss of generality, we assume that the step functions $\lambda, \tilde{\lambda}$ have the same partitions of $[0,1]$. Then

$$(\xi(t), \tilde{\xi}(t))_{H(\cdot)} = \sum_{i,j} (\lambda_i, \tilde{\lambda}_j)_{H(\cdot)} (M(t_{i+1} \wedge t) - M(t_i \wedge t))(M(t_{j+1} \wedge t) - M(t_j \wedge t)).$$

Note that

$$\begin{aligned}
& (M(t_{i+1} \wedge t) - M(t_i \wedge t))(M(t_{j+1} \wedge t) - M(t_j \wedge t)) \\
= & \int_0^t (M(t_{i+1} \wedge s) - M(t_i \wedge s)) 1_{[t_j, t_{j+1})}(s) \mathrm{d}M(s) \\
& + \int_0^t (M(t_{j+1} \wedge s) - M(t_j \wedge s)) 1_{[t_i, t_{i+1})}(s) \mathrm{d}M(s) \\
& + \int_0^t 1_{[t_i, t_{i+1})}(s) 1_{[t_j, t_{j+1})}(s) \mathrm{d}\langle M \rangle_s
\end{aligned}$$

and $(\lambda_i, \tilde{\lambda}_j)_{H(\cdot)} \in \mathcal{F}_{t_i \vee t_j}$ by Lemma 2.7 (ii). The formula (2.14) now follows.

(iv) can be proved similarly. $\square$

We now extend the domain of the integrands to L_2^M. For $\lambda \in L_2^M$, since $\langle M \rangle_t$ is absolutely continuous with respect to $\mathrm{d}t$, by a standard approximation, we may choose a sequence of step functions $\lambda_n \in L_2^M$ such that

$$\lim_{n \to \infty} \|\lambda - \lambda_n\|_{L_2^M}^2 = \lim_{n \to \infty} \mathbb{E} \left(\int_0^1 \|\lambda(t) - \lambda_n(t)\|_{H(\cdot)}^2 \mathrm{d}\langle M \rangle_t \right) = 0.$$

By (ii) of Lemma 2.16, $t \mapsto \|I_t^M(\lambda_n) - I_t^M(\lambda_m)\|_{H(\cdot)}^2$ is a continuous submartingale. So, by Doob's maximal inequality and the isometric property (2.15) we have

$$\begin{aligned}
& \mathbb{E} \left(\sup_{t \in [0,1]} \|I_t^M(\lambda_n) - I_t^M(\lambda_m)\|_{H(\cdot)}^2 \right) \\
\leqslant & 4\mathbb{E} \left(\|I_1^M(\lambda_n) - I_1^M(\lambda_m)\|_{H(\cdot)}^2 \right) \\
= & 4\|\lambda_n - \lambda_m\|_{L_2^M}^2 \to 0,
\end{aligned}$$

as $n, m \to \infty$. It follows then that there is a subsequence of $\{\lambda_n\}$, still denoted by $\{\lambda_n\}$, such that $\{I_t^M(\lambda_n), t \in [0,1]\}$ is uniformly convergent in t with probability one as $n \to \infty$. The limit is hence a continuous adapted process and we define it as the integral of λ with respect to M:

$$I_t^M(\lambda) := \int_0^t \lambda(s) \mathrm{d}M(s) := \lim_{n \to \infty} I_t^M(\lambda_n).$$

It is easily seen that this integral is well-defined, i.e., the limit is independent of the choice of the approximating sequence. By a limit process, we obtain:

Theorem 2.17. *The stochastic integral operator $I_1^M(\lambda)$ on L_2^M is linear and isometric. Moreover, all the conclusions in Lemma 2.16 hold for any $\lambda, \tilde{\lambda} \in L_2^M$.*

We can prove a Burkholder inequality using (2.14).

Theorem 2.18. *If $p \geqslant 1$, then there exist two constants c_p and C_p depending only on p such that for any $t \in [0,1]$*

$$c_p \mathbb{E}\left(\int_0^t \|\lambda(s)\|^2_{H(\cdot)} \mathrm{d}\langle M\rangle_s\right)^p \leqslant \mathbb{E}\left(\|I_t^M(\lambda)\|^{*2p}_{H(\cdot)}\right) \leqslant C_p \mathbb{E}\left(\int_0^t \|\lambda(s)\|^2_{H(\cdot)} \mathrm{d}\langle M\rangle_s\right)^p,$$

for all $\lambda \in L_2^M$, where $\|I_t^M(\lambda)\|^*_{H(\cdot)} := \sup_{0\leqslant s\leqslant t} \|I_s^M(\lambda)\|_{H(\cdot)}$.

Moreover, let W be one dimensional standard Brownian motion, if $\lambda \in L_p^W$ for $p > 2$, then $\{I_t^W(\lambda), t \in [0,1]\}$ has a Hölder continuous version.

Proof. We adopt a standard stochastic calculus proof (cf. e.g. [12]).

We suppose that $p > 1$ since for $p = 1$ the equality holds with $C_p = 1$. Set $\xi(t) := I_t^M(\lambda)$. We have by (2.14) and the usual Itô formula

$$\begin{aligned}
\|\xi(t)\|^{2p}_{H(\cdot)} &= 2p \int_0^t \|\xi(s)\|^{2(p-1)}_{H(\cdot)} (\xi(s), \lambda(s))_{H(\cdot)} \mathrm{d}M(s) \\
&\quad + p \int_0^t \|\xi(s)\|^{2(p-1)}_{H(\cdot)} \|\lambda(s)\|^2_{H(\cdot)} \mathrm{d}\langle M\rangle_s \\
&\quad + 2p(p-1) \int_0^t \|\xi(s)\|^{2(p-2)}_{H(\cdot)} (\xi(s), \lambda(s))^2_{H(\cdot)} \mathrm{d}\langle M\rangle_s.
\end{aligned}$$

Here as usual, set $\xi(s)/\|\xi(s)\|_{H(\cdot)} = 0$ on $\{\|\xi(s)\|_{H(\cdot)} = 0\}$.

Set

$$A(t) := \int_0^t \|\lambda(s)\|^2_{H(\cdot)} \mathrm{d}\langle M\rangle_s.$$

Then,

$$\begin{aligned}
\mathbb{E}\left(\|\xi(t)\|^{2p}_{H(\cdot)}\right) &\leqslant C_p \mathbb{E}\left(\int_0^t \|\xi(s)\|^{2(p-1)}_{H(\cdot)} \mathrm{d}A(s)\right) \\
&\leqslant C_p \mathbb{E}\left(\|\xi(t)\|^{*2(p-1)}_{H(\cdot)} A(t)\right) \\
&\leqslant C_p \left(\mathbb{E}\left(\|\xi(t)\|^{*2p}_{H(\cdot)}\right)\right)^{\frac{p-1}{p}} \left(\mathbb{E}A(t)^p\right)^{\frac{1}{p}},
\end{aligned}$$

which gives

$$\mathbb{E}\left(\|\xi(t)\|^{2p}_{H(\cdot)}\right) \leqslant C_p \mathbb{E}(A(t)^p).$$

To prove the other inequality, we set

$$N(t) := \int_0^t A(s)^{\frac{p-1}{2}} \lambda(s) \mathrm{d}M(s).$$

Then

$$\begin{aligned}
\mathbb{E}\left(\|N(t)\|^2_{H(\cdot)}\right) &= \mathbb{E}\left(\int_0^t A(t)^{p-1} \|\lambda(s)\|^2_{H(\cdot)} \mathrm{d}\langle M\rangle_s\right) \\
&= \mathbb{E}\left(\int_0^t A(t)^{p-1} \mathrm{d}A(s)\right) \\
&= \frac{1}{p} \mathbb{E}\left(A(t)^p\right).
\end{aligned}$$

By the integration by parts formula (2.16) we have

$$\begin{aligned}\xi(t)A(t)^{\frac{p-1}{2}} &= \int_0^t A(s)^{\frac{p-1}{2}}\lambda(s)\mathrm{d}M(s) + \int_0^t \xi(s)\mathrm{d}A(s)^{\frac{p-1}{2}} \\ &= N(t) + \int_0^t \xi(s)\mathrm{d}A(s)^{\frac{p-1}{2}}.\end{aligned}$$

Consequently,

$$\|N(t)\|_{H(\cdot)} \leqslant 2\|\xi(t)\|^*_{H(\cdot)}A(t)^{\frac{p-1}{2}}.$$

Thus we obtain

$$\begin{aligned}\frac{1}{p}\mathbb{E}\left(A(t)^p\right) &= \mathbb{E}\left(\|N(t)\|^2_{H(\cdot)}\right) \\ &\leqslant 4\mathbb{E}\left(\|\xi(t)\|^{*2}_{H(\cdot)}A(t)^{p-1}\right) \\ &\leqslant 4\left(\mathbb{E}\|\xi(t)\|^{*2p}_{H(\cdot)}\right)^{\frac{1}{p}}\left(\mathbb{E}A(t)^p\right)^{\frac{p-1}{p}},\end{aligned}$$

and therefore

$$\mathbb{E}\left(A(t)^p\right) \leqslant c_p\mathbb{E}\left(\|\xi(t)\|^{*2p}_{H(\cdot)}\right).$$

Finally, the Hölder continuity of $t \mapsto I^W_t(\lambda)$ follows from Theorem 2.14 and

$$\mathbb{E}\|I^W_t(\lambda) - I^W_{t'}(\lambda)\|^p_{H(\cdot)} \leqslant C_p\mathbb{E}\left|\int_t^{t'}\|\lambda(s)\|^2_{H(\cdot)}\mathrm{d}s\right|^{p/2} \leqslant C_p\cdot\|\lambda\|^p_{L^W_p}\cdot|t'-t|^{\frac{p}{2}-1}.$$

The proof is then complete. □

2.5. **Stochastic integration with respect to random measures.** Let $(X, \mathcal{X})$ be a measurable space, $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,1]}, P)$ a complete filtered probability space and γ a homogeneous point measure on $\mathbb{X} := [0,1]\times X$ defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,1]}, P)$. By this we mean that γ has the following property:

(**B**1) for each $A \in \mathcal{B}([0,1])\times\mathcal{X}$, $\gamma(\cdot, A)$ takes its value in $\mathbb{Z}\cup\{+\infty\}$;
(**B**2) $\gamma(\omega, \{t\}\times X) \leqslant 1$ for all $t\in[0,1]$ and P-almost all $\omega\in\Omega$;
(**B**3) for every $t\in[0,1]$ and $U\in\mathcal{X}$, $\gamma(\cdot,(0,t]\times U)$ is $\mathcal{F}_t$-measurable;
(**B**4) for P-almost every $\omega\in\Omega$, $\gamma(\omega,\cdot)$ is a σ-finite measure on $(\mathbb{X}, \mathcal{B}([0,1])\times\mathcal{X})$;
(**B**5) there exists a σ-finite measure m on $(X,\mathcal{X})$ such that

$$t \mapsto \tilde{\gamma}(\cdot, t, U) := \gamma(\cdot,(0,t]\times U) - t\cdot m(U)$$

is an $(\mathcal{F}_t)$-martingale for all $U\in\mathcal{X}$.

