Convexity of limits of harmonic measures

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Preliminary version

Abstract

It is shown that, given a point $x \in \mathbb{R}^d$, $d \geq 2$, and open sets $U_1, \dots, U_k$ in $\mathbb{R}^d$ containing x, any convex combination of the harmonic measures $\varepsilon_x^{U_n^c}$ for x with respect to U_n, $1 \leq n \leq k$, is the limit of a sequence $(\varepsilon_x^{W_m^c})_{m \in \mathbb{N}}$ of harmonic measures, where each W_m is an open subset of $U_1 \cup \dots \cup U_k$ containing x. This answers a question raised in connection with Jensen measures.

More generally, we prove that, for arbitrary measures on an open set W, the set of extremal representing measures, with respect to the cone of continuous potentials on W or with respect to the cone of continuous functions on $\overline{W}$ which are superharmonic W, is dense in the compact convex set of all representing measures.

This is achieved approximating balayage on open sets by balayage on unions of balls which are pairwise disjoint and very small with respect to their mutual distances and then shrinking these balls in a suitable manner.

The results are presented simultaneously for the classical case and for the theory of Riesz potentials.

Finally, a characterization of all Jensen measures and of all extremal Jensen measures is given.

Keywords: Harmonic measure, Jensen measure, extremal measure, balayage, Riesz potentials, Brownian motion, stable process, Skorokhod stopping

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1 Introduction and main results

The principal motivation for this paper is the following natural question from classical potential theory which has been raised explicitly in [3] in connection with Jensen measures: Given an open subset Ω in $\mathbb{R}^d$, $d \geq 2$, and a point $x \in \Omega$, is the set of limits of harmonic measures $\varepsilon_x^{U^c}$ convex (where the sets U are supposed to be relatively compact open neighborhoods of x in Ω)? We shall see that the answer is "yes". In fact, we shall prove that even for general measures ν instead of ε_x the extremal

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representing measures are dense in the compact convex set of all representing measures (see Corollary 1.3 and Corollary 1.7).

Our method of sweeping on families of disjoint balls, which are very small with respect to their mutual distances, works as well for the theory of Riesz potentials related to the fractional Laplacian $-(-\Delta)^{\alpha/2}$ on $\mathbb{R}^d$, $0 < \alpha < \min\{2, d\}$. Therefore we shall also cover the case of Riesz potentials from the very beginning. We recall that classical potential theory of the Laplacian is the limiting case $\alpha = 2$. The reader, who is interested in the classical case only, may neglect this generality. He will hardly notice any difference in the presentation except from the additional discussion of the "Poisson kernel" for a ball with respect to Riesz potentials (which has a density with respect to Lebesgue measure on the complement of the ball).

So we shall deal simultaneously with the following two situations:

- Classical case: $\alpha = 2$, X is a non-empty open set in $\mathbb{R}^d$, $d \geq 2$, such that $\mathbb{R}^d \setminus X$ is non-polar, if $d = 2$.
- Riesz potentials: $\alpha < 2$, X is a non-empty open set in $\mathbb{R}^d$, $d \geq 1$, $d > \alpha$.

Given $Y \subset X$, $Y^c := X \setminus Y$ will denote the complement of Y *with respect to* X.

Let $\mathcal{M}(X)$ be the set of all (Radon) measures on X, let $\mathcal{P}(X)$ denote the convex cone of all continuous real potentials on X, and $\mathcal{M}(\mathcal{P}(X))$ be the set of all $\nu \in \mathcal{M}(X)$ such that $\mu(p) < \infty$ for *some* strictly positive $p \in \mathcal{P}(X)$. Let us note that every finite measure on X and hence every $\nu \in \mathcal{M}(X)$ with compact support is contained in $\mathcal{M}(\mathcal{P}(X))$. For every $\nu \in \mathcal{M}(\mathcal{P}(X))$ and for every subset A of X, let ν^A denote the balayage of ν on A *with respect to* X, that is, for every superharmonic function $u \geq 0$ on X,

$$\nu^A(u) := \int u \, d\nu^A = \int \hat{R}_u^A \, d\nu,$$

where $\hat{R}_u^A(x) = \liminf_{y \to x} R_u^A(y)$ and R_u^A is the infimum of all positive superharmonic functions v on X majorizing u on A. In particular,

$$\nu^A(u) \leq \nu(u) \tag{1.1}$$

for every superharmonic function $u \geq 0$ on X. For every $x \in X$, let ε_x denote the Dirac measure at x. It is easily seen that $\nu^A = \int \varepsilon_x^A \, d\nu(x)$. If A is closed and $x \in A^c$, then ε_x^A is the restriction of the harmonic measure for x and the open set $X \setminus A$ on A. Given $A \subset X$, there exists a Borel set (even a G_δ-set) $\tilde{A}$ containing A such that $\nu^{\tilde{A}} = \nu^A$ for every $\nu \in \mathcal{M}(\mathcal{P}(X))$. To discuss extremal representing measures for $\nu \in \mathcal{M}(\mathcal{P}(X))$ we shall also need reduced measures $\mathring{\nu}^A$ for Borel sets $A \subset X$. They are defined by

$$\int u \, d\mathring{\nu}^A = \int R_u^A \, d\nu,$$

$u \geq 0$ superharmonic on X, and related to ν^A by $\mathring{\nu}^A = \nu|_A + (\nu|_{A^c})^A$, since $R_p^A = p$ on A and $R_p^A = \hat{R}_p^A$ on A^c. If A is open or, more generally, if A is not thin at any of its points, then $\mathring{\nu}^A = \nu^A$ for every $\nu \in \mathcal{M}(\mathcal{P}(X))$. We refer to [2] for details.

Let $\mathcal{K}(X)$ denote the linear space of all continuous real functions on X with compact support. We recall that a sequence (μ_m) of Radon measures converges *weakly* to a Radon measure μ on X if $\lim_{m\to\infty} \mu_m(f) = \mu(f)$ for every $f \in \mathcal{K}(X)$. It is this convergence for Radon measures we shall use.

Let us fix a natural number $k \geq 2$ and define

$$\Lambda_k := \{\lambda \in [0,1]^k \colon \sum_{n=1}^k \lambda_n = 1\}.$$

Our fundamental result is the following.

THEOREM 1.1. *Let ν be a measure in $\mathcal{M}(\mathcal{P}(X))$ which is supported by an open subset W of X, let $U_1, \dots, U_k$ be open subsets of W, and $\lambda \in \Lambda_k$. Then there exist finite unions C_m, $m \in \mathbb{N}$, of pairwise disjoint closed balls in $U_1 \cup \dots \cup U_k$ such that*

$$\lim_{m\to\infty} \nu^{C_m \cup W^c} = \sum_{n=1}^k \lambda_n \nu^{U_n \cup W^c}.$$

The key to Theorem 1.1 is the following result concerning balayage on finite families of small balls where, given $\gamma \in [0,1]$ and a closed ball B with center x and radius r, the ball with center x and radius γr is denoted by B^γ (see Proposition 3.3 for a precise formulation). It will be applied to balayage on subsets A of W *with respect to* W in place of X to deal with balayage measures of the form $\nu^{A\cup W^c}$.

