RANDOM JUMPS IN EVOLVING RANDOM ENVIRONMENT

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Abstract. *We consider a particle moving in $\mathbb{R}^d$ accordingly to a jump Markov process and interacting with an evolving random environment. The latter is represented by a stationary Glauber type dynamics in the continuum. Assuming a low activity-high temperature regime for the Glauber dynamics and small coupling between particle and environment, we obtain the large time asymptotics for the particle position distribution.*

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§1. Introduction and main results.

In this paper we study the asymptotics for large time of the distribution of the position of a tagged particle in $\mathbb{R}^d$ interacting with other particles, described by a random Markov point field in $\mathbb{R}^d$. For this field we assume that it is a stationary birth-and-death Markov process, namely, the equilibrium Glauber type stochastic dynamics of a gas of particles, which was studied in [8, 11].

The study of the asymptotic behavior of lattice random walks in random environments is a quite well developed area of modern Mathematical Physics and stochastics. (See [3,4] for the case of evolving environment. For the problem of fixed environment, which is much more studied, we refer to the recent review [15].) The main novelty of the present paper consists in considering a continuous space random walk in interacting with an equilibrium Markov process on a continuous configuration space.

Passing to precise definitions, we take as state space of the model $\mathbb{R}^d \times \Gamma$, where $\mathbb{R}^d$ is the state space of the tagged particle, and Γ, the state space of the random environment, is the space of all locally finite configurations of points in $\mathbb{R}^d$.

The free random walk of the tagged particle is a jump Markov process in $\mathbb{R}^d$ starting from a given position $x_0 \in \mathbb{R}^d$. The intensity of the jumps from x to y is given by a nonnegative function $a(x-y) \geq 0$, which we assume to be even, continuous, and fast decreasing at infinity. The generator of the corresponding stochastic semigroup of the process is a self-adjoint operator in $L_2(\mathbb{R}^d)$ of the form

$$(L_{RW} f)(x) = \int_{\mathbb{R}^d} a(x-y)(f(y) - f(x))dy, \qquad f \in L_2(\mathbb{R}^d). \tag{1.1}$$

The Fourier transform of the function a,

$$\tilde{a}(\lambda) = \int_{\mathbb{R}^d} a(u)\, e^{i(\lambda,u)}\, du, \qquad \lambda \in \mathbb{R}^d, \tag{1.2}$$

is an even real function, and satisfies the relation $|\tilde{a}(\lambda)| \leq \tilde{a}(0)$. Moreover

$$|\tilde{a}(\lambda)| \; < \tilde{a}(0) \qquad \lambda \neq 0, \tag{1.3a}$$

and the Taylor expansions of $\tilde{a}(\lambda)$ in a neighborhood of $\lambda = 0$ is of the form

$$\tilde{a}(\lambda) \;=\; \tilde{a}(0) - \frac{1}{2} \sum_{i,j=1}^{d} a_{ij} \lambda_i \lambda_j + \mathcal{O}(|\lambda|^4), \tag{1.3b}$$

where the matrix $A = \{a_{ij}\}$ is positive definite.

The free evolution of the random environment is a birth-and-death Markov process with state space Γ. The particles do not move, they only randomly appear and disappear in $\mathbb{R}^d$. A particle configuration (a point of Γ) is denoted γ. The rates of the process (see for more detail [2,11]) are

$$d(x,\gamma) \equiv 1 \quad \text{for death of a particle at } x \in \gamma \tag{1.4a}$$

$$b(x,\gamma) = z e^{-\beta \sum_{y\in\gamma} \phi(x-y)} \text{ for birth of a particle at } x \in \mathbb{R}^d. \tag{1.4b}$$

ϕ is an even interaction potential between particles. The activity z and the inverse temperature β are the parameters of the model.

The corresponding stationary Markov process $\{\gamma_t : t \in \mathbb{R}^1\}$ with the rates (1.4a,b) was constructed in [11, 12], where it was also shown that the stationary measures are Gibbsian measures $\mu_{\beta,z}$ generated by the formal Hamiltonian

$$H(\gamma) = \sum_{x,y\in\gamma} \phi(x-y)$$

with parameters β, z.

We now formulate general assumptions on the parameters β, z and the potential ϕ which guarantee existence and uniqueness of the Gibbs measures $\mu_{\beta,z}$.

1. Integrability.

$$\tilde{C}(\beta) = \int_{\mathbb{R}^d} |1 - e^{-\beta\phi(u)}| du \ < \ +\infty \qquad \text{for any} \ \ \beta > 0 \tag{1.5a}$$

2. Positivity.

$$\phi(u) \geq 0 \qquad u \in \mathbb{R}^d \tag{1.5b}$$

3. Low activity-high temperature regime. We assume that z and β are such that

$$\epsilon := z\tilde{C}(\beta) \ << \ 1, \tag{1.5c}$$

i.e., we assume that ϵ is small enough.

The generator of the process $\{\gamma_t\}$ acting on $L_2(\Gamma, \mu_{\beta,z})$ has the form

$$(L_{RE}F)(\gamma) = \sum_{x\in\gamma}(F(\gamma \setminus x) - F(\gamma)) + z \int_{\mathbb{R}^d} e^{-\beta \sum_{y\in\gamma} \phi(x-y)}(F(\gamma \cup x) - F(\gamma))dx \tag{1.6}$$

The operator L_{RE} is defined on the set of the bounded local functions and under our assumptions is an essentially self-adjoint operator in $L_2(\Gamma, \mu_{\beta,z})$ (see details in [11]).

The interaction of the tagged particle with the random environment is given by a term which modulates the intensity of the jumps of the particle in dependence of the field:

$$a(x-y) \left(1 + \kappa \sum_{u\in\gamma} p(x-u)\right). \tag{1.7}$$

Here κ is a constant which will be assumed to be small enough, and the function $p(u)$ is continuous, nonnegative, even, bounded and rapidly decreasing at infinity. Observe that such interaction implies that the particle moves faster in regions with high concentration of points of the environment. We set

$$p_0 := \max_u p(u) < \infty, \qquad\qquad p_1 := \int_{\mathbb{R}^d} p(u)du. \tag{1.8}$$

The first result to prove is an existence theorem.

Theorem 1.1. *Under the assumptions above on the functions $a(u), p(u)$, the constant κ, and (1.5,a,b,c) on ϕ, β and z, for a.a. choices of the initial data (x_0, γ_0), with respect to the measure $dx_0 \times d\mu_{\beta,z}(\gamma_0)$, there is a Markov process $\{(X_t, \gamma_t), t \geq 0\}$ on the space*

$\mathbb{R}^d \times \Gamma$ *starting at* (x_0, γ_0) . *The generator of the corresponding stochastic semigroup* $S(t)$ *on* $\mathcal{H} = L_2(\mathbb{R}^d) \otimes L_2(\Gamma, \mu_{\beta,z})$ *is*

$$(LF)(x,\gamma) = ((L_{RW} \otimes I_2)F)(x,\gamma) + \kappa \int_{\mathbb{R}^d} a(x-y) \sum_{u\in\gamma} p(x-u)(F(y,\gamma) - F(x,\gamma))\, dy$$

$$+ ((I_1 \otimes L_{RE})F)(x,\gamma) \tag{1.9}$$

where I_1 *and* I_2 *are the identity operators in* $L_2(\mathbb{R}^d)$ *and* $L_2(\Gamma, \mu_{\beta,z}) = L_2(\Gamma)$ *respectively.*

Theorem 1.1 is a particular case of a general problem of constructing processes for a system consisting of a particle in interaction with an equilibrium field. The proof is necessarily rather lengthy, and will be fully published in a separate paper. Here we give only a sketch, which deals with a central point for the existence problem, that of obtaining a convenient bound on the number of jumps of the particle.

Sketch of the proof. We will prove that the number of jumps in any finite time interval is a.e. finite under some simplifying assumptions, namely that the function $a(\cdot), p(\cdot)$, defined in (1.1), 1.7), are finite range (without loss of generality we may assume that they have the same range $r > 0$).

Let $\gamma_s : s \in [0,T)$ be a trajectory of the environment, and $\{(X_0, 0), (X_1, t_1), \dots\}$ be a trajectory of the random walk. It is easy to see that the positions $X_0, X_1, \dots$ are a Markov chain with transition probabilities $P(X_n = x_n | X_{n-1} = x_{n-1}) = \tilde{a}(0)^{-1} a(x_n - x_{n-1})$. The jump time at position $x \in \mathbb{R}^d$ at time $s \in \mathbb{R}_+$ has intensity

$$\lambda(x, \gamma_s) = \tilde{a}(0) \left(1 + \kappa \sum_{u\in\gamma_s} p(x-u) \right).$$

To prove that the random walk is well defined for a.a. trajectories of the environment we need to prove that the number of jumps is finite in any fixed time interval $[0,T)$.

We do this by constructing a random walk with the same trajectories, and intensities $\bar{\lambda}_T(x)$, which depend on the set of births of the trajectory γ_s, $s \in [0,T)$, and are such that $\bar{\lambda}_T(x) \geq \lambda(x, \gamma_s)$ for all $s \in [0,T)$. We then prove that for any random walk trajectory $\sum_{j=0}^{\infty} \bar{\lambda}_T^{-1}(X_j) = \infty$, which implies (see [5]) a finite number of jumps up to time T for the process with rates $\bar{\lambda}_T(x)$. The proof will follow by a simple coupling between the latter process and the original one.

In fact, consider a pure birth process with intensity $z \geq \sup_{x,\gamma} b(x,\gamma)$ starting from the initial configuration γ_0. Denoting the new process by $\tilde{\gamma}$, by a trivial coupling, which should be such that the births of γ are a subset of the births of $\tilde{\gamma}$, we have $\gamma(s) \subseteq \tilde{\gamma}(s)$, almost-surely for any $s \in \mathbb{R}_+$. As $\tilde{\gamma}_{s_1} \subseteq \tilde{\gamma}_{s_2}$ for all $s_1 < s_2$, we have, for all $s \in [0,T)$, $\gamma(s) \subseteq \tilde{\gamma}_T = \gamma(0) \cup \hat{\gamma}_T$, where $\hat{\gamma}_T$ is a Poisson process with intensity zT. Therefore

$$\lambda(x, \gamma(s)) \leq \lambda(x, \tilde{\gamma}_T) = \tilde{a}(0)^{-1} \left(1 + \kappa \sum_{u\in\tilde{\gamma}_T} p(x-u) \right) := \bar{\lambda}_T(x), \tag{1.10a}$$

for all $s \in [0,T)$. By a result of the paper [10] we have, a.s., the following bound

$$|\tilde{\gamma}_T \cap B(x,r)| \leq C_{\tilde{\gamma}_T}(r) \ln(2 + |x|), \tag{1.10b}$$

where $B(x,r)$ is the ball with center $x \in \mathbb{R}^d$ and radius $r > 0$ and $C_{\tilde{\gamma}_T}(r)$ is a constant independent of x. Therefore by (1.10a,b) we have

$$\bar{\lambda}_T(x) \leq \tilde{a}(0)^{-1} \left(1 + \kappa p_0 C_{\tilde{\gamma}_T}(r) \ln(2 + |x|)\right).$$

As $a(\cdot)$ is finite range, $|\bar{\lambda}_T(X_n)| \leq a(0)^{-1} + \bar{C}_{\tilde{\gamma}_T} \ln(2 + nr)$, so that $\sum_{j=0}^{\infty} \bar{\lambda}^{-1}(X_j) = \infty$.

