Convexity properties of harmonic measures

WOLFHARD HANSEN and IVAN NETUKA *

Abstract

It is shown that any convex combination of harmonic measures $\mu_x^{U_1}, \dots, \mu_x^{U_k}$, where $U_1, \dots, U_k$ are relatively compact open neighborhoods of a given point $x \in \mathbb{R}^d$, $d \ge 2$, can be approximated by a sequence $(\mu_x^{W_n})_{n \in \mathbb{N}}$ of harmonic measures such that each W_n is an open neighborhood of x in $U_1 \cup \dots \cup U_k$.

This answers a question raised in connection with Jensen measures. Moreover, it implies that, for every Green domain X containing x, the extremal representing measures for x with respect to the convex cone of potentials on X (these measures are obtained by balayage of the Dirac measure at x on Borel subsets of X) are dense in the compact convex set of all representing measures.

This is achieved approximating balayage on open sets by balayage on unions of balls which are pairwise disjoint and very small with respect to their mutual distances and then reducing the size of these balls in a suitable manner.

These results, which are presented simultaneously for the classical potential theory and for the theory of Riesz potentials, can be sharpened if the complements or the boundaries of the open sets have a capacity doubling property. The methods developed for this purpose (continuous balayage on increasing families of compact sets, approximation using scattered sets with small capacity) finally lead to answers even in a very general potential-theoretic setting covering a wide class of second order partial differential operators (uniformly elliptic or in divergence form, or sums of squares of vector fields satisfying Hörmander's condition, for example, sub-Laplacians on stratified Lie algebras).

Keywords: Harmonic measure, Jensen measure, extremal measure, balayage, Riesz potentials, Brownian motion, stable process, Skorokhod stopping, Harnack's inequalities, Green function, capacity density, doubling property, harmonic space, sub-Laplacian

2000 Mathematics Subject Classification: 30C85, 31A05, 31A15, 31B05, 31B15, 31C40, 31D05, 35J15, 35J70, 46A55, 60G52, 60J45, 60J65

1 Introduction and main results

The original motivation for this paper is the following problem on harmonic measures in classical potential theory.

*Research supported in part by the project MSM 0021620839 financed by MSMT, by the grant 201/07/0388 of the Grant Agency of the Czech Republic, and by CRC-701, Bielefeld.

Question 1. Can every convex combination of harmonic measures $\mu_x^{U_1}, \dots, \mu_x^{U_k}$, where $U_1, \dots, U_k$ are relatively compact open neighborhoods of a given point $x \in \mathbb{R}^d$, $d \geq 2$, be approximated by a sequence $(\mu_x^{W_n})_{n\in\mathbb{N}}$ of harmonic measures such that each W_n is an open neighborhood of x in $U_1 \cup \dots \cup U_k$?

Here approximation is understood in the sense of weak convergence of measures, that is, pointwise convergence on continuous functions with compact support. In a slightly less demanding form (where the sets W_n are not required to be contained in the union of the sets $U_1, \dots, U_k$) this problem has been raised in [32, p. 229] and [14, p. 32] as being essential for the understanding of Jensen measures.

We consider only dimensions $d \geq 2$, since the answer would of course be negative on the real line. For every $x \in \mathbb{R}^d$ and $r \geq 0$, let

$$U(x,r) := \{y \in \mathbb{R}^d \colon |y-x| < r\} \quad \text{and} \quad B(x,r) := \{y \in \mathbb{R}^d \colon |y-x| \leq r\}.$$

It may help to illustrate Question 1 by a simple example. Let $d = 2$, $U = U(0,1)$, $V = U(0,R)$, $R > 1$, and $\lambda \in (0,1)$. Given $n \in \mathbb{N}$, let

$$C_n := \{(\cos t, \sin t) \colon \frac{j}{n} \leq \frac{t}{2\pi} \leq \frac{j}{n} + \gamma_n,\ 0 \leq j < n\} \quad \text{and} \quad W_n := V \setminus C_n,$$

where, by continuity, we may choose $\gamma_n \in (0, 1/n)$ in such a way that $\mu_0^{W_n}(C_n) = \lambda$ and hence $\mu_0^{W_n}(\partial V) = 1 - \lambda$ (see Figure 1). Since $\mu_0^{W_n}$ is obviously invariant under rotations by the angle $2\pi/n$, we then obtain that

$$\lim_{n\to\infty} \mu_0^{W_n} = \lambda \mu_0^U + (1-\lambda)\mu_0^V.$$

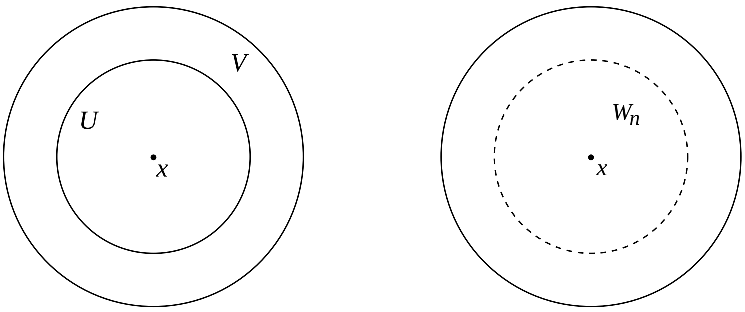

Figure 1. A simple example

Let us note that, by the minimum principle, $\gamma_n < \lambda/n$. Moreover, due to the recurrence in the plane, γ_n is very small if R is very large. In fact, for every $R > 1$, $\lim_{n\to\infty} n\gamma_n = 0$ (cf. Proposition 8.1).

But how can we approximate $\lambda\mu_x^U + (1-\lambda)\mu_x^V$ if $x \in U \setminus \{0\}$? How to proceed in $\mathbb{R}^3$, if $x = 0$, $U = U(0,1)$, and $V = U(0,R)$, $R > 1$?

A problem which is closely related to Question 1 can be formulated in terms of representing measures. Let X be an open set in $\mathbb{R}^d$ such that $\mathbb{R}^d \setminus X$ is non-polar, if $d = 2$. Let $\mathcal{K}(X)$ denote the linear space of all continuous real functions on X with compact support, let $\mathcal{M}(X)$ be the set of all (positive) Radon measures on X, and let $\mathcal{P}(X)$ denote the set of all continuous real potentials on X. Given

$x \in X$, let $\mathcal{M}_x(\mathcal{P}(X))$ denote the set of all representing measures μ for x, that is, of all measures $\mu \in \mathcal{M}(X)$ such that $\mu(p) \le p(x)$ for every $p \in \mathcal{P}(X)$. In terms of Brownian motion (X_t), starting at x and killed upon leaving X, $\mu \in \mathcal{M}_x(\mathcal{P}(X))$ if and only if there is a stopping time T such that μ is the distribution of X_T (see [16, 19, 17]).