PROPOSITION 1.2. *Let $\delta > 0$ be small, let A be a union of finitely many pairwise disjoint closed balls $B_1, \dots, B_m$ in X which are sufficiently small with respect to their mutual distances and to the distance from $\mathbb{R}^d \setminus X$, and let $\nu \in \mathcal{M}(\mathcal{P}(X))$ such that $\nu(A) = 0$. Moreover, let $\lambda \in \Lambda_k$, let $I_1, \dots, I_k$ be a partition of $\{1, \dots, m\}$, and K_n be the union of the balls B_i, $i \in I_n$, $1 \leq n \leq k$.*

Then there exist $\gamma_1, \dots, \gamma_m \in [0,1]$ such that $C := B_1^{\gamma_1} \cup \dots \cup B_m^{\gamma_m}$ satisfies

$$\nu^C(B_i) = (1-\delta) \sum_{n=1}^k \lambda_n \nu^{K_n}(B_i) \qquad \textit{for every } 1 \leq i \leq m.$$

Given two measures μ, ν on X, we shall write $\mu \prec \nu$ provided $\mu(p) \leq \nu(p)$ for every $p \in \mathcal{P}(X)$. For every $\nu \in \mathcal{M}(\mathcal{P}(X))$, let $\mathcal{M}_\nu(\mathcal{P}(X))$ be the set of all measures μ on X such that $\mu(p) \leq \nu(p)$ for every $p \in \mathcal{P}$, that is,

$$\mathcal{M}_\nu(\mathcal{P}(X)) = \{\mu \in \mathcal{M}(\mathcal{P}(X)) \colon \mu \prec \nu\}.$$

$\mathcal{M}_\nu(\mathcal{P}(X))$ is a compact convex set with respect to weak convergence and its set of extreme points is given by

$$\big(\mathcal{M}_\nu(\mathcal{P}(X))\big)_e = \{\mathring{\nu}^A \colon A \text{ Borel subset of } X\} \tag{1.2}$$

(see [9] and [2, VI.12.5]). Moreover, the subset of all ν^U, U open in X, as well as the subset of all $\mathring{\nu}^C$, C compact subset of X, is dense in $\big(\mathcal{M}_\nu(\mathcal{P}(X))\big)_e$ (see [2, VI.1.9]). Therefore, by the theorem of Krein-Milman and taking $W = X$, Theorem 1.1 yields the following.

COROLLARY 1.3. *For every $\nu \in \mathcal{M}(\mathcal{P}(X))$, $\big(\mathcal{M}_\nu(\mathcal{P}(X))\big)_e$ is dense in $\mathcal{M}_\nu(\mathcal{P}(X))$.*

REMARK 1.4. *Let us note that Corollary 1.3 has the following consequence related to Skorokhod stopping* (see [11, 4, 6, 5, 1]). *Let ν be a probability measure on X and let $(X(t))$ be Brownian motion or an α-stable process on X with initial distribution ν. Then, for every measure $\mu \prec \nu$, there exists a sequence (T_m) of hitting times at relatively compact open subsets U_m of X such that the distributions $P^\nu_{X(T_m)}$ converge weakly to μ as $m \to \infty$.*

Next let us consider open subsets $U_1, \dots, U_k$ of X and let ν be a measure in $\mathcal{M}(\mathcal{P}(X))$ which is supported by $W := U_1 \cup \cdots \cup U_k$. Then, for every $1 \le n \le k$, there exist open $(1/m)$-neighborhoods $\tilde U_{nm}$ of U_n^c in X such that $(\nu^{\tilde U_{nm}})$ converges weakly to $\overset{\circ}{\nu}^{U_n^c}$ as $m \to \infty$ (see [2, VI.1.9]). For all n and m, $\tilde U_{nm} = (\tilde U_{nm} \cap W) \cup W^c$ and $\tilde U_{nm} \cap W$ is an open subset of W. Therefore Theorem 1.1 implies as well the following.

COROLLARY 1.5. *Let $U_1, \dots, U_k$ be open subsets of X, ν be a measure in $\mathcal{M}(\mathcal{P}(X))$ which is supported by $W := U_1 \cup \cdots \cup U_k$, and $\lambda \in \Lambda_k$. Then there exist finite unions C_m of pairwise disjoint closed balls in a $(1/m)$-neighborhood of $W \setminus (U_1 \cap \cdots \cap U_k)$ in W such that*

$$\lim_{m\to\infty} \overset{\circ}{\nu}^{W^c \cup C_m} = \sum_{n=1}^{k} \lambda_n \overset{\circ}{\nu}^{U_n^c}. \tag{1.3}$$[1]

If ν is supported by $U_1 \cap \cdots \cap U_k$, then the reduced measures may be replaced by balayage measures.

Given an open subset W of X, let $S(W)$, $H(W)$ denote the set of all continuous functions on X which are $\mathcal{P}(X)$-bounded (that is, bounded in modulus by some $p \in \mathcal{P}(X)$) and superharmonic on W, harmonic on W, respectively. As for $\mathcal{P}(X)$, we have sets of representing measures $\mathcal{M}_\nu(S(W))$ and $\mathcal{M}_\nu(H(W))$. Since semipolar sets are polar and points are polar for both the classical case and for Riesz potentials, we see from [2, VI.9.5] that the following holds for Dirac measures $\nu = \varepsilon_x$ (as customary, we write $\mathcal{M}_x$ instead of $\mathcal{M}_{\varepsilon_x}$); the proof for the general case $\nu \in \mathcal{M}(\mathcal{P}(X))$ will be given in an Appendix.

THEOREM 1.6. *For every open subset W of X and for every $\nu \in \mathcal{M}(\mathcal{P}(X))$ which is supported by W,*

$$\begin{aligned}\mathcal{M}_\nu(S(W)) &= \mathcal{M}_\nu(\mathcal{P}(X)) \cap \mathcal{M}_\nu(H(W)) \\ &= \{\mu \in \mathcal{M}_\nu(\mathcal{P}(X)) \colon \mu^{W^c} = \nu^{W^c}\} = \{\mu \in \mathcal{M}(\mathcal{P}(X)) \colon \nu^{W^c} \prec \mu \prec \nu\}.\end{aligned}$$

Moreover, $\mathcal{M}_\nu(S(W))$ is a closed face of $\mathcal{M}_\nu(\mathcal{P}(X))$ and

$$\big(\mathcal{M}_\nu(S(W))\big)_e = \{\overset{\circ}{\nu}^A \colon W^c \subset A \subset X,\ A \text{ Borel set}\}.$$

[1] In the classical case and for ν supported by $U_1 \cap \cdots \cap U_k$, we may choose C_m in a $(1/m)$-neighborhood of $W \cap (\partial U_1 \cup \cdots \cup \partial U_k)$.

In particular, for every $x \in W$,

$$\big(\mathcal{M}_x(S(W))\big)_e = \{\varepsilon_x^A \colon W^c \subset A \subset X\}.$$

Let us note that, taking $W = X$, we have $W^c = \emptyset$, $H(W) = \{0\}$, and $S(W)$ is $\mathcal{P}(X)$. Since the measures $\nu^{U \cup W^c}$, U open subset of W, are dense in $\big(\mathcal{M}_\nu(S(W))\big)_e$, Theorem 1.1 also yields the following.