Definition 2.19. *A function $f\colon \mathbb{X}\times\Omega \ni (t,x,\omega) \mapsto f(t,x,\omega) \in H(\omega)$ is called adapted if for every $(t,x)\in\mathbb{X}$, $f(t,x)\in\mathcal{M}_t$ and for every n, $(x,\omega)\mapsto (f(t,x,\omega), h_n(\omega))_{H(\omega)}$ is $\mathcal{X}\times\mathcal{F}_t$-measurable. Let $\mathcal{A}_L$ denote the set of all adapted processes such that $\forall(x,\omega)\in X\times\Omega$, $t\mapsto f(t,x,\omega)\in H(\omega)$ is left continuous and let*

$$\begin{aligned}\mathcal{P} := \Big\{&\lambda : \exists\{f_n\}\subset\mathcal{A}_L \textit{ such that } \forall(t,x)\in\mathbb{X}, \\ &\lim_{n\to\infty}\|f_n(t,x,\cdot) - f(t,x,\cdot)\|_{H(\cdot)} = 0 \ a.e.\Big\}.\end{aligned}$$

Elements of $\mathcal{P}$ are called strongly predictable.

Set $\nu(\mathrm{d}t,\mathrm{d}x) := \mathrm{d}t \times m(\mathrm{d}x)$. Define

$$\mathfrak{F}_1 := \left\{ f \in \mathcal{P} : \int_0^1 \int_X \mathbb{E}\left(\|f(s,x,\cdot)\|_{H(\cdot)} \right) \nu(\mathrm{d}s,\mathrm{d}x) < \infty \right\},$$

$$\mathfrak{F}_2 := \left\{ f \in \mathcal{P} : \int_0^1 \int_X \mathbb{E}\left(\|f(s,x,\cdot)\|_{H(\cdot)}^2 \right) \nu(\mathrm{d}s,\mathrm{d}x) < \infty \right\}.$$

For $f \in \mathfrak{F}_1 \cap \mathfrak{F}_2$, by definition we know that $\|f(\cdot,x,\cdot)\|_{H(\cdot)}$ is predictable. By the same argument as in [12, p.62], we have for any $t \in [0,1]$

$$\mathbb{E}\left[\int_0^{t^+} \int_X \|f(s,x,\cdot)\|_{H(\cdot)} \gamma(\cdot,\mathrm{d}s,\mathrm{d}x) \right] = \mathbb{E}\left[\int_0^{t} \int_X \|f(s,x,\cdot)\|_{H(\cdot)} \nu(\mathrm{d}s,\mathrm{d}x) \right]. \tag{2.17}$$

Here and in the sequel, $\int_0^{t^+} := \int_{(0,t]}$. Hence $\mathfrak{F}_1 \subset \mathfrak{F}$, where

$$\mathfrak{F} := \left\{ f \in \mathcal{P} : \int_0^1 \int_X \|f(s,x,\cdot)\|_{H(\cdot)} \gamma(\cdot,\mathrm{d}s,\mathrm{d}x) < \infty \text{ a.e.} \right\}.$$

For $f \in \mathfrak{F}_1$ and P-a.e. $\omega \in \Omega$, we define

$$\begin{aligned} &\int_0^{t^+} \int_X f(s,x,\omega) \tilde{\gamma}(\omega,\mathrm{d}s,\mathrm{d}x) \\ := \ &\int_0^{t^+} \int_X f(s,x,\omega) \gamma(\omega,\mathrm{d}s,\mathrm{d}x) - \int_0^{t} \int_X f(s,x,\omega) \nu(\mathrm{d}s,\mathrm{d}x). \end{aligned}$$

The right hand is understood as Bochner's integrals taking values in $H(\omega)$.

We use the following notations for $f \in \mathfrak{F}_1$:

$$\begin{aligned} \tilde{\gamma}_t(f) &:= \int_0^{t+} \int_X f(s,x) \tilde{\gamma}(\mathrm{d}s,\mathrm{d}x), \\ \gamma_t(f) &:= \int_0^{t+} \int_X f(s,x) \gamma(\mathrm{d}s,\mathrm{d}x), \\ \nu_t(f) &:= \int_0^{t} \int_X f(s,x) \nu(\mathrm{d}s,\mathrm{d}x). \end{aligned}$$

We also simply write $\tilde{\gamma}(f)$, $\gamma(f)$ and $\nu(f)$ for $\tilde{\gamma}_1(f)$, $\gamma_1(f)$ and $\nu_1(f)$ respectively.

Similarly to Lemma 2.16, we have

Theorem 2.20. *(i) For $f_1, f_2 \in \mathfrak{F}_1 \cap \mathfrak{F}_2$ and $a, b \in \mathbb{R}$, we have*

$$\tilde{\gamma}(a \cdot f_1 + b \cdot f_2) = a \cdot \tilde{\gamma}(f_1) + b \cdot \tilde{\gamma}(f_2).$$

(ii) For $f \in \mathfrak{F}_1 \cap \mathfrak{F}_2$, $\{\tilde{\gamma}_t(f), t \in [0,1]\}$ is a right continuous $\mathcal{M}_t$-martingale. Consequently, the real process $\{\|\tilde{\gamma}_t(f)\|_{H(\cdot)}^2, t \in [0,1]\}$ is a right continuous submartingale.

(iii) For $f \in \mathfrak{F}_1 \cap \mathfrak{F}_2$,

$$\|\tilde{\gamma}_t(f)\|_{H(\cdot)}^2 = 2 \int_0^{t+} \int_X (f(s,x), \tilde{\gamma}_{s-}(f))_{H(\cdot)} \tilde{\gamma}(\mathrm{d}s,\mathrm{d}x) + \gamma_t\left(\|f\|_{H(\cdot)}^2 \right). \tag{2.18}$$

In particular, for all $t \in [0,1]$

$$\mathbb{E}\left(\|\tilde{\gamma}_t(f)\|_{H(\cdot)}^2 \right) = \mathbb{E}\left(\nu_t\left(\|f\|_{H(\cdot)}^2 \right) \right). \tag{2.19}$$

Proof. (i) needs no proof.

(ii) can be proved by approximation and similar calculations as in the proof of (ii) of Lemma 2.16.

(iii) First of all we prove the assertion for f of the following form

$$f(s,x)=\sum_{k=0}^{2^n-1}\sum_i f_i(k2^{-n})1_{U_i}(x)1_{(k2^{-n},(k+1)2^{-n}]}(s),$$

where U_i's with $m(U_i)<+\infty$ are disjoint and $f_i(k2^{-n})\in\mathfrak{H}_{k2^{-n}}\subset\mathcal{M}_{k2^{-n}}$.

Set $\tilde{\gamma}_{k,i}(t):=\tilde{\gamma}(k2^{-n}\wedge t,U_i)$, then we have

$$\tilde{\gamma}_t(f)=\sum_{k=0}^{2^n-1}\sum_i f_i(k2^{-n})(\tilde{\gamma}_{k+1,i}(t)-\tilde{\gamma}_{k,i}(t)),$$

and

$$\begin{aligned}\|\tilde{\gamma}_t(f)\|^2_{H(\cdot)} &= \sum_{k,k'}\sum_{i,i'}(f_i(k2^{-n}),f_{i'}(k'2^{-n}))_{H(\cdot)}\\ &\quad\cdot(\tilde{\gamma}_{k+1,i}(t)-\tilde{\gamma}_{k,i}(t))(\tilde{\gamma}_{k'+1,i'}(t)-\tilde{\gamma}_{k',i'}(t)).\end{aligned} \tag{2.20}$$

Using the familiar formula(cf. [6])

$$\tilde{\gamma}_t^2(1_U)=2\int_0^{t+}\tilde{\gamma}_{s-}(1_U)\tilde{\gamma}(\mathrm{d}s,U)+\gamma_t(1_U)$$

and polarization, we obtain

$$\begin{aligned}&(\tilde{\gamma}_{k+1,i}(t)-\tilde{\gamma}_{k,i}(t))(\tilde{\gamma}_{k'+1,i'}(t)-\tilde{\gamma}_{k',i'}(t))\\ =\ &\int_0^{t+}(\tilde{\gamma}_{k+1,i}(s-)-\tilde{\gamma}_{k,i}(s-))1_{(k'2^{-n},(k'+1)2^{-n}]}(s)\tilde{\gamma}(\mathrm{d}s,U_{i'})\\ &+\int_0^{t+}(\tilde{\gamma}_{k'+1,i'}(s-)-\tilde{\gamma}_{k',i'}(s-))1_{(k2^{-n},(k+1)2^{-n}]}(s)\tilde{\gamma}(\mathrm{d}s,U_i)\\ &+\int_0^{t+}1_{(k2^{-n},(k+1)2^{-n}]}(s)1_{(k'2^{-n},(k'+1)2^{-n}]}(s)\gamma(\mathrm{d}s,U_i\cap U_{i'}).\end{aligned}$$

Substituting this into (2.20) yields the desired formula.