The extreme points of the convex set $\mathcal{M}_x(\mathcal{P}(X))$ have been identified almost forty years ago [31]:

$$\big(\mathcal{M}_x(\mathcal{P}(X))\big)_e = \{\varepsilon_x^A \colon A \text{ Borel in } X\}. \tag{1.1}$$

In other words, the extreme points of $\mathcal{M}_x(\mathcal{P}(X))$ are the measures ε_x^A, A Borel in X, obtained by *reducing* (with respect to X) the Dirac measure ε_x at x on A. Viewed probabilistically, ε_x^A is the distribution of the process, starting at x, at the first entry time $D_A := \inf\{t \ge 0 \colon X_t \in A\}$ (for the analytic definition of ε_x^A see Section 2). We note that, for every open subset U of X containing x, the measure $\varepsilon_x^{U^c}$ is the harmonic measure μ_x^U.

The convex set $\mathcal{M}_x(\mathcal{P}(X))$ is compact and metrizable with respect to the topology of weak convergence (see, for example, [6, p. 336]). We recall that, by definition, a sequence (μ_n) in $\mathcal{M}(X)$ converges *weakly* to $\mu \in \mathcal{M}(X)$ if $\lim_{n\to\infty} \mu_n(f) = \mu(f)$ for every $f \in \mathcal{K}(X)$.

The following question is certainly very natural. It remained without any answer even after knowing (1.1).

Question 2. Is the set $\big(\mathcal{M}_x(\mathcal{P}(X))\big)_e$ of extreme points dense in $\mathcal{M}_x(\mathcal{P}(X))$?

Since the set $H_x(X)$ of harmonic measures $\varepsilon_x^{U^c}$, U relatively compact open in X, $x \in U$, is dense in $\big(\mathcal{M}_x(\mathcal{P}(X))\big)_e$ (see Lemma 2.3), the Krein-Milman theorem implies that $\mathcal{M}_x(\mathcal{P}(X))$ is the closed convex hull of $H_x(X)$. Therefore a positive answer to Question 1 immediately yields a positive answer to Question 2.

Our basic idea consists in approximating balayage on arbitrary sets by balayage on finite families of balls which are very small with respect to their mutual distances and then reducing the size of these balls in a suitable way. This approach works as well for the theory of Riesz potentials related to the fractional Laplacian $-(-\Delta)^{\alpha/2}$ on $\mathbb{R}^d$, $0 < \alpha < 2 \wedge d$. Therefore we shall also cover the case of Riesz potentials from the very beginning. We recall that classical potential theory of the Laplacian is the limiting case $\alpha = 2$. The reader, who is interested in the classical case only, may neglect this generality and will hardly notice any difference in the presentation except for the additional discussion of the "Poisson kernel" for a ball with respect to Riesz potentials (which has a density with respect to Lebesgue measure on the *complement* of the ball). So we shall deal simultaneously with the following two situations (for a more general potential-theoretic setting see Section 10):

- Classical case: $\alpha = 2$, X is a non-empty open set in $\mathbb{R}^d$, $d \ge 2$, such that $\mathbb{R}^d \setminus X$ is non-polar, if $d = 2$.

- Riesz potentials: $\alpha < 2$, X is a non-empty open set in $\mathbb{R}^d$, $d \ge 1$, $d > \alpha$.

Given $Y \subset X$, $Y^c := X \backslash Y$ will always denote the complement of Y *with respect to* X. Let $\mathcal{B}(X)$ denote the σ-algebra of all Borel sets in X and let $\mathcal{M}(\mathcal{P}(X))$ be the set of all $\nu \in \mathcal{M}(X)$ such that $\nu(p) < \infty$ for *some* strictly positive $p \in \mathcal{P}(X)$. Obviously, every finite measure on X and hence every $\nu \in \mathcal{M}(X)$ with compact support is contained in $\mathcal{M}(\mathcal{P}(X))$. For all $\nu \in \mathcal{M}(\mathcal{P}(X))$ and $A \in \mathcal{B}(X)$, let ν^A denote the measure obtained *reducing* ν on A *with respect to* X. It can be defined by

$$\nu^A := \int \varepsilon_x^A \, d\nu(x).$$

Let $k \in \mathbb{N}$, $k \geq 2$, and

$$\Lambda_k := \{\lambda \in [0,1]^k \colon \sum_{j=1}^{k} \lambda_j = 1\}.$$

The following main results (Theorem 1.1, Corollaries 1.2 and 1.4) immediately yield positive answers to both Question 1 and Question 2.

THEOREM 1.1. *Let $\nu \in \mathcal{M}(\mathcal{P}(X))$, $A_1, \ldots, A_k \in \mathcal{B}(X)$, and $\lambda \in \Lambda_k$. Further, let A_0 be a Borel subset of $A_1 \cap \cdots \cap A_k$ and let (V_n) be a sequence of open neighborhoods of $(A_1 \cup \cdots \cup A_k) \setminus A_0$ in X. Then there exist finite unions C_n of pairwise disjoint closed balls in V_n, $n \in \mathbb{N}$, such that*

$$\lim_{n \to \infty} \nu^{C_n \cup A_0} = \sum_{j=1}^{k} \lambda_j \nu^{A_j}.$$

In the classical case we have the following consequence (see Figure 2).

COROLLARY 1.2. *Let $\alpha = 2$ (classical case), let U, V be open sets in X, and suppose that $\nu \in \mathcal{M}(\mathcal{P}(X))$ is supported by $U \cap V$. Then, for every $\lambda \in (0,1)$, there exist finite unions C_n of pairwise disjoint closed balls in a $(1/n)$-neighborhood of $(\partial U \cap V) \cup (\partial V \cap U)$ in $U \cup V$ such that*

$$\lim_{n \to \infty} \nu^{((U \cup V) \setminus C_n)^c} = \lambda \nu^{U^c} + (1-\lambda) \nu^{V^c}. \tag{1.2}$$

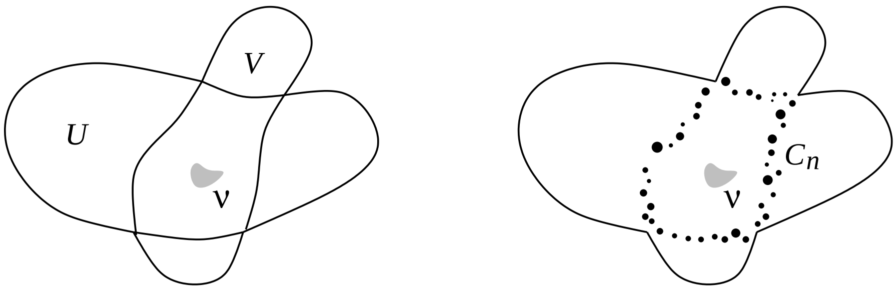

Figure 2. Approximation in the classical case

REMARK 1.3. *If, in addition, the sets $\partial U \cap V$ and $\partial V \cap U$ have a weak capacity doubling property* (*see Section 9*), *then we may choose compact sets C_n in the union of $\partial U \cap V$ and $\partial V \cap U$ such that* (1.2) *holds* (*see Figure 3 and Corollary 9.7*).