COROLLARY 1.7. *Let W be an open subset of X and let ν be a measure in $\mathcal{M}(\mathcal{P}(X))$ which is supported by W. Then $\big(\mathcal{M}_\nu(S(W))\big)_e$ is dense in $\mathcal{M}_\nu(S(W))$.*

Finally, restricting our attention to classical potential theory, let us see how Jensen measures, introduced in function theory in [1] and extensively studied in [3] and [10], fit into our considerations. To that end we fix an open subset Ω of $\mathbb{R}^d$, $d \geq 2$, and a point $x \in \Omega$. A Jensen measure for x with respect to Ω is probability measure μ supported on a compact subset of Ω such that $\int u\, d\mu \leq u(x)$ for every superharmonic function u on Ω. Equivalently, since constants are harmonic and superharmonic functions are increasing limits of continuous superharmonic functions, the set $J_x(\Omega)$ of all Jensen measures for x with respect to Ω is the set of all Radon measures with compact support in Ω such that $\int u\, d\mu \leq u(x)$ for every continuous superharmonic function u on Ω. Clearly, Corollary 1.5 implies by [3, p. 32] that, for every $x \in \Omega$, the set of all harmonic measures $\varepsilon_x^{U^c}$, U open, $x \in U$, $\overline{U}$ compact in Ω, is dense in $J_x(\Omega)$ with respect to the weak*-topology on $\mathcal{C}(\Omega)^*$, that is, Question 1.6 in [3] has a positive answer.

Further, we shall give the following characterization for Jensen measures and extremal Jensen measures.

THEOREM 1.8. *Let Ω be an open subset of $\mathbb{R}^d$, $d \geq 2$, and $x \in \Omega$. Then*

$$J_x(\Omega) = \bigcup \big\{\mathcal{M}_x(S(W)) \colon W \text{ open},\ x \in W,\ \overline{W} \text{ compact in } \Omega\big\}$$

and

$$\begin{aligned}\big(J_x(\Omega)\big)_e &= \bigcup \big\{\big(\mathcal{M}_x(S(W))\big)_e \colon W \text{ open},\ x \in W,\ \overline{W} \text{ compact in } \Omega\big\} \\ &= \big\{\varepsilon_x^{A^c} \colon \overline{A} \text{ compact in } \Omega\big\}.\end{aligned}$$

The paper is organized as follows. In the next section we shall approximate balayage replacing open sets U by unions of small balls contained in U. In Section 3 we shall prove Proposition 1.2. Section 4 will consist of the proof for Theorem 1.1, and in Section 5 we shall establish the results on Jensen measures. The paper is finished by an Appendix where Theorem 1.6 is proven for general $\nu \in \mathcal{M}(\mathcal{P}(X))$.

2 Approximation of $\nu^{U \cup W^c}$

Balayage on open sets can be approximated by balayage on subsets consisting of finitely many balls having radii which are *arbitrarily* small with respect to their mutual distances (see Proposition 2.1). Since this does not seem to be widely known, we include a complete proof.

For every $x \in \mathbb{R}^d$ and $r \geq 0$, let $B(x,r)$ denote the *closed* ball having center x and radius r. Given $x_0 \in \mathbb{R}^d$, $a \in (0,1)$, and $m \in \mathbb{N}$, let

$$\mathcal{Z}_m(x_0,a) := \{B(z,\frac{a}{m})\colon z \in \frac{1}{m}(x_0+\mathbb{Z}^d)\}, \qquad Z_m(x_0,a) := \bigcup_{B\in\mathcal{Z}_m(x_0,a)} B.$$

PROPOSITION 2.1. *Let U,W be open sets, $U \subset W \subset X$, $\nu \in \mathcal{M}(\mathcal{P}(X))$, $x_0 \in \mathbb{R}^d$, and $a \in (0,1)$. For every $m \in \mathbb{N}$, let A_m be the (finite) union of all balls $B \in \mathcal{Z}_m(x_0,a)$ such that $B \subset U \cap B(0,m)$ and* $\operatorname{dist}(B, \mathbb{R}^d \setminus U) \geq 1/m$. *Then*

$$\lim_{m\to\infty} \nu^{A_m\cup W^c}(q) = \nu^{U\cup W^c}(q) \qquad \textit{for every } q \in \mathcal{P}(X). \tag{2.1}$$

For the proof we shall need the following lemma.

Lemma 2.2. *Let $a \in (0,1)$, $x_0 \in \mathbb{R}^d$, $r > 0$, and $\delta > 0$. Then there exists $m_0 \in \mathbb{N}$ such that, for all $m \geq m_0$ and $x \in X$ satisfying $B(x,r) \subset X$,*

$$R_1^{Z_m(x_0,a)\cap B(x,r)}(x) > 1-\delta. \tag{2.2}$$

Proof. 1. Let us first suppose that $d > \alpha$. For every subset A of $\mathbb{R}^d$, we define

$$u_A := \inf\{u\colon u \geq 0 \text{ superharmonic on } \mathbb{R}^d, u \geq 1 \text{ on } A\}$$

(if A is bounded, then $\hat{u}_A$ is the equilibrium potential of A). Let $Z := Z_1(0,a)$. Obviously,

$$\inf u_Z(\mathbb{R}^d) = \inf u_Z([0,1]^d),$$

that is, the continuous superharmonic function u_Z admits a minimum. Therefore u_Z is constant. Since $u_Z = 1$ on Z, we see that u_Z is identically 1. Consequently, the sequence $(u_{Z\cap B(0,k)})_{k\in\mathbb{N}}$ increases to 1 locally uniformly on $\mathbb{R}^d$ as $k \uparrow \infty$. Given $\delta > 0$, we hence may choose $k \in \mathbb{N}$ such that

$$u_{Z\cap B(0,k)} > 1-\frac{\delta}{2} \quad \text{on } [0,1]^d. \tag{2.3}$$

There exists $K \geq k$ with $u_{Z\cap B(0,k)} < \delta/2$ on $\mathbb{R}^d \setminus B(0,K)$. Then the function $v := (u_{Z\cap B(0,k)} - \delta/2)^+$ is subharmonic on $\mathbb{R}^d \setminus Z$ and vanishes outside $B(0,K)$.

Now let $m > (K+1)/\min\{r,1\}$. There exists $z \in \frac{1}{m}(x_0+\mathbb{Z}^d)$ with $x-z \in [0,\frac{1}{m}]^d$. We define

$$w(y) := v(m(y-z)) \qquad (y \in \mathbb{R}^d).$$

Then $w \leq 1$, w is subharmonic on $\mathbb{R}^d \setminus Z_m(x_0,a)$, and $w = 0$ on $\mathbb{R}^d \setminus B(z,K/m)$. Since $B(z,K/m) \subset B(x,r) \subset X$, we see that the restriction of w on X vanishes outside $B(x,r)$ and hence $w|_X \leq R_1^{Z_m(x_0,a)\cap B(x,r)}$. Since $m(x-z) \in [0,1]^d$, we know by (2.3) that $w(x) \geq 1-\delta$. Thus (2.2) holds.

2. Let us finally consider the remaining case $\alpha = d = 2$ (classical case in the plane) and let $\sqrt{2}/m \le r/2$. There exists $z \in Z_m(x_0, a)$ such that $|x - z| < \sqrt{2}/m$. Then $B(z, r/2) \subset B(x, r)$. We define

$$p_m(y) := \min\{1, \frac{\ln(r/2) - \ln|y - z|}{\ln(r/2) + \ln m - \ln a}\} \qquad (y \in B(x, r))$$

Moreover, $p_m = 1$ on $B(z, a/m)$, $p_m \le 0$ on $\partial B(x, r)$, and p_m is harmonic on the open set $\overset{\circ}{B}(x, r) \setminus B(z, a/m)$. Therefore

$$R_1^{Z_m(x_0,a)\cap B(x,r)} \ge p_m \quad \text{on } B(x, r).$$

Since

$$\frac{\ln(r/2) - \ln|x - z|}{\ln(r/2) + \ln m - \ln a} \ge \frac{\ln(r/2) + \ln m - \ln\sqrt{2}}{\ln(r/2) + \ln m - \ln a},$$

we see that $p_m(x) > 1 - \delta$, if m is sufficiently large. This finishes the proof.

