# Capacities and surface measures in locally convex spaces

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**Abstract.** We prove tightness of capacities generated by Sobolev classes of all orders in a wide class of locally convex spaces. We employ these capacities in construction of surface measures on level sets of Sobolev functions and local Sobolev functions.

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## 1 Sobolev classes and capacities

The problem of tightness of capacities generated by Sobolev classes of functions over smooth measures arises in measure theory and stochastic analysis. The problem of tightness of classical Sobolev capacities in infinite dimensional case has been considered in many works, in particular, in [1], [3], [5], [11], [13]. Tightness of capacities generated by the classes $W^{r,p}$ is important for construction of surface measures on infinite dimensional spaces (see Section 2 and the references therein).

In the present work, we consider the Sobolev classes generated by Radon probability measures on a certain class of locally convex spaces that is wider than the one in [8]. The related results concerned with surfaces measures (see Section 2) are also generalized. Then, in Section 3, a new construction based on local Sobolev classes is developed.

### 1.1 Differentiability of measures

Let $X$ be a locally convex space (l.c.s.); let a separable Hilbert space $H$ be continuously embedded into $X$. Denote by $\langle\cdot;\cdot\rangle$ the scalar product in $H$, by $|\cdot|$ the norm in $H$. Then the formula

$$j_H : X^* \to H^* = H, \quad \langle j_H(l); h\rangle = l(h) \quad \forall h \in H,$$

defines a continuous mapping with dense range. If a function $f$ is differentiable at a point $x$ in some sense (Gâteaux or Sobolev), then the vector $D_H f(x) = j_H(f'(x))$ (gradient along $H$) corresponds to the linear functional $f'(x) \in X^*$.

Let $E$ be a separable Hilbert space. Denote by $\mathcal{H}_1(H,E)$ the class of Hilbert–Schmidt operators from $H$ to $E$ endowed with the Hilbert–Schmidt norm

$$\|T\|^2_{\mathcal{H}_1(H,E)} = \sum_{m=1}^{\infty} \|Te_m\|^2_E,$$

where $\{e_m\}$ is an orthonormal basis of $H$. The classes of Hilbert–Schmidt operators of higher orders are defined inductively:

$$\mathcal{H}_n(H,E) = \mathcal{H}_1(H, \mathcal{H}_{n-1}(H,E)), \quad n = 2, 3, ...,$$

equipped with the norms

$$\|T\|^2_{\mathcal{H}_n(H,E)} = \sum_{m_1=1}^{\infty} \dots \sum_{m_n=1}^{\infty} \|T(e_{m_1}, \dots e_{m_n})\|^2_E.$$

For $n = 0$ it is natural to set $\mathcal{H}_0(H,E) = E$.

A function $f : X \to E$ is said to be smooth cylindrical if it has the form

$$f(x) = \sum_{k=1}^{n} u_k(l_1(x), \dots l_m(x)) y_k$$

with $n, m \in \mathbb{N}$, $u_k \in C_b^\infty(\mathbb{R}^m)$, $l_j \in X^*$, $y_k \in E$. Denote the class of such functions by $\mathcal{FC}_b^\infty(X,E)$. It is easy to see that any smooth cylindrical function $f$ is continuous and infinitely Fréchet differentiable along $H$; its gradient of order $r$ along $H$ is defined by

$$D_H^r f(x)(h_1, \dots h_r) = \partial_{h_1} \dots \partial_{h_r} f(x), \quad h_1, \dots h_r \in H,$$

which belongs to $\mathcal{H}_r(H,E)$ at any $x \in X$, and its Hilbert–Schmidt norm $\|D_H^r f(x)\|_{\mathcal{H}_r(H,E)}$ is bounded.

From now on, in the case $E = \mathbb{R}$ we will omit the index $E$ in our notation of classes of functions.

Let $\mu$ be a probability measure on $X$. If $\mu$ is Fomin differentiable along a vector $h \in X$, we denote by $\beta_h^\mu$ the logarithmic derivative of $\mu$ along $h$ (see [6], [3], [5]). The function $\beta_h^\mu$ is determined from the integration by parts formula

$$\int_X \partial_h \varphi(x)\, \mu(dx) = -\int_X \varphi(x) \beta_h^\mu(x)\, \mu(dx)$$

for every smooth cylindrical function $\varphi$ on $X$.

For a non-Gaussian probability measure $\mu$ on a l.c.s. $X$ we introduce the following analog of the Cameron–Martin space:

$$H(\mu) = \big\{ h \in X : \quad \beta_h^\mu \in L^2(\mu) \big\}.$$

This space is Hilbert when endowed by the norm $\|h\|_{H(\mu)} = \|\beta_h^\mu\|_{L^2(\mu)}$; it is continuously embedded in $X$; if $L^2(\mu)$ is separable, then so is $H(\mu)$ (see [5, Ch. 5]).

From now on, we shall assume that the measure $\mu$ is such that, if a function $\mathcal{FC}_b^\infty(X) \ni \varphi = 0$ $\mu$-a. e., then $D_H \varphi = 0$ $\mu$-a. e. In particular, this is the case if supp $\mu = X$.

We shall assume, in addition, that

$$j_H(X^*) \subset H(\mu). \tag{1.1}$$

A measure $\mu$ is (Fomin) differentiable along a vector field $v : X \mapsto X$ if there exists a signed measure of finite variation $d_v\mu$ such that

$$\int \partial_v \varphi(x) \mu(dx) = -\int \varphi(x) d_v\mu(dx) \quad \forall \varphi \in \mathcal{FC}_b^\infty(X),$$

where $\partial_v \varphi(x) = \varphi'(x)(v(x))$; and if one has $d_v\mu \ll \mu$. The density of $d_v\mu$ with respect to $\mu$ is called *divergence* of $v$ and denoted by $\delta v$.

## 1.2 Sobolev classes

**Definition 1.1.** *A function $f \in L^p(\mu, E)$ belongs to the Sobolev class $W^{r,p}(\mu, E)$, $p \geq 1$, $r \in \mathbb{N}$, if there exists a sequence of functions $f_n \in \mathcal{FC}_b^\infty(X, E)$ converging to $f$ in $L^p(\mu, E)$ and fundamental in the norm*

$$\|f\|_{r,p,E} = \|f\|_{L^p(\mu,E)} + \sum_{k=1}^{r} \|D_H^k f\|_{L^p(\mu,\mathcal{H}_k(H,E))} \tag{1.2}$$

*For $m = 1, \dots r$, the $L^p(\mu, \mathcal{H}_m(H, E))$-limit of functions $D_H^m f_n$ is called the gradient of order $m$ of $f$ and denoted by $D_H^m f$.*

Under condition (1.1), if $p \geq 2$, Sobolev gradients do not depend on our choice of a sequence $\{f_n\}$. This implies that if two functions from $W^{r,p}(\mu)$ coincide $\mu$-a.e., then their gradients of orders up to $r$ coincide $\mu$-a.e.

The following statement is derived from Hölder's inequality and from the identity

$$D_H^k(fg) = \sum_{j=0}^{k} \mathrm{C}_k^j D_H^j f \otimes D_H^{k-j} g.$$

**Lemma 1.1.** *(i) If $f \in W^{r,p}(\mu)$, $g \in W^{r,q}(\mu)$, $1 < p, q < \infty$, $1/p + 1/q = 1/s \leq 1$, then $fg \in W^{r,s}(\mu)$ with $\|fg\|_{r,s} \leq 2^r \cdot \|f\|_{r,p} \cdot \|g\|_{r,q}$.*
*(ii) If $f \in W^{r,p}(\mu, H)$, $h \in W^{r,q}(\mu, H)$ with $r \geq 0$, $1/p + 1/q = 1/s \leq 1$, then $\langle f, h\rangle \in W^{r,s}(\mu)$.*

**Lemma 1.2.** *Let $f \in W^{r,p}(\mu)$; let $\varphi \in C_b^{r-1}(\mathbb{R})$ be such that $\varphi^{(r-1)}$ is Lipschitzian. Then $\varphi \circ f \in W^{r,p}(\mu)$, and its gradients of orders up to $r$ are calculated by the chain rule.*

**Lemma 1.3.** *If $j_H(X^*) \subset H(\mu)$, then the Sobolev classes $W^{r,p}(\mu)$, $p \geq 2$, possess the localization property, i.e., if a function $f$ belongs to one of such classes, $A = \{f = 0\}$, then all derivatives of $f$ of orders $k \leq r$ vanish on $A$ $\mu$-almost everywhere (see [5, Lemma 7.3.1]).*