The original random walk can be coupled to the random walk with intensities $\bar{\lambda}_T(x)$ in such a way that the trajectories are the same and at each site of these trajectories the original random walk leaves the site not earlier than the new random walk.

To do this, consider that for a fixed trajectory, the random walk can be determined by assigning to each site x a Poisson process in $[0, \infty)$ which gives the jump times. For the original random walk this is process $\pi_x(s)$ with intensity $\lambda(x, \gamma_s)$, depending on time, and for the new one it is a process $\pi_x^T(s)$ with constant intensity $\bar{\lambda}_T(x)$. We now, for each x in the given trajectory, couple the processes $\pi_x(s)$ and $\pi_x^T(s)$ in such a way that the realizations of the first one are a subset of the realizations of the second one (which is possible because $\lambda(x, \gamma_s) < \lambda_T(x)$). Then the original random walk leaves the starting point x_0 not earlier than the new one, and the same clearly happens at all the subsequent points of the trajectory. Hence the number of jumps of the original random walk is always less than the number of jumps of the new one.

We introduce the normalized displacement $u_t = \frac{X_t - x_0}{\sqrt{t}}$, and for any bounded region $G \subset \mathbb{R}^d$ with piecewise smooth boundary, we consider the probability

$$\begin{aligned} Pr\left(u_t \in G | X_0 = x_0\right) &= \int d\mu_{\beta,z}(\gamma_0) \int \mathbb{I}_G(u_t)\, dP(X_t, \gamma_t | X_0, \gamma_0) \\ &= \int (S(t)\Phi_G)(x_0, \gamma_0) d\mu_{\beta,z}(\gamma_0), \end{aligned} \tag{1.11}$$

where $\Phi_G(u, \gamma) = \mathbb{I}_G(u)$, $\mathbb{I}_G$ is the indicator function of the region G, and $P(X_t, \gamma_t | X_0, \gamma_0)$ is the conditional distribution of the process (X_t, γ_t) at time t under the condition that the initial state is fixed. We want to study the limit of the probability (1.11) as $t \to \infty$.

Theorem 1.2. *Under the conditions above, for κ and ϵ small enough the limit of the probability (1.11) exists and is given by*

$$\lim_{t \to \infty} Pr\left(u_t \in G | X_0 = x_0\right) = \frac{1}{(2\pi)^{\frac{d}{2}} \sqrt{\det \tilde{A}}} \int_G e^{-\frac{1}{2}(\tilde{A}^{-1}\xi, \xi)} d\xi. \tag{1.12a}$$

Here $\tilde{A} = \{\tilde{a}_{ij}\}$ is a real symmetric positive definite matrix, with elements verifying the following estimates

$$\max_{i,j} |\tilde{a}_{ij} - a_{ij}| < C\, \epsilon, \tag{1.12b}$$

where the matrix $\{a_{ij}\}$ is defined by (1.3b), C is an absolute positive constant and ϵ is given by (1.5c).

In what follows we first consider the asymptotics for a more general quantity. Let φ be a bounded function on $\mathbb{R}^d$. Consider the average

$$\mathbb{E}\left(\varphi(u_t)|X_0 = x_0\right) = \int_\Gamma \left(S(t)\Phi^\varphi\right)(x_0, \gamma_0) d\mu_{\beta,z}(\gamma_0) \tag{1.13}$$

where $\Phi^\varphi(u, \gamma) = \varphi(u)$. We first prove that the limit of the quantity $\mathbb{E}(\varphi(u_t)|X_0 = x_0)$ as $t \to \infty$ exists for a sufficiently regular function φ and is given by

$$\lim_{t\to\infty} \mathbb{E}\left(\varphi(u_t)|X_0 = x_0\right) = \frac{1}{(2\pi)^{\frac{d}{2}}\sqrt{\det \tilde{A}}} \int_{\mathbb{R}^d} e^{-\frac{1}{2}(\tilde{A}^{-1}\xi,\xi)}\varphi(\xi)d\xi. \tag{1.14}$$

From this result we then deduce the assertion of Theorem 1.2.

It would be of course interesting to consider as environment process some other Markov processes on the configuration space, such as equilibrium Kawasaki dynamics in the continuum, see [12]. But the problem is that in order to apply the approach developed in our paper one needs first of all a careful analysis of the spectral properties of the environment process. The latter was done in [8] for the case of the Glauber type dynamics in the continuum and this information is used in an essential way in the present paper. Taking into consideration other types of equilibrium infinite particle processes will need, as a preliminary step, an analogous spectral analysis of their generators.

§2. Preliminary constructions.

2.1. Decompositions according to the eigenspaces of the translation group.

We denote the translation operators in $\mathcal{H}$ by $V_s, s \in \mathbb{R}^d$:

$$(V_s f)(x, \gamma) = f(x + s, \gamma + s), \qquad f \in \mathcal{H}, s \in \mathbb{R}^d, \tag{2.1}$$

where $\gamma + s = \{u + s : u \in \gamma\}$ denotes, as usual, the space shift of γ. We have the canonical isomorphism

$$T : \mathcal{H} \to \int_{\mathbb{R}^d} \mathcal{H}_\lambda d\lambda \tag{2.2}$$

and the space $\mathcal{H}_\lambda$ can be identified with the space $L_2(\Gamma, \mu_{\beta,z})$, and T acts according to the formula

$$(T\Phi)_\lambda(\gamma) = \int_{\mathbb{R}^d} \Phi(x, \gamma + x)\, e^{i(\lambda,x)} dx \ := \ \Phi_\lambda(\gamma), \qquad \Phi \in \mathcal{H}. \tag{2.3a}$$

Clearly, the action of the group V_s on $\mathcal{H}_\lambda$ reduces to multiplication by $e^{-i(\lambda,s)}$, i.e.,

$$(TV_s\Phi)_\lambda(\gamma) = e^{-i(\lambda,s)}\Phi_\lambda(\gamma). \tag{2.3b}$$

The inverse transformation T^{-1} acts according to the formula

$$\left(T^{-1}\Phi_\lambda\right)(x, \gamma) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \Phi_\lambda(\gamma - x)e^{-i(\lambda,x)} d\lambda. \tag{2.4}$$

The operators L and $S(t)$ commute with the translation group and are decomposed by the decomposition of the space (2.2), i.e.,

$$L=\int_{\mathbb{R}^d} L_\lambda d\lambda, \qquad S(t)=\int_{\mathbb{R}^d} S_\lambda(t)d\lambda,$$

where the operators L_λ and $S_\lambda(t)$ for $\lambda\in\mathbb{R}^d$ act on $\mathcal{H}_\lambda$, i.e., in $L_2(\Gamma,\mu_{\beta,z})$. Moreover the operators $\{S_\lambda(t), t\geq 0\}$ form a semigroup with generator L_λ given by

$$\begin{aligned}
&(L\Phi)_\lambda(\gamma)=\int_{\mathbb{R}^d} a(u)\left(\Phi_\lambda(\gamma+u)e^{i(\lambda,u)}-\Phi_\lambda(\gamma)\right)du+\sum_{y\in\gamma}\left(\Phi_\lambda(\gamma\setminus\{y\})-\Phi_\lambda(\gamma)\right)+\\
&z\int e^{-\beta\sum_{y\in\gamma}\phi(u-y)}(\Phi_\lambda(\gamma\cup\{u\})-\Phi_\lambda(\gamma))du+\\
&\kappa\sum_{v\in\gamma}p(v)\int a(u)\left(\Phi_\lambda(\gamma+u)e^{i(\lambda,u)}-\Phi_\lambda(\gamma)\right)du \;:=\; (L_\lambda\Phi_\lambda)(\gamma).
\end{aligned}\tag{2.5}$$

2.2. Spectrum of the "free" (unperturbed) generator.

In what follows we will deduce Theorem 1.2 by the analysis of the upper branch of the spectrum of the full generator L. However for a better understanding of the picture of such spectrum we first study the spectrum of the unperturbed generator (for $\kappa=0$)

$$L^{(0)}=L_{RW}\otimes I_2+I_1\otimes L_{RE}\tag{2.6}$$

where I_1, I_2 are the identity operators in the spaces $L_2(\mathbb{R}^d)$ and $L_2(\Gamma,\mu_{\beta,z})$, respectively.

Let by u_s, U_s be the unitary shift operators on $L_2(\mathbb{R}^d)$ and $L_2(\Gamma,\mu_{\beta,z})$, respectively:

$$(u_sf)(x)=f(x+s), \qquad (U_s\Phi)(\gamma)=\Phi(\gamma+s).$$

As it follows immediately from formula (1.1), by Fourier transform the operators L_{RW} and u_s go into multiplication operators, respectively, by the functions $e_0(\lambda):=\tilde{a}(\lambda)-\tilde{a}(0)$ and $e^{-i(\lambda,s)}$, acting on $L_2(\mathbb{R}^d)$. Therefore the spectrum of L_{RW} is the whole interval $[\min_\lambda e_0(\lambda), 0]$.

The spectrum of L_{RE} in $L_2(\Gamma,\mu_{\beta,z})$ (more precisely, its upper branch), was extensively studied in the paper [8]. We briefly recall the results. The operator L_{RE} has the following properties.

1. It has an eigenvector $\Phi_0\equiv 1$ with eigenvalue 0. We denote by $h_0=\{C\Phi_0\}$ the one-dimensional subspace spanned by Φ_0. It is clearly invariant with respect to U_s.

2. It has a "one-particle" subspace h_1, i.e., a subspace invariant with respect to U_s and L_{RE}, on which such operators are both unitarily equivalent to the operators of multiplication by the functions $e^{-i(\lambda,s)}$ and $m(\lambda)$, acting on $L_2(\mathbb{R}^d)$. The function $m(\lambda)$ is a smooth real-valued function which has the form $m(\lambda)=-1+\ell(\lambda)$, where $\max|\ell(\lambda)|<2\epsilon$, and ϵ is the constant in (1.5c).

3. In the orthogonal complement $h_2 = (h_0 + h_1)^\perp$ (also invariant with respect to U_s and L_{RE}) the spectrum of L_{RE} admits for small ϵ the estimate

$$\text{spec } L_{RE}|_{h_2} < -2 + 2\epsilon. \tag{2.7}$$

From the description of the operators L_{RW} and L_{RE}, it follows immediately that the operator $L^{(0)}$ has the following properties.

i) It has an invariant subspace $\mathcal{H}_0 = L_2(\mathbb{R}^d) \otimes h_0 \sim L_2(\mathbb{R}^d)$, invariant also with respect to the group $V_s = u_s \otimes U_s$, on which the operator $L^{(0)}$ acts as the multiplication operator by $e_0(\lambda)$ and the operator V_s acts as the multiplication operator by $e^{i(\lambda,s)}$.

ii) It has an invariant subspace $\mathcal{H}_1 = L_2(\mathbb{R}^d) \otimes h_1 \sim L_2(\mathbb{R}^d \times \mathbb{R}^d)$ in which the operator $L^{(0)}$ acts as multiplication by the function $e_0(\lambda_1) + m(\lambda_2)$, $\lambda_1, \lambda_2 \in \mathbb{R}^d$, and the operator V_s acts as multiplication by $e^{i(\lambda_1+\lambda_2,s)}$.

iii) It has an invariant subspace $\mathcal{H}_2 = L_2(\mathbb{R}^d) \otimes h_2$, invariant both with respect to $L^{(0)}$ and V_s, in which the spectrum of $L^{(0)}$ admits the estimate (2.7).