□

Proof of Proposition 2.1. Let $q \in \mathcal{P}(X)$, $\delta > 0$, and $p := q + \delta p_0$, where $p_0 \in \mathcal{P}(X)$ such that $p_0 > 0$ and $\nu(p_0) \le 1$. We choose an arbitrary sequence (K_n) of compact sets which is increasing to U. For the moment, let us fix $n \in \mathbb{N}$. There exists $0 < r < \frac{1}{2}\mathrm{dist}(K_n, \mathbb{R}^d \setminus U)$ such that, for every $x \in K_n$, $p > q(x)$ on $B(x, r)$. If $m \in \mathbb{N}$ such that $m \ge 1/r$ and $K_n \subset B(0, m)$, then $Z_m(x_0, a) \cap B(x, r)$ is a subset of A_m and hence, for every $x \in K_n$,

$$R_p^{A_m}(x) \ge R_{q(x)}^{Z_m(x_0,a)\cap B(x,r)}(x) = q(x) R_1^{Z_m(x_0,a)\cap B(x,r)}(x).$$

Using Lemma 2.2 we hence obtain $m_n \in \mathbb{N}$ such that, for every $m \ge m_n$,

$$R_p^{A_m} > (1 - \delta) q \qquad \text{on } K_n.$$

By the definition of reduced functions, this implies that, for every $m \ge m_n$,

$$R_q^{U\cup W^c} + \delta p_0 \ge R_p^{U\cup W^c} \ge R_p^{A_m\cup W^c} \ge (1 - \delta) R_q^{K_n\cup W^c}$$

and therefore

$$\hat{R}_q^{U\cup W^c} + \delta p_0 \ge \hat{R}_p^{A_m\cup W^c} \ge (1 - \delta) \hat{R}_q^{K_n\cup W^c}. \tag{2.4}$$

If $n \uparrow \infty$, then $\hat{R}_q^{K_n\cup W^c} \uparrow \hat{R}_q^{U\cup W^c}$ whence

$$\nu^{K_n\cup W^c}(q) = \int \hat{R}_q^{K_n\cup W^c}\, d\nu \uparrow \int R_q^{U\cup W^c}\, d\nu = \nu^{U\cup W^c}(q). \tag{2.5}$$

Since $\nu(p_0) \le 1$, (2.4) and (2.5) imply that (2.1) holds. □

3 Joint shrinking of disjoint small balls

The following simple facts on iterated balayage will be used again and again. If $\nu \in \mathcal{M}(\mathcal{P}(X))$ and $A, \tilde{A}$ are closed sets such that $\tilde{A} \subset A \subset X$ and $\nu(A) = 0$, then,

$$\nu^{\tilde{A}} = \nu^A|_{\tilde{A}} + (\nu^A|_{A \setminus \tilde{A}})^{\tilde{A}} \geq \nu^A|_{\tilde{A}} \tag{3.1}$$

(see [2, VI.9.4] for a far more general statement).

In terms of harmonic kernels H_V for open subsets of X, defined by $H_V(x, \cdot) = \varepsilon_x^{V^c}$ for $x \in V$ and $H_V(x, \cdot) = \varepsilon_x$ for $x \in V^c$, this can be expressed by

$$H_U H_{\tilde{U}} = H_U,$$

whenever $U, \tilde{U}$ are open subsets of X satisfying $U \subset \tilde{U}$. In the classical case this is equivalent to the following property of the generalized Dirichlet solution, that is, the Perron-Wiener-Brelot solution of the Dirichlet problem. If f is a continuous $\mathcal{P}(X)$-bounded function on the boundary $\partial\tilde{U}$ of $\tilde{U}$ in X, then the generalized Dirichlet solution $\tilde{h}$ for $\tilde{U}$ and f coincides on U with the generalized Dirichlet solution h for U and the boundary function g, where $g = f$ on $\partial U \cap \partial\tilde{U}$ and $g = \tilde{h}$ on $\partial U \cap U$ (see [8, Lemma 8.39]).

Moreover, for every $\nu \in \mathcal{M}(\mathcal{P}(X))$ and for all closed sets A, B in X,

$$(\nu^A)^B(B) \leq \nu^B(B). \tag{3.2}$$

Indeed, it suffices to notice that both measures $(\nu^A)^B$ and ν^B are supported by B and that $(\nu^A)^B(1) = \nu^A(\hat{R}_1^B) \leq \nu(\hat{R}_1^B) = \nu^B(1)$ by (1.1).

We recall that, for every $\eta \in [0, 1]$ and every closed ball B in $\mathbb{R}^d$ having center x_B and radius r_B, we denote the ball obtained by shrinking B with the factor η by B^η, that is,

$$B^\eta := x_B + \eta(B - x_B).$$

Lemma 3.1. *Let A be the union of finitely many closed balls $B_1, \dots, B_m$ which are contained in X and pairwise disjoint. For every $t \in [0,1]^m$, let $A_t \subset A$ denote the union of the balls $B_i^{t_i}$, $1 \leq i \leq m$. Moreover, let $\nu \in \mathcal{M}(\mathcal{P}(X))$ such that $\nu(A) = 0$, $\gamma_1, \dots, \gamma_m \in \mathbb{R}_+$, and*

$$\Gamma := \{t \in [0,1]^m \colon \nu^{A_t}(B_i) \leq \gamma_i,\ 1 \leq i \leq m\}.$$

Then there exists $s \in \Gamma$ such that $s \geq t$ for every $t \in \Gamma$. Moreover, $\nu^{A_s}(B_i) = \gamma_i$ for every $i \in \{1, \dots, m\}$ such that $s_i < 1$.

Proof. Let us note first that $\nu^{A_t}(B_i) = \nu^{A_t}(B_i^{t_i})$ for every $t \in \Gamma$ and for every $1 \leq i \leq m$, since ν^{A_t} is supported by the subset A_t of A.

0. Of course, $(0, \dots, 0) \in \Gamma$, since points are polar.

1. If $t, \tilde{t} \in \Gamma$, then $t \vee \tilde{t} \in \Gamma$. Indeed, let us fix $1 \leq i \leq m$. We may assume without loss of generality that $t_i \geq \tilde{t}_i$. Since $A_t \subset A_{t \vee \tilde{t}}$, we conclude by (3.1) that

$$\nu^{A_{t\vee\tilde{t}}}(B_i^{t_i \vee \tilde{t}_i}) = \nu^{A_{t\vee\tilde{t}}}(B_i^{t_i}) \leq \nu^{A_t}(B_i^{t_i}) \leq \gamma_i.$$

2. For every $t < (1, \dots, 1)$, the set A_t is the intersection of all $A_{\tilde t}$, $\tilde t > t$. For every $t > (0, \dots, 0)$ the set A_t is the fine closure of the union of all $A_{\tilde t}$, $\tilde t < t$. This implies that, for every $p \in \mathcal{P}(X)$, the mapping $t \mapsto \nu^{A_t}(p)$ is continuous on $[0,1]^m$. Hence, for every $f \in \mathcal{K}(X)$, the mapping $t \mapsto \nu^{A_t}(f)$ is continuous. Since the closed balls $B_1, \dots, B_m$ are disjoint, we obtain that the mapping

$$t \mapsto \big(\nu^{A_t}(B_1), \dots, \nu^{A_t}(B_m)\big)$$

from $[0,1]^m$ into $[0,1]^m$ is continuous. Therefore Γ is closed.