**Lemma 1.4.** *Let $f \in W^{1,p}(\mu)$. If the measure $\mu$ is differentiable along the vector field $v \in L^q(\mu, H)$ with $\delta v \in L^q(\mu)$, $1/p + 1/q = 1$, then the measure $f\mu$ is differentiable along $v$ as well;*

$$d_v(f\mu) = \big(f \cdot \delta v + \partial_v f\big)\mu.$$

*Proof.* We begin with $f \in \mathcal{FC}_b^\infty(X)$. For any $\varphi \in \mathcal{FC}_b^\infty(X)$ we have

$$\int \partial_v \varphi(x) f(x)\mu(dx) = \int \partial_v(\varphi(x)f(x))\mu(dx) - \int \varphi(x)\partial_v f(x)\mu(dx) =$$

$$= -\int \varphi(x)f(x)d_v\mu(dx) - \int \varphi(x)\partial_v f(x)\mu(dx) =$$

$$= -\int \varphi(x)\Big(f(x)\delta v(x) + \partial_v f(x)\Big)\mu(dx). \tag{1.3}$$

The variation of the measure $(f \cdot \delta v + \partial_v f)\mu$ does not exceed $\|f\|_{L^p(\mu)} \cdot \|\delta v\|_{L^q(\mu)} + \|D_H f\|_{L^p(\mu,H)} \cdot \|v\|_{L^q(\mu,H)}$. For a fixed $v$ it is estimated via $\|f\|_{1,p}$. Hence formula (1.3) remains valid when passing to the limit as $\mathcal{FC}_b^\infty \ni f_n \to f$ in $W^{1,p}(\mu)$. $\square$

## 1.3 Capacities

Let $\mathcal{F}$ be some linear subspace of $L^1(\mu)$ endowed with a norm $\|\cdot\|_0$ such that, if $f = 0$ $\mu$-a. e. then $\|f\|_0 = 0$. The capacity generated by $\mathcal{F}$ is defined as follows.

**Definition 1.2.** *If a set $U \subset X$ is open, then*

$$C_{\mathcal{F}}(U) = \inf\{\|f\|_0 : f \in \mathcal{F},\ f \geq 0;\ f \geq 1 \text{ on } U\ \mu\text{-a. e.}\}.$$

*For an arbitrary set $A \subset X$ let*

$$C_{\mathcal{F}}(A) = \inf\{C_{\mathcal{F}}(U) : A \subset U,\ U \text{ is open}\}.$$

In general, capacities are not additive. The triangle inequality for a norm implies the subadditivity of $C_{\mathcal{F}}$, i.e., for any two sets $A$ and $B$ we have

$$C_{\mathcal{F}}(A \cup B) \leq C_{\mathcal{F}}(A) + C_{\mathcal{F}}(B).$$

**Definition 1.3.** *A function $f$ is said to be $C_{\mathcal{F}}$-quasicontinuous if there exist closed sets $Q_n$ such that $f\,|_{Q_n}$ is continuous for each $n$, and $C_{\mathcal{F}}(X \backslash Q_n) < 1/n$.*

It is known that for any function $f \in W^{r,p}(\mu)$ there exists a $C_{W^{r,p}}$-quasicontinuous $\mu$-version (see [5, Theorem 7.4.6]).

It follows from Lemma 1.1 that for any set $A \subset X$ one has $C_{W^{r,p}}(A) \leq C_{W^{r',p'}}(A)$ when $p \leq p'$, $r \leq r'$.

## 1.4 Convolution of a function with a measure

**Definition 1.4.** *Let $H$ be a separable Hilbert space continuously embedded in $X$. We say that $H$ satisfies condition (T1) if there exists a centered Radon Gaussian measure $\gamma$ on $X$ such that $H \subset H(\gamma)$* (this embedding is automatically continuous).

**Lemma 1.5.** *Let $H \subset X$ be a separable Hilbert space satisfying (T1). Let $f : X \longmapsto \mathbb{R}$ be a universally measurable bounded function. Then the function*

$$F(x) = \int_X f(x+y)\gamma(dy) \tag{1.4}$$

*is infinitely Gâteaux differentiable along $H$, its derivatives $D_H^n F(x)$ are Hilbert–Schmidt operators, and*

$$\|D_H^n F(x)\|_{\mathcal{H}_n(H)} \leq c_n \cdot \|f(x+\cdot)\|_{L^2(\gamma)} \leq c_n \cdot \sup_y |f(y)|,$$

*where $c_n = c^n\sqrt{n!}$, $c$ is the norm of embedding operator $H \hookrightarrow H(\gamma)$.*

*Proof.* Let $H_0$ be the closure of $H$ in $H(\gamma)$, endowed with the norm from $H(\gamma)$. The Hilbert space $H_0$ is separable as well as $H$. Since the embedding $H \hookrightarrow H(\gamma)$ has finite norm $c$, we have

$$\|T\|_{\mathcal{H}_n(H,E)} \leq c^n \cdot \|T\|_{\mathcal{H}_n(H_0,E)}$$

for any Hilbert–Schmidt operator $T : H_0 \longmapsto E$. Therefore, it is sufficient to prove our statement in the case of equal norms of $H$ and $H_0$. The existence of the Gâteaux derivatives follows from [5, Example 2.1.15]. Let $\{e_j\}$ be an orthonormal basis of $H = H_0$.

Let us fix orders of differentiation $k_1, k_2, \ldots$ along $e_1, e_2, \ldots$ respectively; $k_1+\ldots+k_m = n$. Denote $L_m = l.h.\{e_1, \ldots e_m\} = \{\sum_{j=1}^m t_j e_j\}$; $X = X_m \oplus L_m$. Then $\gamma$ can be represented as a product of Gaussian measures on $X_m$ and on $L_m$:

$$\gamma = \gamma_m \otimes \left( \frac{1}{(2\pi)^{m/2}} exp\left[ -\frac{1}{2}\sum_{j=1}^m t_j^2 \right] dt_1 \ldots dt_m \right).$$

By decomposing $y = z + \sum_{j=1}^m t_j e_j$ we find $\partial_{e_1}^{k_1} \ldots \partial_{e_m}^{k_m} F(x) =$

$$= \partial_{t_1}^{k_1} \ldots \partial_{t_m}^{k_m} \int\limits_{X_m}\int\limits_{L_m} f(x+z+\sum_{j=1}^m t_j e_j) \frac{exp\left[-\frac{1}{2}\sum_{j=1}^m t_j^2\right]}{(2\pi)^{m/2}} dt_1 \ldots dt_m \gamma_m(dz) =$$

$$= \int\limits_{X_m}\int\limits_{L_m} f(x+z+\sum_{j=1}^m t_j e_j) \prod_{j=1}^m \left( \partial_{t_j}^{k_j} \frac{exp(-\frac{1}{2}t_j^2)}{\sqrt{2\pi}} \right) dt_1 \ldots dt_m \gamma_m(dz) =$$

$$= (-1)^n \int\limits_X f(x+z+\sum_{j=1}^m t_j e_j) \prod_{j=1}^m \left( \sqrt{k_j!} \cdot H_{k_j}(t_j) \right) \gamma(dy),$$

where $H_k(t)$, $k = 0, 1, 2, \ldots$ are the Hermite polynomials. It is known (see [3]) that the functions $\prod\limits_{j=1}^m H_{k_j}(t_j)$ are orthonormal in $L^2(\gamma)$. By Bessel's inequality we obtain

$$\sum_{i_1=1}^\infty \cdots \sum_{i_n=1}^\infty \left( \partial_{e_{i_1}} \ldots \partial_{e_{i_n}} F(x) \right)^2 =$$

$$= \sum_{m=1}^\infty \sum_{k_1+\ldots+k_m=n} \left( \int\limits_X f(x+y) \prod_{j=1}^m \left( \sqrt{k_j!} \cdot H_{k_j}(t_j) \right) \gamma(dy) \right)^2 \leq$$

$$\leq n! \int\limits_X f(x+y)^2 \gamma(dy) = n! \cdot \|f(x+\cdot)\|^2_{L^2(\gamma)} \leq n! \cdot \sup_y |f(y)|^2.$$

□

**Corollary 1.1.** *The function $F$ constructed in Lemma 1.5 is $\mu$-measurable, and for $p \geq 2$ one has*

$$\|F\|_{L^p(\mu)} \leq \|f\|_{L^p(\mu*\gamma)},$$
$$\|D_H^n F\|_{L^p(\mu,\mathcal{H}_n(H))} \leq c_n \cdot \|f\|_{L^p(\mu*\gamma)}.$$

*Proof.* Let us verify that $F$ is $\mu$-measurable. Let $A \subset X$ be a $\mu$-measurable set. Set

$$\mu_F(A) = \int\limits_A \int f(x+y)\gamma(dy)\mu(dx) \equiv \int f(z)\big((\mu\mid_A) * \gamma\big)(dz).$$

This expression is well-defined since $f$ is universally measurable. The countable additivity of the function $\mu_F$ is easily verified. The measure $\mu_F$ is absolutely continuous with respect to $\mu$, and its Radon–Nikodym density coincides with $F$ $\mu$-a. e.