Looking at equation (2.5) we see that the decomposition (2.2) reduces the operator $L^{(0)}$, as $L^{(0)} = \int L_\lambda^{(0)} d\lambda$, where

$$(L_\lambda^{(0)}\psi)(\gamma) = \int a(u) \left(\psi(\gamma + u)e^{i(\lambda,u)} - \psi(\gamma)\right) du + \sum_{y\in\gamma} (\psi(\gamma \setminus \{y\}) - \psi(\gamma))$$
$$+ z \int e^{-\beta \sum_{y\in\gamma} \phi(u-y)} \left(\psi(\gamma \cup \{u\}) - \psi(\gamma)\right) du.$$

By the description above of the invariant subspaces $\mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_2$ of the operator $L^{(0)}$ the following properties follow for the operators $L_\lambda^{(0)}$.

i') It has an eigenvector $\Phi_\lambda^{(0)} \equiv 1$ with eigenvalue $e_0(\lambda)$.

ii') It has an invariant one-particle subspace $\mathcal{H}_{1,\lambda} \subset L_2(\Gamma, \mu_{\beta,z})$ in which $L_\lambda^{(0)}$ is unitary equivalent to the multiplication operator by the function $e_{1,\lambda}(x) = e_0(x) + m(\lambda - x)$, $x \in \mathbb{R}^d$, acting in the space $L_2(\mathbb{R}^d)$.

iii') It has an invariant subspace $\mathcal{H}_{2,\lambda} \subset L_2(\Gamma, \mu_{\beta,z})$, in which the spectrum of $L_\lambda^{(0)}$ verifies the estimate (2.7).

From the picture drawn above it follows that, for λ in some neighborhood $\mathcal{O}_\delta$ of the point $\lambda = 0$, the operator $L_\lambda^{(0)}$ has an eigenvector with eigenvalue $e_0(\lambda)$, separated from the remaining spectrum and located above it:

$$e_0(\lambda) > r(\lambda) := \max_x e_{1,\lambda}(x), \qquad \max_x e_{1,\lambda}(x) < -1 + 2\epsilon.$$

When $\lambda \notin \mathcal{O}_\delta$ the spectrum of $L_\lambda^{(0)}$ is uniformly separated from zero. We show below that if κ and ϵ are small enough the picture is still valid for the perturbed operator L_λ.

2.3. K-transform formulation.

We now go over to a representation for functions of $L_2(\Gamma, \mu_{\beta,z})$ which is more convenient for computation, and is known as K-transform (see, e.g., [1, 9]).

We denote by Γ_0 the collection of all finite subsets of $\mathbb{R}^d$, and by $\Gamma_0^{(n)}$ the collection of its subsets with n points, $n = 0, 1, \dots$, with $\Gamma_0^{(0)} = \emptyset$, so that $\Gamma_0 = \cup_{n=0}^{\infty} \Gamma_0^{(n)}$. As a set $\Gamma_0^{(n)}$ is equivalent to the factorization

$$\Gamma_0^{(n)} = \widehat{(\mathbb{R}^d)^n}/S_n,$$

where $\widehat{(\mathbb{R}^d)^n} = \{(x_1, \dots, x_n\} : x_k \neq x_j \text{ if } k \neq j\}$, and S_n is the permutation group over $\{1, \dots, n\}$. The space of the finite configurations Γ_0 is equipped with the natural topology of a disjoint union of topological spaces, and the corresponding Borel σ-algebra is denoted $\mathcal{B}(\Gamma_0)$. We define the Lebesgue-Poisson measure on Γ_0 as

$$d\eta = \frac{dx_1 \dots dx_n}{n!}, \qquad \eta = \{x_1, \dots, x_n\} \in \Gamma_0.$$

We say that the function Ψ on the space Γ_0 has bounded support (or is "finite") if one can find a bounded region Λ and an integer non-negative number N such that $\Psi(\eta) = 0$ unless $\eta \subset \Lambda$ and $|\eta| \leq N$. We then consider a mapping (the so-called K-transform) $K : C_{bs}(\Gamma_0) \to L_2(\Gamma, \mu_{\beta,z})$, where $C_{bs}(\Gamma_0)$ is the set of the continuous bounded functions with bounded support on Γ_0:

$$(K\phi)(\gamma) =: G_\phi(\gamma) = \sum_{\eta \subset \gamma} \phi(\eta) \in L_2(\Gamma, \mu_{\beta,z}).$$

Here the notation $\eta \subset \gamma$ denotes the sum over the finite subsets η of the configuration γ. Moreover it was proved (see [9]) that $KerK = \{0\}$, and that, if $\rho(\eta)$ is the correlation function of the measure $\mu_{\beta,z}$ (with the condition $\rho(\emptyset) = 1)$), then

$$\begin{aligned}(G_{\phi_1}, G_{\phi_2})_{L_2(\Gamma,\mu_{\beta,z})} &= \int_{\Gamma_0} (\phi_1 \star \overline{\phi_2})(\eta)\rho(\eta)d\eta \\ &= \int_{\Gamma_0} \sum_{\substack{(\eta_1,\eta_2,\eta_3) \\ \eta=\eta_1\cup\eta_2\cup\eta_3}} \phi_1(\eta_1 \cup \eta_2)\, \overline{\phi_2}(\eta_2 \cup \eta_3)\rho(\eta)d\eta. \qquad (2.8)\end{aligned}$$

The right side of equality (2.8) can be taken as a scalar product in $C_{bs}(\Gamma_0)$, and we denote by $\mathcal{H}^\star$ the closure of $C_{bs}(\Gamma_0)$ with respect to such scalar product. The set of functions in the image $K(C_{bs}(\Gamma_0))$ turns out to be dense in $L_2(\Gamma, \mu_{\beta,z})$. This implies the existence of an extension of K to a canonical unitary transformation between the Hilbert spaces $\mathcal{H}^\star$ and $L_2(\Gamma, \mu_{\beta,z})$ (see [9] for more detail).

Using the notation $\mathbf{1}(\gamma) \equiv 1$ for the function identically equal to 1, observe that (again we refer to [9] for more detail)

$$(K^{-1}\mathbf{1})(\eta) = \delta_{\eta,\emptyset} =: \Psi^{(0)}$$

and therefore, for any $\Phi \in L_2(\Gamma, \mu_{\beta,z})$

$$\int_{\Gamma} \Phi(\gamma) d\mu_{\beta,z}(\gamma) = (\Phi, \mathbf{1})_{L_2(\Gamma,\mu_{\beta,z})} = \int_{\Gamma_0} (K^{-1}\Phi)(\eta)\rho(\eta)d\eta. \tag{2.9}$$

The operator L_λ turns, under the K-transform, into the unitary equivalent of the operator $\hat{L}_\lambda = K^{-1}L_\lambda K$ acting on functions of $C_{bs}(\Gamma_0) \subset \mathcal{H}^\star$ as follows:

$$\begin{aligned}\left(\hat{L}_\lambda \Psi\right)(\eta) = &\int_{\mathbb{R}^d} a(s)\Psi(\eta + s)e^{i(\lambda,s)}ds - \tilde{a}(0)\Psi(\eta) - |\eta|\Psi(\eta) + \\ &z \sum_{\gamma \subseteq \eta} \int_{\mathbb{R}^d} \Psi(\gamma \cup \{\tilde{x}\}) \prod_{v \in \eta \setminus \gamma} (e^{-\beta\phi(\tilde{x}-v)} - 1) \prod_{u \in \gamma} e^{-\beta\phi(\tilde{x}-u)} d\tilde{x} + \\ &\kappa \int_{\mathbb{R}^d} a(s) \left(\sum_{x\in\eta} p(x)\Psi(\eta \setminus \{x\} + s)\right) e^{i(\lambda,s)}ds - \kappa\tilde{a}(0) \left(\sum_{x\in\eta} p(x)\Psi(\eta \setminus \{x\})\right) + \\ &\kappa \sum_{x\in\eta} p(x) \int_{\mathbb{R}^d} a(s)\Psi(\eta + s)e^{i(\lambda,s)}ds - \kappa\tilde{a}(0) \sum_{x\in\eta} p(x)\Psi(\eta).\end{aligned} \tag{2.10}$$

Here $\eta + s$, $s \in \mathbb{R}^d$, denotes the shift of the configuration η. The operator $\hat{L}_\lambda$, for any $\lambda \in \mathbb{R}^d$, is the generator of a stochastic semigroup $\{\hat{S}_\lambda(t), t \geq 0\}$, with $\hat{S}_\lambda(t) = K^{-1}S_\lambda(t)K$.

§3. Basic lemmas and proof of Theorem 1.2.

Let $\delta > 0$ be such that the set $\mathcal{O}_\delta = \{\lambda \in \mathbb{R}^d : e_0(\lambda) > -\delta\}$ is a connected neighborhood of the origin. By conditions (1.3a,b), if δ is small enough, $\mathcal{O}_\delta$ does not contain more than one critical point of the function $e_0(\lambda)$ and $\sup_{\lambda \notin \mathcal{O}_\delta} e_0(\lambda) \leq -\delta$. Clearly if such properties of $\mathcal{O}_\delta$ hold for some value $\delta = \delta_0 > 0$, they also hold for all values $\delta \in (0, \delta_0)$.

Lemma 3.1. *Under the conditions listed above, if κ is small enough, one can find a positive number δ_1 so small that the following assertions hold:*

i) for any $\lambda \in \mathcal{O}_{\delta_1}$ the space $\mathcal{H}^\star$ can be decomposed in a (non-orthogonal) sum of two closed subspaces

$$\mathcal{H}^\star = \mathcal{H}_\lambda^{(0)} + \mathcal{H}_\lambda^{(1)}, \tag{3.1}$$

where $\mathcal{H}_\lambda^{(0)}$ is a one-dimensional eigenspace of the operator $\hat{L}_\lambda$ spanned by an eigenvector of the form

$$h_\lambda^{(0)}(\eta) = \delta_{\eta,\emptyset} + \chi_\lambda(\eta), \tag{3.2a}$$

with eigenvalue $q(\lambda)$, to be described below. Moreover the vector $\chi_\lambda \in \mathcal{H}^\star$ is smooth in λ (or analytic) and such that for some constant C and $\lambda \in \mathcal{O}_{\delta_1}$

$$\|\chi_\lambda\|_{\mathcal{H}^\star} < C\,|\lambda|^2, \qquad \chi_\lambda(\emptyset) = 0. \tag{3.2b}$$

ii) If the vector $\Psi^{(0)}(\eta) = \delta_{\eta,\emptyset}$ is represented as a sum of vectors of $\mathcal{H}_\lambda^{(0)}$ and $\mathcal{H}_\lambda^{(1)}$

$$\Psi^{(0)} = c(\lambda) h_\lambda^{(0)} + \bar{\Psi}_\lambda^{(0)}, \qquad \bar{\Psi}_\lambda^{(0)} \in \mathcal{H}_\lambda^{(1)} \tag{3.3a}$$

then $c(\lambda)$ is a smooth function (or analytic) which for small λ can be represented in the form $c(\lambda) = 1 + \mathcal{O}(|\lambda|^2)$, and the norm of $\bar{\Psi}_\lambda^{(0)}$ is uniformly bounded on $\mathcal{O}_{\delta_1}$:

$$\|\bar{\Psi}_\lambda^{(0)}\| < C, \qquad \lambda \in \mathcal{O}_{\delta_1}. \tag{3.3b}$$

iii) The spectrum of the operator $L_\lambda^{(1)} := \hat{L}_\lambda|_{\mathcal{H}_\lambda^{(1)}}$ on the space $\mathcal{H}_\lambda^{(1)}$ lies in the rectangular region of the complex z-plane

$$\mathcal{R} = \{z : \mathbf{Re}\, z < -\alpha,\ |\mathbf{Im} z| < 1\}$$

where $\alpha > 0$ is a constant, which will be given below .