3. Combining (1) and (2) we see that

$$s := (\sup_{t \in \Gamma} t_1, \dots, \sup_{t \in \Gamma} t_m) \in \Gamma,$$

where of course $s \geq t$ for every $t \in \Gamma$. To finish the proof, let us consider $i \in \{1, \dots, m\}$ such that $s_i < 1$ and suppose that $\nu^{A_s}(B_i) < \gamma_i$. Let us define $\tilde s := (s_1, \dots, s_{i-1}, b, s_{i+1}, \dots, s_m)$, where $s_i < b \leq 1$. By (2), we may choose b in such a way that $\nu^{A_{\tilde s}}(B_i) < \gamma_i$. Since $A_s \subset A_{\tilde s}$, we obtain by (3.1) that $\nu^{A_{\tilde s}}(B_j^{s_j}) \leq \nu^{A_s}(B_j^{s_j}) \leq \gamma_j$ for every $j \in \{1, \dots, m\}$, $j \neq i$. Thus $\tilde s \in \Gamma$, $\tilde s \leq s$, $b = \tilde s_i \leq s_i$, a contradiction. □

Let us note the following simple consequence.

PROPOSITION 3.2. *Let A be the union of finitely many closed balls $B_1, \dots, B_m$ which are contained in X and pairwise disjoint. Moreover, let $\nu \in \mathcal{M}(\mathcal{P}(X))$ such that $\nu(A) = 0$ and let $\beta_1, \dots, \beta_m \in [0,1]$. Then there exist $s_1, \dots, s_m \in [0,1]$ such that the union $\tilde A$ of the shrinked balls $B_1^{s_1}, \dots, B_m^{s_m}$ satisfies*

$$\nu^{\tilde A}(B_i) = \beta_i \nu^A(B_i) \qquad \textit{for every } 1 \leq i \leq m.$$

Proof. It suffices to take $\gamma_i := \beta_i \nu^A(B_i)$, $1 \leq i \leq m$, and to choose $s = (s_1, \dots, s_m)$ in $[0,1]^m$ according to Lemma 3.1. Then $\nu^{\tilde A}(B_i) \leq \beta_i \nu^A(B_i)$ for all $1 \leq i \leq m$. Furthermore, equality holds whenever $s_i < 1$. If, however, $i \in \{1, \dots, m\}$ such that $s_i = 1$, then $\nu^{\tilde A}(B_i) \geq \nu^A(B_i)$ by (3.1) whence as well $\nu^{\tilde A}(B_i) \geq \beta_i \nu^A(B_i)$ (and $\beta_i = 1$ unless $\nu^A(B_i) = 0$). □

In the classical case $\alpha = 2$, the harmonic measure $\varepsilon_y^{U^c}$ for a ball $U = \overset{\circ}{B}(y_0, r)$ and $y \in U$ has the Poisson density

$$\rho_y^U(z) = r^{d-2}(r^2 - |y - y_0|^2)|y - z|^{-d}, \qquad |z - y_0| = r,$$

with respect to normalized surface measure on the boundary of U. For Riesz potentials (the case $0 < \alpha < 2$), $\varepsilon_y^{U^c}$ has a density ρ_y^U with respect to Lebesgue measure on U^c,

$$\rho_y^U(z) = a_\alpha \frac{(r^2 - |y - y_0|^2)^{\alpha/2}}{(|z - y_0|^2 - r^2)^{\alpha/2}} |z - y|^{-d}, \qquad |y - y_0| < r \leq |z - y_0|,$$

where a_α is a constant depending on d and α (see [2, p. 192]). If $y, \tilde{y} \in B(y_0, \eta r)$, $0 < \eta < 1$, then in both cases

$$\rho_y^U(z) / \rho_{\tilde{y}}^U(z) \leq \frac{1}{(1-\eta^2)^{\alpha/2}} \frac{(1+\eta)^d}{(1-\eta)^d} = \frac{(1+\eta)^{d-\frac{\alpha}{2}}}{(1-\eta)^{d+\frac{\alpha}{2}}}. \tag{3.3}$$

If η is small, then the expression on the right side of (3.3) is approximately $1 + 2d\eta$. So there exists $\delta_0 \in (0,1)$ such that

$$\varepsilon_{\tilde{y}}^{U^c} \leq (1+\delta)\varepsilon_y^{U^c}, \tag{3.4}$$

whenever $0 < \delta \leq \delta_0$ and $y, \tilde{y} \in B(y_0, \frac{\delta}{3d} r)$. For the moment, let us fix $\delta \in (0, \delta_0]$ and a closed subset A of X such that $B(y_0, r) \cap A = \emptyset$. We observe that, for every $y \in B(y_0, r)$, $\varepsilon_y^A = (\varepsilon_y^{U^c})^A$. Indeed, in the classical case this follows from (3.1), since then $\varepsilon_y^{U^c}(A) = 0$. In the general case, it follows from $(\varepsilon_y^A)^A = \varepsilon_y^A$ (see [2, VI.5.21]), since trivially $(\varepsilon_y^A)^A \prec (\varepsilon_y^{U^c})^A \prec \varepsilon_y^A$. Therefore (3.4) implies that

$$\varepsilon_{\tilde{y}}^A \leq (1+\delta)\varepsilon_y^A \qquad \text{for all } y, \tilde{y} \in B\big(y_0, \frac{\delta}{3d} r\big). \tag{3.5}$$

Let us say that finite family $\mathcal{B}$ of closed balls, which are contained in X and pairwise disjoint, is a *δ-family in X*, if $0 < \delta < \delta_0$ and the union A of all $B \in \mathcal{B}$ satisfies

$$r_B \leq \frac{\delta}{3d} \operatorname{dist}\big(x_B, (\mathbb{R}^d \setminus X) \cup (A \setminus B)\big) \qquad \text{for every } B \in \mathcal{B}. \tag{3.6}$$

Here is the key to Theorem 1.1. As already indicated, it will be applied to balayage on subsets A of W *with respect to* W in place of X to deal with balayage measures of the form $\nu^{A \cup W^c}$.

PROPOSITION 3.3. *Let A be the union of a δ-family $B_1, \dots, B_m$ in X and let $\nu \in \mathcal{M}(\mathcal{P}(X))$ such that $\nu(A) = 0$. Moreover, let $\lambda \in \Lambda_k$, let $I_1, \dots, I_k$ be a partition of $\{1, \dots, m\}$, and K_n be the union of the balls B_i, $i \in I_n$, $1 \leq n \leq k$.*

Then there exist $s_1, \dots, s_m \in [0,1]$ such that $C := B_1^{s_1} \cup \dots \cup B_m^{s_m}$ satisfies

$$\nu^C(B_i) = (1-\delta) \sum_{n=1}^{k} \lambda_n \nu^{K_n}(B_i) \qquad \textit{for every } 1 \leq i \leq m. \tag{3.7}$$

Proof. Since the measures ν^{K_n} are supported by K_n, the sum on the right side of (3.7) reduces to the term $\lambda_n \nu^{K_n}(B_i)$, when $i \in I_n$. By Lemma 3.1, there exists $s \in [0,1]^m$ such that $C := B_1^{s_1} \cup \dots \cup B_m^{s_m}$ satisfies

$$\nu^C(B_i) \leq (1-\delta)\lambda_n \nu^{K_n}(B_i) \qquad \text{for all } 1 \leq n \leq k,\ i \in I_n, \tag{3.8}$$

with equality whenever $s_i < 1$. We claim that we even have

$$\nu^C(B_i) \geq \lambda_n \nu^{K_n}(B_i) \qquad \text{if } s_i = 1,\ i \in I_n,\ 1 \leq n \leq k. \tag{3.9}$$

and this will clearly finish the proof (in fact, it shows even that s_i cannot be equal to 1 for $i \in I_n$, unless $\lambda_n \nu^{K_n}(B_i) = 0$).

