Existence and uniqueness of nonnegative solutions to the stochastic porous media equation

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Abstract. One proves that the stochastic porous media equation in 3-D has a unique nonnegative solution for nonnegative initial data.

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1 Introduction

Let $\mathcal{O}$ be an open bounded domain of $\mathbb{R}^n$ with smooth boundary $\partial\mathcal{O}$. We consider the linear operator Δ in $L^2(\mathcal{O})$ defined on $H^2(\mathcal{O})\cap H^1_0(\mathcal{O})$. It is well known that $-\Delta$ is self-adjoint positive and anti-compact. So, there exists a complete orthonormal system $\{e_k\}$ in $L^2(\mathcal{O})$ of eigenfunctions of $-\Delta$. We denote by $\{\lambda_k\}$ the corresponding sequence of eigenvalues,

$$\Delta e_k = -\lambda_k e_k, \quad k \in \mathbb{N}.$$

We shall consider a Wiener process in $L^2(\mathcal{O})$ of the following form

$$W(t) = \sum_{k=1}^{\infty} \mu_k \beta_k(t) e_k, \quad t \geq 0,$$

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where $\{\mu_k\}$ is a sequence of positive numbers and $\{\beta_k\}$ a sequence of mutually independent standard Brownian motions on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. To be more specific, we shall assume that $1 \leq n \leq 3$.

In this work we consider the stochastic partial differential equation,

$$\begin{cases} dX(t) - \Delta\beta(X(t))dt = XdW(t), \quad t \geq 0, \\ \\ \beta(X(t)) = 0, \quad \text{on } \partial\mathcal{O}, \quad t \geq 0, \\ \\ X(0,x) = x. \end{cases} \tag{1.1}$$

Here β is a continuous, monotonically increasing function on $\mathbb{R}$ which satisfies the following conditions,

$$\begin{cases} |\beta(r)| \leq \alpha_1|r|^m + \alpha_2|r|, \quad \forall\, r \in \mathbb{R}, \\ \\ j(r) \colon = \displaystyle\int_0^r \beta(s)ds \geq \alpha_3|r|^{m+1} + \alpha_4 r^2, \quad \forall\, r \in \mathbb{R}, \end{cases} \tag{1.2}$$

where $\alpha_i > 0,\ i = 1,2,3,4$ and $m \geq 2$. We note that since β is increasing, the mean value theorem implies that

$$r\beta(r) \geq j(r), \quad r \geq 0. \tag{1.3}$$

Equation (1.1) with additive noise was recently studied in [3],[4],[6], [7],[8], see also [2]. In particular, in [6] was given an existence result under similar conditions on β. Here we consider a multiplicative noise (of a special form, but it would be possible to consider a more general noise $f(X)dW(t)$ with $f(0) = 0$), which is needed in order to ensure positivity of solutions.

As was shown in [11] existence and uniqueness of solutions follow by the general results in [11] (see also [12] for generalizations). In this paper we present an alternative proof, based on the Yosida approximation of $-\Delta\beta$, and prove the positivity of solutions for nonnegative initial data x.

As in deterministic case the Sobolev space $H^{-1}(\mathcal{O})$ is natural for studying equation (1.1). Equation (1.1) can be written in the abstract form

$$\begin{cases} dX(t) + AX(t) = \sigma(X(t))dW(t), \quad t \geq 0, \\ \\ X(0) = x, \end{cases} \tag{1.4}$$

where the operator $A\colon D(A)\subset H^{-1}(\mathcal{O})\to H^{-1}(\mathcal{O})$ is defined by

$$\begin{cases} Ax=-\Delta\beta(x), \quad x\in D(A), \\ \\ D(A)=\{x\in H^{-1}(\mathcal{O})\cap L^1(\mathcal{O}):\ \beta(x)\in H_0^1(\mathcal{O})\}, \end{cases} \tag{1.5}$$

and where

$$\sigma(X)dW(t)=\sum_{k=1}^{\infty}\mu_k Xe_k d\beta_k(t), \quad X\in H^{-1}(\mathcal{O}). \tag{1.6}$$

To give a rigorous sense to this noise term we first note that since $n\le 3$, by Sobolev embedding it follows that

$$\sup_{k\in\mathbb{N}}\frac{1}{\lambda_k}\,|e_k|_\infty<\infty. \tag{1.7}$$

Furthermore, troughout this paper we shall assume that

$$\sum_{k=1}^{\infty}\mu_k^2\lambda_k^2=:C<\infty. \tag{1.8}$$

(1.8) implies for some constant $c_1>0$

$$\sum_{k=1}^{\infty}\mu_k^2|xe_k|_{-1}^2\le c_1\sum_{k=1}^{\infty}\mu_k^2\ \lambda_k^2\ |x|_{-1}^2\le c_1C|x|_{-1}^2, \quad \forall\, x\in H^{-1}(\mathcal{O}), \tag{1.9}$$

because $|xe_k|_{-1}^2\le c_1\lambda_k^2|x|_{-1}^2$ by an elementary calculation, since $n\le 3$ and due to (1.7).

Defining

$$\sigma(x)h:=\sum_{k=1}^{\infty}\mu_k(h,e_k)xe_k, \quad x\in H^{-1}(\mathcal{O}),\ h\in L^2(\mathcal{O}), \tag{1.10}$$

we obtain by (1.9) that $\sigma(x)\in L_2(L^2(\mathcal{O}),H^{-1}(\mathcal{O}))$. Considering $(\beta_k)_{k\in\mathbb{N}}$ as a cylindrical Wiener process on $L^2(\mathcal{O})$, it follows that (1.6) is well defined. Note that since σ is linear we have that $x\to\sigma(x)$ is Lipschitz from $H^{-1}(\mathcal{O})$ to $L_2(L^2(\mathcal{O}),H^{-1}(\mathcal{O}))$ (in particular [10], [11], [12] really apply).

The plan of the paper is the following: main results are stated in §2 and proofs are given in §3.

The following notations will be used troughout in the following.

(i) $H_0^1(\mathcal{O}), H^2(\mathcal{O})$ are standard Sobolev spaces on $\mathcal{O}$ endowed with their usual norms denoted by $|\cdot|_{H_0^1(\mathcal{O})}$ and $|\cdot|_{H^2(\mathcal{O})}$ respectively.

(ii) H is the space $H^{-1}(\mathcal{O})$ (the dual of $H_0^1(\mathcal{O})$) endowed with the norm

$$|x|_H = |x|_{-1} = |-\Delta^{-1}x|_{H_0^1(\mathcal{O})}.$$

(Here $(-\Delta)^{-1}x = y$ is the solution to Dirichlet problem $-\Delta y = x$ in $\mathcal{O}$, $y \in H_0^1(\mathcal{O})$). The scalar product in H is

$$\langle x, z\rangle_{-1} = \int_{\mathcal{O}} (-\Delta)^{-1}\, x\, z\, d\xi, \quad \forall\, x, z \in H_0^1(\mathcal{O}).$$

(iii) The scalar product and the norm in $L^2(\mathcal{O})$ will be denoted by $(\cdot,\cdot)$ and $|\cdot|_2$, respectively and the norm in $L^p(\mathcal{O})$, $1 \le p \le \infty$ by $|\cdot|_p$.

(iv) For two Hilbert spaces H_1, H_2 the space of Hilbert-Schmidt operators from H_1 to H_2 is denoted by $L_2(H_1, H_2)$.