A related notion of regularity, the capacity density condition, has been widely investigated and used in various situations [1, 2, 3, 35, 12, 18, 30, 29]. *It is easily verified that the capacity doubling property is weaker* (*see Proposition 14.2*). *In fact, a result in* [30] *implies that it is much weaker than the capacity density condition: there exists a Cantor set K which is not thin at any of its points such that no point of K satisfies the capacity density condition, whereas K has the capacity doubling property at every point in K* (*see Proposition 14.3*).

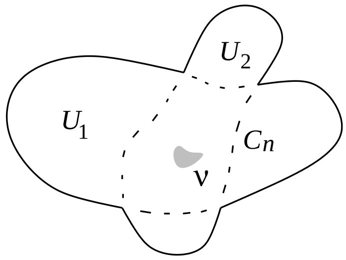

Figure 3. Approximation using the weak capacity doubling property

It is known that, for any measure $\nu \in \mathcal{M}(\mathcal{P}(X))$, the set

$$\mathcal{M}_\nu(\mathcal{P}(X)) := \{\mu \in \mathcal{M}(\mathcal{P}(X))\colon \mu(p) \le \nu(p) \text{ for every } p \in \mathcal{P}(X)\}$$

of representing measures for ν is a metrizable compact convex set and that the set $\big(\mathcal{M}_\nu(\mathcal{P}(X))\big)_e$ of its extreme points consists of all reduced measures ν^A, $A \in \mathcal{B}(X)$ (see [31] or [6, VI.12.4]).

COROLLARY 1.4. *For every $\nu \in \mathcal{M}(\mathcal{P}(X))$, the set $\big(\mathcal{M}_\nu(\mathcal{P}(X))\big)_e$ of extreme points is dense in $\mathcal{M}_\nu(\mathcal{P}(X))$.*

REMARK 1.5. *Let us note that Corollary 1.4 has the following consequence related to Skorokhod stopping* (see [33, 16, 19, 17, 5]). *Let ν be a probability measure on X and let (X_t) be Brownian motion or an α-stable process on X with initial distribution ν. Then, for every measure $\mu \in \mathcal{M}_\nu(\mathcal{P}(X))$, there exists a sequence (T_n) of hitting times at relatively compact open subsets U_n of X such that the distributions $P^\nu_{X_{T_n}}$ converge weakly to μ as $n \to \infty$.*

In fact, Theorem 1.1 implies a more general statement on representing measures. Given a set W in X which is open or, more generally, is finely open and Borel, let $S(W)$ denote the set of all continuous functions on X which are $\mathcal{P}(X)$–bounded (that is, bounded in modulus by some $p \in \mathcal{P}(X)$) and (finely) superharmonic on W. Of course, $\mathcal{P}(X) \subset S(W)$. Let $\nu \in \mathcal{M}(\mathcal{P}(X))$ such that ν is supported by W and $\nu(p) < \infty$ for *every* $p \in \mathcal{P}(X)$. Let $\mathcal{M}_\nu(S(W))$ denote the set of all $\mu \in \mathcal{M}(\mathcal{P}(X))$ such that $\mu(s) \le \nu(s)$ for every $s \in S(W)$. If $W = X$, then $S(X) = \mathcal{P}(X)$ and therefore $\mathcal{M}_\nu(S(X)) = \mathcal{M}_\nu(\mathcal{P}(X))$.

If $\nu = \varepsilon_x$, $x \in X$, it is known by [6, VII.9.5] that the extreme points of $\mathcal{M}_x(S(W))$ are the measures ε_x^A, where $A \in \mathcal{B}(X)$ contains W^c (as customary, we write $\mathcal{M}_x$ instead of $\mathcal{M}_{\varepsilon_x}$). In fact, this holds for any $\nu \in \mathcal{M}(\mathcal{P}(X))$ such that $\nu(p) < \infty$ for every $p \in \mathcal{P}(X)$. Then the set of extreme points of $\mathcal{M}_\nu(S(W))$ consists of all ν^A, $A \in \mathcal{B}(X)$, $W^c \subset A$ (see Section 13, where, in addition, various characterizations of measures in $\mathcal{M}_\nu(S(W))$ are given). So we obtain the following consequence of Theorem 1.1.

COROLLARY 1.6. *Let W be a finely open Borel set in X and let $\nu \in \mathcal{M}(\mathcal{P}(X))$ such that $\nu(p) < \infty$ for every $p \in \mathcal{P}(X)$. Then $\big(\mathcal{M}_\nu(S(W))\big)_e$ is dense in $\mathcal{M}_\nu(S(W))$.*

An important tool for the proof of Theorem 1.1 will be the use of families of compact sets which are very small with respect to their mutual distances. Given $c > 1$, we shall say that a family $(K_i)_{i\in I}$ of pairwise disjoint compact sets in X is a *c-Harnack family in X* provided that, for each $i \in I$ and all compact sets A in the union of $\bigcup_{j\neq i} K_j$,

$$\varepsilon_x^A \le c\,\varepsilon_y^A \qquad \text{for all } x, y \in K_i.$$

For every closed ball B with center x and radius r and every $\gamma \in [0,1]$, let B^γ denote the downsized ball with center x and radius γr. For every $c > 1$, there exists $a \in (0,1)$ such that, for every family $(B_i)_{i\in I}$ of pairwise disjoint closed balls in X, the family $(B_i^a)_{i\in I}$ is a c-Harnack family (see Proposition 3.3).

The key to Theorem 1.1 is the following result on simultaneous dilations of closed balls which may be of independent interest.