Let $p \geq 2$. Then

$$\int |F(x)|^p \mu(dx) \leq \int \|f(x+\cdot)\|^p_{L^1(\gamma)} \mu(dx) \leq$$

$$\leq \int \|f(x+\cdot)\|^p_{L^p(\gamma)} \mu(dx) = \|f\|^p_{L^p(\mu*\gamma)};$$

$$\int \|D^n_H F(x)\|^p_{\mathcal{H}(H)} \mu(dx) \leq c_n^p \cdot \int \|f(x+\cdot)\|^p_{L^2(\gamma)} \mu(dx) \leq$$

$$\leq c_n^p \cdot \int \|f(x+\cdot)\|^p_{L^p(\gamma)} \mu(dx) = \left(c_n \cdot \|f\|_{L^p(\mu*\gamma)}\right)^p,$$

which completes the proof. □

## 1.5 Tightness of Sobolev capacities

**Definition 1.5.** *A capacity $C_{\mathcal{F}}$ is tight if for any $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subset X$ such that $C_{\mathcal{F}}(X \backslash K_\varepsilon) < \varepsilon$.*

The first positive result on tightness of Sobolev capacities in the non-Gaussian case is the following theorem [11, Proposition 3.1].

**Theorem 1.1.** *Let $X$ be a separable Banach space with a probability measure $\mu$ and let $H$ be a separable Hilbert space continuously embedded in $X$. Suppose that Sobolev class $W^{1,p}(\mu)$ is well-defined. Then the capacity $C_{W^{1,p}(\mu)}$ is tight.*

A new result of this paper is this.

**Theorem 1.2.** *Let $X$ be a l.c.s. with a Radon probability measure $\mu$. Suppose that for any $\varepsilon > 0$ there exists a metrizable compact set $K^\varepsilon \subset X$ with $\mu(X \backslash K^\varepsilon) < \varepsilon$. Let $H \subset X$ be a separable Hilbert space with (T1). Then the capacity $C_{r,p} = C_{W^{r,p}(\mu)}$ is tight for any $p \in [1; +\infty)$, $r \in \mathbb{N}$, provided that the Sobolev class $W^{r,p}(\mu)$ is well-defined.*

*Proof.* Choose a centered Radon Gaussian measure $\gamma$ with $H(\gamma) \supset H$. Put $\varepsilon = 10^{-j}$ for some $j \in \mathbb{N}$; let $K^\varepsilon$ be a metrizable compact set with $\mu(X \backslash K^\varepsilon) < \varepsilon$ and let $K^\varepsilon_\gamma$ be an absolutely convex metrizable compact set with $\gamma(X \backslash K^\varepsilon_\gamma) < \varepsilon$. Denote by $K_j$ the convex hull of $K^\varepsilon \cup (-K^\varepsilon) \cup K^\varepsilon_\gamma$.

Consider the function $\varphi_j(x) = \gamma(X \backslash (2K_j - x))$. If $x \in K_j$, then $2K_j - x \supset K_j$, hence $\varphi_j(x) \leq \gamma(X \backslash K_j) < \frac{1}{10}$. On the other hand, if $x \notin 2K_j$, then

$$1 - \varphi_j(x) = \gamma(2K_j - x) = \gamma(-2K_j + x),$$

but the compact sets $2K_j - x$ and $-2K_j + x$ are disjoint: if it were not so, there would exist a point $y$ with $y + x \in 2K_j$, $-y + x \in 2K_j$, hence $x \in 2K_j$. Therefore, $\varphi_j(x) \geq \frac{1}{2}$.

The function $\varphi_j$ has the form (1.4) with $f(x) = I_{X \backslash 2K_j}(x)$. By Lemma 1.5 we obtain the Gâteaux differentiability of any order of $\varphi_j$ along all vectors $h \in H$, and the estimate

$$\|D^n_H \varphi_j(x)\|_H \leq c_n \cdot \|I_{X \backslash (2K_j - x)}\|_{L^2(\gamma)},$$

where $c_n = c^n \sqrt{n!}$, $c$ is the norm of the embedding $H \hookrightarrow H(\gamma)$.

Fix a non-decreasing function $\psi \in C^\infty(\mathbb{R})$ such that $\psi \mid_{[0; \frac{1}{10}]} = 0$, $\psi \mid_{[\frac{1}{2}; 1]} = 1$. Denote $g_j = \psi \circ \varphi_j$. Then $1 \geq g_j \geq 0$, $g_j \mid_{X \backslash 2K_j} = 1$, $g_j \mid_{K_j} = 0$, and we obtain

$$\|g_j\|_{L^p(\mu)} \leq \left(\mu(X \backslash K_j)\right)^{1/p} \longrightarrow 0$$

as $j \to \infty$. Moreover, by estimating the gradients of $g_j$ by the chain rule, we have

$$\int \|D_H^n g_j(x)\|_H^p \mu(dx) \le (c_n \cdot 2^n \sup_{[0;1]} \sum_{k=1}^n |\psi^{(k)}|)^p \int \|I_{X\backslash(2K_j - x)}\|_{L^2(\gamma)}^p \mu(dx) \le$$

$$\le C_n(p)\Big[\int_{X\backslash K_j} \mu(dx) + \int_{K_j} \big(\gamma(X\backslash K_j)\big)^{p/2} \mu(dx)\Big] \le$$

$$\le C_n(p)\big(10^{-j} + 10^{-jp/2}\big) \to 0 \qquad \text{as} \quad j \to \infty.$$

Therefore, we obtain $C_{r,p}(X\backslash 2K_j) \le \|g_j\|_{r,p} < \varepsilon$ for $j$ sufficiently large.

But in order to prove that $g_j \in W^{r,p}$, we have to construct a sequence of smooth cylindrical functions converging to $g_j$ in $L^p(\mu)$ and fundamental in the norm $\|\cdot\|_{r,p}$.

Denote by $Y$ the linear span of $\bigcup\limits_{j=1}^{\infty} K_j$. The linear submanifold $Y$ is a Souslin space, therefore, there exists a continuous injective linear operator $I$ that maps $Y$ into the separable Fréchet space $\Phi = \mathbb{R}^\infty$. The $I$-image of $K_j$ is an absolutely convex compact set $Q \subset \Phi$. It is known that a convex compact set $Q$ in a separable Fréchet space $\Phi$ can be represented as the countable intersection of closed half-spaces of the form $\{l_n \le 1\}$, $l_n \in \Phi^*$. Therefore, there exists a sequence of decreasing closed convex cylinders $C_n = \bigcap_{i=1}^n \{l_i \le 1\}$ with $\bigcap_{n=1}^\infty C_n = Q$. The linear functionals $l_i \circ I : X \mapsto \mathbb{R}$ are continuous on $X$. Let

$$B_n = \{x \in X : l_i(I(x)) \le 1, i = 1, \dots, n\}\,; \qquad \text{then} \qquad \bigcap_{n=1}^{\infty} B_n \cap Y = K_j.$$

Consider the functions

$$f_n(x) = \gamma(X\backslash(2B_n - x)).$$

It is easy to check that $f_n$ has the form

$$f_n(x) = u(l_1(x), \dots, l_n(x)), \qquad u \in C_b^\infty(\mathbb{R}^n).$$

Therefore, $f_n \in \mathcal{FC}_b^\infty(X)$. For all $x \in Y$, hence for $\mu$-a. e. $x \in X$, one has $f_n(x) \to \varphi_j(x)$, whence due to the uniform boundedness of the functions $f_n$ it follows by the Lebesgue dominated convergence theorem that $\|f_n - \varphi_j\|_{L^p(\mu)} \to 0$. Let $n > m$. By Corollary 1.1 we obtain

$$\|D_H^n f_n - D_H^n f_m\|_{L^p(\mu,H)} \le C_n \cdot \|I_{B_m\backslash B_n}\|_{L^{\max\{p;2\}}(\mu*\gamma)} \longrightarrow 0$$

uniformly in $n > m$ as $m \to \infty$. So, $\{f_n\}$ is an approximating sequence for $\varphi_j$. Finally, $\{\psi \circ f_n\} \subset \mathcal{FC}_b^\infty(X)$ is an approximating sequence for $g_j = \psi \circ \varphi_j$. □

**Remark 1.1.** The proofs in this paper remain valid if we replace condition (T1) on the Hilbert subspace $H$ by the following condition (T): *there exists a centered Radon probability measure $\lambda$ on $X$ with $H \subset H(\lambda)$.* This condition is slightly weaker than (T1), but in many spaces both conditions are equivalent (e.g., if $X$ is Hilbert or nuclear, or if $X = \mathcal{S}$ or $X = \mathcal{S}'$). Only the proof of Lemma 1.5 becomes longer in this case, see [8].