Notice that $s\mapsto(f(s,x),\tilde{\gamma}_{s-}(f))_{H(\cdot)}$ is a left continuous and adapted real process by Lemma 2.7. Using (2.17) and taking expectation for (2.18) gives (2.19). Lastly, by standard approximation, we complete the proof. □

Now we define the stochastic integral of any element $f\in\mathfrak{F}_2$. Choose $U_n\in\mathcal{B}(X)$ such that $U_n\uparrow X$ and $m(U_n)<\infty$ for every n. Let

$$f_n(s,x,\omega):=1_{[0,n)}(s)(\|f(s,x,\omega)\|_{H(\omega)})1_{U_n}(x)f(s,x,\omega).$$

Then $f_n\in\mathfrak{F}_1\cap\mathfrak{F}_2$. Hence for every n, $\tilde{\gamma}_t(f_n)$ is well defined. Moreover, by Doob's maximal inequality and (ii) of Theorem 2.20

$$\mathbb{E}\left[\sup_{t\in[0,1]}\|\tilde{\gamma}_t(f_n)-\tilde{\gamma}_t(f_m)\|^2_{H(\cdot)}\right]\leqslant C\mathbb{E}\left(\nu_t\left(\|f_n-f_m\|^2_{H(\cdot)}\right)\right)\to 0,\ m,n\to\infty.$$

Hence extracting a subsequence if necessary we may and will assume that almost surely $\tilde{\gamma}_t(f_n)$ converge uniformly in $t\in[0,1]$. We then define the limit as the integral of f:

$$\tilde{\gamma}_t(f):=\lim_{n\to\infty}\tilde{\gamma}_t(f_n).$$

It is easily seen that the isometry property (2.19) is preserved.

We have also the following moment estimates on stochastic integrals, analogous to Theorem 2.18.

Theorem 2.21. *Let f be in $\mathfrak{F}_1 \cap \mathfrak{F}_2$. Set $\|\tilde{\gamma}_t(f)\|^*_{H(\cdot)} := \sup_{0\leqslant s\leqslant t} \|\tilde{\gamma}_s(f)\|_{H(\cdot)}$. Then*

(i) For any $p \geqslant 1$ there exists a constant C_p such that

$$\begin{aligned}\mathbb{E}\left(\|\tilde{\gamma}_t(f)\|^{*2p}_{H(\cdot)}\right) &\leqslant C_p \mathbb{E}\left(\gamma_t\left(\|f\|^2_{H(\cdot)}\right)\right)^p \\ &\leqslant C_p\left(\mathbb{E}\left(\nu_t\left(\|f\|^2_{H(\cdot)}\right)\right)^p + \mathbb{E}\left(\tilde{\gamma}_t\left(\|f\|^2_{H(\cdot)}\right)\right)^p\right). \end{aligned} \tag{2.21}$$

(ii) For any positive integer k there exists a constant C_k such that

$$\mathbb{E}\left(\|\tilde{\gamma}_t(f)\|^{*2^k}_{H(\cdot)}\right) \leqslant C_k \sum_{i=1}^{k} \mathbb{E}\left[\nu_t\left(\|f\|^{2^i}_{H(\cdot)}\right)\right]^{2^{k-i}}. \tag{2.22}$$

Proof. By the BDG's inequality(see [5]), we have

$$\begin{aligned} &\mathbb{E}\left|\sup_{0\leqslant t'\leqslant t}\int_0^{t'+}\int_X (f(s,x),\tilde{\gamma}_{s-}(f))_{H(\cdot)}\tilde{\gamma}(\mathrm{d}s,\mathrm{d}x)\right|^p \\ \leqslant\ & C_p\mathbb{E}\left(\int_0^t\int_X |(f(s,x),\tilde{\gamma}_{s-}(f))_{H(\cdot)}|^2\gamma(\mathrm{d}s,\mathrm{d}x)\right)^{p/2} \\ \leqslant\ & C_p\mathbb{E}\left[\|\tilde{\gamma}_t(f)\|^{*p}_{H(\cdot)}\left(\int_0^t\int_X \|f(s,x)\|^2_{H(\cdot)}\gamma(\mathrm{d}s,\mathrm{d}x)\right)^{p/2}\right] \\ \leqslant\ & C_p\left[\mathbb{E}\left(\|\tilde{\gamma}_t(f)\|^{*2p}_{H(\cdot)}\right)\right]^{1/2}\cdot\left[\mathbb{E}\left(\gamma_t\left(\|f\|^2_{H(\cdot)}\right)\right)^p\right]^{1/2}. \end{aligned}$$

So by formula (2.18) we have

$$\begin{aligned}\mathbb{E}\left(\|\tilde{\gamma}_t(f)\|^{*2p}_{H(\cdot)}\right) \leqslant\ & C_p\left[\mathbb{E}\left(\|\tilde{\gamma}_t(f)\|^{*2p}_{H(\cdot)}\right)\right]^{1/2}\cdot\left[\mathbb{E}\left(\gamma_t\left(\|f\|^2_{H(\cdot)}\right)\right)^p\right]^{1/2} \\ & + C_p\mathbb{E}\left(\gamma_t\left(\|f\|^2_{H(\cdot)}\right)\right)^p, \end{aligned}$$

which yields by Young's inequality that

$$\mathbb{E}\left(\|\tilde{\gamma}_t(f)\|^{*2p}_{H(\cdot)}\right) \leqslant C_p\mathbb{E}\left(\gamma_t\left(\|f\|^2_{H(\cdot)}\right)\right)^p.$$

(2.22) is easily deduced from (2.21). □

3. Review on Malliavin calculus on configuration space

This section is modelled on [2, 7]. Continuing subsection 2.2, let $m(\mathrm{d}x)$ denote the volume element on Riemannian manifold X, and assume that $m(X) = \infty$. Set $\nu(\mathrm{d}t, \mathrm{d}x) = \mathrm{d}t \times m(\mathrm{d}x)$. The gradient operator on X is denoted by ∇^X. Recalling that $\mathbb{X} := [0,1] \times X$, on $(\Gamma_{\mathbb{X}}, \mathcal{F}^\Gamma)$ we have a unique Poisson probability measure π_ν such that(cf. [12])

(**D**1) $\pi_\nu(\gamma \in \Gamma_{\mathbb{X}} : \#(\gamma \cap \{t\} \times X) \geqslant 2, \exists t \in [0,1]) = 0$.

(**D**2) For each $B \in \mathcal{B}(\mathbb{X})$ with $\nu(B) < +\infty$, $\gamma(B)$ is Poisson distributed, i.e.,

$$\pi_\nu(\gamma(B) = n) = \nu(B)^n \exp(-\nu(B))/n!, n = 0, 1, 2, \cdots.$$

(**D**3) If $B_1, \cdots, B_n \in \mathcal{B}(\mathbb{X})$ are disjoint, then $\gamma(B_1)$, $\gamma(B_2)$, $\cdots, \gamma(B_n)$ are mutually independent.

The completions of $\mathcal{F}^\Gamma$ and $\mathcal{F}^\Gamma_{t+}$ with respect to π_ν are still denoted by $\mathcal{F}^\Gamma$ and $\mathcal{F}^\Gamma_t$ respectively. Then it is not hard to verify that $(\Gamma_{\mathbb{X}}, \mathcal{F}^\Gamma, \pi_\nu; \mathcal{F}^\Gamma_t)$ satisfy (**B**1)-(**B**5).

As in (2.8) and (2.9), we define

$$\mathfrak{A} := \int_{\Gamma_{\mathbb{X}}}^{\oplus} H(\gamma) \mathrm{d}\pi_\nu(\gamma) := \left\{ V \in \mathcal{M} : \ \|V\|^2_{\mathfrak{A}} := \int_{\Gamma_{\mathbb{X}}} \|V(\gamma)\|^2_{H(\gamma)} \mathrm{d}\pi_\nu(\gamma) < \infty \right\},$$

and

$$\mathfrak{A}_t = \int_{\Gamma_{\mathbb{X}}}^{\oplus} H_t(\gamma) \mathrm{d}\pi_\nu(\gamma) := \left\{ V \in \mathcal{M}_t : \ \|V\|^2_{\mathfrak{A}_t} := \int_{\Gamma_{\mathbb{X}}} \|V(\gamma)\|^2_{H_t(\gamma)} \mathrm{d}\pi_\nu(\gamma) < \infty \right\}.$$

These spaces will play a basic role in the rest of the paper.

We denote by $C_p^\infty(\mathbb{R}^k)$(resp. $C_b^\infty(\mathbb{R}^k)$) the space of infinitely differentiable functions on $\mathbb{R}^n$, whose derivatives of all orders are of polynomial growth(resp. bounded).

We denote by $\mathbb{D}$ the set of all measurable functions $\phi : \mathbb{X} \mapsto \mathbb{R}$ such that $\phi(t, \cdot) \in C_0^\infty(X)$ for each $t \geqslant 0$. As in [2] introduce $\mathcal{F}C_p^\infty(X, \mathbb{D})$ (test function space) as the set of all random variables of the form:

$$f(\gamma) = F(\gamma(\phi_1), \cdots, \gamma(\phi_k)), \tag{3.1}$$

where $F \in C_p^\infty(\mathbb{R}^k)$, $\phi_i \in \mathbb{D}$, $i = 1, \cdots, k$. For abbreviation we set $\mathcal{G} := \mathcal{F}C_p^\infty(X, \mathbb{D})$. The elements in $\mathcal{G}$ are called cylindrical functions.

For any map $v : [0,1] \mapsto \mathcal{V}_0(X)$, let $\epsilon \mapsto \varphi_\epsilon^v(t, x)$ be the integral curve of the vector field $v(t, \cdot)$ for each $t \in [0,1]$. The natural action of φ^v_ϵ on the configuration space $\Gamma_{\mathbb{X}}$ is given by

$$\Gamma_{\mathbb{X}} \ni \gamma \mapsto \varphi^v_\epsilon(\gamma) := \{(t, \varphi^v_\epsilon(t, x)), (t, x) \in \gamma\} \in \Gamma_{\mathbb{X}}.$$

For $f \in \mathcal{G}$ of the form (3.1), the gradient of f is defined by

$$\nabla^\Gamma_{t,x} f(\gamma) := \sum_{i=1}^k \partial_i F(\gamma(\phi_1), \cdots, \gamma(\phi_k)) \nabla^X_x \phi_i(t, \cdot) \in T_\gamma(\Gamma_{\mathbb{X}}) =: H(\gamma),$$

where ∂_i denotes the usual partial derivative with respect to the i-th variable. In the sequel, we sometimes write ∇^Γ instead of $\nabla^\Gamma_{t,x}$.

Then for any $v : \mathbb{R}_+ \mapsto \mathcal{V}_0(X)$ we have

$$\begin{aligned} \nabla^\Gamma_v f(\gamma) := (\nabla^\Gamma f(\gamma), v)_{H(\gamma)} &= \int_0^1 \int_X \langle \nabla^\Gamma_{t,x} f(\gamma), v(t,x) \rangle_{T_x X} \gamma(\mathrm{d}t, \mathrm{d}x) \\ &= \lim_{\epsilon \to 0} \frac{f(\varphi^v_\epsilon(\gamma)) - f(\gamma)}{\epsilon}. \end{aligned}$$

For $p \geqslant 2$, we define the norm on $\mathcal{G}$ as follows(cf. [2]):

$$\|f\|_{\mathcal{H}^1_p} := \|f\|_{L^p(\Gamma_{\mathbb{X}}, \pi_\nu)} + \| \|\nabla^\Gamma f\|_{H(\cdot)} \|_{L^p(\Gamma_{\mathbb{X}}, \pi_\nu)}.$$

We can now give the following definition of Sobolev spaces on $(\Gamma_{\mathbb{X}}, \mathcal{F}^\Gamma, \pi_\nu)$.