For $z \notin \mathcal{R}$ the resolvent $(L_\lambda^{(1)} - z)^{-1}$ is uniformly bounded, i.e.,

$$\|(L_\lambda^{(1)} - z)^{-1}\| < C, \qquad \lambda \in \mathcal{O}_{\delta_1}, \quad z \notin \mathcal{R}, \tag{3.4}$$

where C is a constant independent of λ and z.

iv) The eigenvalue $q(\lambda)$, for $\lambda \in \mathcal{O}_{\delta_1}$, is a smooth real function (or analytic) with expansion

$$q(\lambda) = -\frac{1}{2} \sum_{i,j=1}^{d} \hat{a}_{ij} \lambda_i \lambda_j + \mathcal{O}(|\lambda|^4), \qquad \lambda \in \mathcal{O}_{\delta_1}, \tag{3.5a}$$

where the matrix $\widehat{\mathcal{A}} = \{\hat{a}_{ij}\}$ is positive definite and for some absolute constant $C > 0$

$$\max_{i,j} |\hat{a}_{ij} - a_{ij}| < C\,\epsilon, \tag{3.5b}$$

the elements $\{a_{i,j}\}$ being defined by (1.3b). Moreover $h_\lambda^{(0)}$ is a real eigenvector.

The functions appearing on lemma 3.1 for which we write an alternative are either smooth or analytic according to whether the operator $\hat{L}_\lambda$ in (2.10) is smooth or analytic in λ (see [13,14]). It is easy to see that $\hat{L}_\lambda$ inherits the properties of $\tilde{a}(\lambda)$.

For the proof of Lemma 3.1, see §4

Corollary 1. *For $\lambda \in \mathcal{O}_{\delta_1}$ the operators $S_\lambda^{(1)}(t) := \hat{S}_\lambda(t)|_{\mathcal{H}_\lambda^{(1)}}$ are contracting for t large enough:*

$$\|S_\lambda^{(1)}(t)\Psi\| \le C\, e^{-\bar{\alpha}t}\|\Psi\|, \qquad \Psi \in \mathcal{H}_\lambda^{(1)}, \tag{3.6}$$

where the positive constants C and $\bar{\alpha} > \frac{\alpha}{2}$ are independent of λ, t and Ψ.

Proof of the Corollary. We use the resolvent representation of the semigroup $S_\lambda^{(1)}(t)$:

$$S_\lambda^{(1)}(t)\Psi = \frac{1}{2\pi i}\int_{\mathcal{S}} e^{tz}\left(L_\lambda^{(1)} - z\,I\right)^{-1}\Psi\, dz. \tag{3.7}$$

Here I is the identity operator and the integral is along a contour $\mathcal{S}$ in the complex plane which goes around the spectrum of the operator $L_\lambda^{(1)}$. According to point iii) of the preceding lemma, we choose the contour as the boundary of the larger rectangle

$$\mathcal{S} = \left\{\mathbf{Re}\; z = -\frac{\alpha}{2}, |\mathbf{Im}\; z| < 2\right\} \cup \left\{\mathbf{Re}\; z < -\frac{\alpha}{2}, |\mathbf{Im}\; z| = 2\right\}.$$

It is easy to deduce the estimate

$$\begin{aligned}\left\|\int_{\mathcal{S}} e^{zt}\left(L_\lambda^{(1)} - z\,I\right)^{-1}\Psi dz\right\| \;&<\; C\,\|\Psi\|\int_{\mathcal{S}}\mathbf{Re}\; e^{zt}|dz| \;<\; C\,\|\Psi\|\left(2e^{-\frac{\alpha}{2}t} + 2\int_{\frac{\alpha}{2}}^{\infty} e^{-tx}dx\right)\\ &\le\; \bar{C}\,\|\Psi\|e^{-\frac{\alpha}{2}t},\end{aligned}$$

for some positive constant $\bar{C}$. This implies (3.6).

Lemma 3.2. *For $\lambda \notin \mathcal{O}_{\delta_1}$ the spectrum of the operator $\hat{L}_\lambda$ in $\mathcal{H}^\star$ lies inside the rectangular region*

$$\tilde{\mathcal{R}} = \{\mathbf{Re}\; z < -\alpha', |\mathbf{Im}\; z| < \beta'\},$$

for some positive constants α', β', and the resolvent $(\hat{L}_\lambda - z)^{-1}$ is uniformly bounded for z outside $\tilde{\mathcal{R}}$:

$$\left\|(\hat{L}_\lambda - z\,I)^{-1}\right\| \;<\; C', \tag{3.8}$$

where the constant C' does not depend on $\lambda \notin \mathcal{O}_{\delta_1}$ and $z \notin \tilde{\mathcal{R}}$.

For the proof of Lemma 3.2, see §4

Corollary 2. *The operator $\hat{S}_\lambda(t)$, for $\lambda \notin \mathcal{O}_{\delta_1}$ and t large enough , is contracting:*

$$\|\hat{S}_\lambda(t)\Psi\| < \; C\, e^{-\frac{\alpha' t}{2}}\|\Psi\|, \tag{3.9}$$

Proof. The proof follows from Lemma 3.2, in analogy with the proof of Corollary 1.

Proof of Theorem 1.2. The conditional average (1.13) may be written in the form

$$\mathbb{E}(\varphi(u_t)|X_0 = x_0) = \int d\mu_{\beta,z}(\gamma)\int_{\mathbb{R}^d}(S_\lambda(t)\hat{\Phi}^\varphi_\lambda)(\gamma - x_0)e^{-i(\lambda,x_0)}d\lambda, \tag{3.10a}$$

where, denoting by $\tilde{\varphi}$ the Fourier transform of φ, we have set

$$\hat{\Phi}^\varphi_\lambda(\gamma) = (T\Phi^\varphi)_\lambda(\gamma) = \tilde{\varphi}(\lambda)\mathbf{1}(\gamma), \tag{3.10b}$$

where Φ^φ is as in formula (1.13).

We go over to the space $\mathcal{H}^\star_\lambda$ by applying the K-transform, and, by formula (2.9) the integral on the right in (3.10a) can be written as

$$\int \rho(\eta)d\eta\int_{\mathbb{R}^d}\left(\hat{S}_\lambda(t)\Psi^\varphi_\lambda\right)(\eta - x_0)e^{-i(\lambda,x_0)}d\lambda \tag{3.11a}$$

$$\Psi^\varphi_\lambda(\eta) = \left(K^{-1}\Phi^\varphi_\lambda\right)(\eta) = \tilde{\varphi}(\lambda)\Psi^{(0)}(\eta). \tag{3.11b}$$

Let w_1, w_2 be two smooth nonnegative functions which provide a decompostion of unity relative to the neighborhood of the origin $\mathcal{O}_{\delta_1} \subset \mathbb{R}^d$ mentioned in Lemma 3.1, i.e., such that $w_1(\lambda) + w_2(\lambda) = 1$ and

$$w_1(\lambda) = \begin{cases} 1 & \lambda \in \mathcal{O}_{\frac{\delta_1}{2}} \\ 0 & \lambda \notin \mathcal{O}_{\delta_1}\end{cases} \qquad\qquad w_2(\lambda) = \begin{cases} 0 & \lambda \in \mathcal{O}_{\frac{\delta_1}{2}} \\ 1 & \lambda \notin \mathcal{O}_{\delta_1}\end{cases}.$$

The integral (3.11a) becomes

$$\begin{aligned}&\int w_1(\lambda)e^{-i(\lambda,x_0)}\tilde{\varphi}(\lambda)d\lambda\int_{\Gamma_0}\left(\hat{S}_\lambda(t)\Psi^{(0)}\right)(\eta - x_0)\rho(\eta)d\eta\\ &+\int w_2(\lambda)e^{-i(\lambda,x_0)}\tilde{\varphi}(\lambda)d\lambda\int_{\Gamma_0}\left(\hat{S}_\lambda(t)\Psi^{(0)}\right)(\eta - x_0)\rho(\eta)d\eta.\end{aligned} \tag{3.12}$$

The first integral is over $\lambda \in \mathcal{O}_{\delta_1}$ and we can use the expansion (3.3a) so that, as $h^{(0)}_\lambda$ is an eigenvector of $\hat{S}_\lambda(t)$ with eigenvalue $e^{q(\lambda)t}$, we find

$$\left(\hat{S}_\lambda(t)\Psi^{(0)}\right)(\eta) = c(\lambda)e^{q(\lambda)t}h^{(0)}_\lambda(\eta) + \left(\hat{S}_\lambda(t)\bar{\Psi}^{(0)}_\lambda\right)(\eta). \tag{3.13}$$

By translation invariance of the Gibbs measure, by Corollary 1 and (3.3b), the contribution of the second term in (3.13) can be estimated as

$$\left|\int \left(\hat{S}_\lambda(t)\bar{\Psi}_\lambda^{(0)}\right)(\eta - x_0)\rho(\eta)d\eta\right| = \left|\int \left(\hat{S}_\lambda(t)\bar{\Psi}_\lambda^{(0)}\right)(\eta)\rho(\eta)d\eta\right| = \left|\left(\hat{S}_\lambda(t)\bar{\Psi}_\lambda^{(0)}, \Psi^{(0)}\right)_{\mathcal{H}^\star}\right|$$
$$\leq \|\hat{S}_\lambda(t)\bar{\Psi}_\lambda^{(0)}\| \, \|\Psi^{(0)}\| \leq \bar{C}e^{-\frac{\alpha}{2}t},$$

where $\bar{C}$ is an absolute constant. Therefore, setting $g(\lambda) = \int h_\lambda(\eta)\rho(\eta)d\eta$, the first term in (3.12) is equal to

$$\int_{\mathcal{O}_{\delta_1}} w_1(\lambda)e^{-i(\lambda,x_0)}\tilde{\varphi}(\lambda)c(\lambda)g(\lambda)e^{q(\lambda)t}e^{-i(\lambda,x)}d\lambda + \mathcal{O}(e^{-\frac{\alpha}{2}t}). \tag{3.14}$$

Observe that, in force of (2.9) and (3.2a,b), we have $g(0) = 1$.

The second integral in (3.12), the one containing w_2, is easily estimated with the help of Corollary 2, and gives a term which falls off exponentially in time.

By applying to the integral (3.14) the standard methods of proof of the integral limit theorem (see [6]), and using the expansion (3.5a) of $q(\lambda)$, we find that the limit as $t \to \infty$ of the first term in (3.12) has the form (1.14). This proves relation (1.14).

The proof of the theorem is obtained by choosing two smooth functions φ_+, φ_-, which approximate the indicator function $\mathbb{I}_G$ from above and from below. As one can choose such function close enough to each other, the integrals (1.14) of such functions are as close as we want, and we can conclude that the limit exists and is given by the integral on the right of (1.12a).

§4. Proof of Lemmas 3.1 and 3.2.