In the case of a Fréchet space and $r = 1$ it is possible to drop condition (T1) on $H$. Namely, the following theorem generalizes the result obtained by Röckner and Schmuland [11, Proposition 3.1].

**Theorem 1.3.** *Let $X$ be a Fréchet space with a Radon probability measure $\mu$ and let $H$ be a separable Hilbert space continuously embedded in $X$. Suppose that the Sobolev class $W^{1,p}(\mu)$ is well-defined. Then the capacity $C_{1,p}$ is tight.*

See the proof in [8].

# 2 Surface measures

Surface measures in infinite dimensional spaces are interesting for analytical and geometrical measure theory itself, as well as for its applications in non-linear analysis, theory of random processes, and the theory of differential equations with respect to functions and measures on infinite dimensional spaces.

Two approaches to construction of surface measures in infinite dimensional spaces are known. The first one originated in the monograph [12] by A.V.Skorohod and substantially developed in papers [14], [15], [4], [7] is based on construction of a local surface measure in quite small neighborhoods of points. A completely different approach was realized by H.Airault and P.Malliavin [1] and further developped in [2], [9], [10]. By this method a measure is constructed at once on the whole surface, whereas the surface-determining function is smooth in Sobolev sense.

## 2.1 The absolute continuity of images of measures

**Theorem 2.1.** *Suppose that $X$ is a l.c.s. with a finite (possibly, signed) Radon measure $\mu$ and let $H$ be a separable Hilbert space continuously embedded in $X$. If a function $F : X \to \mathbb{R}$ possesses the following properties:*

*a) $F \in W^{2,p}(\mu)$, $p > 4$,*

*b) $\frac{1}{|D_H F|} \in L^p(\mu)$,*

*c) $\mu$ is differentiable along the vector field $v = D_H F$ with $\delta v \in L^2(\mu)$,*

*then the image measure $\mu \circ F^{-1}$ is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}$, and its density $k$ has a continuous version of bounded variation.*

*Proof.* In order to prove the existence of a desired density, we shall construct a measure $\lambda_1$ on $X$ such that for any function $\varphi \in C_b^1(\mathbb{R})$, one has the equality

$$\int \varphi'(t)\mu \circ F^{-1}(dt) \equiv \int \varphi'(F(x))\mu(dx) = -\int \varphi(F(x))\lambda_1(dx), \tag{2.1}$$

which by [5, Proposition 2.2.3(ii)] implies the Skorohod differentiability of the measure $\mu \circ F^{-1}$. We have

$$\int \varphi'(F(x))\mu(dx) = \int \varphi'(F(x))\partial_v F(x)\frac{\mu(dx)}{\partial_v F(x)} =$$

$$= \int \partial_v\big(\varphi(F(x))\big)\frac{\mu(dx)}{|D_H F(x)|^2} = -\int \varphi(F(x)) \cdot d_v\left(\frac{\mu}{|D_H F|^2}\right)(dx). \tag{2.2}$$

In order to justify the last identity, we observe that $|D_H F|^2 \in W^{1,p/2}(\mu)$ due to Lemma 1.1, therefore, the functions $G_n = \big(|D_H F|^2 + \frac{1}{n}\big)^{-1}$ belong to the same class according to Lemma 1.2. We have $G_n \nearrow |D_H F|^{-2} \in L^{p/2}(\mu)$, hence $G_n \to |D_H F|^{-2}$ in the norm of $L^{p/2}(\mu)$. Moreover,

$$D_H G_n = \frac{-D_H \langle D_H F, D_H F\rangle}{(|D_H F|^2 + \frac{1}{n})^2} = \left(\frac{|D_H F|^2}{|D_H F|^2 + \frac{1}{n}}\right)^2 D_H |D_H F|^{-2},$$

where the first factor increases to 1 for $\mu$-a. e. $x$, and the norm of the second factor (independent of $n$) is estimated by

$$\frac{2|D_H^2 F(D_H F)|}{|D_H F|^4} \leq \frac{2\|D_H^2 F\|_{\mathcal{H}_2(H)}}{|D_H F|^3} \in L^{p/4}(\mu).$$

Therefore, the function $|D_HF|^{-2} = \lim_{n\to\infty} G_n$ belongs to $W^{1,2}(\mu)$. By Lemma 1.4, the measure $|D_HF|^{-2}\mu$ is differentiable along $v$, and its derivative is

$$\lambda_1 = d_v\Big(\frac{\mu}{|D_HF|^2}\Big) = \Big(-\frac{2D_H^2F(D_HF, D_HF)}{|D_HF|^4} + \frac{\delta v}{|D_HF|^2}\Big)\mu.$$

The function $\varphi\circ F$ is bounded and belongs to the class $W^{1,p}(\mu)$ (see Lemma 1.2), hence we can choose a sequence of uniformly bounded functions $f_n \in \mathcal{FC}_b^\infty$ converging to $\varphi\circ F$ in the norm $\|\cdot\|_{1,p}$ and $\mu$-a.e. as well. Then, in the integration by parts formula

$$\int \partial_v f_n(x)\frac{\mu}{|D_HF|^2}(dx) = -\int f_n(x)\lambda_1(dx),$$

we can pass to the limit as $n\to\infty$ (by using the Lebesgue theorem) and obtain the same formula for the function $\varphi\circ F$. Identity (2.2) is proved.

Thus, the measure $\mu\circ F^{-1}$ is Skorohod differentiable on $\mathbb{R}$ and $d_1(\mu\circ F^{-1}) = \lambda_1\circ F^{-1}$. This implies (see [5, Example 2.2.1(ii)]) that some version of the density of $\mu\circ F^{-1}$ with respect to Lebesgue measure is of finite variation (not exceeding $\|\lambda_1\|$).

Since $k(t)\to 0$ as $t\to\pm\infty$, one has $\sup_t |k(t)| \le \frac{1}{2}Var\ k \le \frac{1}{2}\|\lambda_1\|$.

But since $\lambda_1 \ll \mu$, we obtain $d_1(\mu\circ F^{-1}) \ll (\mu\circ F^{-1})$, therefore, the measure $\mu\circ F^{-1}$ on $\mathbb{R}$ is Fomin differentiable as well, and for the measure $d_1(\mu\circ F^{-1})$ there exists a density $k'$ with respect to Lebesgue measure such that

$$k(t) = \int_{-\infty}^{t} k'(u)du \qquad \forall t\in\mathbb{R}.$$

In particular, $k(t)$ is continuous. $\square$

**Corollary 2.1.** *Suppose the hypotheses of Theorem 2.1 are fulfilled. Let $g\in W^{1,r}(\mu)$ with some $r>0$ and let $\frac{2}{p}+\frac{1}{r}\le\frac{1}{2}$. Then the measure $g\mu$ is differentiable along $v = D_HF$, and its image $(g\mu)\circ F^{-1}$ has a continuous density $k_g$ whose variation does not exceed* const $\cdot\, \|g\|_{1,r}$.

See the proof in [10, Corollary 1].

## 2.2 Construction and properties of surface measures

**Theorem 2.2.** *Let $X$ be a l.c.s., $\mu$ be a Radon probability measure concentrated on a sequence of metrizable compact sets and let $H\subset X$ be a separable Hilbert space satisfying (T1). Let $F\in W^{2,8}(\mu)$, $|D_HF|^{-1}\in L^8(\mu)$. Suppose that $\mu$ is differentiable along the vector field $v = D_HF$ with $\delta v\in L^2(\mu)$. Then there is a unique Radon measure $\nu$ such that*

$$\int \varphi(x)\nu(dx) = k_\varphi(0), \qquad \forall\varphi\in\mathcal{FC}_b^\infty(X).$$

See the proof in [10, Theorem 2]. In that paper $X$ was supposed to be a separable Fréchet space, but now we can repeat the reasoning from [10] in our more general setting, due to the results in Section 1.

Now let the measure $\mu$ be nonnegative. Let us introduce some additional notation. Let $\nu^{(0)}\equiv\nu$ and $\nu^{(a)}$ be the measures constructed in Theorem 2.2 for the mapping $F-a$ in place of $F$; for all those $a$ with $k(a)\ne 0$ we put $\mu_\sigma^{(a)} = \nu^{(a)}/k(a)$; if $k(a)=0$, we put $\mu_\sigma^{(a)}\equiv 0$.

**Theorem 2.3.** *Let the hypotheses of Theorem 2.2 be fulfilled. Let $F$ be $C_{1,4}$-quasicontinuous. Then the measures $\mu_\sigma^{(a)}$ are conditional measures for $\mu$ with respect to the mapping $F$, i.e.*

*1) for $\mu \circ F^{-1}$-a.e. $a \in \mathbb{R}$, the measure $\mu_\sigma^{(a)}$ is a probability measure concentrated on by the set $F^{-1}(a)$;*

*2) for any Borel set $B \subset X$ one has*

$$\mu(B) = \int_{-\infty}^{+\infty} \mu_\sigma^{(a)}(B) \mu \circ F^{-1}(da).$$

See the proof in [10, Theorem 3].