Definition 3.1. *Let $p \geqslant 2$. The Sobolev space $\mathcal{H}^1_p$ is defined as the completion of $\mathcal{G}$ with respect to the norm $\| \cdot \|_{\mathcal{H}^1_p}$.*

We set

$$\mathcal{H}^1_\infty := \cap_{2 \leqslant p < \infty} \mathcal{H}^1_p.$$

By the integration by parts formula proved in [2] (see also the [7, Lemma 3.3]), we know that these spaces are naturally embedded into $L^p(\Gamma_{\mathbb{X}}, \pi_\nu)$ and that the gradients extend uniquely to all functions in $\mathcal{H}^1_p$. Also the following is obvious:

Proposition 3.2. *If $f \in \mathcal{H}_2^1$, then $\nabla^\Gamma f$ is a measurable function of (t, x, γ) which will again be denoted by $\nabla^\Gamma f$ in the sequel. Moreover, $\nabla^\Gamma f \in L^2(\mathbb{X} \times \Gamma_{\mathbb{X}}, \nu(\mathrm{d}s \times \mathrm{d}x) \times \mathrm{d}\pi_\nu)$.*

E.g. by using the integration by parts formula from [2] one easily sees that the following is true.

Proposition 3.3. *Let $f \in \mathcal{H}_2^1$ be measurable with respect to $\mathcal{F}_t^\Gamma$. Then we have*

$$\nabla_{s,x}^\Gamma f(\gamma) = 0 \text{ for any } s > t \text{ and } x \in X,\ \pi_\nu - a.e.\ \gamma.$$

Moreover, $\nabla^\Gamma f \in \mathfrak{A}_t$.

4. Kusuoka-Stroock formula

We are about to establish a commutation formula between the gradient and stochastic integrals. On Wiener space this formula is given by Kusuoka and Stroock in [13].

Remember that $\mathbb{X} := [0,1] \times X$. The following is immediate from the definition of ∇^Γ by suitable approximation.

Theorem 4.1. *Let $f : \mathbb{X} \times \Gamma_{\mathbb{X}} \to \mathbb{R}$ be a measurable function such that for all $\gamma \in \Gamma_{\mathbb{X}}$, $\mathrm{supp} f(\cdot, \cdot, \gamma) \subset [0,1] \times K$, where K is a compact subset of X, $f(s, x, \cdot) \in \mathcal{H}_2^1$ for all $(s, x) \in \mathbb{X}$, and $\nu(\|f\|^2_{\mathcal{H}_2^1}) < \infty$. Then*

$$\nabla^\Gamma(\nu(f)) = \nu(\nabla^\Gamma f),\ \pi_\nu - a.e..$$

We introduce $\tilde{\mathcal{G}}$ as the set of all functions of the form:

$$f(x, \gamma) = F(x; \gamma(\phi_1), \cdots, \gamma(\phi_k)), \tag{4.1}$$

where $\phi_i \in \mathbb{D}$, $i = 1, \cdots, k$ and $F \in C^\infty(X \times \mathbb{R}^k)$ satisfying that for each $x \in X$, $F(x; \cdot) \in C_p^\infty(\mathbb{R}^k)$ and there is a compact set K such that $F(x; \cdot) = 0$ for all $x \in K^c$.

For $p \geqslant 2$, we define the norm on $\tilde{\mathcal{G}}$ as follows:

$$\|f\|_{\tilde{\mathcal{H}}_p^1} := \|f\|_{L^p(m \times \pi_\nu)} + \|\|\nabla^\Gamma f\|_{H(\cdot)}\|_{L^p(m \times \pi_\nu)} + \||\nabla^X f|_{T_x X}\|_{L^p(m \times \pi_\nu)}.$$

We will need the following Sobolev spaces on $(X \times \Gamma_{\mathbb{X}}, \mathcal{B}(X) \times \mathcal{F}^\Gamma, m \times \pi_\nu)$.

Definition 4.2. *For $p \geqslant 2$, the Sobolev space $\tilde{\mathcal{H}}_p^1$ is defined as the completion of $\tilde{\mathcal{G}}$ with respect to the norm $\| \cdot \|_{\tilde{\mathcal{H}}_p^1}$.*

By a standard integration by parts argument we can prove that $\tilde{\mathcal{H}}_p^1$ embeds into $L^p(X \times \Gamma_{\mathbb{X}}, m \times \pi_\nu)$ and that the corresponding gradient operators are well defined (cf. e.g., [2, 7, 6]).

Remark 4.3. *For $g \in \tilde{\mathcal{H}}_p^1$, it is clear that for $m(\mathrm{d}x)$-a.e. $x \in X$, $g(x, \cdot) \in \mathcal{H}_p^1$.*

In the following, the expectation $\mathbb{E}$ is taken with respect to π_ν and set

$$\tilde{\mathcal{H}}_\infty^1 := \cap_{2 \leqslant p < \infty} \tilde{\mathcal{H}}_p^1.$$

In order to prove the Kusuoka-Stroock formula. We first prepare the following lemma.

Lemma 4.4. *For fixed $t_0 \in \mathbb{R}_+$, let $g(x, \gamma)$ be a $\mathcal{B}(X) \times \mathcal{F}_{t_0}^\Gamma$ measurable function. Assume that $g(\cdot, \cdot) \in \tilde{\mathcal{H}}_\infty^1$ and for all $\gamma \in \Gamma_{\mathbb{X}}$, $\mathrm{supp}\ g(\cdot, \gamma) \subset K$, where K is a compact subset of X. Set $f(s, x, \gamma) := 1_{[t_0,1]}(s) g(x, \gamma)$, then $\tilde{\gamma}_t(f) \in \mathcal{H}_\infty^1$ and*

$$\nabla^\Gamma(\tilde{\gamma}_t(f)) = \tilde{\gamma}_t(\nabla^\Gamma f) + (\nabla^X f(\cdot, \gamma)) 1_{[t_0,t]} \in \mathfrak{A}_t. \tag{4.2}$$

Proof. For $k \in \mathbb{N}$, let $g_n \in \tilde{\mathcal{G}}$ be such that

$$\lim_{n\to\infty} \|g_n - g\|_{\tilde{\mathcal{H}}^1_{2^k}} = 0.$$

Set $f_n(s,x,\gamma) := 1_{[t_0,1]}(s) g_n(x,\gamma)$. Then by the definition of ∇^Γ, we have for each n

$$\nabla^\Gamma\left(\tilde{\gamma}_t(f_n)\right) = \tilde{\gamma}_t(\nabla^\Gamma f_n) + (\nabla^X g_n(\cdot,\gamma)) 1_{[t_0,t]}.$$

Now by inequality (2.22), we have for any $k \in \mathbb{N}$

$$\begin{aligned}
\mathbb{E}\left(\|\tilde{\gamma}_t(\nabla^\Gamma f_n - \nabla^\Gamma f)\|^{2^k}_{H(\cdot)}\right) &\leqslant C \sum_{i=1}^{k} \mathbb{E}\left[\nu_t\left(\|f_n - f\|^{2^i}_{H(\cdot)}\right)\right]^{2^{k-i}} \\
&\leqslant C_{K,k} \|g_n - g\|^{2^k}_{\tilde{\mathcal{H}}^1_{2^k}} \to 0 \text{ as } n \to \infty.
\end{aligned}$$

On the other hand, set

$$V_n(s,x,\gamma) := \nabla^X (g_n - g)(\cdot,\gamma) 1_{[t_0,t]}(s) \in H(\gamma)$$

and

$$h_n(s,x,\gamma) := |\nabla^X (g_n - g)(\cdot,\gamma)|^2_{T_x X} 1_{[t_0,t]}(s).$$

Then

$$\|V_n(\cdot,\cdot,\gamma)\|^2_{H(\gamma)} = \gamma(h_n(\cdot,\cdot,\gamma)).$$

So by (2.22)

$$\begin{aligned}
&\mathbb{E}\left(\|V_n(\cdot,\cdot,\gamma)\|^{2^k}_{H(\gamma)}\right) = \mathbb{E}|\gamma(h_n)|^{2^{k-1}} \\
\leqslant\ & C_r \left(\mathbb{E}|\tilde{\gamma}(h_n)|^{2^{k-1}} + \mathbb{E}|\nu(h_n)|^{2^{k-1}}\right) \\
\leqslant\ & C_r \left(\sum_{i=1}^{k-1} \mathbb{E}\left(\nu(|h_n|^{2^i})\right)^{2^{k-1-i}} + \mathbb{E}|\nu(h_n)|^{2^{k-1}}\right) \\
\leqslant\ & C_{K,r} \|g_n - g\|^{2^k}_{\tilde{\mathcal{H}}^1_{2^k}} \to 0 \text{ as } n \to \infty,
\end{aligned}$$

and the proof is complete. $\square$

Lemma 4.5. *Let $f(s,x,\gamma) : \mathbb{X} \times \Gamma_{\mathbb{X}} \mapsto \mathbb{R}$ be a measurable function satisfying the following conditions:*

(i) $\mathrm{supp} f(\cdot,\cdot,\gamma) \subset [0,1] \times K$ *for some compact subset K of X and all $\gamma \in \Gamma_{\mathbb{X}}$;*
(ii) for each $s \in [0,1]$, $f(s,\cdot,\cdot) \in \tilde{\mathcal{H}}^1_\infty$;
(iii) for each x, $(s,\gamma) \mapsto f(s,x,\gamma), \nabla^X_x f(s,\cdot,\gamma)$ are left-continuous and adapted process;
(iv) for any $p \geqslant 2$

$$\int_0^1 \|f(s,\cdot,\cdot)\|^p_{\tilde{\mathcal{H}}^1_p} \mathrm{d}s < +\infty.$$

Then $\tilde{\gamma}_t(f) \in \mathcal{H}^1_\infty$ for all $t \in [0,1]$ and

$$\nabla^\Gamma\left(\tilde{\gamma}_t(f)\right) = \tilde{\gamma}_t(\nabla^\Gamma f) + (\nabla^X f) 1_{[0,t]} \in \mathfrak{A}_t. \tag{4.3}$$

Proof. We first construct the approximating sequence f^ϵ of f. Let $\rho(t)$ be a C^∞-function with support in $[0,1]$ and with total mass 1. Set

$$f^\epsilon(s,x,\gamma) := \int_0^s f(t,x,\gamma) \rho_\epsilon(s-t) \mathrm{d}t, \ (s,x,\gamma) \in \mathbb{X} \times \Gamma_{\mathbb{X}},$$

where $\rho_\epsilon(s) = \frac{1}{\epsilon}\rho(s/\epsilon)$ is a regularizing sequence. Then f^ϵ is a continuous adapted process. Secondly, we define

$$f^\epsilon_n(s, x, \gamma) := \sum_{k=0}^{2^n-1} f^\epsilon(k2^{-n}, x, \gamma)1_{(k2^{-n},(k+1)2^{-n}]}(s),$$

and write

$$\gamma_t(f^\epsilon_n) = \gamma_{k2^{-n}}(f^\epsilon_n) + \int_{k2^{-n}}^{t+}\int_X f^\epsilon(k2^{-n}, x)\gamma(\mathrm{d}s, \mathrm{d}x),$$

where $t \in (k2^{-n}, (k+1)2^{-n}]$.