2 The main result

To begin with let us define the solution concept we shall work with. Formally, a solution to (1.1) (equivalently (1.4)) might be an H–valued continuous adapted process such that $X, AX \in C_W([0,T]; L^2(\Omega; H))$ and

$$X(t) = x - \int_0^t AX(s)ds + \int_0^t \sigma(X(s))dW(s), \quad t \in [0,T]. \tag{2.1}$$

(By $C_W([0,T]; L^2(\Omega; H))$ we mean the Banach space of all the processes X in $(\Omega, \mathcal{F}, \mathbb{P})$ with values in H which are adapted and mean square continuous, endowed with the norm

$$\|X\|^2_{C_W([0,T];L^2(\Omega;H))} := \sup_{t\in[0,T]} \mathbb{E}|X(t)|_H^2.$$

Spaces $L^p_W([0,T]; L^2(\Omega; H))$, $p \in [1, \infty]$, are defined similarly.)

However, such a concept of solution might fail to exist for equation (1.1) and so we shall confine to a weaker one inspired by [6] and [10].

Definition 2.1 *An H-valued continuous $\mathcal{F}_t$-adapted process X is called a solution to* (1.1) *on* $[0,T]$ *if* $X \in L^{m+1}(\Omega \times (0,T) \times \mathcal{O})$ *and*

$$
\begin{aligned}
(X(t), e_j) \quad &= \quad (x, e_j) + \int_0^t \int_{\mathcal{O}} \beta(X(s)) \Delta e_j d\xi ds \\
&\quad + \sum_{k=1}^{\infty} \mu_k \int_0^t (X(s)e_k, e_j)\beta_k(s), \quad \forall\, j \in \mathbb{N},\ t \in [0,T].
\end{aligned}
\tag{2.2}
$$

Taking into account that $-\Delta e_j = \lambda_j e_j$ in $\mathcal{O}$ we may equivalently write (2.2) as follows

$$
\begin{aligned}
\langle X(t), e_j \rangle_{-1} \quad &= \quad \langle x, e_j \rangle_{-1} - \int_0^t \int_{\mathcal{O}} \beta(X(s)) e_j d\xi ds \\
&\quad + \sum_{k=1}^{\infty} \mu_k \int_0^t \langle X(s)e_k, e_j \rangle_{-1} d\beta_k(s), \quad \forall\, j \in \mathbb{N},
\end{aligned}
$$

i.e.

$$
d\langle X(t), e_j \rangle_{-1} + (\beta(X(t)), e_j)_2 dt = \sum_{k=1}^{\infty} \mu_k \langle X(s)e_k, e_j \rangle_{-1} d\beta_k(s).
$$

Recalling (1.6) we see that

$$
\sum_{k=1}^{\infty} \mu_k (X(t)e_k, e_j)\beta_k(t) = (\sigma(X(t))W(t), e_j), \quad j \in \mathbb{N}.
$$

We also note that since by assumption (1.2), $\beta(X) \in L^{\frac{m+1}{m}}((0,T) \times \Omega \times \mathcal{O})$, the integral arising in the right hand side of (2.2) makes sense because $e_j \in C^\infty(\overline{\mathcal{O}})$ for all $j \in \mathbb{N}$. Of course, one might derive a vector valued version of Definition 2.1 as in [6]. Now we are ready to formulate the main results.

Theorem 2.2 *Assume that* (1.2) *and* (1.8) *hold. Then for each* $x \in H^{-1}(\mathcal{O})$ *there is a unique solution X to* (1.1)*. Moreover, if* $x \in L^4(\mathcal{O})$ *is nonnegative a.e. on* $\mathcal{O}$ *then* $X \in L^\infty_W(0,T; L^4(\Omega; L^4(\mathcal{O})))$ *and* $X \geq 0$ *a.e. on* $(0,\infty) \times \mathcal{O}$, $\mathbb{P}$*-a.s. If* $x \in H^{-1}(\mathcal{O})$ *is such that* $x \geq 0$*, i.e. x is a positive measure, then* $X(t) \geq 0$ *for all* $t \geq 0$, $\mathbb{P}$*-a.s.*

The positivity of the solution X to (1.1) will be proven below by choosing an appropriate Lyapunov function.

3 Proof of Theorem 2.2

As mentionned before the existence and uniqueness part of Theorem 2.2 follows from [10], [11] which is based on finite dimensional Galerkin approximations. However, for later purpose to prove positivity, we shall use an alternative approach based on the Yosida approximation of the monotone operator $-\Delta\beta$ which may have an intrinsic interest.

We mention that in our estimates in the sequel constants may change from line to line though we do not express this in our notation.

We recall that the operator A, defined by (1.5), is maximal monotone in H (see e.g. [5]). Then we consider the Yosida approximation

$$A_\varepsilon(x) = \frac{1}{\varepsilon}\,(x - J_\varepsilon(x)) = A(1+\varepsilon A)^{-1}(x), \quad \varepsilon > 0,\ x \in H,$$

where $J_\varepsilon(x) = (1+\varepsilon A)^{-1}(x)$. The operator A_ε is monotone and Lipschitzian on H. Then, by (1.9) it follows by standard existence theory for stochastic equations in the Hilbert spaces (see e.g. [9]) that the approximating equation

$$\left\{\begin{array}{l} dX_\varepsilon(t) + A_\varepsilon X_\varepsilon(t)dt = \sigma(X_\varepsilon(t))dW(t), \quad t \ge 0, \\ \\ X_\varepsilon(0) = x, \end{array}\right. \tag{3.1}$$

has a unique solution $X_\varepsilon \in C_W([0,T];L^2(\Omega;H))$ with $A_\varepsilon X_\varepsilon \in C_W([0,T];L^2(\Omega;H))$.

A priori estimates. By Itô's formula we have

$$\begin{array}{l} \displaystyle\frac{1}{2}\,d|X_\varepsilon(t)|_{-1}^2 + \langle A_\varepsilon X_\varepsilon(t), X_\varepsilon(t)\rangle_{-1}dt \\ \\ \displaystyle= \langle \sigma(X_\varepsilon(t))dW(t), X_\varepsilon(t)\rangle_{-1} + \frac{1}{2}\,\sum_{k=1}^\infty \mu_k^2 |X_\varepsilon e_k|_{-1}^2 dt. \end{array} \tag{3.2}$$

This yields (see (1.9))

$$\begin{array}{l} \displaystyle\frac{1}{2}\,\mathbb{E}|X_\varepsilon(t)|_{-1}^2 + \mathbb{E}\int_0^t \langle A_\varepsilon X_\varepsilon(s), X_\varepsilon(s)\rangle_{-1}ds \\ \\ \displaystyle\le \frac{1}{2}\,|x|_{-1}^2 + C\mathbb{E}\int_0^t |X_\varepsilon(s)|_{-1}^2 ds \end{array}$$

and therefore

$$\frac{1}{2}\,\mathbb{E}|X_\varepsilon(t)|^2_{-1}+\mathbb{E}\int_0^t\langle A_\varepsilon X_\varepsilon(s),X_\varepsilon(s)\rangle_{-1}ds\le C|x|^2_{-1},\quad\forall\,\varepsilon>0. \tag{3.3}$$

We set $Y_\varepsilon(t)=J_\varepsilon(X_\varepsilon(t))$ (see (3.1)). Then

$$\begin{aligned}&\frac{1}{2}\,\mathbb{E}|X_\varepsilon(t)|^2_{-1}+\mathbb{E}\int_0^t\int_{\mathcal{O}}j(Y_\varepsilon(s))dsd\xi\\&+\frac{1}{\varepsilon}\,\mathbb{E}\int_0^t|X_\varepsilon(s)-Y_\varepsilon(s)|^2_{-1}ds\le C|x|^2_{-1},\quad\forall\,\varepsilon>0.\end{aligned} \tag{3.4}$$