**Remark 2.1.** Theorems 2.2 and 2.3 can be easily generalized to signed measures. Of course, in the definitions of Sobolev classes and capacities, the measure $\mu$ should be replaced by the measure $|\mu|$. Moreover, in that case $k(t)$ will denote the density of the measure $|\mu| \circ F^{-1}$ with respect to Lebesgue measure.

## 2.3 Gauss–Ostrogradskii formula

Let $\mu$ be a Radon probability measure on a l.c.s. $X$ with a sequence of metrizable compacts $K_n$ with $\mu(K_n) \to 1$ and let a separable Hilbert space $H \subset X$ satisfy condition (T1). Suppose that the Sobolev classes $W^{r,p}(\mu)$ are well-defined for $p$ sufficiently large and $r = 1$, 2.

Let $U \subset X$ be a $\mu$-measurable set such that there exists a $C_{1,4}$-quasicontinuous function $F \in W^{2,12}(\mu)$ with $U = F^{-1}\big((-\infty; 0)\big)$ and $|D_H F|^{-1} \in L^{12}(\mu)$. Let $\mu$ be differentiable along the vector field $D_H F$ and let $\delta(D_H F)$ belong to $L^2(\mu)$.

We shall call the set $\Sigma = F^{-1}(0)$ the *surface* of $U$.

The function $F$ and the measure $\mu$ satisfy the hypotheses of Theorems 2.2 and 2.3. On $\Sigma$, we have the conditional measure $\mu_\sigma^{(0)}$. We define the measure

$$\mu_\sigma^0(dx) = |D_H F(x)| \nu^{(0)}(dx) = k(0) |D_H F(x)| \cdot \mu_\sigma^{(0)}(dx),$$

where a version of $|D_H F|^2 \in W^{1,6}(\mu)$ is chosen to be $C_{W^{1,6}}$-quasicontinuous. This measure has bounded variation.

We shall call the vector field $\{n(x) = |D_H F(x)|^{-1} D_H F(x) : \ x \in \Sigma\}$ the (outward) *normal vector* of $\Sigma$.

**Theorem 2.4.** *Let $u \in W^{1,12}(\mu, H)$ be a vector field having the divergence $\delta u$ with respect to $\mu$. Then the function $\langle n(x); u(x) \rangle$ has a $C_{W^{1,6}}$-quasicontinuous version that is integrable against the measure $\mu_\sigma^0$, and the following Gauss–Ostrogradskii formula holds true:*

$$\int_U \delta u(x) \mu(dx) = \int_\Sigma \langle n(x); u(x) \rangle \, \mu_\sigma^0(dx). \qquad (2.3)$$

*In particular, the right-hand side of (2.3) does not depend on our choice of such a version.*

See the proof in [9, Theorem 4.1].

**Corollary 2.2.** *Consider the vector field*

$$u(x) = \varphi(x) h, \quad \varphi \in W^{1,12}(\mu), \quad h \in H \cap H(\mu).$$

*Then $\delta u$ exists, and the formula (2.3) takes the form*

$$\int\limits_U \Big(\partial_h\varphi(x)+\varphi(x)\beta_h^\mu(x)\Big)\mu(dx)=\int\limits_\Sigma \varphi(x)\,\langle n(x);h\rangle\,\mu_\sigma^0(dx),$$

*where the expression on the right-hand side is $C_{W^{1,6}}$-quasicontinuous.*

**Corollary 2.3.** *Let $H$ be embedded into $H(\mu)$ by a Hilbert–Schmidt operator. Let $f\in W^{2,12}(\mu)$. Then we have the first Green's formula:*

$$\int\limits_U \Delta f(x)\mu(dx)+\int\limits_U\sum_{j=1}^\infty \partial_{e_j}f(x)\beta_{e_j}^\mu(x)\mu(dx)=\int\limits_\Sigma \partial_{n(x)}f(x)\mu_\sigma^0(dx),$$

*where $\Delta f=\sum_j\partial_{e_j}^2 f\in L^1(\mu)$, and the version of $\partial_n f$ is chosen to be $C_{W^{1,6}}$-quasicontinuous.*

See the proof in [9].

**Corollary 2.4.** *Let $\mu$ be a Radon probability measure on a l.c.s. $X$. Suppose that $j_H(X^*)\subset H(\mu)$. Let $F$ and $G$ be two $C_{1,4}$-quasicontinuous functions satisfying the conditions specified at the beginning of this subsection, corresponding to one and the same $U$ as well as $\Sigma$ (up to sets of zero $C_{1,4}$ capacity). Let $\{e_j\}_{j=1}^\infty$ be an orthonormal basis of $H$ such that $e_j\in j_H(X^*)$. Then for all vector fields of the form $u=\varphi e_j$, $\varphi\in\mathcal{FC}_b^\infty(X)$, the formula (2.3) yields*

$$\int\limits_{F^{-1}(0)}\langle n_F(x);u(x)\rangle\,\mu_\sigma^{0(F)}(dx)=\int\limits_{G^{-1}(0)}\langle n_G(x);u(x)\rangle\,\mu_\sigma^{0(G)}(dx),$$

*therefore, for any $j=1,2,\dots$ the following measures coincide:*

$$\langle n_F(x),e_j\rangle\,\mu_\sigma^{0(F)}(dx)=\langle n_G(x),e_j\rangle\,\mu_\sigma^{0(G)}(dx)$$

*(on the set $F^{-1}(0)\triangle G^{-1}(0)$ they vanish). Denote these measures by $\xi_j$. Consider the measure $\lambda=\mu_\sigma^{0(F)}+\mu_\sigma^{0(G)}$. Then $\xi_j=f_j\lambda$. Define the measure $\mu_\sigma^0$ by the formula*

$$\mu_\sigma^0(dx)=\Big(\sum_{j=1}^\infty f_j^2(x)\Big)^{1/2}\lambda(dx).$$

*It is easy to see that $\mu_\sigma^0(dx)=|n_F(x)|\mu_\sigma^{0(F)}(dx)=|n_G(x)|\mu_\sigma^{0(G)}(dx)$, and, since $|n_F|=1\quad\mu_\sigma^{0(F)}$-a.e., $|n_G|=1\quad\mu_\sigma^{0(G)}$-a.e., this implies that $\mu_\sigma^{0(F)}=\mu_\sigma^{0(G)}=\mu_\sigma^0$. The normalizing factor $|D_HF|$ was introduced just for this purpose.*

# 3 Surface measures generated by local Sobolev functions

In applications we often deal with functions that possess Sobolev properties locally. The purpose of this section is to generalize the results obtained in [9], [10] to locally Sobolev functions and surfaces determined by them.

## 3.1 Localizing sequences

Let $\mu$ be a probability measure on a l.c.s. $X$.

**Definition 3.1.** *Let $\mathcal{K}$ be a sequence of compact sets in a l.c.s. $X$. We shall call $\mathcal{K}$ a localizing sequence of order $r$ if:*

*1) $\forall K_1, K_2 \in \mathcal{K}\ \ \exists K_3 \in \mathcal{K}$: $K_1 \cup K_2 \subset K_3$,*

*2) $\mu(X \setminus \bigcup\limits_{K\in\mathcal{K}} K) = 0$,*

*3) for any $K \in \mathcal{K}$ there exists a function $\zeta_K \in W^{r,\infty}(\mu) = \bigcap\limits_{p\geq 1} W^{r,p}(\mu)$ that is $C_{W^{r,p}}$-quasicontinuous for all $p$, and $0 \leq \zeta_K \leq 1$, $\zeta_K\ |_K = 1$, and there exists a compact set $K' \in \mathcal{K}$ such that $\zeta_K\ |_{X\setminus K'} = 0$.*

*We shall call the sequence of functions $\{\zeta_K \mid K \in \mathcal{K}\}$ localizing of order $r$ as well.*

**Lemma 3.1.** *Let $\mathcal{K}$ be a sequence of increasing symmetric convex metrizable compact sets, such that $\mu(X \setminus \bigcup\limits_{K\in\mathcal{K}} K) = 0$, and for any $K \in \mathcal{K}$ there exists $K' \in \mathcal{K}$ such that $2K \subset K'$. Then the sequence $\mathcal{K}$ of compact sets is localizing of any order $r \in \mathbb{N}$ if $H \subset H(\gamma)$ for some centered Radon Gaussian measure $\gamma$, and $\gamma(X \setminus \bigcup\limits_{K\in\mathcal{K}} K) = 0$.*

*Proof.* It follows from our hypotheses that the capacities $C_{r,p}$ are tight for all $r \in \mathbb{N}$, $p \geq 1$. Take the function $g = 1 - g_j$, where $g_j$ is the function constructed in the proof of Theorem 1.2 with $g(x) = 1$ for $x \notin 2K$, $g(x) = 0$ for $x \in K$, $g \in W^{r,p}$ for any $r$, $p$. We can choose a $\mu$-version of the function $g$ to be $C_{r,p}$-quasicontinuous for all $p$ (see [8, Corollary 5.2]). Then we can take $K' \supset 2K$. $\square$

**Definition 3.2.** *A function $f$ is said to belong to the class $W^{r,p}_{Loc}(\mu)$, $p > 2$, if there exists a localizing sequence of functions $\{\zeta_K\}$ of order $r$ such that*

$$\zeta f \in \bigcap_{q<p} W^{r,q}(\mu) \qquad \forall \zeta \in \{\zeta_K\}.$$

The derivatives of $f$ at a point $x \in K \in \mathcal{K}$ are defined by the formula

$$D^k_H f(x) = D^k_H(\zeta_K f)(x), \qquad k = 1, \ldots r.$$

By Lemma 1.3, if we choose another localizing sequence, this value is $\mu$-almost everywhere unchanged.