By induction and Lemma 4.4, we obtain that

$$\nabla^\Gamma\left(\tilde{\gamma}_t(f^\epsilon_n)\right) = \tilde{\gamma}_t(\nabla^\Gamma f^\epsilon_n) + (\nabla^X f^\epsilon_n)1_{[0,t]}$$

and $\gamma_t(f^\epsilon_n) \in \mathcal{H}^1_\infty$ for each n, t, ϵ. Applying inequality (2.22), and passing to the limit we prove (4.3). □

Now we can state and prove our main result.

Theorem 4.6. *(Kusuoka-Stroock formula) If (i) and (iv) in Lemma 4.5 are replaced by*

(iv)′ For any $r \in \mathbb{N}$

$$\sum_{i=0}^{k}\mathbb{E}\left(\nu\left(\|\nabla^\Gamma f\|^{2^i}_{H(\cdot)} + |\nabla^X f|^{2^i}_{TX} + |f|^{2^i}\right)\right)^{2^{k-i}} < +\infty, \tag{4.4}$$

then the conclusion of Lemma 4.5 still holds.

Proof. Let $B_n \uparrow X$ be an exhausting sequence of compact sets such that there exist smooth functions $\psi_n : X \to [0, 1]$ satisfying(cf. [3])

$$\psi_n(z) = \begin{cases} 1, z \in B_n \\ 0, z \notin B_{2n}, \end{cases}$$

and

$$|\nabla^X\psi_n(x)|_{T_xX} \leqslant C \quad \text{for any } x \in X, \tag{4.5}$$

where the constant C does not depend on n. Then we set

$$f_n(s, x, \gamma) = \psi_n(x)f(s, x, \gamma).$$

It is clear that each f_n satisfies all the conditions in Lemma 4.5 and f_n(resp. $\nabla^\Gamma f_n$) pointwisely converge to f (resp. $\nabla^\Gamma f$). By (2.22), (4.4) and the dominated convergence theorem, we have for any $k \in \mathbb{N}$

$$\lim_{n\to\infty}\mathbb{E}|\tilde{\gamma}_t(f_n - f)|^{2^k} \leqslant C\lim_{n\to\infty}\sum_{i=1}^{k}\mathbb{E}\left[\nu_t\left(|f_n - f|^{2^i}\right)\right]^{2^{k-i}} = 0,$$

and as $n \to \infty$

$$\mathbb{E}\left(\left\|\tilde{\gamma}_t\left(\nabla^\Gamma f_n - \nabla^\Gamma f\right)\right\|^{2^k}_{H(\cdot)}\right) \leqslant C\sum_{i=1}^{k}\mathbb{E}\left[\nu_t\left(\|\nabla^\Gamma f_n - \nabla^\Gamma f\|^{2^i}_{H(\cdot)}\right)\right]^{2^{k-i}} \to 0.$$

As in Lemma 4.4, set

$$V_n(s, x, \gamma) := (\nabla^X(f_n - f)(s, \cdot, \gamma))(x)1_{[t_0,t]}(s) \in H(\gamma),$$

and

$$\alpha_n(s, x, \gamma) := |\nabla^X(f_n - f)(s, \cdot, \gamma)|^2_{T_xX}1_{[t_0,t]}(s).$$

Then $\alpha_n(s,x,\gamma)$ is strongly predictable and

$$|V_n(\cdot,\cdot,\gamma)|^2_{H(\gamma)} = \gamma(\alpha_n(\cdot,\cdot,\gamma)).$$

Noticing that

$$\nabla^X_x f_n(s,\cdot,\gamma) = f(s,x,\gamma)\nabla^X_x \psi_n + \psi_n(x)\nabla^X_x f(s,\cdot,\gamma),$$

by (4.5) (4.4) and the dominated convergence theorem, we hence have

$$\begin{aligned}
&\mathbb{E}\left(\|V_n(\cdot,\cdot,\gamma)\|^{2^k}_{H(\gamma)}\right) = \mathbb{E}|\gamma(\alpha_n)|^{2^{k-1}} \\
\leqslant\;& C\left(\mathbb{E}|\tilde{\gamma}(\alpha_n)|^{2^{k-1}} + \mathbb{E}|\nu(\alpha_n)|^{2^{k-1}}\right) \\
\leqslant\;& C\left(\sum_{i=1}^{k-1}\mathbb{E}\left(\nu(|\alpha_n|^{2^i})\right)^{2^{k-1-i}} + \mathbb{E}|\nu(\alpha_n)|^{2^{k-1}}\right) \\
\to\;& 0 \text{ as } n\to\infty,
\end{aligned}$$

and the proof is thus complete. □

Remark 4.7. *Under the same assumptions as in Theorem 4.6, we also have that $\gamma_t(f) \in \mathcal{H}^1_\infty$ for all $t\in[0,1]$ and*

$$\nabla^\Gamma(\gamma_t(f)) = \gamma_t(\nabla^\Gamma f) + (\nabla^X f)1_{[0,t]} \in \mathfrak{A}_t.$$

5. Quasi-regular Dirichlet forms on Wiener-Poisson space

First of all, let us recall the definition of a quasi-regular Dirichlet form according to [14]. Let E be a separable Hausdorff topological space. $\mathcal{B}(E)$ denotes its Borel σ-algebra. We fix a finite positive measure m on $(E,\mathcal{B}(E))$.

Let $(\mathcal{E},\mathcal{D}(\mathcal{E}))$ be a Dirichlet form on $L^2(E,m)$, and we assume $1\in\mathcal{D}(\mathcal{E})$. The notion of capacity associated with $(\mathcal{E},\mathcal{D}(\mathcal{E}))$ is defined as follows:

Definition 5.1. *(i) For an arbitrarily open set A, define*

$$\begin{aligned}
\mathcal{V}^A &:= \{f\in\mathcal{D}(\mathcal{E}); f\geqslant 1 \ m\text{-a.e. on } A\},\\
\mathrm{Cap}(A) &:= \inf_{f\in\mathcal{V}^A}\mathcal{E}_1(f,f)^{1/2},
\end{aligned}$$

where $\mathcal{E}_1(f,f) := (f,f)_{L^2} + \mathcal{E}(f,f)$, and for any set $A\subset E$, we let

$$\mathrm{Cap}(A) := \inf\{\mathrm{Cap}(B); A\subset B\subset E, B \text{ open}\}.$$

We use e_A to denote its equilibrium potential.

(ii) We say that a statement holds $\mathcal{E}$-quasi everywhere (abbreviated $\mathcal{E}$-q.e.) if there exists a set N of zero capacity such that the statement is true for every $x\in N^c$.

(iii) A function $f: E\to\mathbb{R}$ is called $\mathcal{E}$-quasi continuous if for any $\epsilon>0$ there exists an open set $G\subset E$ such that $\mathrm{Cap}(G)<\epsilon$ and f restricted on G^c is continuous.

The following definition is taken from [14].

Definition 5.2. *A Dirichlet form $(\mathcal{E},\mathcal{D}(\mathcal{E}))$ is called quasi-regular if:*

(i) There exists an increasing sequence $(K_n)_{n\in\mathbb{N}}$ of compact subsets of E such that

$$\lim_{n\to\infty}\mathrm{Cap}(K_n^c) = 0.$$

(ii) There exists an $\mathcal{E}_1^{1/2}$-dense subset of $\mathcal{D}(\mathcal{E})$ whose elements have $\mathcal{E}$-quasi continuous m-versions.

(iii) There exist $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{D}(\mathcal{E})$, having $\mathcal{E}$-quasi continuous m-version $\tilde{f}_n$, and a zero capacity set $N \subset E$ such that $\{\tilde{f}_n, n \in \mathbb{N}\}$ separates the points of N^c.

5.1. **Dirichlet form on Poisson space.** On the Poisson space $(\Gamma_{\mathbb{X}}, \mathcal{F}^\Gamma, \pi_\nu)$, we define the following symmetric bilinear form

$$\mathcal{E}^\Gamma(\Phi, \Psi) := \mathbb{E}^{\pi_\nu}\left(\langle \nabla^\Gamma \Phi, \nabla^\Gamma \Psi\rangle_{H(\gamma)}\right);\ \Phi, \Psi \in \mathcal{G}.$$

Then we have

Proposition 5.3. *$(\mathcal{E}^\Gamma, \mathcal{G})$ is a closable Dirichlet form on $L^2(\Gamma_{\mathbb{X}}, \pi_\nu)$.*

Using the integration by parts formula from [2], the proof of this proposition is entirely standard (see [14]), and the closure of $(\mathcal{E}^\Gamma, \mathcal{G})$ is denoted by $(\mathcal{E}^\Gamma, \mathcal{D}(\mathcal{E}^\Gamma))$. Note that $\mathcal{D}(\mathcal{E}^\Gamma) = \mathcal{H}_2$.

In [15] the authors constructed diffusions on configuration spaces applying the theory of quasi-regular Dirichlet forms. Their arguments easily go through to the setting of the present paper and, in particular, we have that the Dirichlet form $(\mathcal{E}^\Gamma, \mathcal{D}(\mathcal{E}^\Gamma))$ on $L^2(\Gamma_{\mathbb{X}}, \pi_\nu)$ is quasi-regular in the sense of Definition 5.2.