THEOREM 1.7. *Let $\delta > 0$ and let $L_1, \dots, L_k$ be pairwise disjoint sets such that $L_1 \cup \dots \cup L_k$ is the union of a $(1+\delta)$-Harnack family of closed balls $B_1, \dots, B_m$ in X.*

Then, for every $\lambda \in \Lambda_k$ and every measure $\nu \in \mathcal{M}(\mathcal{P}(X))$ which does not charge the centers of $B_1, \dots, B_m$, there exist $\gamma_1, \dots, \gamma_m \in [0,1]$ such that the union C of the downsized balls $B_1^{\gamma_1}, \dots, B_m^{\gamma_m}$ satisfies

$$\nu^C(B_i) = (1+\delta)^{-1} \sum_{j=1}^{k} \lambda_j \nu^{L_j}(B_i) \qquad \text{for every } 1 \le i \le m.$$

Theorem 1.7 will be applied using balayage relative to an open subset W of X and the fact that balayage on Borel sets can be approximated by balayage on $(1+\delta)$-Harnack families of balls (see Proposition 5.2).

To establish the result stated in Remark 1.3, that is, to obtain an approximation using compact sets C_n contained in the *boundaries* of the open sets U, V, we can no longer use balls. We have to enlarge our toolkit to deal with arbitrary compact sets instead of balls.

In Section 7, we shall see that, for *any* compact set K in X not containing atoms of the measure ν, there is an increasing family $(K^t)_{0\le t\le 1}$ of compact sets in K such that ν^{K^t}, $0 \le t \le 1$, varies continuously from 0 to ν^K (Proposition 7.1). This will allow us to obtain an analogue of Theorem 1.7, dealing with downsizing of disjoint balls, for *arbitrary* Harnack families.

Assuming a capacity doubling property of the relevant part of the boundaries and proving a Faraday cage result, we obtain the necessary approximation of the balayage on U^c and V^c using Harnack families contained in $\partial U \cap V$ and $\partial V \cap U$, respectively (Sections 8 and 9).

The methods developed in these three sections are general enough to be applied to harmonic spaces (Section 12). This will cover second order elliptic partial differential operators of the form

$$\sum_{i,j=1}^{d} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i \frac{\partial}{\partial x_i} + c \quad \text{or} \quad \sum_{j=1}^{d} \frac{\partial}{\partial x_j} \Big(\sum_{i=1}^{d} a_{ij} \frac{\partial}{\partial x_i} + d_i \Big) + \sum_{i=1}^{d} b_i \frac{\partial}{\partial x_i} + c,$$

and even degenerate operators $\sum_{j=1}^{r} X_j^2 + Y$, where the vector fields $X_1, \dots, X_r$ satisfy Hörmander's condition of hypoellipticity (see Examples 10.1). Additional ingredients are intrinsic metrics on harmonic spaces related to Green functions (Section 10) and corresponding scaling invariant Harnack's inequalities obtained using Moser's trick (Section 11).

In the last Section, we discuss the relation between the capacity density condition, which has been studied extensively in the literature, and the weak capacity doubling property we use in Section 9.

2 Some facts on reduced measures

In this section, we collect some basic facts we shall need. To begin with, let us recall the analytic definition of reduced measures (see [6, Chapter VI] for further details). For every open set U in X, let $\mathcal{S}^+(U)$ denote the set of all superharmonic functions $v \geq 0$ on U. Given $\nu \in \mathcal{M}(\mathcal{P}(X))$ and $A \in \mathcal{B}(X)$, let ν^A denote the measure obtained *reducing* ν on A *with respect to* X, that is, for every $v \in \mathcal{S}^+(X)$,

$$\nu^A(v) := \int v \, d\nu^A = \int R_v^A \, d\nu,$$

where R_v^A is the infimum of all functions in $\mathcal{S}^+(X)$ majorizing v on A.

We stress that in [6] such a reduced measure is denoted by $\mathring{\nu}^A$, whereas there ν^A denotes the *swept* measure defined by $\nu^A(v) = \int \hat{R}_v^A \, d\nu$, $v \in \mathcal{S}^+(X)$, using the regularized function $x \mapsto \hat{R}_v^A(x) := \liminf_{y \to x} R_v^A(y)$.

If A is open, then $R_v^A \in \mathcal{S}^+(X)$ for every $v \in \mathcal{S}^+(X)$. Of course, $\nu^A(v) \leq \nu(v)$. Moreover,

$$\nu^A = \int \varepsilon_x^A \, d\nu(x) = \nu|_A + (\nu|_{A^c})^A \tag{2.1}$$

and ν^A is supported by the closure of A. Further, by [6, VI.1.7],

$$R_v^{A_n} \uparrow R_v^A, \tag{2.2}$$

whenever $A_1, A_2, \ldots$ are subsets of X such that $A_n \uparrow A$.

Let $\mathcal{P}_\nu(X)$ denote the set of all $q \in \mathcal{P}(X)$ such that $\nu(q) < \infty$.

LEMMA 2.1. *For all $\nu \in \mathcal{M}(\mathcal{P}(X))$ and $\sigma_1, \sigma_2, \ldots, \sigma_\infty \in \mathcal{M}_\nu(\mathcal{P}(X))$ the following holds. If $\lim_{n\to\infty} \sigma_n = \sigma_\infty$, then $\lim_{n\to\infty} \sigma_n(p) = \sigma_\infty(p)$ for every $p \in \mathcal{P}_\nu(X)$. Conversely, there exists a sequence (q_m) in $\mathcal{P}_\nu(X)$ such that $\lim_{n\to\infty} \sigma_n = \sigma_\infty$ provided $\lim_{n\to\infty} \sigma_n(q_m) = \sigma_\infty(q_m)$ for every $m \in \mathbb{N}$.*

Proof. Let $p \in \mathcal{P}_\nu(X)$. There exists a strictly positive $q \in \mathcal{P}_\nu(X)$ such that p/q vanishes at infinity (see [6, p. 321]). Let $\varepsilon > 0$ and $f := (p - \varepsilon q)^+$. Then $f \in \mathcal{K}^+(X)$ and $f \le p \le f + \varepsilon q$. So, for every $n \in \mathbb{N} \cup \{\infty\}$,

$$\sigma_n(f) \le \sigma_n(p) \le \sigma_n(f) + \varepsilon \sigma_n(q) \le \sigma_n(f) + \varepsilon \nu(q)$$

and therefore $|\sigma_n(p) - \sigma_\infty(p)| < \varepsilon \nu(q)$.