It follows from Lemma 1.1 that for any $r$ and $p$ one has $W^{r,p}(\mu) \subset W^{r,p}_{Loc}(\mu)$.

**Lemma 3.2.** *If $f \in W^{k,p}_{Loc}(\mu)$, $g \in W^{k,q}_{Loc}(\mu)$, $1/p + 1/q = 1/s < 1/2$, then*

*1) $fg \in W^{k,s}_{Loc}(\mu)$,*

*2) if $k > 0$, then $\langle D_H f; D_H g\rangle \in W^{k-1,s}_{Loc}(\mu)$.*

*Proof.* Let $\{\zeta^f_n\}$ be a localizing sequence of functions of order $k$ for $f$ and let $\{\zeta^g_n\}$ be an analogous sequence for $g$. Then the functions $\zeta_n := \zeta^f_n \zeta^g_n$ provide a localizing sequence for both functions $f$ and $g$. Fix $\zeta = \zeta_n$. Take $s' \in (2; s)$. There exists a function $\xi = \zeta_N$ taking the value 1 on supp$\zeta$. Then $\zeta \equiv \zeta\xi$.

1) Since $\zeta fg \equiv \zeta f \cdot \xi g$, $\zeta f \in W^{k,ps'/s}(\mu)$ and $\xi g \in W^{k,qs'/s}(\mu)$, it follows from by Lemma 1.1(i) that $\zeta fg \in W^{k,s'}(\mu)$.

2) The function $\zeta \langle D_H f; D_H g\rangle$, which coincides $\mu$-almost everywhere with the function $\zeta \langle D_H(\xi f); D_H(\xi g)\rangle$ by the localization property, belongs to the class $W^{k-1,s'}(\mu)$ by Lemma 1.1(ii), since

$$D_H(\xi f) \in W^{k-1,ps'/s}(\mu, H), \quad D_H(\xi g) \in W^{k-1,qs'/s}(\mu, H).$$

The lemma is proved. $\square$

## 3.2 One-dimensional images of measures

Now we can generalize Theorem 2.1 to the case of local Sobolev classes.

**Theorem 3.1.** *Let $\mu$ be a Radon probability measure on a locally convex space $X$; let $H$ be a separable Hilbert space, $H \subset H(\mu)$. Let $F \in W^{2,p}_{Loc}(\mu)$, $|D_H F|^{-1} \in L^p_{Loc}(\mu) \equiv W^{0,p}_{Loc}(\mu)$, where $p > 4$. Suppose there exists a localizing sequence $\mathcal{L}$ of functions of order 2 for $F$ and $|D_H F|^{-1}$. If for any $\zeta \in \mathcal{L}$ the measure $\zeta\mu$ is differentiable along the vector field $D_H F$, and the density $\rho_\zeta$ of the measure $d_{D_H F}(\zeta\mu)$ with respect to the measure $\mu$ belongs to $L^2(\mu)$, then the measure $(\zeta\mu)\circ F^{-1}$ on $\mathbb{R}$ possesses a continuous density $k^\zeta$ (with respect to Lebesgue measure) of bounded variation.*

*Proof.* Let $\zeta, \xi \in \mathcal{L}$ be such that $\xi = 1$ on the support of the function $\zeta$. Then $\xi|D_H F|^2 \in W^{1,2+\varepsilon}(\mu)$. Therefore, the functions $G_n = \xi^2/(\xi|D_H F|^2 + \frac{1}{n})$ belong to $W^{1,2}(\mu)$ by Lemma 1.1 and [5, Lemma 7.1.13]. Applying the same reasoning as in Theorem 2.1, we obtain that the function $\xi|D_H F|^{-2} = \lim_{n\to\infty} G_n$ belongs to the class $W^{1,2}(\mu)$. Therefore, by Lemma 1.4 we obtain the measure

$$\lambda_\zeta = d_{D_H F}\left(\frac{\xi\zeta\mu}{|D_H F|^2}\right) = \left(-\frac{\partial^2_{D_H F} F}{|D_H F|^4}\zeta + \frac{\rho_\zeta}{|D_H F|^2}\right)\mu.$$

Its image $\lambda_\zeta \circ F^{-1}$ on $\mathbb{R}$ is the Fomin derivative of the measure $(\zeta\mu)\circ F^{-1}$, whence the existence of a desired density follows. □

Similarly to Corollary 2.1 we obtain the following result.

**Corollary 3.1.** *Let the hypotheses of Theorem 3.1 be fulfilled and let a constant $r > 1$ satisfy the inequality*

$$\frac{2}{p} + \frac{1}{r} < \frac{1}{2}. \tag{3.1}$$

*Then for any $g \in W^{1,r}_{Loc}(\mu)$, the measure $(\zeta g\mu)\circ F^{-1}$ possesses a density $k^\zeta_g$ of bounded variation if $\zeta \in \mathcal{L}$, where $\mathcal{L}$ is a localizing sequence of functions for $F$ and $g$.*

**Lemma 3.3.** *If $f \in W^{r,p}_{Loc}(\mu)$, then for any $p' < p$ the function $f$ has a $C_{W^{r,p'}}$-quasicontinuous version, provided that $\inf_{K\in\mathcal{K}} C_{W^{r,p'}}(X\backslash K) = 0$.*

*Proof.* Fix $p' < p$. Choose a sequence of compact sets $K_n \in \mathcal{K}$ and functions $\zeta_n = \zeta_{K_n}$ with the following properties: $\mathrm{supp}\zeta_n \subset K_{n+1}$, $C_{W^{r,p'}}(X\backslash K_n) < 2^{-n}$. Put $\xi_1 = \zeta_1$; $\xi_n = \zeta_n - \zeta_{n-1}$ for $n > 1$. For the function $\zeta_{n+1} f \in W^{r,p'}(\mu)$ we choose a $C_{W^{r,p'}}$-quasicontinuous version $g_n$. Then $f_n := \xi_n g_n = \xi_n f$ $\mu$-almost everywhere; $f_n$ vanishes on $K_{n-1}$ and is $C_{W^{r,p'}}$-quasicontinuous, i.e., one can choose closed sets $Q^m_n$, $m \in \mathbb{N}$, such that $Q^m_n \subset Q^{m+1}_n$, $C_{W^{r,p'}}(X\backslash Q^m_n) < 2^{-n-m}$, and $f_n$ is continuous on each $Q^m_n$. Put $f_o(x) = \sum_{n=1}^\infty f_n(x)$. On $K_n$ this sum has only $n$ nonzero summands, and $\mu$-almost everywhere $f_o = \lim_{n\to\infty} \zeta_n f = f$. Moreover, for each $m \in \mathbb{N}$, the function $f_o$ is continuous on the closed set

$$F^m = \bigcap_{n=1}^\infty Q^m_n \cap K_m; \quad F^m \subset F^{m+1}; \quad C_{W^{r,p'}}(X\backslash F^m) < 2^{1-m}.$$

Hence $f_o$ is the desired version of $f$. □

## 3.3 Surface measures

The next theorem generalizes Theorems 2.2 and 2.3:

**Theorem 3.2.** *Let $X$ be a l.c.s. and let the hypotheses of Theorem 3.1 be fulfilled. Then there is a unique family of Radon measures $\{\nu_\zeta^{(a)} \mid \zeta \in \mathcal{L}, a \in \mathbb{R}\}$ such that*

$$\int \varphi(x)\nu_\zeta^{(a)}(dx) = k_\varphi^\zeta(a) \quad \forall \varphi \in \mathcal{FC}_b^\infty(X).$$

*These measures vanish on the sets of zero $C_{1,r}$ capacity with $r$ satisfying condition (3.1).*

*Moreover, if $\inf\limits_{K\in\mathcal{K}} C_{W^{1,2r}}(X\backslash K) = 0$, and if a version of the function $g \in W_{Loc}^{1,2r+\varepsilon}(\mu)$ is chosen to be $C_{W^{1,2r}}$-quasicontinuous, then*

$$\int g(x)\nu_\zeta^{(a)}(dx) = k_g^\zeta(a). \tag{3.2}$$

The proof of this theorem is analogous to the reasoning in the proofs of Theorems 2.2 and 2.3.