5.2. **Dirichlet forms on Wiener space.** Let $(W, \mathcal{F}^W, \mu; \mathbb{H}; (\mathcal{F}^W_t)_{t\in[0,1]})$ be the classical Wiener space of the d-dimensional Brownian motion on $[0,1]$. Namely, W is the space of continuous functions $w : [0,1] \to \mathbb{R}^d$ satisfying $w(0) = 0$, endowed with the topology of uniform convergence. $\mathbb{H}$ is the Cameron-Martin space. $\mathcal{F}^W$ denotes the Borel σ-algebra over W, and for each $0 \leqslant t \leqslant 1$, $\mathcal{F}^W_t$ denotes the σ-algebra $\sigma\{w(s) : 0 \leqslant s \leqslant t\}$. μ is the standard Wiener measure on $(W, \mathcal{F}^W)$.

Let $\mathcal{C}$ be the set of cylindrical functions on W, i.e.

$$\mathcal{C} := \left\{f(w) := F(w(t_1), \cdots, w(t_n)), 0 \leqslant t_1 \leqslant \cdots \leqslant t_n \leqslant 1, F \in C_0^\infty(\mathbb{R}^{n\times d})\right\}.$$

For $f \in \mathcal{C}$, we define $\nabla^W f(w) \in \mathbb{H}$ by

$$\nabla^W f(w)(s) = \sum_{i=1}^n \partial_i F(w(t_1), \cdots, w(t_n)) \cdot (s \wedge t_i), \quad s \in [0,1].$$

And we set

$$\mathcal{E}^W(\Phi, \Psi) := \mathbb{E}^\mu((\nabla^W \Phi, \nabla^W \Psi)_{\mathbb{H}});\ \Phi, \Psi \in \mathcal{C}.$$

Then $(\mathcal{E}^W, \mathcal{C})$ is closable in $L^2(W, \mu)$, and the closure is denoted by $(\mathcal{E}^W, \mathcal{D}(\mathcal{E}^W))$. The following result is well known.

Proposition 5.4. *$(\mathcal{E}^W, \mathcal{D}(\mathcal{E}^W))$ is a quasi-regular Dirichlet form on $L^2(W, \mu)$.*

5.3. **Product of quasi-regular Dirichlet forms.** Bouleau and Hirsch in [4, p.200] defined the product of two Dirichlet forms. This naturally leads to the definition of Dirichlet forms on Wiener-Poisson space.

Definition 5.5. *Let $(\Omega, \mathcal{F}, P; \mathcal{F}_t) := (\Gamma_{\mathbb{X}} \times W, \overline{\mathcal{F}^\Gamma \times \mathcal{F}^W}, \pi_\nu \times \mu; \overline{\mathcal{F}^\Gamma_t \times \mathcal{F}^W_t})$ be the completed product probability space. The Dirichlet form on $L^2(\Omega, P)$ is defined as*

$$\mathcal{E}(\Phi, \Psi) := \int_{\Gamma_{\mathbb{X}}} \mathcal{E}^W(\Phi(\gamma, \cdot), \Psi(\gamma, \cdot))\pi_\nu(\mathrm{d}\gamma) + \int_W \mathcal{E}^\Gamma(\Phi(\cdot, w), \Psi(\cdot, w))\mu(\mathrm{d}w),$$

where Φ and Ψ belong to $\mathcal{D}(\mathcal{E})$, and

$$\mathcal{D}(\mathcal{E}) \ := \ \Big\{\Phi \in L^2(\Omega, P) : \gamma \mapsto \Phi(\gamma, \cdot) \text{ belongs to } L^2(\Gamma_{\mathbb{X}}, \pi_\nu; \mathcal{D}(\mathcal{E}^W))$$

$$\textit{and } w \mapsto \Phi(\cdot, w) \textit{ belongs to } L^2(W, \mu; \mathcal{D}(\mathcal{E}^\Gamma))\Big\}.$$

It is easy to check that $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ in Definition 5.5 is indeed a Dirichlet form. We define the gradient for $\Phi \in \mathcal{D}(\mathcal{E})$ by

$$(\nabla\Phi)(\gamma, w) := (\nabla^\Gamma \Phi(\gamma, w), \nabla^W \Phi(\gamma, w)) \in H(\gamma) \times \mathbb{H},$$

and for $\Phi, \Psi \in \mathcal{D}(\mathcal{E})$

$$(\nabla\Phi, \nabla\Psi)_{H(\gamma)\times\mathbb{H}} := (\nabla^\Gamma \Phi(\gamma, w), \nabla^\Gamma \Psi(\gamma, w))_{H(\gamma)} + (\nabla^W \Phi(\gamma, w), \nabla^W \Psi(\gamma, w))_{\mathbb{H}}.$$

Then

$$\mathcal{E}(\Phi, \Psi) = \mathbb{E}^P((\nabla\Phi, \nabla\Psi)_{H(\cdot)\times\mathbb{H}}).$$

We have the following result.

Theorem 5.6. *The Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E})$ is quasi-regular in the sense of Definition 5.2.*

Proof. Let $(K_n)_{n\in\mathbb{N}}$(resp.$(U_n)_{n\in\mathbb{N}}$) be an increasing sequence of compact sets of $\Gamma_{\mathbb{X}}$ (resp. W) such that $\lim_{n\to\infty} \mathrm{Cap}^\Gamma(K_n^c) = 0$ (resp. $\lim_{n\to\infty} \mathrm{Cap}^W(U_n^c) = 0$). We need to prove that

$$\lim_{n\to\infty} \mathrm{Cap}((K_n \times U_n)^c) = 0,$$

where Cap^Γ, Cap^W and Cap are the capacities corresponding to $(\mathcal{E}^\Gamma, \mathcal{D}(\mathcal{E}^\Gamma))$, $(\mathcal{E}^W, \mathcal{D}(\mathcal{E}^W))$ and $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$.

Let $e_{K_n^c}$ and $e_{U_n^c}$ be the respectively equilibrium potentials of K_n^c and U_n^c. By the above definition, we know that

$$e_{K_n^c} \cdot 1_W \in \mathcal{D}(\mathcal{E}),\ 1_{\Gamma_{\mathbb{X}}} \cdot e_{U_n^c} \in \mathcal{D}(\mathcal{E}),$$

and

$$\begin{aligned}
\mathcal{E}_1(e_{K_n^c} \cdot 1_W, e_{K_n^c} \cdot 1_W) &= \mathcal{E}_1^\Gamma(e_{K_n^c}, e_{K_n^c}),\\
\mathcal{E}_1(1_{\Gamma_{\mathbb{X}}} \cdot e_{U_n^c}, 1_{\Gamma_{\mathbb{X}}} \cdot e_{U_n^c}) &= \mathcal{E}_1^W(e_{U_n^c}, e_{U_n^c}).
\end{aligned}$$

Hence

$$\begin{aligned}
& \mathrm{Cap}((K_n \times U_n)^c)\\
\leqslant\ & \mathrm{Cap}(K_n^c \times W) + \mathrm{Cap}(\Gamma_{\mathbb{X}} \times U_n^c)\\
\leqslant\ & \mathcal{E}_1(e_{K_n^c} \cdot 1_W, e_{K_n^c} \cdot 1_W)^{1/2} + \mathcal{E}_1(1_{\Gamma_{\mathbb{X}}} \cdot e_{U_n^c}, 1_{\Gamma_{\mathbb{X}}} \cdot e_{U_n^c})^{1/2}\\
=\ & \mathcal{E}_1^\Gamma(e_{K_n^c}, e_{K_n^c})^{1/2} + \mathcal{E}_1^W(e_{U_n^c}, e_{U_n^c})^{1/2}\\
=\ & \mathrm{Cap}^\Gamma(K_n^c) + \mathrm{Cap}^W(U_n^c)\\
\to\ & 0, \text{ as } n \to \infty,
\end{aligned}$$

which gives (i) of Definition 5.2. (ii) and (iii) of Definition 5.2 are easily verified, and we complete the proof. □

As a consequence of Theorem 5.6 and [14, pp.153 Theorem 1.11], we now obtain

Theorem 5.7. *There exists a conservative (strong Markov) diffusion process*

$$M = (\Omega', \mathcal{F}', (\mathcal{F}'_t)_{t\geqslant 0}, (X_t)_{t\geqslant 0}, (P'_\omega)_{\omega\in\Omega})$$

on Ω which is properly associated with $(\mathcal{E}, D(\mathcal{E}))$. In particular, M is P-symmetric.

Henceforth, we shall work on the Wiener-Poisson space $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$.

6. Fractional Sobolev spaces and capacities on Wiener-Poisson spaces

For $p \geqslant 2$, define a norm on $\mathcal{D}(\mathcal{E})$ as follows:

$$\|f\|_{\mathcal{D}_p^1} := \|f\|_{L^p} + \|\nabla f\|_{L^p}.$$

We set

$$\mathcal{D}_p^1 := \{f \in \mathcal{D}(\mathcal{E}) \cap L^p; \|f\|_{\mathcal{D}_p^1} < \infty\}.$$

Then $\mathcal{D}_p^1$ is a Banach space.

Applying the K-method in real interpolation theory (see [23]), for $0 < r < 1$ we define the fractional Sobolev spaces:

$$\mathcal{D}_p^r := (L^p, \mathcal{D}_p^1)_{r,p} = \left\{ f \in L^p : \|f\|_{\mathcal{D}_p^r} := \left(\int_0^1 [\epsilon^{-r} K(\epsilon, f)]^p \frac{\mathrm{d}\epsilon}{\epsilon} \right)^{\frac{1}{p}} < \infty \right\},$$

where

$$K(\epsilon, f) := \inf_{f = f_1 + f_2} \{\|f_1\|_{L^p} + \epsilon \|f_2\|_{\mathcal{D}_p^1}\}.$$

Let $\mathcal{L}$ be the generator of the Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$. We define another type of Sobolev space by

$$\tilde{\mathcal{D}}_p^r := (I - \mathcal{L})^{-\frac{r}{2}}(L^p).$$

Then we have

Theorem 6.1. *For $0 < r < 1$, $0 < \epsilon < r$ and $p \geqslant 2$,*

$$\mathcal{D}_p^r \subset \tilde{\mathcal{D}}_p^{r-\epsilon}.$$

Proof. By a general result of [8, Theorem 1], there is a constant $C_{p,\theta}$ such that

$$\|(I - \mathcal{L})^{\frac{\theta}{2}} f\|_{L^p} \leqslant C_{p,\theta}(\|\nabla f\|_{L^p} + \|f\|_{L^p})$$

for $p \geqslant 2$ and $0 < \theta < 1$. This means that

$$\mathcal{D}_p^1 \subset \tilde{\mathcal{D}}_p^\theta.$$

Hence, by [23, Theorem 1.15.2] we have for any $0 < \theta < 1$ and $0 < \epsilon < r\theta$

$$\mathcal{D}_p^r \subset (L^p, \tilde{\mathcal{D}}_p^\theta)_{r,p} = (L^p, \tilde{\mathcal{D}}_p^1)_{r\theta,p} \subset (L^p, \tilde{\mathcal{D}}_p^1)_{r\theta-\epsilon,1} \subset \tilde{\mathcal{D}}_p^{r\theta-\epsilon}.$$

The result then follows. □

Since $\mathcal{D}_p^r$ is uniformly convex (see e.g. [11, Lemma 3.5]), we may define the (p, r)-capacity on $\mathcal{D}_p^r$ similar to Definition 5.1.