(Here we have used the equality

$$\langle A_\varepsilon x,x\rangle_{-1}=\langle AJ_\varepsilon x,J_\varepsilon x\rangle_{-1}+\frac{1}{\varepsilon}\,|x-J_\varepsilon(x)|^2_{-1},$$

and (1.3).) Taking into account estimate (1.2) we conclude that $\{Y_\varepsilon\}$ is bounded in $C_W([0,T];L^2(\Omega;H))$ and $L^{m+1}(\Omega\times(0,T)\times\mathcal{O})$. Then on a subsequence, again denoted $\{\varepsilon\}\to 0$, we have

$$Y_\varepsilon\to X\quad\text{weakly in }\ L^{m+1}(\Omega\times(0,T)\times\mathcal{O}). \tag{3.5}$$

It is also clear that $X(t)$ is adapted (because so are Y_ε). Moreover, since

$$\lim_{\varepsilon\to 0}\,\mathbb{E}\int_0^t|X_\varepsilon(s)-Y_\varepsilon(s)|^2_{-1}ds=0,$$

we infer that

$$X_\varepsilon\to X\quad\text{weakly in }\ L^2_W(\Omega;L^2(0,T;H)) \tag{3.6}$$

and by the weak lower semicontinuity of the convex functional

$$Y\to\mathbb{E}\int_0^t\int_{\mathcal{O}}j(Y(s))dsd\xi$$

we conclude by virtue of (3.4) and (3.5) that

$$\mathbb{E}|X(t)|^2_{-1}+\mathbb{E}\int_0^t\int_{\mathcal{O}}j(X(s))dsd\xi\le C|x|^2_{-1},\quad\text{a.e. }\ t\in[0,T]. \tag{3.7}$$

Now we consider equation

$$\begin{cases} d\widetilde{X}_\varepsilon(t) + A_\varepsilon \widetilde{X}_\varepsilon(t)dt = \sigma(X(t))dW(t), \quad t \geq 0, \\ \\ \widetilde{X}_\varepsilon(0) = x. \end{cases} \tag{3.8}$$

Equivalently,

$$\begin{cases} d\widetilde{X}_\varepsilon(t) - \Delta\beta(\widetilde{Y}_\varepsilon(t))dt = \sigma(X(t))dW(t), \quad t \geq 0, \\ \\ \widetilde{X}_\varepsilon(0) = x, \end{cases} \tag{3.9}$$

where

$$\widetilde{Y}_\varepsilon = (1+\varepsilon A)^{-1}\widetilde{X}_\varepsilon.$$

Comparing (3.1) and (3.8) we see that by virtue of (3.6) and monotonicity of A_ε we have

$$\lim_{\varepsilon\to 0}(X_\varepsilon(t) - \widetilde{X}_\varepsilon(t)) = 0, \quad \mathbb{P}\text{-a.s. } \forall\, t \in [0,T], \tag{3.10}$$

in the weak topology of H.

On the other hand, for equation (3.8) we have the same estimates as for (3.1). In fact by Itô's formula we get (see (3.7))

$$\mathbb{E}|\widetilde{X}_\varepsilon(t)|_{-1}^2 + \mathbb{E}\int_0^t \int_{\mathcal{O}} j(\widetilde{X}_\varepsilon(s))dsd\xi \leq C|x|_{-1}^2. \tag{3.11}$$

which by virtue of assumption (1.2) implies that

$$\mathbb{E}\int_0^T \int_{\mathcal{O}} |\beta(\widetilde{Y}_\varepsilon(s))|^{\frac{m+1}{m}}dsd\xi \leq C|x|_{-1}^2, \quad \varepsilon > 0$$

and so along a subsequence, we have

$$\beta(\widetilde{Y}_\varepsilon) \to \eta \quad \text{weakly in } L^{\frac{m+1}{m}}((0,T)\times\Omega\times\mathcal{O}). \tag{3.12}$$

On the other hand, we have by (3.9) that for a.e. $t \in [0,T]$

$$\langle \widetilde{X}_\varepsilon(t), e\rangle_{-1} + \int_0^t \int_{\mathcal{O}} \eta(s)ed\xi ds = \langle x, e\rangle_{-1} + \int_0^t \langle \sigma(X(s))dW(s), e\rangle_{-1}ds,$$

for all $e \in L^m(\mathcal{O})$. Then letting ε tend to 0 we get for a.e. $t \in [0,T]$

$$\langle X(t), e\rangle_{-1} + \int_0^t \int_{\mathcal{O}} \eta(s) e d\xi ds = \langle x, e\rangle_{-1} + \int_0^t \langle \sigma(X(s))dW(s), e\rangle_{-1} ds. \tag{3.13}$$

We note that since by estimate (3.11) $\widetilde{X}_\varepsilon \to X$ weakly in $C_W([0,T]; L^2(\Omega; H))$ we have by (3.10) that

$$\left\{ \begin{array}{l} \widetilde{X}_\varepsilon \to X \quad \text{weakly in } \ C_W([0,T]; L^2(\Omega; H)) \\ \\ \widetilde{Y}_\varepsilon \to X \quad \text{weakly in } \ C_W([0,T]; L^2(\Omega; H)) \cap L^{m+1}(\Omega \times (0,T) \times \mathcal{O}). \end{array} \right. \tag{3.14}$$

Taking into account (3.13), to conclude the proof of existence it suffices to show that

$$\eta(t,\xi,\omega) = \beta(X(t,\xi,\omega)) \quad \text{a.e. } (\omega,t,\xi) \in \Omega \times (0,T) \times \mathcal{O}. \tag{3.15}$$

Indeed, in such a case we may take in (3.13) $e = \Delta e_j$ for $j \in \mathbb{N}$.

To this end we consider the operator

$$F\colon L^m(\Omega \times (0,T) \times \mathcal{O}) \to L^{\frac{m}{m+1}}(\Omega \times (0,T) \times \mathcal{O}) = (L^m(\Omega \times (0,T) \times \mathcal{O}))',$$

defined by

$$(Fx)(t,\xi,\omega) = \beta(x(t,\xi,\omega)) \quad \text{a.e. } (\omega,t,\xi) \in \Omega \times (0,T) \times \mathcal{O}.$$

This operator is maximal monotone and more precisely, it is the subgradient of the convex function $\Phi : L^m(\Omega \times (0,T) \times \mathcal{O} \to \mathbb{R}$ defined as,

$$\Phi(x) = \mathbb{E} \int_0^T \int_{\mathcal{O}} j(x(t,\xi,\omega)) dt d\xi.$$

For each $Z \in L^m(\Omega \times (0,T)\mathcal{O})$ we have

$$\Phi(\widetilde{Y}_\varepsilon) - \varphi(Z) \le \mathbb{E} \int_0^T \int_{\mathcal{O}} \beta(\widetilde{Y}_\varepsilon(t,\xi,\omega))(\widetilde{Y}_\varepsilon(t,\xi,\omega) - Z(t,\xi,\omega)) dt d\xi$$

Letting ε tend to 0 we have by (3.12), (3.14) and by the weak lower semi-continuity of φ

$$\Phi(X) - \Phi(Z) \le \liminf_{\varepsilon\to 0} \mathbb{E} \int_0^T \int_{\mathcal{O}} \beta(\widetilde{Y}_\varepsilon(t,\xi,\omega)) \widetilde{Y}_\varepsilon(t,\xi,\omega) dt d\xi - \mathbb{E} \int_0^T \int_{\mathcal{O}} \eta Z dt d\xi.$$

To prove (3.15) it suffices to show that

$$\liminf_{\varepsilon\to 0}\mathbb{E}\int_0^T\int_{\mathcal{O}}\beta(\widetilde{Y}_\varepsilon(t,\xi,\omega))\widetilde{Y}_\varepsilon(t,\xi,\omega)dtd\xi\le\mathbb{E}\int_0^T\int_{\mathcal{O}}\eta X dtd\xi. \tag{3.16}$$