**Definition 3.3.** *Let the hypotheses of Theorem 3.2 be fulfilled. Let $A \in \mathcal{B}(X)$. If $A \subset K$ for some $K \in \mathcal{K}$ ($\mathcal{K}$ is a localizing sequence of compact sets of order 2 for the functions $F$ and $|D_H F|^{-1}$), and* $\mathrm{supp}\zeta_K \subset K' \in \mathcal{K}$, *then we set*

$$\nu^{(a)}(A) = \nu_{\zeta_{K'}}^{(a)}(A);$$

*for $p > 12$ we also set*

$$\mu_\sigma^{(a)}(A) = \int\limits_A |D_H F(x)|\nu_{\zeta_{K'}}^{(a)}(dx),$$

*where for the function $|D_H F|^2$ a $C_{W^{1,6}}$-quasicontinuous version is chosen. If $A$ is arbitrary, put*

$$\nu^{(a)}(A) = sup\Big\{\nu^{(a)}(A\cap K): \ K \in \mathcal{K}\Big\};$$

*and similarly for $\mu_\sigma^{(a)}$.*

**Statement 3.1.** *The functions $\nu^{(a)}, \mu_\sigma^{(a)} : \mathcal{B}(X) \to [0; +\infty]$ are well-defined Radon $\sigma$-finite positive measures that are finite on compact sets from $\mathcal{K}$.*

*Proof.* 1) Let $A \subset K_1$, $A \subset K_2$; $\mathrm{supp}\zeta_{K_j} \subset K_j'$, $\xi_j = \zeta_{K_j'}$, $j = 1, 2$. Then $\xi_1 = \xi_2 = 1$ on $K_1' \cap K_2'$. Put $\psi = \xi_1\xi_2$. Let $g$ denote the function $\zeta_{K_1}\zeta_{K_2}$ (taking the value 1 on $K_1 \cap K_2$ and 0 outside $K_1' \cap K_2'$). Since for $j = 1, 2$ $\xi_j \geq \psi$, we have for any $B \in \mathcal{B}(X)$ the estimate $\nu_{\xi_j}^{(a)}(B) \geq \nu_\psi^{(a)}(B)$, hence

$$\int g(x)\nu_{\xi_j}^{(a)}(dx) \geq \int g(x)\nu_\psi^{(a)}(dx), \qquad j = 1, 2,$$

and an equality is possible only if these measures coincide on the set $\{g > 0\}$ that contains $A$. But $g\xi_j \equiv g\psi$, and by Corollary 1, $k_g^{\xi_j}(t) = k_g^\psi(t)$ for almost all, hence, by continuity, for all $t \in \mathbb{R}$, whence by Theorem 3.2 we have

$$\int g(x)\nu_{\xi_j}^{(a)}(dx) = k_g^{\xi_j}(a) = k_g^\psi(a) = \int g(x)\nu_\psi^{(a)}(dx).$$

Therefore, $\nu^{(a)}_{\xi_1}(A) = \nu^{(a)}_{\xi_2}(A)$, so the measure $\nu^{(a)}$ is well-defined at the set $A \subset K \in \mathcal{K}$, hence at an arbitrary set as well.

2) For any $K \in \mathcal{K}$, the Radon measures $\nu^{(a)} |_K = \nu^{(a)}_{\zeta_{K'}} |_K$ are finite. The measures $\mu^{(a)}_\sigma$ are finite on compact sets from $\mathcal{K}$ as well if $p > 12$, since for any $\zeta = \zeta_K$, the function $|D_H F|^2 \in W^{6+\varepsilon}_{Loc}(\mu)$ is integrable with respect to the measure $\nu^{(a)}_\zeta$ by Theorem 3.2. It follows automatically that the measures $\nu^{(a)}$ and $\mu^{(a)}_\sigma$ are $\sigma$-finite and nonnegative.

3) The finite additivity of the measure $\nu^{(a)}$ is obvious, and so is its $\sigma$-additivity in the case of $A = \bigcup_{n=1}^\infty A_n \subset K$, $K \in \mathcal{K}$. Let us verify the $\sigma$-additivity in the general case. Let $A_1, A_2, \ldots$ be disjoint Borel sets, $\nu^{(a)}(A_n) < \infty$, $A = \bigcup_{n=1}^\infty A_n$; let $A_{n,1} \subset A_{n,2} \subset \ldots \subset A_n$, $A_{n,m}$ be subsets of compact sets from $\mathcal{K}$; $\nu^{(a)}(A_{n,m}) \geq \nu^{(a)}(A_n) - 2^{-n-m}$. Then $B_N = \bigcup_{n=1}^N A_{n,N}$ is a subset of a compact set from $\mathcal{K}$, $B_N \subset A$, and

$$\nu^{(a)}(B_N) \nearrow \sum_{n=1}^\infty \nu^{(a)}(A_n) \leq +\infty$$

as $N \to \infty$. Therefore, $\nu^{(a)}(A) \geq \sum\limits_{n=1}^\infty \nu^{(a)}(A_n)$. In the other hand, if $B \subset A$, $B \subset K$, $K \in \mathcal{K}$, then

$$\nu^{(a)}(B) = \sum_{n=1}^\infty \nu^{(a)}(B \cap A_n) \leq \sum_{n=1}^\infty \nu^{(a)}(A_n),$$

which implies the inverse assertion for $\nu^{(a)}(A)$. The proof for the measure $\mu^{(a)}_\sigma$ is just the same. $\square$

Now we can obtain several corollaries from Theorem 3.2.

**Corollary 3.2.** *Let the hypotheses of Theorem 2.2 be fulfilled. Then the measures $\nu^{(a)}$ and $\mu^{(a)}_\sigma$ constructed in this paragraph coincide with the corresponding measures from Theorem 2.2.*

**Corollary 3.3.** *If $\mathcal{K}_1$ and $\mathcal{K}_2$ are two localizing sequences of compact sets of order 2 that satisfy the hypotheses of Theorem 3.2, then the measures $\nu^{(a)}_1$ and $\nu^{(a)}_2$ associated with them are identical. The same is true for the measures $\mu^{(a)}_\sigma$.*

In order to prove this statement, we construct a family of compact sets

$$\mathcal{K}_3 = \mathcal{K}_1 \bigcup \mathcal{K}_2 \bigcup \{K_1 \cup K_2 \mid \quad K_1 \in \mathcal{K}_1, K_2 \in \mathcal{K}_2\}$$

with localizing functions of the form $\zeta_{K_1}, \zeta_{K_2}$, and $\phi(\zeta_{K_1} + \zeta_{K_2})$, where $\phi \in C_b^\infty(\mathbb{R})$ is monotonic, $\phi(0) = 0$, $\phi(t) = 1$ for $t \geq 1$. Then the identities $\nu^{(a)}_1 = \nu^{(a)}_3$, $\nu^{(a)}_2 = \nu^{(a)}_3$ follow from Statement 3.1 (step 1 of the proof), and from the estimates of the localized surface measures via capacities.

**Corollary 3.4.** *Let the hypotheses of Theorem 3.2 be fulfilled. Then for almost every $a \in \mathbb{R}$, the measure $\nu^{(a)}$ is finite, and for any $\varphi \in \mathcal{F}C_b^\infty(X)$ one has*

$$(\varphi\mu) \circ F^{-1}(da) = \int \varphi(x) \nu^{(a)}(dx) da.$$

Indeed, choose an arbitrary subsequence $\{\zeta_n\}\subset\mathcal{L}$ such that $\zeta_n\nearrow 1$ in $L^1(\mu)$. Then for a nonnegative $\varphi\in\mathcal{FC}_b^\infty(X)$ we have

$$\|(\zeta_n\varphi\mu)\circ F^{-1}-(\varphi\mu)\circ F^{-1}\|\le\|\zeta_n\varphi\mu-\varphi\mu\|\to 0\quad\text{as } n\to\infty.$$

The function $\varrho(t)=\lim\limits_{n\to\infty}k_\varphi^{\zeta_n}(t)\le+\infty$ is the $L^1(\mathbb{R})$-limit of the functions $k_\varphi^{\zeta_n}$, since by the monotone convergence theorem one has

$$\int\limits_{\mathbb{R}}\varrho(t)dt=\lim_{n\to\infty}\int\limits_{\mathbb{R}}k_\varphi^{\zeta_n}(t)dt=\lim_{n\to\infty}\int\limits_X\varphi\zeta_n d\mu<\infty.$$

Therefore, $(\varphi\mu)\circ F^{-1}(dt)=\varrho(t)dt$. On the other hand, we have

$$\int\varphi(x)\nu^{(a)}(dx)=\sup\Big\{\int\varphi(x)\nu_\zeta^{(a)}(dx)\ :\ \zeta\in\mathcal{L}\Big\}=$$

$$=\sup\Big\{k_\varphi^\zeta(a)\ :\ \zeta\in\mathcal{L}\Big\}=\varrho(a)$$

for almost all $a\in\mathbb{R}$.