Indeed, let us suppose, for example, that $s_l = 1$ for some $l \in I_1$ and let $I_1' := I_1 \setminus \{l\}$. Then $B := B_l = B_l^{s_l}$, that is, B is a subset of C, and we get by (3.1) that

$$\nu^B = \nu^C|_B + (\nu^C|_{C\setminus B})^B, \tag{3.10}$$

where

$$\nu^C|_{C\setminus B} = \sum_{i\in I_1'} \nu^C|_{B_i} + \sum_{n=2}^{k} \sum_{i\in I_n} \nu^C|_{B_i}. \tag{3.11}$$

By (3.5),

$$(\nu^C|_{B_i})^B \le (1+\delta)(1-\delta)\lambda_1(\nu^{K_1}|_{B_i})^B \le \lambda_1(\nu^{K_1}|_{B_i})^B \qquad \text{for all } i \in I_1'.$$

Similarly, $(\nu^C|_{B_i})^B \le \lambda_n(\nu^{K_n}|_{B_i})^B$ for all $i \in I_n$, $2 \le n \le k$. Taking sums we see that

$$\sum_{i\in I_1'} (\nu^C|_{B_i})^B \le \lambda_1(\nu^{K_1}|_{K_1\setminus B})^B \quad \text{ and } \quad \sum_{i\in I_n} (\nu^C|_{B_i})^B \le \lambda_n(\nu^{K_n})^B$$

for every $2 \le n \le k$. Therefore (3.10) and (3.11) imply the inequality

$$\nu^B(B) \le \nu^C(B) + \lambda_1(\nu^{K_1}|_{K_1\setminus B})^B(B) + \sum_{n=2}^{k} \lambda_n(\nu^{K_n})^B(B), \tag{3.12}$$

where $(\nu^{K_n})^B(B) \le \nu^B(B)$ by (3.2). Hence

$$\lambda_1\nu^B(B) \le \nu^C(B) + \lambda_1(\nu^{K_1}|_{K_1\setminus B})^B(B).$$

Knowing that $\nu^B = \nu^{K_1}|_B + (\nu^{K_1}|_{K_1\setminus B})^B$ by (3.1), we thus get the inequality $\lambda_1\nu^{K_1}(B) \le \nu^C(B)$, and the proof is finished. $\square$

Let us note that a combination of Proposition 3.2 and Proposition 3.3 yields the following stronger result (which will not be needed in the sequel).

COROLLARY 3.4. *Under the assumptions of Proposition 3.3 and given any real numbers $\beta_1, \dots, \beta_m \in [0,1]$, there exist $\gamma_1, \dots, \gamma_m \in [0,1]$ such that the union C of the shrinked balls $B_1^{\gamma_1}, \dots, B_m^{\gamma_m}$ satisfies*

$$\nu^C(B_i) = (1-\delta)\beta_i\lambda_n\nu^{K_n}(B_i) \qquad \textit{for all } 1 \le n \le k,\ i \in I_n.$$

4 Proof of Theorem 1.1

1. Let $U_1, \dots, U_k$ be open subsets of an open set W in X, let $\nu \in \mathcal{M}(\mathcal{P}(X))$ such that $\nu(W^c) = 0$, and $\lambda \in \Lambda_k$. We fix a strictly positive $p \in \mathcal{P}(X)$ such that $\nu(p) < \infty$. To prove the weak convergence of a sequence (μ_n) to μ, it is sufficient to check the

convergence of the sequence $(\mu_n(f))$ to $\mu(f)$ for each functions f from a suitable countable subset of $\mathcal{K}(X)$. For every $f \in \mathcal{K}(X)$ and for every $\eta > 0$, there exist $q, q' \in \mathcal{P}(X)$ which are bounded by a multiple of p such that $|f - (q - q')| \le \eta p$ (see [2, I.1.2, III.6.10]). To prove Theorem 1.1, it is therefore sufficient to show the following. Let $\mathcal{Q}$ be a finite set of potentials $q \in \mathcal{P}(X)$ which are bounded by p and let $0 < \eta < 1$. Then there exists a union C of pairwise disjoint closed balls in $U := U_1 \cup \cdots \cup U_k$ such that

$$\big|\nu^{C\cup W^c}(q) - \sum_{n=1}^{k} \lambda_n \nu^{U_n \cup W^c}(q)\big| < \eta \qquad \text{for every } q \in Q. \tag{4.1}$$

By [2, VI.1.9], we may assume without loss of generality that all $\overline{U}_n$, $1 \le n \le k$, are compact subsets of W and that $p \ge 1$ on $\overline{U}$. We then define $\delta := (6\nu(p) + 3k)^{-1}\eta$. There exists $0 < \delta' \le \delta$ such that

$$|q(y) - q(z)| < \delta, \qquad \text{whenever } q \in \mathcal{Q} \text{ and } y, z \in \overline{U},\ |y - z| < \delta'. \tag{4.2}$$

2. By Proposition 2.1, we are able to replace each U_n, $1 \le n \le k$, by a finite union K_n of very small closed balls. We then want to shrink these balls using Proposition 3.3. This, however, not only requires that $K_1, \dots, K_k$ be pairwise disjoint, but that they are separated well enough to obtain a δ-family. Moreover, taking $A := K_1 \cup \cdots \cup K_k$ we shall have to replace ν by the measure $\tilde{\nu} := 1_{X\setminus A}\nu$ not charging A, and therefore $\nu(A)$ will have to be small. This can be achieved considering $\mathcal{Z}_m(x_0, a)$ for sufficiently many points $x_0 \in \mathbb{R}^d$. We fix a natural number $N > k + \nu(p)/\delta$ and define

$$a := \min\{\frac{\delta}{4dN}, \frac{\delta'}{2}\}, \qquad x_j := \big(\frac{j}{N}, 0, \dots, 0\big), \quad 1 \le j \le N.$$

For every $M \in \mathbb{N}$, $1 \le n \le k$, and $1 \le j \le N$, let $A_M(n,j) \subset Z_M(x_j, a)$ be the union of all balls $B \in \mathcal{Z}_M(x_j, a)$ such that $B \subset U_n$ and $\mathrm{dist}(B, \mathbb{R}^d \setminus U_n) \ge 1/M$. By Proposition 2.1, there exists $M \in \mathbb{N}$ such that, for every $q \in \mathcal{Q}$ and for all $1 \le n \le k$, $j \in \{1, \dots, N\}$,

$$|\nu^{A_M(n,j)\cup W^c}(q) - \nu^{U_n \cup W^c}(q)| < \delta. \tag{4.3}$$

By our definition of a and $x_1, \dots, x_N$, the sets $Z_M(x_1, a), \dots, Z_M(x_N, a)$ are pairwise disjoint and hence

$$\sum_{j=1}^{N} \nu(1_{Z_M(x_j,a)} p) \le \nu(p) < (N - k)\delta.$$

Therefore at least k of terms of the sum must be strictly smaller than δ, that is, there exist k different $j_1, \dots, j_k \in \{1, \dots, N\}$ such that

$$\nu(1_{Z_M(x_{j_n},a)} p) < \delta \qquad \text{for every } 1 \le n \le k. \tag{4.4}$$

We define

$$K_1 := A_M(1, j_1),\ \dots\ , K_k := A_M(k, j_k), \quad A := K_1 \cup \cdots \cup K_k, \quad \tilde{\nu} = 1_{X\setminus A}\nu.$$