Definition 6.2. *For an open set A of Ω, we define*

$$\mathcal{V}_{p,r}^A := \{f \in \mathcal{D}_p^r; f \geqslant 1 \ m\text{-a.e. on } A\},$$
$$\mathrm{Cap}_{p,r}(A) := \inf_{f \in \mathcal{V}_{p,r}^A} \|f\|_{\mathcal{D}_p^r},$$

and for any set $A \subset \Omega$, we let

$$\mathrm{Cap}_{p,r}(A) := \inf \left\{ \mathrm{Cap}_{p,r}(B); A \subset B \subset \Omega, B \text{ is an open set} \right\}.$$

We also have the respective notions of redefinition, equilibrium potential, etc.(cf. e.g. [11]).

Using Theorem 5.7, the following result is proved in [19].

Theorem 6.3. *If $0 < r \leqslant 1, pr > 2$, then there exists a constant $C = C(p, r)$ such that*

$$[\mathrm{Cap}_{2,1}(A)]^2 \leqslant C \cdot [\mathrm{Cap}_{p,r}(A)]^p, \ \forall A \subset \Omega.$$

7. Regularities of local times with jumps

Consider a semimartingale of the following form:

$$U_t = U_0 + \int_0^t M_s \mathrm{d}W_s + \int_0^{t+} \int_X g(s, x) \tilde{\gamma}(\mathrm{d}s, \mathrm{d}x) + \int_0^t N_s \mathrm{d}s,$$

where $U_0 \in \mathbb{R}$, $\{W_s\}_{0 \leqslant s \leqslant 1}$ is a standard one dimensional Brownian motion, and M_s, N_s, $g(s, x)$ satisfy the following regularity hypotheses:

(**H**1) M_s is an $(\mathcal{F}_s)$-measurable adapted real-valued process, N_s is an $(\mathcal{F}_s)$-adapted real-valued process and $g(s, x)$ is an $(\mathcal{F}_s)$-predictable real-valued process for each x.

(**H**2) For each s, x, $M_s, N_s, g(s, x) \in \cap_{p \geqslant 2} \mathcal{D}_p^1$, and $\int_0^1 \|M_s\|_{\mathcal{D}_p^1}^p + \|N_s\|_{\mathcal{D}_p^1}^p \mathrm{d}s < +\infty$ for each $p \geqslant 2$. Moreover, for any $p \geqslant 2$

$$g(s, \cdot, \cdot) \in \tilde{\mathcal{H}}_p^1,$$

and for any $k \in \mathbb{N}$

$$\sum_{i=0}^k \mathbb{E} \left(\nu \left(\|\nabla^\Gamma g\|_{H(\gamma)}^{2^i} + \|\nabla^W g\|_{\mathbb{H}}^{2^i} + |\nabla^X g|_{TX}^{2^i} + |g|^{2^i} \right) \right)^{2^{k-i}} < +\infty.$$

(**H**3) There exists an $(\mathcal{F}_s)$-measurable adapted process ξ_s such that $\int_0^1 \mathbb{E}|\xi_s|^p \mathrm{d}s < +\infty$ for each $p \geqslant 2$, and $|N_s| \vee \int_X |g(s, x)| m(\mathrm{d}x) \leqslant |M_s||\xi_s|$ a.e. for each $s \in [0, 1]$.

By the Kusuoka-Stroock formula in Theorem 4.6 and Theorem 2.21, it is trivial to see that under these hypotheses we have for any $p \geqslant 2$

$$\mathbb{E} \left(\sup_{0 \leqslant t \leqslant 1} \left(|U_t|^p + \|\nabla U_t\|_{H(\cdot) \otimes \mathbb{H}}^p \right) \right) < +\infty. \tag{7.1}$$

By Tanaka's formula (see [10]), the local time of U_t is given by

$$\begin{aligned} \frac{1}{2} L_t^a &= (U_t - a)^+ - (U_0 - a)^+ - \int_0^{t+} 1_{(a,\infty)}(U_{s-}) \mathrm{d}U_s \\ &\quad - \sum_{0 < s \leqslant t} \left[1_{(a,\infty)}(U_{s-})(U_s - a)^- + 1_{(-\infty,a]}(U_{s-})(U_s - a)^+ \right]. \end{aligned}$$

It will be convenient to write the above sum as an integral. By (**D**1), for π_ν-almost all $\gamma \in \Gamma_{\mathbb{X}}$, γ can be regarded as a point function from $[0, 1]$ to X. For such γ, we have $U_s - U_{s-} = g(s, x)$, where $x = \gamma(s)$. Noting that

$$y^+ - x^+ - 1_{(a,\infty)}(x)(y - x) = \begin{cases} y^-, x > a \\ y^+, x \leqslant a, \end{cases}$$

we have for π_ν-almost all $\gamma \in \Gamma_{\mathbb{X}}$

$$\begin{aligned} &\sum_{0 < s \leqslant t} \left[1_{(a,\infty)}(U_{s-})(U_s - a)^- + 1_{(-\infty,a]}(U_{s-})(U_s - a)^+ \right] \\ &= \int_0^{t+} \int_X \left[(U_{s-} + g(s, x) - a)^+ - (U_{s-} - a)^+ - 1_{(a,\infty)}(U_{s-}) g(s, x) \right] \gamma(\mathrm{d}s, \mathrm{d}x). \end{aligned}$$

By $(\mathbf{H}3)$, we know that $g(s,x) \in \mathfrak{F}_1 \cap \mathfrak{F}_2$. Thereby we obtain that

$$
\begin{aligned}
\frac{1}{2} L_t^a &= \left[(U_t - a)^+ - (X_0 - a)^+\right] \\
&\quad - \int_0^t 1_{(a,\infty)}(U_s) M_s \mathrm{d}W_s - \int_0^t 1_{(a,\infty)}(U_s) N_s \mathrm{d}s \\
&\quad - \int_0^{t+} \int_X \left[(U_{s-} + g(s,x) - a)^+ - (U_{s-} - a)^+\right] \gamma(\mathrm{d}s, \mathrm{d}x) \\
&\quad + \int_0^t \int_X g(s,x) 1_{(a,\infty)}(U_s) \nu(\mathrm{d}s, \mathrm{d}x) \\
&:= I_1(t) - I_2(t) - I_3(t) - I_4(t) + I_5(t).
\end{aligned}
$$

We can now state and prove our main result of this section.

Theorem 7.1. *For fixed $t \in [0,1]$, $a \in \mathbb{R}$, if $p \geqslant 2$ and $r < \frac{1}{2}$, then we have $L_t^a \in \mathcal{D}_p^r$ and $L_t^a \in \tilde{\mathcal{D}}_p^r$.*

Proof. By Theorem 6.1, it suffices to prove that $L_t^a \in \mathcal{D}_p^r$ for any $r < \frac{1}{2}$ and $p \geqslant 2$.

In the following, without loss of generality, we may assume that $a = 0$ and $p = 2^k$ for $k \in \mathbb{N}$. Let us first construct some approximating functions. Set for $\epsilon > 0$

$$
F_\epsilon(x) := \begin{cases} 1, & x > \epsilon, \\ \frac{1}{2\epsilon}(x + \epsilon), & |x| \leqslant \epsilon, \\ 0, & x < -\epsilon \end{cases}
$$

and $f_\epsilon(x) := \int_{-\infty}^x F_\epsilon(y)\mathrm{d}y$. Then $F_\epsilon(x) \to 1_{(0,+\infty)}(x) + \frac{1}{2} 1_{\{0\}}(x)$ and $f_\epsilon(x) \to f(x) := x^+$ as $\epsilon \downarrow 0$. Since $f(x) = x^+$ is a Lipschitz function, it is not hard to see that for fixed $t \in [0,1]$

$$
\sup_{x \in \mathbb{R}} \mathbb{E}|L_t^x|^p < +\infty \tag{7.2}
$$

and $I_1(t) \in \mathcal{D}_p^1$. For $I_4(t)$, define

$$
\begin{aligned}
I_4^\epsilon(t) &:= \int_0^{t+} \int_X \left[f_\epsilon(U_{s-} + g(s,x)) - f_\epsilon(U_{s-})\right] \gamma(\mathrm{d}s, \mathrm{d}x) \\
&= \int_0^{t+} \int_X g(s,x) \left(\int_0^1 F_\epsilon(U_{s-} + \theta g(s,x)) \mathrm{d}\theta \right) \gamma(\mathrm{d}s, \mathrm{d}x).
\end{aligned}
$$

By Remark 4.7 and (7.1), we clearly have

$$
\sup_\epsilon \|I_4^\epsilon(t)\|_{\mathcal{D}_p^1} < +\infty.
$$

Moreover, by the elementary inequality

$$
|x^+ - (x-y)^+ - f_\epsilon(x) - f_\epsilon(x-y)| \leqslant 2\epsilon|y|,
$$

we may deduce that

$$
\|I_4^\epsilon(t) - I_4(t)\|_{L^p} \leqslant C\epsilon.
$$

Hence, $I_4(t) \in \mathcal{D}_p^1$.