To this end we come back to equation (3.9) and notice that by Itô's formula we have

$$\begin{aligned}&\frac{1}{2}\,\mathbb{E}|\widetilde{X}_\varepsilon(t)|_{-1}^2+\mathbb{E}\int_0^t\int_{\mathcal{O}}\beta(\widetilde{Y}_\varepsilon(s))\widetilde{X}_\varepsilon(s)dsd\xi\\ &=\frac{1}{2}\,|x|_{-1}^2+\frac{1}{2}\,\sum_{k=1}^\infty\mathbb{E}\int_0^t\mu_k^2|X(s)e_k|_{-1}^2ds.\end{aligned}$$

Equivalently,

$$\begin{aligned}&\frac{1}{2}\,\mathbb{E}|\widetilde{X}_\varepsilon(t)|_{-1}^2+\mathbb{E}\int_0^t\int_{\mathcal{O}}\beta(\widetilde{Y}_\varepsilon(s))\widetilde{Y}_\varepsilon(s)dsd\xi\\ &+\mathbb{E}\int_0^t\int_{\mathcal{O}}\beta(\widetilde{Y}_\varepsilon(s))(\widetilde{X}_\varepsilon(s)-\widetilde{Y}_\varepsilon(s))dsd\xi\\ &=\frac{1}{2}\,|x|_{-1}^2+\frac{1}{2}\,\sum_{k=1}^\infty\mathbb{E}\int_0^t\mu_k^2|X(s)e_k|_{-1}^2ds.\end{aligned}$$

Taking into account (3.14) and that

$$\int_{\mathcal{O}}\beta(\widetilde{Y}_\varepsilon(s))(\widetilde{X}_\varepsilon(s)-\widetilde{Y}_\varepsilon(s))dsd\xi=\langle A_\varepsilon\widetilde{X}_\varepsilon,\widetilde{X}_\varepsilon-J_\varepsilon(\widetilde{X}_\varepsilon)\rangle_{-1}=\varepsilon|A_\varepsilon\widetilde{X}_\varepsilon|_{-1}^2$$

we obtain that

$$\begin{aligned}&\liminf_{\varepsilon\to 0}\mathbb{E}\int_0^t\int_{\mathcal{O}}\beta(\widetilde{Y}_\varepsilon(s)\widetilde{Y}_\varepsilon(s)dsd\xi+\frac{1}{2}\,\mathbb{E}|X(t)|_{-1}^2\\ &\le\frac{1}{2}\,|x|_{-1}^2+\frac{1}{2}\,\sum_{k=1}^\infty\mathbb{E}\int_0^t\mu_k^2|X(s)e_k|_{-1}^2ds.\end{aligned} \tag{3.17}$$

(Here we have also used (3.14) and the weak lower continuity of the H-norm.)

On the other hand, by (3.13) we see via Itô's formula that for all $j \in \mathbb{N}$ and a.e. $t \in [0,T]$,

$$
\begin{aligned}
&\frac{1}{2}\,\mathbb{E}|\langle X(t), e_j\rangle_{-1}|^2 + \mathbb{E}\int_0^t \langle \eta_s, e_j\rangle\langle X(s), e_j\rangle_{-1} ds \\
&= \frac{1}{2}\,\langle x, e_j\rangle_{-1}^2 + \frac{1}{2}\,\mathbb{E}\sum_{k=1}^{\infty}\mu_k^2\int_0^t \langle X(s)e_k, e_j\rangle^2 ds
\end{aligned}
$$

and summing up on j we obtain

$$
\begin{aligned}
&\frac{1}{2}\,\mathbb{E}|X(t)|_{-1}^2 + \mathbb{E}\int_0^t\int_{\mathcal{O}} \eta(s)X(s)dsd\xi \\
&= \frac{1}{2}\,|x|_{-1}^2 + \frac{1}{2}\,\sum_{k=1}^{\infty}\mu_k^2\mathbb{E}\int_0^t |X(s)e_k|_{-1}^2 ds.
\end{aligned}
\tag{3.18}
$$

We notice that the integral in the left hand side makes sense since by (3.4), $X \in L^{m+1}((0,T)\times\Omega\times\mathcal{O})$ while $\eta \in L^{\frac{m+1}{m}}((0,T)\times\Omega\times\mathcal{O})$.

Comparing (3.17) and (3.18) we infer that

$$
\liminf_{\varepsilon\to 0}\mathbb{E}\int_0^T\int_{\mathcal{O}} \beta(\widetilde{Y}_\varepsilon(t))\widetilde{Y}_\varepsilon(t)dtd\xi \le \mathbb{E}\int_0^T\int_{\mathcal{O}} \eta(t)X(t)dtd\xi,
$$

as claimed. Hence X is a solution to equation (1.1). A little problem arises, however, because $X(t)$ as constructed before might not be H-continuous. However, arguing as in [10], [11] we may replace it by an H-continuous version defined by

$$
\widetilde{X}(t) = x + \int_0^t \Delta\eta(s)ds + \int_0^t \sigma(X(s))dW(s).
$$

It follows that $X = \widetilde{X}$ a.e. and that $\widetilde{X}$ is also an $\mathcal{F}_t$-adapted process. Moreover, the Itô formula from ([10, Theorem I-3-2]) holds. This completes the proof of existence.

Uniqueness. Let X_1, X_2 be two solutions to equation (1.1). We have (see (2.2))

$$
d\langle X_1 - X_2, e_j\rangle_{-1} + \int_{\mathcal{O}}(\beta(X_1)-\beta(X_2))e_j d\xi dt = \sum_{k=1}^{\infty}\mu_k\langle (X_1-X_2)e_k, e_j\rangle_{-1} d\beta_k.
$$

By Itô's formula we obtain

$$\frac{1}{2}\,\mathbb{E}|\langle X_1(t)-X_2(t),e_j\rangle_{-1}|^2$$

$$+\mathbb{E}\int_0^t(\beta(X_1)-\beta(X_2),e_j)\langle X_1(s)-X_2(s),e_j\rangle_{-1}ds$$

$$=\frac{1}{2}\,\mathbb{E}\int_0^t\sum_{k=1}^{\infty}\mu_k^2\langle(X_1(s)-X_2(s))e_k,e_j\rangle_{-1}^2ds$$

Summing up on j we see that

$$\frac{1}{2}\,\mathbb{E}|X_1(t)-X_2(t)|_{-1}^2+\mathbb{E}\int_0^t(\beta(X_1)-\beta(X_2),X_1(s)-X_2(s)\;ds$$

$$=\frac{1}{2}\,\mathbb{E}\int_0^t\sum_{j,k=1}^{\infty}\mu_k^2\langle(X_1(s)-X_2(s))e_k,e_j\rangle_{-1}^2ds.$$

Then by Gronwall's lemma we obtain that $X_1-X_2=0$ as claimed.

Positivity. We shall assume now that $x\in L^4(\mathcal{O})$ and $x(\xi)\geq 0$ a.e. in $\mathcal{O}$. We shall prove that

$$X\geq 0\quad\text{a.e. in }\;(0,T)\times\mathcal{O}\times\Omega. \tag{3.19}$$

We shall first assume in addition that β is strictly monotone, i.e.

$$(\beta(r)-\beta(\bar r))(r-\bar r)\geq\alpha(r-\bar r)^2,\quad\forall\,r,\bar r\in\mathbb{R}, \tag{3.20}$$

where $\alpha>0$. Below we shall use the following lemma.