By (4.4), $(\nu-\tilde\nu)(p)<k\delta$. The set A is a union of pairwise disjoint balls $B_1,\dots,B_m$ from the union of $\mathcal{Z}(x_{j_1},a),\ \dots,\ \mathcal{Z}(x_{j_k},a)$. Hence, for every $1\le i\le m$,

$$\operatorname{dist}\bigl(x_{B_i},(\mathbb{R}^d\setminus W)\cup(A\setminus B_i)\bigr)\ge\frac{1}{M}\Bigl(\frac{1}{N}-2a\Bigr)\ge\Bigl(\frac{4d}{\delta}-2\Bigr)\frac{a}{M}\ge\frac{3d}{\delta}\frac{a}{M}=\frac{3d}{\delta}r_{B_i}.$$

So $B_1,\dots,B_m$ is a δ-family in W and, of course, $\tilde\nu(A)=0$. Thus we may apply Proposition 3.3 to W in place of X and to $\tilde\nu$ in place of ν. Denoting balayage of $\tilde\nu$ on compact subsets L of W *relative to* W by ${}^W\tilde\nu^L$, we obtain $s_1,\dots,s_m\in[0,1]$ such that the union C of the shrinked balls $B_1^{s_1},\dots,B_m^{s_m}\subset A$ satisfies

$${}^W\tilde\nu^C(B_i)=(1-\delta)\sum_{n=1}^k\lambda_n\,{}^W\tilde\nu^{K_n}(B_i)\qquad\text{for every }1\le i\le m.$$

By [2, VI.2.9], this means that defining $\tilde\mu:=\sum_{n=1}^k\lambda_n\tilde\nu^{K_n\cup W^c}$ we have

$$\tilde\nu^{C\cup W^c}(B_i)=(1-\delta)\tilde\mu(B_i)\qquad\text{for every }1\le i\le m. \tag{4.5}$$

3. We now fix $q\in\mathcal{Q}$ and claim first that

$$\bigl|\tilde\nu^{C\cup W^c}(1_Wq)-\tilde\mu(1_Wq)\bigr|<4\nu(p)\delta. \tag{4.6}$$

Indeed, let $g:=\sum_{i=1}^m q(x_{B_i})1_{B_i}$. By (4.5),

$$\tilde\nu^{C\cup W^c}(g)=(1-\delta)\tilde\mu(g). \tag{4.7}$$

Since $0\le g\le 2p$, we know by (1.1) that, for every $1\le n\le k$,

$$0\le\tilde\nu^{K_n\cup W^c}(g)\le 2\tilde\nu^{K_n\cup W^c}(p)\le 2\nu(p).$$

Moreover, $|g-1_Wq|<\delta p$ on $\overline{U}\cup W^c$ by (4.2). Therefore

$$\bigl|\tilde\nu^{C\cup W^c}(g)-\tilde\nu^{C\cup W^c}(1_Wq)\bigr|\le\delta\tilde\nu^{C\cup W^c}(p)\le\nu(p)\delta$$

and, for every $1\le n\le k$,

$$\bigl|\tilde\nu^{K_n\cup W^c}(g)-\tilde\nu^{K_n\cup W^c}(1_Wq)\bigr|<\delta\tilde\nu^{K_n\cup W^c}(p)\le\nu(p)\delta.$$

Thus (4.7) implies (4.6) (and the proof would readily be finished in the case $W=X$).

4. It may be surprising that (4.5), which merely indicates that $\tilde\mu$ is a good approximation for $\tilde\nu^{C\cup W^c}$ on W, also implies that $\tilde\mu$ approximates $\tilde\nu^{C\cup W^c}$ nicely on $X\setminus W$. We claim that

$$\rho:=\tilde\nu^{C\cup W^c}|_{W^c}-\tilde\mu|_{W^c}\ge 0,\qquad\text{and}\qquad\rho(p)\le 2\nu(p)\delta. \tag{4.8}$$

Indeed, by (3.1),

$$\tilde\nu^{C\cup W^c}|_{W^c}+(\tilde\nu^{C\cup W^c}|_W)^{W^c}=\tilde\nu^{W^c}=\tilde\mu|_{W^c}+(\tilde\mu|_W)^{W^c}.$$

Defining $\sigma := \tilde{\mu}|_W$ and $\tau := \tilde{\nu}^{C\cup W^c}|_W$ we hence see that

$$\rho = \sigma^{W^c} - \tau^{W^c}.$$

By (4.5) and (3.5), for each $B \in \{B_1, \dots, B_m\}$,

$$(1_B\tau)^{W^c} \le (1-\delta)(1+\delta)(1_B\sigma)^{W^c} \le (1_B\sigma)^{W^c},$$

$$(1_B\tau)^{W^c} \ge \frac{1-\delta}{1+\delta}(1_B\sigma)^{W^c} \ge (1-2\delta)(1_B\sigma)^{W^c}.$$

Taking the sum we obtain that $0 \le \rho \le 2\delta\sigma^{W^c}$ where $\sigma^{W^c}(p) \le \tilde{\mu}^{W^c}(p) \le \tilde{\mu}(p) \le \nu(p)$ by (1.1). Thus (4.8) holds and we conclude that

$$\left|\tilde{\nu}^{C\cup W^c}(1_{W^c}q) - \tilde{\mu}(1_{W^c}q)\right| = \rho(q) \le \rho(p) \le 2\nu(p)\delta. \tag{4.9}$$

5. Combining (4.6) and (4.9) we get $\left|\tilde{\nu}^{C\cup W^c}(q) - \tilde{\mu}(q)\right| < 6\nu(p)\delta$. This implies

$$\Big|\nu^{C\cup W^c}(q) - \sum_{n=1}^{k} \lambda_n \nu^{K_n\cup W^c}(q)\Big| < 6\nu(p) + 2k\delta,$$

since $\tilde{\nu} \le \nu$ and $(\nu - \tilde{\nu})(p) < k\delta$. Together with (4.3), this estimate finally yields

$$\Big|\nu^{C\cup W^c}(q) - \sum_{n=1}^{k} \lambda_n \nu^{U_n\cup W^c}(q)\Big| < 6\nu(p)\delta + 3k\delta = \eta,$$

that is, (4.1) holds and the proof is finished.

5 Proof of Theorem 1.8

Let us return to classical potential theory. We fix an open subset Ω of $\mathbb{R}^d$, $d \ge 2$, and a point $x \in \Omega$. If W is a bounded open subset of Ω, then the measures in $\mathcal{M}_x(S(W))$ are supported by $\overline{W}$ and $\mathcal{M}_x(S(W))$ is independent of the choice of the Greenian domain X containing W (of course, we would simply take $X = \Omega$ if $d \ge 3$ or if, in the case $d = 2$, the complement of Ω is non-polar).

PROPOSITION 5.1.

$$J_x(\Omega) = \bigcup \big\{\mathcal{M}_x(S(W)) \colon W \textit{ open},\ x \in W,\ \overline{W} \textit{ compact in } \Omega\big\}.$$

Proof. 1. Let W be a bounded open set such that $x \in W$, $\overline{W} \subset \Omega$, and let $\mu \in \mathcal{M}_x(S(W))$. Then (in contrast to the situation for Riesz potentials) μ is supported by $\overline{W}$. Let X be a bounded domain such that $\overline{W} \subset X \subset \Omega$. If v is a continuous superharmonic function on Ω, then there exists a $\mathcal{P}(X)$-bounded continuous function on X such that $\tilde{v} = v$ on $\overline{W}$ whence $\tilde{v} \in S(W)$ and $\int v\, d\mu = \int \tilde{v}\, d\mu \le \tilde{v}(x) = v(x)$. Thus $\mu \in J_x(\Omega)$.