We introduce the auxiliary Banach space $\mathcal{L}_M$, for $M = \max\{4, \frac{1}{\tilde{C}(\beta)}\}$, where $\tilde{C}(\beta)$ is defined by (1.5a), as the closure of the space $C_{bs}(\Gamma_0)$ with respect to the norm

$$\|G\|_M := \int \sup_{\substack{\eta\in\Gamma_0 \\ \eta\cap\xi=\emptyset}} \left[\frac{1}{3^{|\eta|}}(|\eta| + |\xi|)|G(\eta\cup\xi)|\right] M^{|\xi|}d\xi + |G(\emptyset)|. \tag{4.1}$$

Proposition 4.1. *The space $\mathcal{L}_M$ is an everywhere dense subset of $\mathcal{H}^\star$ and*

$$\|G\|_{\mathcal{H}^\star} \leq \|G\|_M, \qquad G \in \mathcal{L}_M. \tag{4.2}$$

Proof. For the proof see [8].

If $\mathcal{D}_\lambda \subset \mathcal{H}^\star$ is the domain of the operator $\hat{L}_\lambda$ in $\mathcal{H}^\star$, the domain of $\hat{L}_\lambda$ in $\mathcal{L}_M$ is

$$\tilde{\mathcal{D}}_\lambda := \{G \in \mathcal{L}_M \cap \mathcal{D}_\lambda : \hat{L}_\lambda G \in \mathcal{L}_M\} \subset \mathcal{L}_M.$$

As $C_{bs}(\Gamma_0) \subset \tilde{\mathcal{D}}_\lambda$, the domain $\tilde{\mathcal{D}}_\lambda$ is everywhere dense in $\mathcal{H}^\star$

We represent $\mathcal{L}_M$ as the direct sum of two subspaces

$$\mathcal{L}_M = \mathcal{L}^{(0)} + \mathcal{L}^{\geq 1} \tag{4.3}$$

where $\mathcal{L}^{(0)} := \text{span}\{\Psi^{(0)}\}$ is the one-dimensional span of the vector $\Psi^{(0)}(\eta) = \delta_{\eta,\emptyset}$ and $\mathcal{L}^{\geq 1} := \{\Psi \in \mathcal{L}_M : \Psi(\emptyset) = 0\}$. The operator $\hat{L}_\lambda$ is then represented as a matrix:

$$\hat{L}_\lambda = \begin{pmatrix} L_\lambda^{11} & L_\lambda^{12} \\ L_\lambda^{21} & L_\lambda^{22} \end{pmatrix} \tag{4.4}$$

where $L_\lambda^{11} : \mathcal{L}^{(0)} \to \mathcal{L}^{(0)}$, $L_\lambda^{12} : \mathcal{L}^{\geq 1} \to \mathcal{L}^{(0)}$, etc. By the representation (2.10) we have

$$L_\lambda^{11}\Psi^{(0)}(\eta) = e_0(\lambda)\Psi^{(0)}(\eta) \tag{4.5a}$$

$$\left(L_\lambda^{12}\Psi\right)(\emptyset) = z\int_{\mathbb{R}^d} \Psi(\{u\})du, \qquad \Psi \in \mathcal{L}^{\geq 1}, \tag{4.5b}$$

$$\left(L_\lambda^{21}\Psi\right)(\eta) = \begin{cases} \kappa p(x)e_0(\lambda)\Psi(\emptyset), & \Psi \in \mathcal{L}^{(0)}, \quad \eta = \{x\} \\ 0 & \Psi \in \mathcal{L}^{(0)}, \quad |\eta| \geq 2 \end{cases}. \tag{4.5c}$$

As for L_λ^{22} we have (here $|\eta| \geq 1$!)

$$\begin{aligned}
&\left(L_\lambda^{22}\Psi\right)(\eta) = \int a(s)\Psi(\eta+s)e^{i(\lambda,s)}ds - \tilde{a}(0)\Psi(\eta) \; - \; |\eta|\Psi(\eta) + \\
&z\sum_{\gamma\subseteq\eta}\int \Psi(\gamma\cup\{\tilde{x}\}) \prod_{v\in\eta\setminus\gamma}(e^{-\beta\phi((\tilde{x}-v)}-1)\prod_{u\in\gamma}e^{-\beta\phi(\tilde{x}-u)}d\tilde{x} + \\
&\kappa\, \mathbb{I}_{\{|\eta|>1\}}(\eta)\left[\int a(s)\left(\sum_{x\in\eta}p(x)\Psi(\eta\setminus\{x\}+s)\right)e^{i(\lambda,s)}ds - \tilde{a}(0)\sum_{x\in\eta}p(x)\Psi(\eta\setminus\{x\})\right] + \\
&\kappa\sum_{x\in\eta}p(x)\left(\int a(s)\Psi(\eta+s)e^{i(\lambda,s)}ds - \tilde{a}(0)\Psi(\eta)\right),
\end{aligned} \tag{4.5d}$$

$\mathbb{I}_{\{\cdot\}}$ being the indicator function. For $\eta = \{x\}$ the third line disappears and the last line is

$$\kappa p(x)\left(\int a(s)\Psi(\{x\}+s)e^{i(\lambda,s)}ds - \tilde{a}(0)\Psi(\{x\})\right). \tag{4.5e}$$

To prove Lemma 3.1, we need to find an eigenvector of $\hat{L}_\lambda$ of the form $h_\lambda^{(0)} = \Psi^{(0)} + \chi_\lambda$, with $\chi_\lambda(\emptyset) = 0$. We shall prove that there is a unique function $\chi_\lambda \in \mathcal{L}_M$ such that $\Psi^{(0)} + \chi_\lambda$ is an eigenvector of $\hat{L}_\lambda$. In fact, the eigenvalue equation, in analogy to what is done in [8], leads to the following equation for χ_λ

$$\chi_\lambda = -(L_\lambda^{22})^{-1}L_\lambda^{21}\Psi^{(0)} + (L_\lambda^{11}\Psi^{(0)} + L_\lambda^{12}\chi_\lambda)(\emptyset)(L_\lambda^{22})^{-1}\chi_\lambda \tag{4.6a}$$

and the eigenvalue has the expression

$$q(\lambda) = \left(L_\lambda^{11}\Psi^{(0)} + L_\lambda^{12}\chi_\lambda\right)(\emptyset). \tag{4.6b}$$

To prove the existence of χ_λ we estimate the norms of the operators in equation (4.6a).

Lemma 4.2. *For all $\lambda \in \mathcal{O}_{\delta_1}$, ϵ small enough, and κ such that*

$$\kappa < \frac{\tilde{C}(\beta)}{\tilde{a}(0)(\tilde{a}(0)+1)(8p_0\tilde{C}(\beta)+4p_1)} \tag{4.7}$$

we have

$$|||L_\lambda^{11}||| = |e_0(\lambda)| \tag{4.8a}$$

$$|||L_\lambda^{12}||| < \epsilon \tag{4.8b}$$

$$|||L_\lambda^{21}||| = \kappa|\tilde{a}(\lambda) - \tilde{a}(0)|\left(\frac{p_0}{3} + Mp_1\right) = C_1|e_0(\lambda)| \tag{4.8c}$$

$$|||(L_\lambda^{22})^{-1}||| < C_2 \tag{4.8d}$$

where $|||\cdot|||$ denotes the operator norm generated by the norm $\|\cdot\|_M$ in the Banach space $\mathcal{L}_M$, and C_1, C_2 are constants which do not depend on $\lambda \in \mathcal{O}_{\delta_1}$.

Proof. The proof is deferred to the Appendix.

We denote by $\mathcal{F}_\lambda(\chi)$ the right side of (4.6a) and consider it as a map of $\mathcal{L}^{\geq 1}$ into itself, $\mathcal{F}_\lambda(\chi) : \mathcal{L}^{\geq 1} \to \mathcal{L}^{\geq 1}$. By $\mathcal{B}_r \subset \mathcal{L}^{\geq 1}$ we denote the open ball of radius r:

$$\mathcal{B}_r = \{\chi \in \mathcal{L}^{\geq 1} : \|\chi\|_M < r\}.$$

In what follows $\delta > 0$ is a number so small that $\mathcal{O}_\delta$ satisfies the properties required at the beginning of §3.

Lemma 4.3 *For all ϵ and κ small enough one can find $0 < \delta_1 = \delta_1(\epsilon) < \delta$, and for all $\lambda \in \mathcal{O}_{\delta_1}$ a ball $\mathcal{B}_r \subset \mathcal{L}^{\geq 1}$ of radius r, invariant with respect to the map $\mathcal{F}_\lambda$, and such that for $\chi_1, \chi_2 \in \mathcal{B}_r$ the inequality*

$$\|\mathcal{F}_\lambda(\chi_1) - \mathcal{F}_\lambda(\chi_2)\|_M \leq c\,\|\chi_1 - \chi_2\|_M, \tag{4.9}$$

holds for some constant $c \in (0,1)$, independent of $\lambda \in \mathcal{O}_{\delta_1}$.

Proof. From the expression (4.6a) of $\mathcal{F}_\lambda$ and the estimates (4.7), (4.8a,b,c,d), one can see that if r and $\lambda \in \mathcal{O}_{\delta_1}$ verify the inequality

$$C_1C_2|e_0(\lambda)| + C_2(|e_0(\lambda)| + \epsilon r)\, r \ < \ r$$

then the ball $\mathcal{B}_r$ is mapped by $\mathcal{F}_\lambda$ into itself. Moreover if $C_2\,(|e_0(\lambda)| + 2\epsilon\, r) \equiv c < 1$, then the map $\mathcal{F}_\lambda$ is a contraction in $\mathcal{B}_r$ with contraction constant c.

Lemma 4.3 is proved.

By Lemma 4.3 there is a unique solution of Equation (4.6a) for ϵ and κ small enough. As for the proof on Inequality (3.2b), it is not hard to find a constant C_3 (say, $C_3 = 2C_1C_2$) and a value of δ_1 such that both inequalities are satisfied for $r = C_3|e_0(\lambda)|$ and for all $\lambda \in \mathcal{O}_{\delta_1}$. Therefore, by the expansion (1.3b), Lemma 4.3 implies the estimate (3.2b).

As the family of operators $\hat{L}_\lambda$ on $\mathcal{L}_M$ depends smoothly (analytically) on λ, the family of operators $\mathcal{F}_\lambda$ possesses the same property, and the same holds for their fixed points (see [13, 14]).