Lemma 3.1 *Let $y\in D(A)$ and $g:\mathbb{R}\to\mathbb{R}$ Lipschitz and increasing. Then*

$$\langle\nabla\beta(y),\nabla g(y)\rangle_{\mathbb{R}^n}\geq 0,\quad\textit{a.e. on }\;\mathcal{O}.$$

Proof. First note that by definition of $D(A)$ we have that $y,\beta(y)\in H_0^1(\mathcal{O})$. Using a Dirac sequence we can find mollifiers $g_k\in C^1(\mathbb{R})$, $g_k'\geq 0$, $k\in\mathbb{N}$, such that

$$\nabla g(y)=\lim_{k\to\infty}g_k'(y)\nabla y\quad\text{in }L^2(\mathcal{O}).$$

So, it suffices to prove that

$$\langle \nabla \beta(y), \nabla g(y) \rangle_{\mathbb{R}^n} \geq 0, \quad \text{a.e. on } \mathcal{O}.$$

But

$$\langle \nabla \beta(y), \nabla y \rangle_{\mathbb{R}^n} = \langle \nabla \beta(y), \nabla \beta^{-1} \beta(y) \rangle_{\mathbb{R}^n}.$$

Since β is strictly monotone, β^{-1} is Lipschitz, so applying the above mollifier argument with β^{-1} replacing g, we prove the assertion. $\square$

We shall use the approximating equation (3.1) whose solution X_ε is weakly conve1gent to X in $L^2_W(\Omega; L^2(0,T;H))$ (see (3.6)). Namely, we have for $Y_\varepsilon(t) := J_\varepsilon(X_\varepsilon(t))$, $t \geq 0$,

$$dX_\varepsilon(t) - \Delta\beta(Y_\varepsilon(t))dt = \sigma(X_\varepsilon(t))dW(t), \quad t \geq 0. \tag{3.21}$$

We note that equation (3.1) can be equivalently written as

$$\begin{cases} dX_\varepsilon(t) + \dfrac{1}{\varepsilon}\, X_\varepsilon(t)dt = \dfrac{1}{\varepsilon}\, J_\varepsilon(X_\varepsilon(t))dt + \sigma(X_\varepsilon(t))dW(t), \quad t \geq 0, \\ \\ X_\varepsilon(0) = x, \end{cases} \tag{3.22}$$

Fix $x \in H$ and set

$$y = J_\varepsilon(x) = (1 - \varepsilon\Delta\beta)^{-1}x,$$

i.e.

$$y - \varepsilon\Delta\beta(y) = x \tag{3.23}$$

Then $y \in D(A)$. Since β is strictly monotone, β^{-1} is Lipschitz. Therefore, since $\beta(y) \in H^1_0(\mathcal{O})$, also $y \in H^1_0(\mathcal{O}) \subset L^4(\mathcal{O})$. Now assume $x \in L^4(\mathcal{O})$. By multiplying both sides of (3.23) by $\frac{y^3}{1+\lambda y^2}$ and integrating over $\mathcal{O}$ we get by Lemma 3.1

$$\int_{\mathcal{O}} \frac{y^4}{1+\lambda y^2}\, d\xi \leq \int_{\mathcal{O}} \frac{y^3 x}{1+\lambda y^2}\, d\xi.$$

Then, letting $\lambda \to 0$ we find the estimate

$$|y|_4^4 \leq \int_{\mathcal{O}} y^3 x d\xi \leq |y|_4^3\, |x|_4. \tag{3.24}$$

Hence

$$|J_\varepsilon(x)|_4 \leq |x|_4, \quad \forall\, x \in L^4(\mathcal{O}), \tag{3.25}$$

and therefore,

$$|A_\varepsilon(x)|_4 = \frac{1}{\varepsilon}\,|x - J_\varepsilon(x)|_4 \le \frac{2}{\varepsilon}\,|x|_4, \quad \forall\, x \in L^4(\mathcal{O}).$$

(3.23) and (3.25) imply that J_ε is continuous from $L^4(\mathcal{O})$ into itself.

Lemma 3.2 *For each $x \in L^2(\mathcal{O})$ equation* (3.22) *has a unique solution $X_\varepsilon \in C_W([0,T]; L^2(\Omega; L^2(\mathcal{O})))$.*

Proof. Let us first prove that $J_\varepsilon = (1 - \varepsilon\Delta\beta)^{-1}$ is Lipschitz continuous in $L^2(\mathcal{O})$. Indeed, by the equation

$$J_\varepsilon(x) - \varepsilon\Delta\beta(J_\varepsilon(x)) = x, \quad \text{in } \mathcal{O},$$

(taking into account that $\beta(J_\varepsilon(x)) \in H^1_0(\mathcal{O})$) we have for $x, \bar{x} \in L^2(\mathcal{O})$

$$\int_{\mathcal{O}} (J_\varepsilon(x) - J_\varepsilon(\bar{x}))(\beta(J_\varepsilon(x)) - \beta(J_\varepsilon(\bar{x})))d\xi$$

$$+\varepsilon \int_{\mathcal{O}} |\nabla(\beta(J_\varepsilon(x)) - \beta(J_\varepsilon(\bar{x}))|^2 d\xi \le \int_{\mathcal{O}} (x - \bar{x})(\beta(J_\varepsilon(x)) - \beta(J_\varepsilon(\bar{x})))d\xi.$$

This yields, recalling (3.20)

$$\alpha|J_\varepsilon(x) - J_\varepsilon(\bar{x})|_2^2 + \varepsilon|\beta(J_\varepsilon(x)) - \beta(J_\varepsilon(\bar{x}))|^2_{H^1_0(\mathcal{O})} \le |x - \bar{x}|_2\,|\beta(J_\varepsilon(x)) - \beta(J_\varepsilon(\bar{x}))|_2.$$

On the other hand, by the Poincaré inequality there exists $C > 0$ such that

$$|\beta(J_\varepsilon(x)) - \beta(J_\varepsilon(\bar{x}))|_2^2 \le C|\beta(J_\varepsilon(x)) - \beta(J_\varepsilon(\bar{x}))|^2_{H^1_0(\mathcal{O})}.$$

Therefore

$$\alpha|J_\varepsilon(x) - J_\varepsilon(\bar{x}))|_2^2 + \frac{\varepsilon}{2}\,|\beta(J_\varepsilon(x)) - \beta(J_\varepsilon(\bar{x}))|^2_{H^1_0(\mathcal{O})} + \frac{\varepsilon}{2C}\,|\beta(J_\varepsilon(x)) - \beta(J_\varepsilon(\bar{x}))|_2^2$$

$$\le \frac{C}{2\varepsilon}\,|x - \bar{x}|_2^2 \;+ \frac{\varepsilon}{2C}\,|\beta(J_\varepsilon(x)) - \beta(J_\varepsilon(\bar{x}))|_2^2,$$

and consequently

$$\alpha|J_\varepsilon(x) - J_\varepsilon(\bar{x}))|_2^2 + \frac{\varepsilon}{2}\,|\beta(J_\varepsilon(x)) - \beta(J_\varepsilon(\bar{x}))|^2_{H^1_0(\mathcal{O})} \le \frac{C}{2\varepsilon}\,|x - \bar{x}|_2.$$

So, J_ε is Lipschitz continuous in $L^2(\mathcal{O})$ as claimed. Consequently $A_\varepsilon = \frac{1}{\varepsilon}(1-J_\varepsilon)$ is Lipschitz continuous in $L^2(\mathcal{O})$ as well. Moreover, since

$$\|\sigma(x)\|_{L_2(L^2(\mathcal{O}),L^2(\mathcal{O}))} \le \sum_{k=1}^{\infty} \mu_k^2 |xe_k|_2^2 \le \sum_{k=1}^{\infty} \mu_k^2 |e_k|_{L^\infty(\mathcal{O})}^2 \; |x|_2^2 \le C_1 \sum_{k=1}^{\infty} \mu_k^2 \lambda_k^2 \; |x|_2^2$$

we infer by standard existence theory for stochastic PDEs that for each $x \in L^2(\mathcal{O})$ equation (3.22) has a unique solution in $X_\varepsilon \in C_W([0,T];L^2(\Omega;L^2(\mathcal{O})))$ (see e.g. [9]). $\square$