Now we give an example of how the localization method of constructing surface measures works.

**Example 3.1.** Let $X=\mathbb{R}^\infty$ be endowed with the topology of coordinate-wise convergence and let $H=\ell^2$. Consider the probability measure

$$\mu(dx)=\bigotimes_{n=1}^\infty\frac{dx_n}{\pi(1+x_n^2)}.$$

The measure $\mu$ is Radon since $X$ is a separable Fréchet space. One has $H(\mu)=\ell^2$ (see [5, Corollary 3.1.2] or [6, §4.1]). Consider the function $F(x)=\sum_{n=1}^\infty n^{-4}x_n^2$. The functions $F$ and $|D_HF|$ are not integrable, though finite $\mu$-almost everywhere. Let us take the sequence of compact sets

$$\mathcal{K}=\Big\{K_t=\{x\in X:|x_n|\le n^{5/4}t\},\quad t\in\mathbb{N}\Big\}.$$

We have

$$\mu(K_t)=\prod_{n=1}^\infty\frac{2}{\pi}\arctan(n^{5/4}t)\ge 1-\sum_{n=1}^\infty\frac{2}{\pi n^{5/4}t}\to 1\quad\text{as } t\to\infty.$$

The sequence of compact sets $\mathcal{K}$ is localizing of any order. This follows from Lemma 3.1, since $H=H(\gamma)$, where $\gamma$ is a countable product of standard Gaussian measures on the real line. Let $\mathcal{L}=\{\zeta_{K_t},\ t\in\mathbb{N}\}$, be localizing functions of order 2, $\mathrm{supp}\zeta_{K_t}\subset K_{2t}$. We have

$$\mu\{x:|D_HF(x)|<\varepsilon\}\le\mu\Big\{\prod_{n=1}^\infty(-\frac{n^4\varepsilon}{2};\frac{n^4\varepsilon}{2})\Big\}\le\prod_{n=1}^p\frac{n^4\varepsilon}{\pi}=\mathrm{const}(p)\cdot\varepsilon^p$$

for any $p\in\mathbb{N}$, therefore, $|D_HF|^{-1}\in\bigcap\limits_{p\ge1}L^p(\mu)$.

For any $t\in\mathbb{N}$, the functions $F$, $|D_HF|$, and $\|D_H^2F\|_{\mathcal{H}(H)}$ are bounded on $K_{2t}$, and for any $p\ge2$, the function $F$ is approximated with respect to the norm $\|\cdot\|_{2,p}(\mu\mid_{K_{2t}})$ by the cylindrical functions $f_N(x)=\sum_{n=1}^N n^{-4}x_n^2$. Therefore, for all $\zeta\in\mathcal{L}$ one has

$\zeta F \in W^{2,\infty}(\mu)$. The measure $\zeta\mu$ is differentiable along the vector field $D_H F$, since by the uniform boundedness of $\beta^{\mu}_{e_j}$, the series

$$\sum_{j=1}^{\infty}\left(\partial^2_{e_j} F + \partial_{e_j} F \frac{\partial_{e_j}\zeta}{\zeta} + \partial_{e_j} F \cdot \beta^{\mu}_{e_j}\right)\zeta\mu$$

converges in variation, and its sum is the measure $d_{D_H F}(\zeta\mu)$. We have $\rho_\zeta = d_{D_H F}(\zeta\mu)/\mu \in L^2(\mu)$. Therefore, the function $F$, the localizing sequence of functions $\mathcal{L}$ and the measure $\mu$ satisfy the hypotheses of Theorem 3.2, and for all $a > 0$, there exist $\sigma$-finite surface measures $\nu^{(a)}$ and $\mu^{(a)}_\sigma$; obviously, they are zero for $a \leq 0$.

Since $C_{W^{1,p}}(X\backslash K_t) \to 0$ for any $p \geq 1$, formula (3.2) is true for any $C_{W^{1,4}}$-quasicontinuous functions $g \in W^{1,r}_{Loc}(\mu)$ with $r > 4$.

The Gauss–Ostrogradskii formula can be extended to locally Sobolev surfaces and vector fields, too.

**Theorem 3.3.** *Let $X$ be a Fréchet space with a Radon probability measure $\mu$ and let $H \subset H(\mu)$ be a separable Hilbert space. Suppose that $F \in W^{2,p}_{Loc}(\mu)$, $|D_H F|^{-1} \in L^p_{Loc}(\mu)$, $p > 12$, and let $\mathcal{L}$ be their common localizing sequence of functions of order 2, where for any function $\zeta \in \mathcal{L}$, the measure $\zeta\mu$ is differentiable along the vector field $D_H F$ and $d_{D_H F}(\zeta\mu)$ possesses a square-integrable density with respect to $\mu$. Suppose that for the functions $F$ and $|D_H F|^2$ some $C_{W^{1,6}}$-quasicontinuous versions are chosen. Set $U = F^{-1}((-\infty; 0))$, $\Sigma = F^{-1}(0)$. Then for any vector field $u \in W^{1,12}_{Loc}(\mu, H)$ and $\zeta \in \mathcal{L}$ such that the divergence $\delta(\zeta u)$ with respect to $\mu$ exists, the following formula holds true:*

$$\int_U \delta(\zeta u)(x)\mu(dx) = \int_\Sigma \zeta \left\langle u(x), n(x)\right\rangle \mu^{(0)}_\sigma(dx), \tag{3.3}$$

*where for the function $\langle u, n\rangle$ a $C_{W^{1,6}}$-quasicontinuous version is chosen.*

The proof is analogous to that of Theorem 2.4, with the measure $\mu$ replaced by the measure $\xi\mu$, $\xi \in \mathcal{L}$, $\xi\,|_{\mathrm{supp}\zeta} = 1$.

**Corollary 3.5.** *If $u \in W^{1,12}(\mu, H)$, the measure $\mu$ is differentiable along $u$,*

$$\sup\{|D_H\zeta(x)| :\ \zeta \in \mathcal{L},\ x \in X\} = L < \infty,$$

*and $\inf\limits_{K\in\mathcal{K}} C_{W^{1,4}}(X\backslash K) = 0$, then formula (3.3) implies that*

$$\int_U \delta u(x)\mu(dx) = \int_\Sigma \left\langle u(x), n(x)\right\rangle \mu^{(0)}_\sigma(dx), \tag{3.4}$$

*provided that some version of $\langle u(x), n(x)\rangle$ is $\mu^{(0)}_\sigma$-integrable.*

*Proof.* Let $\zeta_n \in \mathcal{L}$ be such that $\zeta_n = 1$ on $K_n$, $\mathrm{supp}\zeta_n = K_{n+1}$, and $C_{1,4}(X\backslash K_n) < 2^{-n}$. For any $x \in K_n$ we have $\delta(\zeta_n u)(x) = \delta u(x)$, therefore, $\delta(\zeta_n u)(x) \to \delta u(x)$ as $n \to \infty$ $\mu$-a.e. Since

$$|\delta(\zeta_n u)(x)| = |\zeta_n \delta u(x) + \left\langle u(x), D_H\zeta_n(x)\right\rangle| \leq |\delta u(x)| + L \cdot |u(x)| \in L^1(\mu),$$

we obtain by the Lebesgue dominated convergence theorem that the left-hand side of (3.3) converges to the left-hand side of (3.4) as $n \to \infty$.

The difference of the right-hand sides is estimated as follows:

$$\left|\int_\Sigma (\zeta_n(x)-1)\,\langle u(x),n(x)\rangle\,\mu_\sigma^{(0)}(dx)\right| \le \int_{\Sigma\cap K_n} |\,\langle u(x),n(x)\rangle\,|\mu_\sigma^{(0)}(dx) \longrightarrow 0$$

as $n\to\infty$ since $\mu_\sigma^{(0)}(X\backslash K_n)\le \mathrm{const}\cdot C_{1,4}(X\backslash K_n)\to 0$. □

Applying formulas (3.3) or (3.4) to constant vector fields $u=e_1$, $u=e_2$,… from some basis of $H$, we obtain the following result.

**Corollary 3.6.** *The measure $\mu_\sigma^{(0)}$ does not depend on our choice of the function $F$ that satisfies the conditions specified and determines the given $U$ and $\Sigma$ up to a set of zero $C_{W^{1,4}}$ capacity.*

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