We now show that there is an invariant space $\mathcal{H}_\lambda^{(1)}$ under the transformation $\hat{L}_\lambda$ such that the expansion (3.1) is valid. In fact the adjoint operator $\hat{L}_\lambda^* := K^{-1}L_\lambda^* K$ has the form

$$\begin{aligned}
\left(\hat{L}_\lambda^*\Psi\right)(\eta) = & \int a(u)\Psi(\eta+u)e^{i(\lambda,u)}du - \tilde{a}(0)\Psi(\eta) - |\eta|\Psi(\eta)+ \\
& z\sum_{\gamma\subseteq\eta}\int_{\mathbb{R}^d}\Psi(\gamma\cup\{\tilde{x}\})\prod_{v\in\eta\setminus\gamma}(e^{-\beta\phi(\tilde{x}-v)}-1)\prod_{u\in\gamma}e^{-\beta\phi(\tilde{x}-u)}d\tilde{x}+ \\
& \kappa\int a(s)e^{i(s,\lambda)}\sum_{x\in\eta}p(x+s)\Psi(\eta\setminus\{x\}+s)ds - \kappa\tilde{a}(0)\sum_{x\in\eta}p(x)\Psi(\eta\setminus\{x\})+ \\
& \kappa\int a(s)e^{i(s,\lambda)}\sum_{x\in\eta}p(x+s)\Psi(\eta+s)ds - \kappa\tilde{a}(0)\sum_{x\in\eta}p(x)\Psi(\eta).
\end{aligned} \tag{4.10}$$

Repeating for $\hat{L}_\lambda^*$ almost the same considerations as for $\hat{L}_\lambda$, we see that it has an eigenvector h_λ^* of the form

$$h_\lambda^*(\eta) = \delta_{\eta,\emptyset} + \chi_\lambda^* \tag{4.11}$$

and the function $\chi_\lambda^* \in \mathcal{L}_M$ has the same properties as χ_λ, in particular it is a smooth (analytic) function of λ. It is immediate that the space

$$\mathcal{H}_\lambda^{(1)} := \{\Psi \in \mathcal{H}^\star : (\Psi, h_\lambda^*) = 0\} \tag{4.12}$$

is invariant with respect to $\hat{L}_\lambda$. Moreover for any vector $\Psi \in \mathcal{H}^\star$ the vector

$$\bar{\Psi} = \Psi - \frac{(\Psi, h_\lambda^*)}{(h_\lambda^{(0)}, h_\lambda^*)}h_\lambda^{(0)} \tag{4.13a}$$

belongs to $\mathcal{H}_\lambda^{(1)}$, so that we get the decomposition of Ψ as

$$\Psi = \bar{\Psi} + c_\Psi(\lambda)h_\lambda^{(0)} \qquad c_\Psi(\lambda) = \frac{(\Psi, h_\lambda^*)}{(h_\lambda^{(0)}, h_\lambda^*)} \tag{4.13b}$$

Observe that by (3.2a), the estimate (3.2b), (4.11) and the analogues for h_λ^*, we have

$$(h_\lambda^{(0)}, h_\lambda^*) = 1 + (\chi_\lambda, \Psi^{(0)}) + (\Psi^{(0)}, \chi_\lambda^*) + (\chi_\lambda^{(0)}, \chi_\lambda^*) = 1 + \mathcal{O}(|\lambda|^2). \tag{4.14a}$$

For $\lambda = 0$, by (3.2a,b) and similar properties for h^*_λ, we have $h^*_0 = h^{(0)}_0 = \Psi^{(0)}$ so that

$$c_{\Psi^{(0)}}(0) = 1. \tag{4.14b}$$

Therefore the first assertion of Lemma 3.1 is proved.

For the second assertion, observe that (3.3a) follows from (4.13b) with $c(\lambda) = c_{\Psi^{(0)}}(\lambda)$, and the stated property of $c(\lambda)$ from (4.11) and (4.14a). The estimate (3.3b) follows from (4.11) and (4.13a) for $\Psi = \Psi^{(0)}$, by using (4.13b) and (4.14a) for $c_{\Psi^{(0)}}(\lambda)$.

We pass to the proof of the third assertion. Let $P_{\geq 1}$ be the projector in $\mathcal{L}_M$ on the subspace $\mathcal{L}^{\geq 1}$. For $\Psi \in \mathcal{L}_M$ we set $P_{\geq 1}\Psi := \Psi_1$ so that

$$\Psi = \Psi_0 + \Psi_1, \qquad \Psi \in \mathcal{L}_M, \quad \Psi_0 \in \mathcal{L}^{(0)}, \quad \Psi_1 \in \mathcal{L}^{\geq 1}.$$

It is clear that $|||P_{\geq 1}||| \leq 1$. We set $\mathcal{H}^{(1)}_{\lambda,M} = \mathcal{H}^{(1)}_\lambda \cap \mathcal{L}_M$. Any vector $\Psi \in \mathcal{H}^{(1)}_{\lambda,M}$ can be written as

$$\Psi = \Psi_1 + c(\Psi_1)\Psi^{(0)} := \Psi_1 + T_\lambda \Psi_1, \qquad \Psi_1 \in \mathcal{L}^{\geq 1}, \qquad c(\Psi_1) = -\frac{(\Psi_1, h^*_\lambda)}{(\Psi^{(0)}, h^*_\lambda)}. \tag{4.15}$$

We denote $P_{\geq 1,\lambda} = P_{\geq 1}|_{\mathcal{H}^{(1)}_\lambda}$. By (4.15) it follows that the inverse of the operator $P_{\geq 1,\lambda}$ is

$$(P_{\geq 1,\lambda})^{-1} = I + T_\lambda, \quad P^{-1}_{\geq 1,\lambda} : \mathcal{L}^{\geq 1} \to \mathcal{H}^{(1)}_{\lambda,M}, \tag{4.16a}$$

where I is the identity operator in $\mathcal{L}^{\geq 1}$. By (4.11), $|(\Psi^{(0)}, h^*_\lambda)| \geq 1 - |(\Psi^{(0)}, \chi^*_\lambda)| \geq 1 - \|\chi^*_\lambda\|_M$, so that the following inequality holds

$$\|T_\lambda \Psi_1\|_M = \frac{|(\Psi_1, h^*_\lambda)|}{|(\Psi^{(0)}, h^*_\lambda)|} < \frac{|(\Psi_1, \Psi^{(0)})| + \|\Psi_1\|_M \|\chi^*_\lambda\|_M}{1 - \|\chi^*_\lambda\|_M}. \tag{4.16b}$$

For the first term on the right in (4.16b), using, as in [8], the Ruelle bound $\rho(\eta) \leq z^{|\eta|}$, it is easy to check that if $\epsilon = z\tilde{C}(\beta)$ is small enough, we have

$$|(\Psi_1, \Psi^{(0)})| \leq \int_{|\eta| \geq 1} |\Psi_1(\eta)| \rho(\eta) d\eta \leq \sum_{n=1}^{\infty} \frac{z}{M} \int_{|\eta|=n} |\Psi_1(\eta)| M^n d\eta < \frac{z}{M} \|\Psi_1\|_M. \tag{4.16c}$$

By inequalities (4.16b,c), one can find $\delta_1 > 0$ such that $|||T_\lambda||| < \frac{1}{2}$ for $\lambda \in \mathcal{O}_{\delta_1}$. Furthermore for any vector $\Psi = \Psi_0 + \Psi_1 \in \mathcal{H}^{(1)}_{\lambda,M}$ with $\Psi_0 \in \mathcal{L}^0$, $\Psi_1 \in \mathcal{L}^{\geq 1}$, we have, by (4.15),

$$\left(\hat{L}_\lambda \Psi\right)_1 = L^{21}_\lambda \Psi_0 + L^{22}_\lambda \Psi_1 = \left(L^{21}_\lambda T_\lambda + L^{22}_\lambda\right) \Psi_1, \tag{4.17a}$$

so that

$$\hat{L}_\lambda|_{\mathcal{H}^{(1)}_{\lambda,M}} := L^{(1)}_\lambda = P^{-1}_{\geq 1,\lambda} \left(L^{21}_\lambda T_\lambda + L^{22}_\lambda\right) P_{\geq 1,\lambda}. \tag{4.17b}$$

Therefore the resolvent is

$$\left(L_\lambda^{(1)} - \xi I_{\mathcal{H}^{(1)}_{\lambda,M}}\right)^{-1} = P^{-1}_{\geq 1,\lambda}\left(L_\lambda^{21}T_\lambda + L_\lambda^{22} - \xi I_{\mathcal{L}^{\geq 1}}\right)^{-1} P_{\geq 1,\lambda}. \tag{4.18}$$

(Here $I_{\mathcal{L}^{\geq 1}}$ and $I_{\mathcal{H}^{(1)}_{\lambda,M}}$ are the unit operators in the corresponding spaces.) We write L_λ^{22} as a sum, $L_\lambda^{22} = L_\lambda^{22,(0)} + L_\lambda^{22,(1)}$ (see (4.5d,e)), where

$$\left(L_\lambda^{22,(0)}\Psi\right)(\eta) = -(|\eta| + \tilde{a}(0))\Psi(\eta), \qquad \Psi \in \mathcal{L}^{\geq 1}, \tag{4.19}$$

and $L_\lambda^{22,(1)} = L_\lambda^{22} - L_\lambda^{22,(0)}$ is the difference. One can write the resolvent as

$$\begin{aligned}\left(L_\lambda^{22} + L_\lambda^{21}T_\lambda - \xi I_{\mathcal{L}^{\geq 1}}\right)^{-1} = &\left(I_{\mathcal{L}^{\geq 1}} + \left(L_\lambda^{22,(0)} - \xi I_{\mathcal{L}^{\geq 1}}\right)^{-1}\left(L_\lambda^{22,(1)} + L_\lambda^{21}T_\lambda\right)\right)^{-1} \cdot \\ &\cdot \left(L_\lambda^{22,(0)} - \xi I_{\mathcal{L}^{\geq 1}}\right)^{-1},\end{aligned} \tag{4.20a}$$

and, as $\tilde{a}(0) > 0$, it is clear that for any $\alpha \in (0,1)$ we have

$$|||\left(L_\lambda^{22,(0)} - \xi I_{\mathcal{L}^{\geq 1}}\right)^{-1}||| \leq \frac{1}{\min_{n\geq 1}|\xi + n + \tilde{a}(0)|} \leq \begin{cases} \frac{1}{1+\tilde{a}(0)-\alpha} & \mathbf{Re}\,\xi > -\alpha \\ 1 & |\mathbf{Im}\,\xi| > 1 \end{cases}. \tag{4.20b}$$

Lemma 4.4. *If $\xi \notin \mathcal{R} := \{\xi \in \mathbb{C}^1 : \mathbf{Re}\,\xi < -\frac{1}{2}, |\mathbf{Im}\,\xi| < 1\}$, then for $\lambda \in \mathcal{O}_{\delta_1}$ and κ and ϵ small enough, the following inequality holds:*

$$|||\left(L_\lambda^{22,(0)} - \xi I_{\mathcal{L}^{\geq 1}}\right)^{-1}\left(L_\lambda^{22,(1)} + L_\lambda^{21}T_\lambda\right)||| \ < \ 1. \tag{4.21}$$

Proof. The proof is deferred to the Appendix.

From Lemma 4.4, Inequality (4.20b), the inequality $|||P^{-1}_{\geq 1}||| < \frac{3}{2}$, which follows from (4.16a), and the inequality $|||T_\lambda||| < \frac{1}{2}$, we find that, for $\xi \notin \mathcal{R}$ and $\lambda \in \mathcal{O}_{\delta_1}$, the norm of the resolvent is bounded:

$$|||\left(L_\lambda^{(1)} - \xi I_{\mathcal{H}^{(1)}_{\lambda,M}}\right)^{-1}||| \ < \ C_1. \tag{4.22}$$

We will also need the following assertion.

Proposition 4.4. *Let A be a self-adjoint operator acting on the Hilbert space $\mathcal{H}$, and let $\mathcal{L} \subset \mathcal{H}$ be a Banach space everywhere dense in $\mathcal{H}$ and with norm $\|\cdot\|_{\mathcal{L}}$ such that*

$$\|h\|_{\mathcal{H}} \ \leq \ \|h\|_{\mathcal{L}}, \qquad h \in \mathcal{L}. \tag{4.23}$$

Suppose furthermore that A acts on $\mathcal{L}$ as a bounded operator and that $A\mathcal{L} \subseteq \mathcal{L}$. Then A is a bounded operator in $\mathcal{H}$ and $\|A\| \leq |||A|||$, where $\|A\|$ and $|||A|||$ are the operator norms in $\mathcal{H}$ and $\mathcal{L}$ respectively.