For $R > 0$ define

$$K_R^\alpha := \{X \in L_W^\infty(0,T;L^4(\Omega\times\mathcal{O})) : \; e^{-4\alpha t}\mathbb{E}|X(t)|_4^4 \le R^4 \quad \text{for a.e. } t \in [0,T]\}$$

Lemma 3.3 *Let $T > 0$ and $x \in L^4(\mathcal{O})$. Then for the solution X_ε of* (3.1) *(or equivalently* (3.22)*) we have $X_\varepsilon \in L_W^\infty(0,T;L^2(\Omega\times\mathcal{O}))$.*

Proof. Obviously, K_R is a closed subset of $L_W^\infty(0,T;L^4(\Omega\times\mathcal{O}))$. Since by (3.22) X_ε is a fixed point of the map F

$$X \mapsto e^{-\frac{t}{\varepsilon}}x + \frac{1}{\varepsilon}\int_0^t e^{-\frac{(t-s)}{\varepsilon}} J_\varepsilon(X(s))ds + \int_0^t e^{-\frac{(t-s)}{\varepsilon}}\sigma(X(s))dW(s), \; t \in [0,T],$$

obtained by iteration in $C_W(0,T;L^2(\Omega\times\mathcal{O}))$, it suffices to prove that this map leaves K_R^α invariant for R and α large enough. But for $X \in K_R^\alpha$ we have by (3.25) for $t \ge 0$

$$\left(e^{-4\alpha t}\mathbb{E}\left|e^{-\frac{t}{\varepsilon}}x + \frac{1}{\varepsilon}\int_0^t e^{-\frac{(t-s)}{\varepsilon}} J_\varepsilon(X(s))ds\right|_4^4\right)^{1/4} \le e^{-(\frac{1}{\varepsilon}+\alpha)t}|x|_4 + \frac{R}{1+\alpha\varepsilon}.$$

Now we set

$$Y(t) = \int_0^t e^{-\frac{(t-s)}{\varepsilon}}\sigma(X(s))dW(s), \quad t \ge 0.$$

Then

$$\begin{cases} dY(t) + \dfrac{1}{\varepsilon}\,Y(t)dt = X(t)dW(t), \quad t \ge 0, \\[2ex] Y(0) = 0. \end{cases}$$

Let $\lambda > 0$. Applying Itô's formula to the function

$$\Psi_\lambda(y) := \frac{1}{4}\,|(1+\lambda A_0)^{-1}y|_4^4, \quad y \in L^2(\mathcal{O}),$$

(see the beginning of the proof of the next lemma for a detailed justification) we obtain via Hölder's inequality that

$$
\begin{aligned}
&\mathbb{E}[\Psi_\lambda(Y(t))] + \frac{1}{\varepsilon}\,\mathbb{E}\int_0^t\int_{\mathcal{O}} |(1+\lambda A_0)^{-1}Y(s)|_4^4 d\xi\,ds \\
&= \frac{3}{2}\sum_{k=1}^{\infty}\mu_k^2\,\mathbb{E}\int_0^t\int_{\mathcal{O}} |(1+\lambda A_0)^{-1}Y(s)|^2 \\
&\qquad \times |(1+\lambda A_0)^{-1}(X(s)e_k)|^2 d\xi\,ds \\
&\le \frac{3C}{2}\,\mathbb{E}\int_0^t |(1+\lambda A_0)^{-1}Y(s)|_4^2\,|X(s)|_4^2 ds \\
&\le \frac{1}{2\varepsilon}\,\mathbb{E}\int_0^t |(1+\lambda A_0)^{-1}Y(s)|_4^4 ds + \frac{9C^2\varepsilon}{8}\,\mathbb{E}\int_0^t |X(s)|_4^4 ds \\
&\le \frac{1}{2\varepsilon}\,\mathbb{E}\int_0^t |(1+\lambda A_0)^{-1}Y(s)|_4^4 ds + \frac{9C^2\varepsilon(e^{4\alpha t}-1)}{32\alpha}\,R^4.
\end{aligned}
$$

Then letting $\lambda\to\infty$, we see by Fatou's lemma that for a.e. $t\in[0,T]$ we have for C_1 independent of ε

$$
e^{-4\alpha t}\mathbb{E}|Y(t)|_4^4 \le \frac{C_1\varepsilon}{\alpha}R^4, \quad \forall\, t\in[0,T].
$$

This means that for α large enough and $R>2|x|_4$ the map leaves K_R^α invariant as claimed.

Consider now the function

$$
\varphi(x) = \frac{1}{4}\,|x^-|_4^4.
$$

For any $x\in L^4(\mathcal{O})$, φ is Gâteaux differentiable and its differential $D\varphi\colon L^4(\mathcal{O})\to L^{4/3}(\mathcal{O})$ is given by

$$
D\varphi(x) = -(x^-)^3,
$$

while the second Gâteaux derivative $D^2\varphi(x)\in L(L^4(\mathcal{O});L^{4/3}(\mathcal{O}))$ is given by

$$
(D^2\varphi(x)h,g) = 3\int_{\mathcal{O}} h\,g\,|x^-|^2 d\xi, \quad \forall\, h,g,x\in L^4(\mathcal{O}).
$$

Lemma 3.4 *Let $n \leq 3$. For each $x \in L^4(\mathcal{O})$ we have*

$$\begin{aligned}
&\mathbb{E}[\varphi(X_\varepsilon(t))] + \mathbb{E}\int_0^t (A_\varepsilon X_\varepsilon(s), D\varphi(X_\varepsilon(s))ds \\
&= \varphi(x) + \frac{3}{2}\sum_{k=1}^{\infty} \mu_k^2\, \mathbb{E}\int_0^t \int_{\mathcal{O}} |X_\varepsilon^-(s)e_k|^2 |X_\varepsilon^-(s)|^2 ds d\xi.
\end{aligned} \tag{3.26}$$

Proof. We note first that since $X_\varepsilon \in L^\infty_W(0,T;L^4(\Omega;L^4(\mathcal{O})))$ the above formula makes sense. Next we approximate φ by

$$\varphi_\lambda(x) = \varphi((1+\lambda A_0)^{-1}x), \quad A_0 = -\Delta,\ \ D(A_0) = H^2(\mathcal{O}) \cap H^1_0(\mathcal{O}),\ \lambda > 0.$$

Since $\varphi \in C^2(C(\overline{\mathcal{O}}))$ and $(1+\lambda A_0)^{-1}$ is linear continuous from $L^2(\mathcal{O})$ to $C(\overline{\mathcal{O}})$ (due to our assumption $n \leq 3$) we infer that $\phi_\lambda \in C^2(L^2(\mathcal{O}))$ and its first order and second order differentials are given, respectively, by

$$D\varphi_\lambda(x) = D\varphi((1+\lambda A_0)^{-1}x))(1+\lambda A_0)^{-1},$$

$$(D^2\varphi_\lambda(x)h,k) = (D^2\varphi((1+\lambda A_0)^{-1}x))((1+\lambda A_0)^{-1}h, (1+\lambda A_0)^{-1}k)$$

for $h,k \in L^2(\mathcal{O}), x \in L^2(\mathcal{O})$. Note that if $x \in L^4(\mathcal{O})$, then

$$D\varphi_\lambda(x) = -(1+\lambda A_0)^{-1}(((1+\lambda A_0)^{-1}x)^-)^3.$$

So, for $\lambda \to 0$ we have $\varphi_\lambda(x) \to \varphi(x)$ and $D\varphi_\lambda(x) \to D\varphi(x)$ in $L^{4/3}(\mathcal{O})$. Next we write Itô's formula for φ_λ in the space $L^2(\mathcal{O})$ which makes sense by Lemma 3.2.