Set

$$
I_2^\epsilon(t) := \int_0^t F_\epsilon(U_s) M_s \mathrm{d}W_s.
$$

Then

$$\begin{aligned}\nabla^W I_2^\epsilon(t) &= \int_0^\cdot F_\epsilon(U_s)M_s\mathrm{d}s + \int_0^t \nabla^W F_\epsilon(U_s)M_s\mathrm{d}s + \int_0^t F_\epsilon(U_s)\nabla^W M_s\mathrm{d}W_s \\ &= I_{21}(t) + I_{22}(t) + I_{23}(t),\end{aligned}$$

and

$$\begin{aligned}\nabla^\Gamma I_2^\epsilon(t) &= \int_0^t \nabla^\Gamma F_\epsilon(U_s)M_s\mathrm{d}W_s + \int_0^t F_\epsilon(U_s)\nabla^\Gamma M_s\mathrm{d}W_s \\ &= I_{24}(t) + I_{25}(t).\end{aligned}$$

By (**H**2), we obviously have

$$\mathbb{E}\|I_{21}(t)\|_{\mathbb{H}}^p + \mathbb{E}\|I_{23}(t)\|_{\mathbb{H}}^p + \mathbb{E}\left(\|I_{25}(t)\|_{H(\cdot)}^p\right) \leqslant C.$$

By Hölder's inequality, we have for any $q > 1$

$$\begin{aligned}\mathbb{E}\|I_{22}(t)\|_{\mathbb{H}}^p &\leqslant C\mathbb{E}\left(\int_0^t \|\nabla^W F_\epsilon(U_s)\cdot M_s\|_{\mathbb{H}}^2\mathrm{d}s\right)^{\frac{p}{2}} \\ &\leqslant C\epsilon^{-p}\mathbb{E}\left(\int_0^t 1_{(-\epsilon,\epsilon)}(U_s)|M_s|^{\frac{2}{q}}\|\nabla^W U_s\|_{\mathbb{H}}^2|M_s|^{2(1-\frac{1}{q})}\mathrm{d}s\right)^{\frac{p}{2}} \\ &\leqslant C\epsilon^{-p}\mathbb{E}\left\{\left(\int_0^t 1_{(-\epsilon,\epsilon)}(U_s)|M_s|^2\mathrm{d}s\right)^{\frac{p}{2q}}\left(\int_0^t \|\nabla^W U_s\|_{\mathbb{H}}^{\frac{2q}{q-1}}|M_s|^2\mathrm{d}s\right)^{\frac{p}{2}(1-\frac{1}{q})}\right\} \\ &\leqslant C\epsilon^{-p}\left(\mathbb{E}\left(\int_0^t 1_{(-\epsilon,\epsilon)}(U_s)\mathrm{d}\langle X\rangle_s\right)^{\frac{p}{q}}\right)^{\frac{1}{2}} \Big[\text{ by the occupation time formula}\Big] \\ &\quad\cdot\left(\mathbb{E}\left(\int_0^t \|\nabla^W U_s\|_{\mathbb{H}}^{\frac{2q}{q-1}}|M_s|^2\mathrm{d}s\right)^{p(1-\frac{1}{q})}\right)^{\frac{1}{2}} \\ &\leqslant C\epsilon^{-p}\left(\mathbb{E}\left(\int_{-\epsilon}^\epsilon L_t^0\mathrm{d}x\right)^{\frac{p}{q}}\right)^{\frac{1}{2}} \quad \Big[\text{ by (7.1)}\Big] \\ &\leqslant C\epsilon^{-p}\left(\epsilon^{p-1}\int_{-\epsilon}^\epsilon \mathbb{E}|L_t^0|^p\mathrm{d}x\right)^{\frac{1}{2q}} \quad \Big[\text{ by (7.2)}\Big] \\ &\leqslant C\epsilon^{p(-1+\frac{1}{2q})}.\end{aligned}$$

Similar arguments lead to

$$\mathbb{E}\left(|I_{24}(t)|_{H(\cdot)}^p\right) \leqslant C\epsilon^{p(-1+\frac{1}{2q})}.$$

Hence

$$\|I_2^\epsilon(t)\|_{\mathcal{D}_p^1} \leqslant C\cdot\epsilon^{-1+\frac{1}{2q}}.$$

In the same way, let

$$I_3^\epsilon(t) := \int_0^t F_\epsilon(U_s)N_s\mathrm{d}s.$$

Then by (**H**3) we also have

$$\|I_3^\epsilon(t)\|_{\mathcal{D}_p^1} \leqslant C\cdot\epsilon^{-1+\frac{1}{2q}}.$$

Now we deal with I_5. Let

$$I_5^\epsilon(t) := \int_0^t \int_X g(s,x) F_\epsilon(U_s) \nu(\mathrm{d}s, \mathrm{d}x).$$

We have

$$\begin{aligned} \nabla^W I_5^\epsilon(t) &= \int_0^t \int_X \nabla^W g(s,x) F_\epsilon(U_s) \nu(\mathrm{d}s, \mathrm{d}x) \\ &\quad + \int_0^t \int_X g(s,x) \nabla^W F_\epsilon(U_s) \nu(\mathrm{d}s, \mathrm{d}x) \\ &= I_{51}(t) + I_{52}(t). \end{aligned}$$

By (**H**2), we have

$$\mathbb{E}\|I_{51}(t)\|^p \leqslant \mathbb{E}\left(\int_0^t \int_X \|\nabla^W g(s,x) F_\epsilon(U_s)\|_{\mathbb{H}} \nu(\mathrm{d}s, \mathrm{d}x) \right)^p \leqslant C.$$

By (**H**3), as in the estimate for $I_{22}(t)$, we have

$$\begin{aligned} \mathbb{E}\|I_{52}(t)\|^p &\leqslant \mathbb{E}\left(\int_0^t \int_X \|g(s,x) \nabla^W F_\epsilon(U_s)\|_{\mathbb{H}} m(\mathrm{d}x)\mathrm{d}s \right)^p \\ &\leqslant \mathbb{E}\left(\int_0^t |\xi_s||M_s| \cdot \|\nabla^W F_\epsilon(U_s)\|_{\mathbb{H}} \mathrm{d}s \right)^p \leqslant C\epsilon^{-p/2}, \end{aligned}$$

consequently,

$$\mathbb{E}\|\nabla^W I_5^\epsilon(t)\|_{\mathbb{H}}^p \leqslant \epsilon^{-p/2}.$$

In the same way

$$\mathbb{E}\left(\|\nabla^\Gamma I_5^\epsilon(t)\|_{H(\cdot)}^p \right) \leqslant \epsilon^{-p/2},$$

hence

$$\|\nabla I_5^\epsilon(t)\|_{\mathcal{D}_p^1} \leqslant C \cdot \epsilon^{-\frac{1}{2}}.$$

Moreover, obviously

$$\mathbb{E}\left(\|I_2^\epsilon(t) - I_2(t)\|^p + \|I_3^\epsilon(t) - I_3(t)\|^p + \|I_5^\epsilon(t) - I_5(t)\|^p \right) \leqslant C\epsilon^{p/2}.$$

By the definition of $\mathcal{D}_p^r$, the result now follows. □

As a direct consequence of Theorem 6.3 and Theorem 7.1, we have

Theorem 7.2. *For any $t \in [0,1], a \in \mathbb{R}$, L_t^a admits an $\mathcal{E}$-quasi continuous modification.*

Lastly, we study the regularity of L_t^a in t, a. Let $C([0,1])$ denote the Banach space of continuous functions defined on $[0,1]$ and taking values in $\mathbb{R}$, endowed with the topology of uniform convergence. Assume that
(**H**4) For any $p \geqslant 2$,

$$\mathbb{E}\left| \int_0^1 \int_X |g(s,x)|^{1/2} \gamma(\mathrm{d}s, \mathrm{d}x) \right|^p < +\infty.$$

We have

Theorem 7.3. *The local time L_t^a satisfies that for any $p \geqslant 2$*

$$\mathbb{E}\left(\sup_{t\in[0,1]} |L_t^a - L_t^b|^p\right) \leqslant C|a-b|^{p/2}.$$

In particular, the process $a \mapsto L_\cdot^a \in C([0,1])$ admits a continuous modification.

Proof. Let us only deal with $I_2(t)$ and $I_4(t)$, the others being analogous. Assume that $b > a$. For $I_2(t)$, we have by (7.2) and Hölder's inequality

$$\begin{aligned}
\mathbb{E}\left(\sup_{t\in[0,1]}\left|\int_0^t 1_{(a,b]}(U_s)M_s\mathrm{d}W_s\right|^p\right) &\leqslant \mathbb{E}\left(\int_0^1 1_{(a,b]}(U_s)|M_s|^2\mathrm{d}s\right)^{p/2}\\
&= \mathbb{E}\left(\int_{\mathbb{R}} 1_{(a,b]}(x)L_1^x\mathrm{d}x\right)^{p/2}\\
&\leqslant C|b-a|^{p/2}.
\end{aligned}$$

By the elementary inequality,

$$\begin{aligned}
&(x-a)^+ - (y-a)^+ - (x-b)^+ + (y-b)^+\\
\leqslant\ & |b-a| \wedge |x-y| \leqslant |b-a|^{1/2}\cdot|x-y|^{1/2}.
\end{aligned}$$

we have for $I_4(t)$

$$\begin{aligned}
&\mathbb{E}\Bigg(\sup_{t\in[0,1]}\Bigg|\int_0^{t+}\int_X [(U_{s-}+g(s,x)-a)^+ - (U_{s-}-a)^+\\
&\qquad -(U_{s-}+g(s,x)-b)^+ + (U_{s-}-b)^+]\gamma(\mathrm{d}s,\mathrm{d}x)\Bigg|^p\Bigg)\\
\leqslant\ & (b-a)^{p/2}\mathbb{E}\left|\int_0^{1+}\int_X |g(s,x)|^{1/2}\gamma(\mathrm{d}s,\mathrm{d}x)\right|^p\\
\leqslant\ & C(b-a)^{p/2}.
\end{aligned}$$

So, the first assertion is proved. The second follows from this by Kolmogorov's criterion for Banach space valued processes, see e.g. [20]. □

Using [19] and arguments similar to [11, Theorem 2.25], we have finally

Theorem 7.4. *There is a set $A \subset \Omega$ such that*

(i) $\mathrm{Cap}_{2,1}(A) = 0$,

(ii) the occupation time formula

$$\int_0^t \phi(U_s(\omega))\mathrm{d}\langle X\rangle_s(\omega) = \int_{\mathbb{R}} \phi(x)L_t^a(\omega)\mathrm{d}x$$

holds for every $\omega \in A^c$, every $t \in [0,1]$ and every positive Borel function ϕ.

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