The converse follows from the separability of $\mathcal{K}(X)$ and a standard approximation result (see [6, I.1.3]). □

A potential $p \in \mathcal{P}(X)$ is called *strict* provided that $\rho = \nu$, whenever $\rho, \nu \in \mathcal{M}(X)$ such that $\rho(p) = \nu(p) < \infty$ and $\rho(q) \le \nu(q)$ for every $q \in \mathcal{P}(X)$. For every $\nu \in \mathcal{M}(\mathcal{P}(X))$, there exists a strict $p \in \mathcal{P}_\nu(X)$ (see [6, p. 321]). The following result on convergence of reduced measures will be very useful.

LEMMA 2.2. *Let $\nu \in \mathcal{M}(\mathcal{P}(X))$ and let $A, A_1, A_2, \ldots \in \mathcal{B}(X)$ such that*

$$\lim_{n\to\infty} \nu^{A_n \cap A}(p) = \lim_{n\to\infty} \nu^{A_n \cup A}(p) = \nu^A(p)$$

for some strict $p \in \mathcal{P}_\nu(X)$. Then $\lim_{n\to\infty} \nu^{A_n} = \nu^A$.

Proof. Of course, $\lim_{n\to\infty} \nu^{A_n}(p) = \nu^A(p)$, since $\nu^{A_n \cap A}(p) \le \nu^{A_n}(p) \le \nu^{A_n \cup A}(p)$ for every $n \in \mathbb{N}$. Since $\mathcal{M}_\nu(\mathcal{P}(X))$ is a metrizable compact set, we may assume without loss of generality that the sequences $(\nu^{A_n \cap A})$, (ν^{A_n}), and $(\nu^{A_n \cup A})$ are convergent. Let

$$\rho := \lim_{n\to\infty} \nu^{A_n \cap A}, \quad \sigma := \lim_{n\to\infty} \nu^{A_n}, \quad \tau := \lim_{n\to\infty} \nu^{A_n \cup A}.$$

By our assumption and Lemma 2.1,

$$\rho(p) = \sigma(p) = \tau(p) = \nu^A(p) \le \nu(p) < \infty.$$

Let $q \in \mathcal{P}(X)$, $m \in \mathbb{N}$, and $q_m := q \wedge (mp)$. Then $\nu(q_m) \le m\nu(p) < \infty$,

$$\nu^{A_n \cap A}(q_m) \le \nu^A(q_m) \le \nu^{A_n \cup A}(q_m), \qquad \nu^{A_n \cap A}(q_m) \le \nu^{A_n}(q_m) \le \nu^{A_n \cup A}(q_m).$$

Hence, by Lemma 2.1,

$$\rho(q_m) \le \nu^A(q_m) \le \tau(q_m) \quad \text{and} \quad \rho(q_m) \le \sigma(q_m) \le \tau(q_m).$$

Letting m tend to infinity, we obtain that

$$\rho(q) \le \nu^A(q) \le \tau(q) \quad \text{and} \quad \rho(q) \le \sigma(q) \le \tau(q).$$

Thus $\rho = \sigma = \tau = \nu^A$, since p is strict. □

LEMMA 2.3. *For all $\nu \in \mathcal{M}(\mathcal{P}(X))$ and $A \in \mathcal{B}(X)$, there exists a sequence (K_n) of compact sets in A and a sequence (V_n) of open neighborhoods of A in X such that*

$$\lim_{n\to\infty} \nu^{K_n} = \lim_{n\to\infty} \nu^{V_n} = \nu^A. \tag{2.3}$$

In particular, for all $x \in X$ and $A \in \mathcal{B}(X)$, there exists a sequence (U_n) of relatively compact open neighborhoods of x in X such that

$$\lim_{n\to\infty} \varepsilon_x^{U_n^c} = \varepsilon_x^A. \tag{2.4}$$

Proof. The first part follows immediately from [6, VI.1.9].

So let $x \in X$ and $A \in \mathcal{B}(X)$. By (2.3), it suffices to consider the case, where A is compact. Let (W_n) be an increasing sequence of relatively compact open neighborhoods of x in X such that $\bigcup_{n\in\mathbb{N}} W_n = X$ and let $U_n := W_n \setminus A$, $n \in \mathbb{N}$. Let $q \in \mathcal{P}(X)$. For every $n \in \mathbb{N}$,

$$R_q^A \le R_q^{U_n^c} = R_q^{A\cup W_n^c} \le R_q^A + R_q^{W_n^c},$$

where $\lim_{n\to\infty} R_q^{W_n^c} = 0$, since the greatest harmonic minorant of q is 0. Evaluating at x, we hence see that $\lim_{n\to\infty} \varepsilon_x^{U_n^c}(q) = \varepsilon_x^A(q)$. So (2.4) holds. □

LEMMA 2.4. *Let $\tilde{A}, A, B \in \mathcal{B}(X)$ such that $\tilde{A} \subset A$, and let $q \in \mathcal{P}(X)$. Then*

$$0 \le R_q^{A\cup B} - R_q^{\tilde{A}\cup B} \le R_q^A - R_q^{\tilde{A}}.$$

Proof. By Lemma 2.3, it suffices to show that, for all open sets $\tilde{U}, U, V$ and all compact sets $\tilde{K}, K, L$ in X such that $\tilde{K} \subset \tilde{U} \subset U$, $\tilde{K} \subset K \subset U$, and $L \subset V$,

$$w := R_q^{\tilde{U}\cup V} + R_q^U - R_q^{K\cup L} - R_q^{\tilde{K}} \ge 0. \tag{2.5}$$

If $x \in K$, then $w(x) = R_q^{\tilde{U}\cup V}(x) + q(x) - q(x) - R_q^{\tilde{K}}(x) \ge 0$. If $x \in L$, then $w(x) = q(x) + R_q^U(x) - q(x) - R_q^{\tilde{K}}(x) \ge 0$. So $w \ge 0$ on $K \cup L$. By [6, VI.2.6], the function w is superharmonic on $X \setminus (K \cup L)$ and lower semicontinuous on X. Therefore $w \ge 0$ on X by the minimum principle. □

If $\nu \in \mathcal{M}(\mathcal{P}(X))$, $x \in X$, and $r > 0$ such that $B(x,r) \subset X$, then $\nu^{U(x,r)} = \nu^{B(x,r)}$ (see [6, pp. 276, 277]) and hence, by Lemma 2.4, for every $A \in \mathcal{B}(X)$,

$$\nu^{A\cup U(x,r)} = \nu^{A\cup B(x,r)}. \tag{2.6}$$

LEMMA 2.5. *Let $\nu \in \mathcal{M}(\mathcal{P}(X))$ and $A, B \in \mathcal{B}(X)$. Then*

$$\nu^B = (\nu^{A\cup B})^B = \nu^{A\cup B}|_B + (\nu^{A\cup B}|_{B^c})^B. \tag{2.7}$$

Moreover, $(\nu^A)^B(X) \le \nu^B(X)$.