We get

$$\mathbb{E}[\varphi_\lambda(X_\varepsilon(t))] + \mathbb{E}\int_0^t (A_\varepsilon(X_\varepsilon(s)), D\varphi_\lambda(X_\varepsilon(s)))ds = \varphi_\lambda(x)$$

$$+\frac{3}{2}\sum_{k=1}^{\infty} \mu_k^2\, \mathbb{E}\int_0^t \int_{\mathcal{O}} |((1+\lambda A_0)^{-1}(X_\varepsilon(s)e_k)|^2\ |((1+\lambda A_0)^{-1}X_\varepsilon(s))^-|^2 d\xi\, ds.$$

This yields

$$
\begin{aligned}
&\mathbb{E}[\varphi_\lambda(X_\varepsilon(t))] - \mathbb{E}\int_0^t \int_{\mathcal{O}} (1+\lambda A_0)^{-1}(A_\varepsilon(X_\varepsilon(s)))(((1+\lambda A_0)^{-1}X_\varepsilon(s))^-)^3 d\xi ds \\
&= \varphi_\lambda(x) \\
&+\frac{3}{2}\sum_{k=1}^\infty \mu_k^2\, \mathbb{E}\int_0^t \int_{\mathcal{O}} |((1+\lambda A_0)^{-1}X_\varepsilon(s))^-|^2 |(1+\lambda A_0)^{-1}(X_\varepsilon(s)e_k)|^2 d\xi\, ds.
\end{aligned}
\tag{3.27}
$$

We know that for $\lambda \to 0$, $(1+\lambda A_0)^{-1}X_\varepsilon(s) \to X_\varepsilon(s)$ strongly in $L^4(\mathcal{O})$ a.e. in $\Omega \times (0,T)$ and

$$
|(1+\lambda A_0)^{-1}X_\varepsilon|_4 \le |X_\varepsilon|_4, \quad \text{a.e. in } \Omega \times (0,T).
$$

Then by the Lebesgue dominated convergence theorem we have

$$
\lim_{\lambda\to 0}(1+\lambda A_0)^{-1}X_\varepsilon = X_\varepsilon \quad \text{strongly in } L^4(\Omega \times (0,T) \times \mathcal{O}). \tag{3.28}
$$

Similarly, since $A_\varepsilon(X_\varepsilon) \in L^4(\Omega \times (0,T) \times \mathcal{O})$ we have for $\lambda \to 0$

$$
(1+\lambda A_0)^{-1}(A_\varepsilon(X_\varepsilon)) \to A_\varepsilon(X_\varepsilon), \quad \text{strongly in } L^4(\Omega \times (0,T) \times \mathcal{O}).
$$

and

$$
((1+\lambda A_0)^{-1}X_\varepsilon)^- \to X_\varepsilon^-, \quad \text{strongly in } L^4(\Omega \times (0,T) \times \mathcal{O}).
$$

This yields

$$
\begin{aligned}
&\lim_{\lambda\to 0}\mathbb{E}\int_0^t \int_{\mathcal{O}} (1+\lambda A_0)^{-1}(A_\varepsilon(X_\varepsilon(s)))(((1+\lambda A_0)^{-1}X_\varepsilon(s))^-)^3 d\xi ds \\
&= \int_0^t \int_{\mathcal{O}} A_\varepsilon(X_\varepsilon(s))(X_\varepsilon^-(s))^3 d\xi ds.
\end{aligned}
\tag{3.29}
$$

Then, if $x \in L^4(\mathcal{O})$ letting $\lambda \to 0$ in (3.27) we get (since by Fatou's lemma $\mathbb{E}\varphi(X_\varepsilon(t)) \le \liminf_{\lambda\to 0}\mathbb{E}\varphi_\lambda(X_\varepsilon(t))$, $\forall\, t \ge 0$)

$$
\begin{aligned}
&\mathbb{E}[\varphi(X_\varepsilon(t))] - \mathbb{E}\int_0^t \int_{\mathcal{O}} A_\varepsilon(X_\varepsilon(s))(X_\varepsilon^-(s))^3 d\xi ds \\
&= \varphi(x) + \frac{3}{2}\sum_{k=1}^\infty \mu_k^2\, \mathbb{E}\int_0^t \int_{\mathcal{O}} |X_\varepsilon(s)e_k|^2\, |X_\varepsilon^-(s)|^2 d\xi ds,
\end{aligned}
$$

and so (3.26) follows. $\square$

We have by (3.26) and the definition of Y_ε that for $x \in L^4(\mathcal{O}), x \geq 0$,

$$\mathbb{E}[\varphi(X_\varepsilon(t))] + \mathbb{E}\int_0^t \int_{\mathcal{O}} \Delta\beta(Y_\varepsilon(s))(X_\varepsilon^-(s))^3 ds d\xi$$

$$= \frac{3}{2}\sum_{k=1}^\infty \mu_k^2\, \mathbb{E}\int_0^t \int_{\mathcal{O}} |X_\varepsilon^-(s)e_k|^2\, |X_\varepsilon^-(s)|^2 d\xi ds$$

$$\leq \frac{3C}{2}\, \mathbb{E}\int_0^t |X_\varepsilon^-(s)|_4^4 ds.$$

(Recall that$A_\varepsilon(X_\varepsilon) = -\Delta\beta(Y_\varepsilon)$.)

We therefore have, taking into account that $\Delta\beta(Y_\varepsilon) = \frac{1}{\varepsilon}(Y_\varepsilon - X_\varepsilon)$,

$$\begin{aligned}&\frac{1}{4}\, \mathbb{E}|X_\varepsilon^-(t)|_4^4 + \frac{1}{\varepsilon}\, \mathbb{E}\int_0^t \int_{\mathcal{O}} (Y_\varepsilon(s) - X_\varepsilon(s))(X_\varepsilon^-(s))^3 d\xi ds \\ &\leq \frac{3C}{2}\, \mathbb{E}\int_0^t |X_\varepsilon^-(s)|_4^4 ds.\end{aligned} \tag{3.30}$$

We have

$$|Y_\varepsilon^-(t)|_4^4 \leq \int_{\mathcal{O}} X_\varepsilon(t)(-Y_\varepsilon^-(t))^3 d\xi, \quad \mathbb{P}\text{-a.s.}. \tag{3.31}$$

To see this, analogously to deriving (3.24) for $x \in L^4(\mathcal{O})$, we multiply (3.23) by $g(y)$ where

$$g(y){:}\ = \frac{-(y^-)^3}{1+\lambda(y^-)^2},$$

to get (after integration by parts) that

$$\int_{\mathcal{O}} \frac{(y^-)^4}{1+\lambda(y^-)^2}\, d\xi + \varepsilon \int_{\mathcal{O}} \langle \nabla\beta(y), \nabla g(y)\rangle_{\mathbb{R}^n}\, d\xi = \int_{\mathcal{O}} \frac{x(-y^-)^3}{1+\lambda(y^-)^2}\, d\xi.$$