2. Let us now suppose conversely that $\mu \in J_x(\Omega)$. Let K be a compact neighborhood of the support of μ such that $x \in K$ and each bounded component of $\mathbb{R}^d \setminus K$ meets $\mathbb{R}^d \setminus \Omega$. Further, let W be a bounded open neighborhood of K such that $\overline{W} \subset \Omega$ and $v \in S(W)$. By [7, Theorem 6.1], there exists a continuous superharmonic function $\tilde{v}$ on Ω such that $\tilde{v} = v$ on K whence $\int v\, d\mu = \int \tilde{v}\, d\mu \le \tilde{v}(x) = v(x)$. Thus $\mu \in \mathcal{M}_x(S(W))$. $\square$

A consequence is a full characterization of extremal Jensen measures which has been asked for in [3].

COROLLARY 5.2. *The set of all extremal Jensen measures for x with respect to Ω is given by*

$$\begin{aligned}\big(J_x(\Omega)\big)_e &= \bigcup\big\{\big(\mathcal{M}_x(S(W))\big)_e \colon W \textit{ open},\ x \in W,\ \overline{W} \textit{ compact in } \Omega\big\} \\ &= \big\{\varepsilon_x^{A^c} \colon \overline{A} \textit{ compact in } \Omega\big\}.\end{aligned}$$

Proof. 1. If $W \subset \tilde{W}$, then $S(\tilde{W}) \subset S(W)$, $\mathcal{M}_x(S(W)) \subset \mathcal{M}_x(S(\tilde{W}))$, and $\mathcal{M}_x(S(W))$ is a closed face of the compact convex set $\mathcal{M}_x(S(\tilde{W}))$ (see Theorem 1.6). By Proposition 5.1, this immediately yields the first identity.

2. To prove the second identity, let $\mu \in \big(\mathcal{M}_x(S(W))\big)_e$ for some open set W such that $x \in W$ and $\overline{W}$ is a compact subset of Ω. By Theorem 1.6, $\mu = \varepsilon_x^B$ for some set B containing the complement of W. Taking $A := B^c$ we have $\mu = \varepsilon_x^{A^c}$ and $\overline{A} \subset \overline{W} \subset \Omega$.

Conversely, let A be a bounded set in $\mathbb{R}^d$ such that $\overline{A} \subset \Omega$ and consider $\mu = \varepsilon_x^{A^c}$. Let W be a bounded open set such that $x \in W$, $\overline{A} \subset W$, and $\overline{W} \subset \Omega$. Then $\mu \in \big(\mathcal{M}_x(S(W))\big)_e$ by Theorem 1.6. $\square$

6 Appendix

Finally, let us give a proof for Theorem 1.6 in the general case. For the moment, we fix a closed set A in X and recall that the base $b(A)$ of A is the set of all points $x \in A$ such that A is not thin at x, that is, $\varepsilon_x^A = \varepsilon_x$. Since in our case of classical potential theory or Riesz potentials every semi-polar set is polar, the set $A \setminus b(A)$ is polar, hence $b(b(A)) = b(A)$ and $\mu^{b(A)} = \mu^A$ for every $\mu \in \mathcal{M}(\mathcal{P}(X))$ (see [2, VI.5.12]). Moreover, $\beta(A)$, which can be characterized as being the largest subset $\tilde{A}$ of $b(A)$ such that $b(\tilde{A}) \subset \tilde{A}$, coincides with $b(A)$ (see [2, VI.6.1, VI.6.6]). Hence $\mu^{\beta(W^c)} = \mu^{W^c}$ for every $\mu \in \mathcal{M}(\mathcal{P}(X))$ and every open W in X.

THEOREM 6.1. *Let W be an open subset of X and $\nu \in \mathcal{M}(\mathcal{P}(X))$ such that $\nu(W^c) = 0$. Then*

$$\begin{aligned}\mathcal{M}_\nu(S(W)) &= \mathcal{M}_\nu(\mathcal{P}(X)) \cap \mathcal{M}_\nu(H(W)) \\ &= \{\mu \in \mathcal{M}_\nu(\mathcal{P}(X)) \colon \mu^{W^c} = \nu^{W^c}\} = \{\mu \in \mathcal{M}(\mathcal{P}(X)) \colon \nu^{W^c} \prec \mu \prec \nu\}.\end{aligned}$$

Moreover, $\mathcal{M}_\nu(S(W))$ is a closed face of $\mathcal{M}_\nu(\mathcal{P}(X))$ and

$$\big(\mathcal{M}_\nu(S(W))\big)_e = \{\mathring{\nu}^A \colon W^c \subset A \subset X,\ A \textit{ Borel set}\}.$$

Proof. Replacing the measure ε_x in the proof of [2, VI.9.5] by ν (and using that $\mu^{\beta(W^c)} = \mu^{W^c}$), we obtain immediately the first three identities, the fact that $\mathcal{M}_\nu(S(W))$ is a closed face of $\mathcal{M}_\nu(\mathcal{P}(X))$, and that every measure $\mathring{\nu}^A$, where A is a Borel set with $W^c \subset A \subset X$, is contained in $\big(\mathcal{M}_\nu(S(W))\big)_e$.

Conversely, let $\mu \in \big(\mathcal{M}_\nu(S(W))\big)_e$. Of course, $\mu \in \big(\mathcal{M}_\nu(\mathcal{P}(X))\big)_e$, since the set $\mathcal{M}_\nu(S(W))$ is a closed face of $\mathcal{M}_\nu(\mathcal{P}(X))$. So, by (1.2), there exists a Borel subset A of X such that $\mu = \mathring{\nu}^A$. We intend to show that $\mu = \mathring{\nu}^{A\cup W^c}$. This will finish the proof, since $A \cup W^c$ is a Borel subset of X containing W^c.

By the characterization of $\mathcal{M}_\nu(S(W))$ given above, $\nu^{W^c} \prec \mu$. By [2, VI.1.9], this implies that, for every $p \in \mathcal{P}(X)$,

$$\nu^{W^c}(p) = \nu^{W^c}(R_p^{W^c}) = \inf_{U \text{open} \supset W^c} \nu^{W^c}(R_p^U) \le \inf_{U \text{open} \supset W^c} \mathring{\nu}^A(R_p^U) = \mathring{\nu}^A(R_p^{W^c}),$$

that is,

$$\nu^{W^c} \prec \mathring{\nu}^A|_{W^c} + (\mathring{\nu}^A|_W)^{W^c} \tag{6.1}$$

(see the proof of [2, VI.9.9]). In addition,

$$\mathring{\nu}^{A\cup W^c} + \mathring{\nu}^A|_{W^c} + (\mathring{\nu}^A|_W)^{W^c} \prec \mathring{\nu}^A + \nu^{W^c}. \tag{6.2}$$

Indeed, if $\nu(A) = 0$, this follows from [2, VI.9.8]. And if $\nu(A^c) = 0$, then $\mathring{\nu}^{A\cup W^c} = \mathring{\nu}^A = \nu$ and (6.1) reduces to the trivial statement $\nu + \nu^{W^c} \prec \nu + \nu^{W^c}$. The general case follows decomposing ν into $1_{A^c}\nu$ and $1_A\nu$.

Combining (6.1) and (6.2), we see that $\mathring{\nu}^{A\cup W^c} \prec \mathring{\nu}^A$. Since trivially $\mathring{\nu}^A \prec \mathring{\nu}^{A\cup W^c}$, we conclude that $\mu = \mathring{\nu}^A = \mathring{\nu}^{A\cup W^c}$ as claimed above, and the proof is finished. $\square$

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