Proof. Let $\rho := \nu^{A\cup B}$. By Lemma 2.3, there exists a sequence (V_n) of open neighborhoods of B such that $\lim_{n\to\infty}\nu^{V_n} = \nu^B$ and $\lim_{n\to\infty}\rho^{V_n} = \rho^B$. Let $q \in \mathcal{P}(X)$. Trivially,

$$R_q^B \le R^{A\cup B}_{R_q^{V_n}} \le R_q^{V_n} \qquad (n \in \mathbb{N}).$$

Integrating with respect to ν we obtain that, for every $n \in \mathbb{N}$,

$$\nu^B(q) \le \rho(R_q^{V_n}) \le \nu^{V_n}(q),$$

where $\rho(R_q^{V_n}) = \rho^{V_n}(q)$. Letting n tend to ∞, we hence see that $\nu^B(q) = \rho^B(q)$ which together with (2.1) proves (2.7).

To prove that $(\nu^A)^B(X) \le \nu^B(X)$ we may assume that B is relatively compact (see (2.2)). Then we may suppose that all sets V_n are contained in a compact neighborhood K of B. For every $n \in \mathbb{N}$, $(\nu^A)^B(X) = \nu^A(R_1^B) \le \nu^A(R_1^{V_n}) \le \nu(R_1^{V_n}) = \nu^{V_n}(X)$. Since the measures ν^{V_n} are supported by K, we finally conclude that $(\nu^A)^B(X) \le \nu^B(X)$. □

In particular, formula (2.7) on iterated reduction of measures will be used again and again. In the classical case and for $\nu = \varepsilon_x$, A, B closed, and $x \in (A\cup B)^c$, it is equivalent to the following property of the Perron-Wiener-Brelot solution to the generalized Dirichlet problem for the open sets $U := B^c$ and $V := (A\cup B)^c$. If φ is a continuous $\mathcal{P}(X)$-bounded function on the boundary ∂U, then the PWB-solution h for U and φ coincides on V with the PWB-solution for V and the boundary function ψ, where $\psi = \varphi$ on $\partial U \cap \partial V$ and $\psi = h$ on $U \cap \partial V$ (see [4, Theorem 6.3.6]).

Let us also note that the strong Markov property of the corresponding process and a consideration of the entry times involved immediately would yield a probabilistic proof of (2.7).

3 Harnack families of closed balls

We recall the following definition from the Introduction. Given $c > 1$, a family $(K_i)_{i\in I}$ of pairwise disjoint compact sets in X is a *c-Harnack family in X* provided that, for each $i \in I$ and all compact sets A in $\bigcup_{j\ne i} K_j$,

$$\varepsilon_x^A \le c\,\varepsilon_y^A \qquad \text{for all } x, y \in K_i. \tag{3.1}$$

For later use of (3.1), let us observe the following.

LEMMA 3.1. *Let $A, B \in \mathcal{B}(X)$ and $c > 1$ such that $\varepsilon_x^A \le c\,\varepsilon_y^A$ for all $x, y \in B$. Then, for all measures σ, τ which are supported by B,*

$$\sigma(B)\tau^A \le c\tau(B)\sigma^A. \tag{3.2}$$

Proof. Fixing $y \in B$ and integrating the inequality $\varepsilon_x^A \le c\,\varepsilon_y^A$ with respect to τ, we obtain that $\tau^A \le c\tau(B)\varepsilon_y^A$. Integrating next with respect to σ, (3.2) follows. □

The following result is useful for the discussion of examples.

LEMMA 3.2. *Let $c > 1$ and let $(K_i)_{i\in I}$ be a family of compact sets in X such that there exist pairwise disjoint open neighborhoods V_i of K_i, $i \in I$, which are relatively compact in X and satisfy*

$$\varepsilon_x^{V_i^c} \le c\, \varepsilon_y^{V_i^c} \qquad \text{for all } x, y \in K_i. \tag{3.3}$$

Then $(K_i)_{i\in I}$ is a c-Harnack family in X.

Proof. Let $i \in I$ and let A be a closed subset of $\bigcup_{j\neq i} K_j$. Defining $\sigma_x := \varepsilon_x^{V_i^c}$ we know by (2.7) that

$$\varepsilon_x^A = \sigma_x^A \qquad (x \in K_i). \tag{3.4}$$

Obviously, (3.1) follows from (3.3) and (3.4). ☐

We recall that, for every $a \in [0, 1]$ and every closed ball B in $\mathbb{R}^d$ having center x_B and radius r_B, we denote by B^a the ball obtained by scaling of B with the factor a, that is,

$$B^a := x_B + a(B - x_B).$$

PROPOSITION 3.3. *Let $c > 1$ and $a \in (0, 1)$ such that $(1+a)^{d-\frac{\alpha}{2}} \le c(1-a)^{d+\frac{\alpha}{2}}$. Let V be the interior of a closed ball B contained in X. Then $\varepsilon_x^{V^c} \le c\,\varepsilon_y^{V^c}$ for all $x, y \in B^a$.*

In particular, for every family $(B_i)_{i\in I}$ of pairwise disjoint closed balls in X, the downsized balls B_i^a, $i \in I$, form a c-Harnack family in X.

Proof. In the classical case $\alpha = 2$, the harmonic measure $\varepsilon_x^{V^c}$ has the Poisson density

$$\rho_x^V(z) = r_B^{d-2}(r_B^2 - |x - x_B|^2)|x - z|^{-d} \qquad (z \in \partial B)$$

with respect to normalized surface measure on ∂B. For Riesz potentials (the case $0 < \alpha < 2$), $\varepsilon_x^{V^c}$ has a density ρ_x^V with respect to Lebesgue measure on B^c (see [6, p. 192 and VI.2.9]). More precisely, there exists $c_\alpha > 0$ such that

$$\rho_x^V(z) = c_\alpha \frac{(r_B^2 - |x - x_B|^2)^{\alpha/2}}{(|z - x_B|^2 - r_B^2)^{\alpha/2}}\, |z - x|^{-d} \qquad (z \in B^c).$$

If $a \in (0, 1)$ and $x, y \in B^a$, then in both cases

$$\frac{\rho_x^V(z)}{\rho_y^V(z)} \le \frac{1}{(1 - a^2)^{\alpha/2}} \frac{(1 + a)^d}{(1 - a)^d} = \frac{(1 + a)^{d-\frac{\alpha}{2}}}{(1 - a)^{d+\frac{\alpha}{2}}} \le c \tag{3.5}$$

and hence $\varepsilon_x^{V^c} \le c\,\varepsilon_y^{V^c}$. An application of Lemma 3.2 finishes the proof. ☐