Proof. For the proof see [7, 14].

If A is not self-adjoint, but A and A^* act as bounded operators on $\mathcal{L}$, making use of the representation

$$A = \frac{A + A^*}{2} + i\frac{A - A^*}{2i},$$

we see that, by inequality (4.23), we get

$$\|A\| \; \leq \; |||A||| \; + \; |||A^*|||. \tag{4.24}$$

Consider now the operator $\hat{L}^*_\lambda$, given by equation (4.10), adjoint of the operator $\hat{L}_\lambda$ on $\mathcal{H}^\star$, and its matrix representation, analogous to the representation (4.4)

$$L^*_\lambda = \begin{pmatrix} \overline{L^{11}_\lambda} & \overline{L^{12}_\lambda} \\ \overline{L^{21}_\lambda} & \overline{L^{22}_\lambda} \end{pmatrix}. \tag{4.25}$$

Then the operator $(L^{(1)}_\lambda)^*$ on the space $\mathcal{H}^{(1)}_{\lambda,M}$, adjoint to the operator $L^{(1)}_\lambda$, has a representation analogous to (4.17b):

$$\left(L^{(1)}_\lambda\right)^* = P^{-1}_{\geq 1,\lambda} \left(\overline{L^{21}_\lambda} T_\lambda + \overline{L^{22}_\lambda}\right) P_{\geq 1,\lambda}.$$

The resolvent is then written as

$$\begin{aligned}\left(\left(L^{(1)}_\lambda - \xi I_{\mathcal{H}^{(1)}_{\lambda,M}}\right)^{-1}\right)^* &= \left(\left(L^{(1)}_\lambda\right)^* - \overline{\xi} I_{\mathcal{H}^{(1)}_{\lambda,M}}\right)^{-1} \\ &= P^{-1}_{\geq 1,\lambda} \left(\overline{L^{21}_\lambda} T_\lambda + \overline{L^{22}_\lambda} - \overline{\xi} I_{\mathcal{L}^{\geq 1}}\right)^{-1} P_{\geq 1,\lambda}.\end{aligned}$$

Making use of the explicit representation (4.10) of the operator L^*_λ and repeating the preceding considerations, we can show that, uniformly for all $\lambda \in \mathcal{O}_{\delta_1}$ and $\xi \notin \mathcal{R}$, the norm

$$|||\left(\left(L^{(1)}_\lambda - \xi I_{\mathcal{H}^{(1)}_{\lambda,M}}\right)^{-1}\right)^*||| \tag{4.26}$$

is bounded. From (4.22) and (4.26) we get inequality (3.4).

Let us now prove the fourth assertion of Lemma 3.1. Observe that

$$q(\lambda) = e_0(\lambda) + \left(L^{12}_\lambda \chi_\lambda\right)(\emptyset). \tag{4.27}$$

Relation (3.5b) then follows from the estimate (4.8b). Moreover the considerations above show that $q(\lambda)$ depends smoothly (analytically) on λ, is an even function, and its second derivatives at the origin

$$\frac{\partial^2 q}{\partial\lambda_i\partial\lambda_j}|_{\lambda=0} := \tilde{a}_{ij}, \qquad \lambda = (\lambda_1, \dots, \lambda_\nu)$$

differ by quantities of the order ϵ from the derivatives $\frac{\partial^2 e_0}{\partial\lambda_i\partial\lambda_j}|_{\lambda=0} = a_{ij}$.

We now prove that $q(\lambda)$ is a real function. Let R be the reflection operator in $\mathcal{H}$:

$$(R\Phi)(x,\gamma) = \Phi(-x,-\gamma), \qquad -\gamma = \{x : -x \in \gamma\}.$$

The operator R commutes with L, as it follows from definitions of §1, in particular from the symmetry of the functions $a(s), p(s)$. As for the shift operator U_s we have

$$RU_s = U_{-s}R.$$

Therefore it is easy to see that the operators L_λ and $L_{-\lambda}$ are equal. On the other hand, as it follows from (2.5), for any real function Φ we have $L_{-\lambda}\Phi = \overline{L_\lambda\Phi}$, so that $L_\lambda\Phi = \overline{L_\lambda\Phi}$. The equality holds when we pass to the space $\mathcal{H}^\star$ as well, so that the eigenvector $h_\lambda \in \mathcal{H}^\star$, and consequently its component $\chi_\lambda \in \mathcal{L}^{\geq 1}$ are real.

Lemma 3.1 is proved

Proof of Lemma 3.2.

We write the resolvent as

$$\left(\hat{L}_\lambda - \xi\, I\right)^{-1} = \begin{pmatrix} L_\lambda^{11} - \xi & L_\lambda^{12} \\ L_\lambda^{21} & L_\lambda^{22} - \xi\, I \end{pmatrix}^{-1} =$$
$$\left[\begin{pmatrix} 1 & 0 \\ 0 & I_{\mathcal{L}^{\geq 1}} \end{pmatrix} + \begin{pmatrix} L_\lambda^{11} - \xi & 0 \\ 0 & L_\lambda^{22,(0)} - \xi\, I \end{pmatrix}^{-1} \begin{pmatrix} 0 & L_\lambda^{12} \\ L_\lambda^{21} & L_\lambda^{22,(1)} \end{pmatrix}\right]^{-1} \cdot$$
$$\begin{pmatrix} L_\lambda^{11} - \xi & 0 \\ 0 & L_\lambda^{22,(0)} - \xi\, I \end{pmatrix}^{-1} \tag{4.28}$$

with $I = I_{\mathcal{L}^{\geq 1}}$. Furthermore we have

$$\begin{pmatrix} L_\lambda^{11} - \xi & 0 \\ 0 & L_\lambda^{22,(0)} - \xi\, I \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{e_0(\lambda)-\xi} & 0 \\ 0 & (L_\lambda^{22,(0)} - \xi\, I)^{-1} \end{pmatrix} \tag{4.29a}$$

$$\begin{pmatrix} L_\lambda^{11} - \xi & 0 \\ 0 & L_\lambda^{22,(0)} - \xi\, I \end{pmatrix}^{-1} \cdot \begin{pmatrix} 0 & L_\lambda^{12} \\ L_\lambda^{21} & L_\lambda^{22,(1)} \end{pmatrix}$$
$$= \begin{pmatrix} 0 & \frac{L_\lambda^{12}}{e_0(\lambda)-\xi)} \\ (L_\lambda^{22,(0)} - \xi\, I)^{-1} L_\lambda^{21} & (L_\lambda^{22,(0)} - \xi\, I)^{-1}\, L_\lambda^{22,(1)} \end{pmatrix}. \tag{4.29b}$$

Observe that, by inequality (A.1) in the Appendix, the estimate (4.8b) is valid for all λ. Hence if $\xi \notin \tilde{\mathcal{R}}$ we have

$$|||\frac{1}{e_0(\lambda) - \xi}\ L_\lambda^{12}||| < \frac{\epsilon}{|e_0(\lambda) - \xi|} < \frac{2\epsilon}{\delta_1}.$$

The estimate for $|||(L_\lambda^{22,(0)} - \xi\ I)^{-1}\ L_\lambda^{21}|||$ follows from (A.2) in the Appendix and is also independent of λ.

As for $|||(L_\lambda^{22,(0)} - \xi\ I)^{-1}\ L_\lambda^{22,(1)}|||$ observe that by inequality (A.5) in the Appendix, which is also independent of λ, we find that

$$|||(L_\lambda^{22,(0)} - \xi\ I)^{-1}\ L_\lambda^{22,(1)}||| \le \max_{|\eta|>1} \frac{\tilde{a}(0) + \frac{1}{2}}{|\tilde{a}(0) + |\eta| + \xi|\ (\tilde{a}(0) + 1)}.$$

From this it is easy to see that $|||(L_\lambda^{22} - \xi\ I)^{-1}||| < C$ for $\xi \notin \tilde{\mathcal{R}}$ and $\lambda \notin \mathcal{O}_{\delta_1}$, where the constant C does not depend on ξ and λ.

By applying similar considerations to the estimate for the resolvent $(\hat{L}_\lambda^* - \xi\ I)^{-1}$ and using the estimate (4.24), we find that the norm $\|(L_\lambda - \xi I)^{-1}\|$ is bounded uniformly for $\lambda \notin \mathcal{O}_{\delta_1}$ and $\xi \notin \tilde{\mathcal{R}}$.

Lemma 3.2 is proved.

APPENDIX

Proof of Lemma 4.2. By (4.5a), L_λ^{11}, for any λ, is a multiplication operator and equality (4.8a) is satisfied. For the norm of the operator L_λ^{12} we find, by (4.5b)

$$\|L_\lambda^{12}\Psi\|_M \leq z\left|\int \Psi(u)du\right| \leq \frac{z}{M}\int |\Psi(u)|Mdu \leq \epsilon\|\Psi\|_M, \tag{A.1}$$

so that (4.8b) is also valid for all λ. Moreover the representation (4.5c) implies

$$\begin{aligned}\|L_\lambda^{21}\Psi\|_M &= \frac{\kappa}{3}|e_0(\lambda)|(\sup_x p(x))|\Psi(\emptyset)| + \kappa|e_0(\lambda)|\,|\Psi(\emptyset)|\int p(x)Mdx \\ &= \kappa|e_0(\lambda)|\left(\frac{1}{3}(\sup_x p(x)) + M\int p(x)dx\right)|\Psi(\emptyset)|,\end{aligned}$$

and, again for all λ, if κ satisfies the bound (4.7), as $M = \max\{4;\ \tilde{C}(\beta)^{-1}\}$, we have

$$|||L_\lambda^{21}||| = \kappa|e_0(\lambda)|\ \left(\frac{p_0}{3} + Mp_1\right) = C_1|e_0(\lambda)|. \tag{A.2}$$

Let us now estimate the norm of the operator $(L_\lambda^{22})^{-1}$. As in §4 we split L_λ^{22} as a sum of two terms, $L_\lambda^{22} = L_\lambda^{22,(0)} + L_\lambda^{22,(1)}$, with $L_\lambda^{22,(0)}$ given by (4.19). We then write

$$(L_\lambda^{22})^{-1} = \left(L_\lambda^{22,(0)} + L_\lambda^{22,(1)}\right)^{-1} = \left(I + (L_\lambda^{22,(0)})^{-1}L_\lambda^{22,(1)}\right)^{-1}(L_\lambda^{22,(0)})^{-1}. \tag{A.3}$$

We now show that $|||(L_\lambda^{22,(0)})^{-1}L_\lambda^{22,(1)}||| < 1$, so that $(I+(L_\lambda^{22,(0)})^{-1}L_\lambda^{22,(1)})^{-1}$ is a bounded operator, and therefore $(L_\lambda^{22})^{-1}$ is also bounded.