Note that g as a composition of two decreasing Lipschitz functions is Lipschitz and increasing. So, we can apply Lemma 3.1 to obtain

$$\int_{\mathcal{O}} \frac{(y^-)^4}{1+\lambda(y^-)^2}\, d\xi \leq \int_{\mathcal{O}} \frac{x(-y^-)^3}{1+\lambda(y^-)^2}\, d\xi$$

and (3.31) follows by taking $\lambda \to \infty$. By (3.31) we have

$$-|Y_\varepsilon^-(t)|_4^4 \geq \int_{\mathcal{O}} (X_\varepsilon^+(t) - X_\varepsilon^-(t))(Y_\varepsilon^-(t))^3 d\xi \geq -\int_{\mathcal{O}} X_\varepsilon^-(t)(Y_\varepsilon^-(t))^3 d\xi$$

and therefore $|Y_\varepsilon^-(t)|_4^4 \leq |X_\varepsilon^-(t)|_4 \; |Y_\varepsilon^-(t)|_4^3$. Hence $|Y_\varepsilon^-(t)|_4 \leq |X_\varepsilon^-(t)|_4$ and so

$$\int_{\mathcal{O}} Y_\varepsilon^-(t)(X_\varepsilon^-(t))^3 d\xi \leq |X_\varepsilon^-(t)|_4^3 \; |Y_\varepsilon^-(t)|_4 \leq |X_\varepsilon^-(t)|_4^4.$$

Inserting the latter into(3.30) and taking into account that $Y_\varepsilon X_\varepsilon^- \geq -Y_\varepsilon^- X_\varepsilon^-$ we see that $\mathbb{E}|X_\varepsilon^-(t)|_4^4 = 0$, a.e. $t \geq 0$ i.e, $X_\varepsilon^-(t) = 0$ a.e. and therefore $X_\varepsilon(t) \geq 0$ a.e.. Taking into account (3.6) we infer that $X \geq 0$. This completes the proof in the case when β is strictly monotone. □

To treat the general case of β satisfying (1.2) we shall associate to (1.4) the equation

$$\begin{cases} dX^\lambda(t) + A^\lambda X^\lambda(t) = \sigma(X^\lambda(t))dW(t), \quad t \geq 0, \\ \\ X^\lambda(0) = x, \end{cases} \tag{3.32}$$

where

$$A^\lambda(x) = -\Delta(\beta(x) + \lambda x), \quad \lambda > 0$$

and

$$D(A^\lambda) = \{x \in H^{-1}(\mathcal{O}) \cap L^1(\mathcal{O}) : \; \beta(x) + \lambda x \in H_0^1(\mathcal{O})\}$$

According to the first part of the proof, for each $x \in L^4(\mathcal{O}), x \geq 0$ and $\lambda > 0$, equation (3.32) has a unique strong solution X^λ which is nonnegative a.e. on $\Omega \times (0,T) \times \mathcal{O}$.

On the other hand, applying the Itô formula from [10, Theorem I 3.2] to the equation

$$d(X^\lambda(t) - X(t)) + (A^\lambda X^\lambda(t) - AX(t))dt = (X^\lambda(t) - X(t))dW(t)$$

where X is the solution to (1.1), we get after some calculations that

$$\frac{1}{2}\,\mathbb{E}|X^\lambda(t) - X(t)|_{-1}^2 + \lambda\mathbb{E}\int_0^t \langle X^\lambda(s), X^\lambda(s) - X(s)\rangle_{-1} ds$$

$$\leq \frac{1}{2}\,\sum_{k=1}^\infty \mu_k^2\,\mathbb{E}\int_0^t |(X^\lambda(s) - X(s))e_k|_{-1}^2 ds.$$

This yields (see (1.9)), since

$$\langle X^\lambda(s), X^\lambda(s) - X(s)\rangle_{-1} \geq \langle X(s), X^\lambda(s) - X(s)\rangle_{-1},$$

$$\mathbb{E}|X^\lambda(t) - X(t)|_{-1}^2 \leq C\mathbb{E}\int_0^t |X^\lambda(s) - X(s)|_{-1}^2 ds + \lambda^2 \mathbb{E}\int_0^t |X(s)|_{-1}^2 ds.$$

Since $X \in C_W([0,T]; L^2(\Omega, L^2(\mathcal{O}))$, we infer via Gronwall's lemma that

$$\lim_{X^\lambda \to 0} X^\lambda = X \quad \text{in } \ C_W([0,T]; L^2(\Omega, L^2(\mathcal{O}))$$

and so $X \geq 0$ a.e. in $\Omega \times (0,T) \times \mathcal{O}$ as claimed.

The final part of the assertion in Theorem 2.2 follows by the continuity of sample paths, since $L^4(\mathcal{O})$ is dense in $H^{-1}(\mathcal{O})$ and the continuity of solutions $X = X(t,x)$ with respect to the initial data x (which follows via Itô's formula in the proof of uniqueness). $\square$

4 Concluding remarks

1) Assumption $1 \leq n \leq 3$ is unnecessarily strong and was taken for convenience only. As a matter of fact, under suitable conditions of the form (1.8) we expect that Theorem 2.2 can be established for any dimension n. This will be the subject of a forthcoming paper.

2) Theorem 2.2 and its proof remain valid for time–dependent nonlinear functions $\beta = \beta(t,x)$ where β is monotonically increasing in x, satisfies (1.2) uniformly with respect to t and is continuous in t.

3) One might speculate however that nonnegativity of $X(t,x)$ for $x \geq 0$ follows directly in $H^{-1}(\mathcal{O})$ by taking instead of $\varphi(x) = \frac{1}{4}|x|_4^4$ a suitable C^2-function on $H^{-1}(\mathcal{O})$ which is zero on the cone of positive $x \in H^{-1}(\mathcal{O})$ but so far we failed to find such a function.

References

[1] D.G. Aronson, *The porous medium equation,* Lecture Notes Math. Vol. 1224, Springer, Berlin, 1–46, 1986.

[2] V. Barbu and G. Da Prato *The two phase stochastic Stefan problem*, Probab. Theory Relat. Fields, **124**, 544–560, 2002.

[3] V. Barbu, V.I. Bogachev, G. Da Prato and M. Röckner, *Weak solution to the stochastic porous medium equations: the degenerate case,* J. Functional Analysis, **235** (2), 430-448, 2006.

[4] V.I. Bogachev, G. Da Prato and M. Röckner, *Invariant measures of stochastic porous medium type equations*, Doklady Math. Russian Acad. Sci. 396, 1, 7–11 (Russian), 2004.

[5] H. Brézis, *Operatéurs maximaux monotones*, North-Holland, 1973.

[6] G. Da Prato, M. Röckner, B. L. Rozovskii and Feng-Yu Wang, *Strong Solutions of Stochastic Generalized Porous Media Equations: Existence, Uniqueness and Ergodicity*, Comm. Partial Differential Equations **31**, no. 1-3, 277–291, 2006

[7] G. Da Prato and M. Röckner, *Weak solutions to stochastic porous media equations,* J. Evolution Equ. 4, 249–271, 2004.

[8] G. Da Prato and M. Röckner, *Invariant measures for a stochastic porous medium equation,* Advanced Studies in Pure Mathematics 41, 2004. Stochastic Analysis and related topics, 13–29, 2004.

[9] G. Da Prato and J. Zabczyk, *Stochastic equations in infinite dimensions*, Cambridge University Press, 1992.

[10] N.V. Krylov and B.L. Rozovskii, *Stochastic evolution equations,* Translated from Itogi Naukii Tekhniki, Seriya Sovremennye Problemy Matematiki 14(1979), 71–146, Plenum Publishing Corp. 1981.

[11] C. Prevot and M. Röckner, *A concise course on stochastic partial differential equations,* Monograph 2006, publication in preparation.

[12] J. Ren, M. Röckner and Feng-Yu Wang, *Stochastic generalized porous media and fast diffusions equations*, BiBoS-preprint 2006.