4 Simultaneous dilation of disjoint balls

Let A be a union of disjoint closed balls $B_1, \ldots, B_m$ in X and let us suppose that ν is a measure in $\mathcal{M}(\mathcal{P}(X))$ which does not charge the set

$$M_A := \{x_{B_1}, \ldots, x_{B_m}\}$$

of the centers of the balls $B_1, \ldots, B_m$. Moreover, we define

$$A_t := B_1^{t_1} \cup \cdots \cup B_m^{t_m}, \qquad t = (t_1, \ldots, t_m) \in [0,1]^m.$$

LEMMA 4.1. *The mapping $t \mapsto \nu^{A_t}$ is continuous on $[0,1]^m$.*

Proof. Let $p \in \mathcal{P}_\nu(X)$ be strict. By (2.3), (2.2), and (2.6),

$$\nu^{A_t}(p) = \lim_{t' \downarrow t} \nu^{A_{t'}}(p) = \lim_{t' \uparrow t} \nu^{A_{t'}}(p). \tag{4.1}$$

Since $A_s \cup A_t = A_{s \vee t}$ and $A_s \cap A_t = A_{s \wedge t}$, an application of Lemma 2.2 yields that $\lim_{s \to t} \nu^{A_s} = \nu^{A_t}$. $\square$

LEMMA 4.2. *Let $\gamma_1, \ldots, \gamma_m \in [0, \infty)$ and*

$$\Gamma := \{t \in [0,1]^m \colon \nu^{A_t}(B_i) \le \gamma_i,\ 1 \le i \le m\}.$$

Then there exists $s \in \Gamma$ such that $s \ge t$ for every $t \in \Gamma$. Moreover, $\nu^{A_s}(B_i) = \gamma_i$ for every $i \in \{1, \ldots, m\}$ such that $s_i < 1$.

Proof. Let us note first that $\nu^{A_t}(B_i) = \nu^{A_t}(B_i^{t_i})$ for every $t \in \Gamma$ and for every $1 \le i \le m$, since ν^{A_t} is supported by the subset A_t of A.

0. Of course, $(0, \ldots, 0) \in \Gamma$, since $\nu(M_A) = 0$.

1. If $t, \tilde{t} \in \Gamma$, then $t \vee \tilde{t} \in \Gamma$. Indeed, let us fix $1 \le i \le m$. We may assume without loss of generality that $t_i \ge \tilde{t}_i$. Since $A_t \subset A_{t \vee \tilde{t}}$, we conclude by (2.7) that

$$\nu^{A_{t \vee \tilde{t}}}(B_i^{t_i \vee \tilde{t}_i}) = \nu^{A_{t \vee \tilde{t}}}(B_i^{t_i}) \le \nu^{A_t}(B_i^{t_i}) \le \gamma_i.$$

By Lemma 4.1, for every $f \in \mathcal{K}(X)$, the mapping $t \mapsto \nu^{A_t}(f)$ is continuous. Since the closed balls $B_1, \ldots, B_m$ are disjoint, we obtain that the mapping

$$t \mapsto \big(\nu^{A_t}(B_1), \ldots, \nu^{A_t}(B_m)\big)$$

is continuous on $[0,1]^m$. Therefore Γ is closed.

2. Combining the previous two parts of the proof, we see that

$$s := (\sup_{t \in \Gamma} t_1, \ldots, \sup_{t \in \Gamma} t_m) \in \Gamma.$$

Of course, $s \ge t$ for every $t \in \Gamma$.

To finish the proof, let us consider $i \in \{1, \ldots, m\}$ such that $s_i < 1$ and suppose that $\nu^{A_s}(B_i) < \gamma_i$. Let us define $\tilde{s} := (s_1, \ldots, s_{i-1}, b, s_{i+1}, \ldots, s_m)$, where $s_i < b \le 1$. By continuity, we may choose b in such a way that $\nu^{A_{\tilde{s}}}(B_i) < \gamma_i$. Since $A_s \subset A_{\tilde{s}}$, we obtain by (2.7) that $\nu^{A_{\tilde{s}}}(B_j^{s_j}) \le \nu^{A_s}(B_j^{s_j}) \le \gamma_j$ for every $j \in \{1, \ldots, m\}$, $j \ne i$. Thus $\tilde{s} \in \Gamma$, $\tilde{s} \le s$, $b = \tilde{s}_i \le s_i$, a contradiction. $\square$

Let us note the following simple consequence.

PROPOSITION 4.3. *Let $\beta_1, \dots, \beta_m$ be arbitrary numbers in $[0,1]$. Then there exist $s_1, \dots, s_m \in [0,1]$ such that the union C of the scaled balls $B_1^{s_1}, \dots, B_m^{s_m}$ satisfies*

$$\nu^C(B_i) = \beta_i \nu^A(B_i) \qquad \text{for every } 1 \le i \le m.$$

Proof. It suffices to take $\gamma_i := \beta_i \nu^A(B_i)$, $1 \le i \le m$, and to choose $s = (s_1, \dots, s_m)$ in $[0,1]^m$ according to Lemma 4.2. Then $\nu^C(B_i) \le \beta_i \nu^A(B_i)$ for all $1 \le i \le m$. Furthermore, equality holds whenever $s_i < 1$. If, however, $i \in \{1, \dots, m\}$ such that $s_i = 1$, then $\nu^C(B_i) \ge \nu^A(B_i)$ by (2.7) whence as well $\nu^C(B_i) \ge \beta_i \nu^A(B_i)$ (and $\beta_i = 1$ unless $\nu^A(B_i) = 0$). □

Here is the key to Theorem 1.1 (cf. Theorem 1.7).