By (4.19) we have $((L_\lambda^{22,(0)})^{-1}\Psi)(\eta) = -(|\eta| + \tilde{a}(0))^{-1}\Psi(\eta)$ $(|\eta| \geq 1)$, and therefore

$$|||(L_\lambda^{22,(0)})^{-1}||| \leq \frac{1}{1+\tilde{a}(0)}. \tag{A.4a}$$

$L_\lambda^{22,(1)}$ is the sum of several terms, of all terms in expression (4.5d) which are not included in $L_\lambda^{22,(0)}$. Starting with the first line of (4.5d), we have

$$\|\int_{\mathbb{R}^d} a(s)\Psi(\eta+s)e^{i(\lambda,s)}ds\|_M \leq \|\Psi\|_M\int_{\mathbb{R}^d} a(s)ds = \|\Psi\|_M\ \tilde{a}(0)$$

as it follows from the positivity of $a(s)$ and translation invariance of the norm $\|\cdot\|_M$. Therefore by (A.4a)

$$\|(L_\lambda^{22,(0)})^{-1}\int a(s)\Psi(\eta+s)e^{i(\lambda,s)}ds\|_M \leq \frac{\tilde{a}(0)}{1+\tilde{a}(0)}\|\Psi\|_M. \tag{A.4b}$$

Passing to the last line of (4.5d), setting $(A^{(1)}\Psi)(\eta) = \frac{\sum_{x\in\eta} p(x)\Psi(\eta)}{\tilde{a}(0)+|\eta|}$ we find

$$\|A^{(1)}\Psi\|_M = \int \sup_{\eta_1} \left(\left(\frac{1}{3}\right)^{|\eta_1|} \frac{|\eta_1|+|\eta_2|}{|\eta_1|+|\eta_2|+\tilde{a}(0)} \left(\sum_{x\in\eta_1\cup\eta_2} p(x) \right) |\Psi(\eta_1\cup\eta_2)| \right) M^{|\eta_2|} d\eta_2$$

$$\leq p_0 \int \sup_{\eta_1} \left(\left(\frac{1}{3}\right)^{|\eta_1|} (|\eta_1|+|\eta_2|)|\Psi(\eta_1\cup\eta_2)| \right) M^{|\eta_2|} d\eta_2 \leq p_0 \|\Psi\|_M. \tag{A.4c}$$

The other term arising from the last line of (4.5d) is estimated in a similar way:

$$\left\| \frac{1}{|\eta|+\tilde{a}(0)} \sum_{x\in\eta} p(x) \int a(s)\Psi(\eta+s)e^{i(\lambda,s)}ds \right\|_M =$$

$$\int \sup_{\eta_1} \left(\left(\frac{1}{3}\right)^{|\eta_1|} \frac{|\eta_1|+|\eta_2|}{|\eta_1|+|\eta_2|+\tilde{a}(0)} \sum_{x\in\eta_1\cup\eta_2} p(x) \int a(s)\, |\Psi(\eta_1\cup\eta_2+s)|\, ds \right) M^{|\eta_2|} d\eta_2 \leq$$

$$p_0 \int\int \sup_{\eta_1} \left(\left(\frac{1}{3}\right)^{|\eta_1|} (|\eta_1|+|\eta_2|)\, |\Psi(\eta_1\cup\eta_2+s)| \right) M^{|\eta_2|} a(s)ds\; d(\eta_2+s)$$

$$\leq p_0\tilde{a}(0)\; \|\Psi\|_M. \tag{A.4d}$$

From the third line of (4.5d) we get the following estimate

$$\left\| \frac{\tilde{a}(0)}{|\eta|+\tilde{a}(0)} \sum_{x\in\eta} p(x)\Psi(\eta\setminus\{x\}) \right\|_M =$$

$$\tilde{a}(0) \int \sup_{\eta_1} \left(\left(\frac{1}{3}\right)^{|\eta_1|} \frac{|\eta_1|+|\eta_2|}{|\eta_1|+|\eta_2|+\tilde{a}(0)} \left| \sum_{x\in\eta_1\cup\eta_2} p(x)\Psi((\eta_1\cup\eta_2)\setminus\{x\}) \right| \right) M^{|\eta_2|} d\eta_2$$

$$\leq \tilde{a}(0) \int \sup_{|\eta_1|>0} \left(\left(\frac{1}{3}\right)^{|\eta_1|} \sum_{x\in\eta_1} p(x)\, |\Psi(\eta_2\cup(\eta_1\setminus\{x\}))| \right) M^{|\eta_2|} d\eta_2 +$$

$$\tilde{a}(0) \int \sup_{\eta_1} \left(\left(\frac{1}{3}\right)^{|\eta_1|} \sum_{x\in\eta_2} p(x)\, |\Psi(\eta_1\cup(\eta_2\setminus\{x\}))| \right) M^{|\eta_2|} d\eta_2 \leq$$

$$\tilde{a}(0)\frac{2}{3}p_0\|\Psi\|_M + \tilde{a}(0) \int \left(\sum_{x\in\eta_2} p(x) \right) \sup_{\eta_1} \left(\left(\frac{1}{3}\right)^{|\eta_1|} |\Psi(\eta_1\cup(\eta_2\setminus\{x\}))| \right) M^{|\eta_2|} d\eta_2$$

$$= \tilde{a}(0)\frac{2}{3}p_0\|\Psi\|_M + \tilde{a}(0)M \int p(x)dx \int \sup_{\eta_1} \left(\left(\frac{1}{3}\right)^{|\eta_1|} |\Psi(\eta_1\cup\eta_2)| \right) M^{|\eta_2|} d\eta_2$$

$$= \|\Psi\|_M \tilde{a}(0) \left(\frac{2}{3}p_0 + Mp_1 \right). \tag{A.4e}$$

The other term on the same line is estimated in a similar way:

$$\left\| \frac{1}{|\eta|+\tilde{a}(0)} \int a(s) \left(\sum_{x\in\eta} p(x)\Psi(\eta\setminus\{x\}+s) \right) e^{i(\lambda,s)}ds \right\|_M \leq \tilde{a}(0) \left(\frac{2}{3}p_0 + Mp_1 \right) \|\Psi\|_M. \tag{A.4f}$$

Finally, for the terms of the second line of (4.5d) we find, using results from [8],

$$\left\| \frac{z}{|\eta|+\tilde{a}(0)} \sum_{\gamma\subseteq\eta} \int \Psi(\gamma\cup\{\tilde{x}\}) \prod_{v\in\eta\setminus\gamma} (e^{-\beta\Phi(\tilde{x}-v)}-1) \prod_{u\in\gamma} e^{-\beta\phi(\tilde{x}-u)} d\tilde{x} \right\|_M \leq \; Cz\,\tilde{C}(\beta)\,\|\Psi\|_M \tag{A.4g}$$

where C is a constant.

The above estimates imply, for κ satisiying (4.7) and ϵ small enough, that

$$\begin{aligned} \|(L^{22,(0)})^{-1}L^{22,(1)}\Psi\|_M &\leq \left(\frac{\tilde{a}(0)}{1+\tilde{a}(0)} + 2\kappa p_0 \tilde{a}(0) + 2\kappa \left(\frac{2}{3}p_0 + Mp_1\right)\tilde{a}(0) + C\,\epsilon \right) \|\Psi\|_M \\ &\leq \frac{\tilde{a}(0)+\frac{1}{2}}{\tilde{a}(0)+1}\|\Psi\|_M. \end{aligned} \tag{A.5}$$

Therefore $|||(L^{22,(0)})^{-1}L^{22,(1)}||| \; < \; 1$, so that

$$|||\,(L^{22}_\lambda)^{-1}||| \; < \; C_2, \tag{A.6}$$

where the constant C_2 does not depend on λ.

This argument completes the proof of Lemma 4.2.

Proof of Lemma 4.4. By repeating the same arguments as in the proof of the preceding lemma we get that for $\mathbf{Re}\,\xi > -\frac{1}{2}$, if ϵ and κ are small enough, then

$$\left\| \left(L^{22,(0)}_\lambda - \xi\, I\right)^{-1} L^{22,(1)}_\lambda \right\| < \frac{\tilde{a}(0)+\frac{1}{4}}{\tilde{a}(0)+\frac{1}{2}}.$$

Furthermore, using the estimate (A.1) for $\|L^{21}_\lambda\|$ and the estimate for the norm $\|T_\lambda\|_M$ given by (4.16c), and the fact that $\|(L^{22,(0)}_\lambda - \xi\; I)^{-1}\|_M < \frac{1}{\frac{1}{2}+\tilde{a}(0)}$ for $\mathbf{Re}\,\xi > -\frac{1}{2}$, for κ and ϵ small enough, we get the proof of Lemma 4.4.

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REFERENCES

[1] Berezansky, Yu.M., Kondratiev, Yu.G., Kuna, T., Lytvynov, E.: On a spectral representation for correlation measures in configuration space analysis, Methods Funct. Anal. Topology, 5, no. 4, 87-100 (1999).

[2] Bertini, L., Cancrini, N., Cesi, F.: The spectral gap for a Glauber-type dynamics in a continuous gas, Ann. Inst. H. Poincaré Probab. Statist., 38, 91108 (2002)

[3] Boldrighini C., Ignatyuk, I.A., Malyshev, V.A., Pellegrinotti A.: "Random walk in dynamic environment with mutual influence", Stochastic Processes and their Applications, 41, 157-177 (1992).

[4] Boldrighini, C., Minlos, R.A., Pellegrinotti, A.: " Central limit theorem for the random walk of one or two particles in a random environment", Advances of Soviet Mathematics, 20 : "Probability Contributions to Statistical Mechanics", R.L.Dobrushin, ed. , 21-75 (1994).

[5] Breiman, L.: "Probability", Classics in Applied Mathematics, 7. Soc. for Industrial and Appl. Mathematics, Philadephia, 1992.

[6] I.I. Gihman, A.V. Skorohod: *The Theory of Stochastic Processes I.* Grundlehren der mathematischen Wissenschaften, **210**. Springer Verlag (1974).

[7] Yu. G. Kondratiev, R. A. Minlos: One-particle subspaces in the stochastic XY model, Journal of Stat. Physics, 87(3/4): 613-642 (1997).

[8] Kondratiev Yu. G., Minlos, R. A., Zhizhina, E. A.: "One particle subspace of the Glauber Dynamics Generator for continuous Particle Systems", Reviews in Math. Physics, 16, n. 9, 1073-1114 (2004)

[9] Kondratiev, Yu.G., Kuna, T.: Harmonic analysis on conøguration spaces I. General theory, Infin. Dimens. Anal. Quantum Probab. Relat. Top., 5, 201233 (2002)

[10]Yu. Kondratiev, T. Kuna, O. Kutoviy, On relations between a priori bounds for measures on configuration spaces, Infinite Dimensional Analysis, Quantum Probability and Related Topics, Vol.7 (2), p. 195-213, 2004.

[11] Kondratiev, Yu., Lytvynov, E.: Glauber dynamics of continuous particle systems, Ann. Inst. H.Poincare - PR 41, 685-702 (2005)

[12] Kondratiev, Yu., Roeckner, M, Lytvynov, E.: Equilibrium Glauber and Kawasaki dynamics of continuous particle systems, BiBoS-Preprint (2005),

available at www.arxiv.org/math.PR/0503042

[13] Malyshev, V.A., Minlos, R.A.: Linear infinite-particle operators, American Mathematical Society Publ., 1995

[14] R. A. Minlos, Invariant subspaces of the stochastic Ising high-temperature dynamics, Markov Processes and Related Fields, 2 (2): 263-284, (1996).

[15] Zeitouni, O., Random Walks in Random Environment. Ecole d' Eté de Probabilités de St. Flour, XXXI- 2001. Lecture Notes in Mathematics 1837. Springer Verlag, 2004.