THEOREM 4.4. *Let $\delta > 0$ and let $L_1, \dots, L_k$ be pairwise disjoint sets such that $L_1 \cup \dots \cup L_k$ is the union of a $(1+\delta)$-Harnack family of closed balls $B_1, \dots, B_m$ in X such that $\nu \in \mathcal{M}(\mathcal{P}(X))$ does not charge the centers of $B_1, \dots, B_m$. Moreover, let $\lambda \in \Lambda_k$ and $\beta_1, \dots, \beta_m \in [0, (1+\delta)^{-1}]$.*

Then there exist $s_1, \dots, s_m \in [0,1]$ such that $C := B_1^{s_1} \cup \dots \cup B_m^{s_m}$ satisfies

$$\nu^C(B_i) = \beta_i \sum_{j=1}^{k} \lambda_j \nu^{L_j}(B_i) \qquad \text{for every } 1 \le i \le m. \tag{4.2}$$

Proof. Since the measures ν^{L_j} are supported by L_j, the sum on the right side of (4.2) reduces to the term $\lambda_j \nu^{L_j}(B_i)$ if $B_i \subset L_j$. For every $1 \le j \le k$, let I_j denote the set of all $1 \le i \le m$ such that $B_i \subset L_j$. Of course, $I_1, \dots, I_k$ is a partition of $\{1, \dots, m\}$. By Lemma 4.2, there exists $s \in [0,1]^m$ such that $C := B_1^{s_1} \cup \dots \cup B_m^{s_m}$ satisfies

$$\nu^C(B_i) \le \beta_i \lambda_j \nu^{L_j}(B_i) \qquad \text{for all } i \in I_j,\ 1 \le j \le k, \tag{4.3}$$

with equality whenever $s_i < 1$. We claim that we have

$$\nu^C(B_i) \ge \lambda_j \nu^{L_j}(B_i), \qquad \text{if } s_i = 1,\ i \in I_j,\ 1 \le j \le k, \tag{4.4}$$

and this will clearly finish the proof, since $\beta_i < 1$ (in fact, it shows even that s_i cannot be equal to 1 for $i \in I_j$, unless $\lambda_j \nu^{L_j}(B_i) = 0$).

Indeed, let us suppose, for example, that $s_n = 1$ for some $n \in I_1$ and let $I_1' := I_1 \setminus \{n\}$. Then $B := B_n = B_n^{s_n}$, that is, B is a subset of C, and we get by (2.7) that

$$\nu^B = \nu^C|_B + (\nu^C|_{C \setminus B})^B, \tag{4.5}$$

where

$$\nu^C|_{C \setminus B} = \sum_{i \in I_1'} \nu^C|_{B_i} + \sum_{j=2}^{k} \sum_{i \in I_j} \nu^C|_{B_i}. \tag{4.6}$$

Since $\beta_i \leq (1+\delta)^{-1}$, (4.3), (3.1), and Lemma 3.1 imply that

$$(\nu^C|_{B_i})^B \leq (1+\delta)\beta_i\lambda_1(\nu^{L_1}|_{B_i})^B \leq \lambda_1(\nu^{L_1}|_{B_i})^B \qquad \text{for all } i \in I_1'.$$

Similarly, $(\nu^C|_{B_i})^B \leq \lambda_j(\nu^{L_j}|_{B_i})^B$ for all $i \in I_j$, $2 \leq j \leq k$. Taking sums we see that

$$\sum_{i\in I_1'}(\nu^C|_{B_i})^B \leq \lambda_1(\nu^{L_1}|_{L_1\setminus B})^B \quad \text{and} \quad \sum_{i\in I_j}(\nu^C|_{B_i})^B \leq \lambda_j(\nu^{L_j}|_{L_j})^B$$

for every $2 \leq j \leq k$. Therefore, by (4.5) and (4.6),

$$\nu^B(B) \leq \nu^C(B) + \lambda_1(\nu^{L_1}|_{L_1\setminus B})^B(B) + \sum_{j=2}^{k}\lambda_j(\nu^{L_j}|_{L_j})^B(B), \tag{4.7}$$

where $(\nu^{L_j}|_{L_j})^B(B) \leq \nu^B(B)$ by Lemma 2.5. Hence

$$\lambda_1\nu^B(B) \leq \nu^C(B) + \lambda_1(\nu^{L_1}|_{L_1\setminus B})^B(B).$$

By (2.7), $\nu^B = \nu^{L_1}|_B + (\nu^{L_1}|_{L_1\setminus B})^B$. Thus $\lambda_1\nu^{L_1}(B) \leq \nu^C(B)$ and the proof is finished. $\square$

5 Approximation by balayage on small balls

Balayage on open sets can be approximated by balayage on subsets consisting of finitely many balls having radii which are *arbitrarily* small with respect to their mutual distances (see Proposition 5.2). Since this does not seem to be widely known, we include a complete proof.

Let $a \in (0, 1/2)$ (for example, $a = 10^{-P}$, P being the largest known prime number) and let Z denote the union of all closed balls $B(z, a)$, $z \in \mathbb{Z}^d$. For every $n \in \mathbb{N}$, let $Z(n)$ be the union of all $B(z, a)$, $z \in \mathbb{Z}^d \cap B(0, n-1)$ (see Figure 4), and let v_n denote the equilibrium potential of $Z(n)$ with respect to $U(0, n)$, that is,

$$v_n := \inf\{v \in \mathcal{S}^+(U(0,n)) \colon v \geq 1 \text{ on } Z(n)\}.$$

We extend each v_n by 0 on $\mathbb{R}^d$.

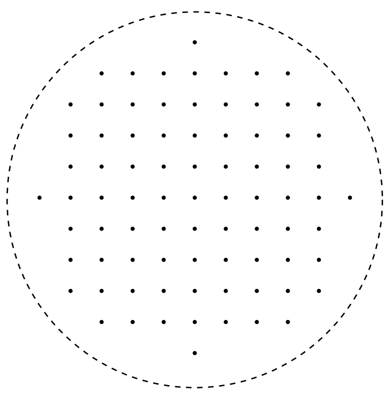

Figure 4. The set $Z(6)$

LEMMA 5.1. *The sequence (v_n) is locally uniformly increasing to* 1.

Proof. Each v_n is superharmonic on $U(0,n)$, and the sequence (v_n) is increasing. Therefore $v := \sup_{n\in\mathbb{N}} v_n$ is superharmonic on $\mathbb{R}^d$. Of course, $v = 1$ on Z.

If $d = 2$, we conclude immediately that v is identically 1 and that hence (v_n) converges locally uniformly to 1.

So let us consider the case $d \geq 3$. We claim first that v attains a minimum on $\mathbb{R}^d$. Indeed, let T denote the translation by some $y \in \mathbb{Z}^d$ with $|y| = 1$. Then $v_{n+1} \geq v_n \circ T$ and $v_{n+1} \circ T \geq v_n$. Therefore $v \circ T = v$. So there exists $x \in [0,1]^d$ such that $v(x) \leq v$ on $\mathbb{R}^d$. Thus v is constant, $v \equiv 1$ on $\mathbb{R}^d$. $\square$

PROPOSITION 5.2. *Let U, W be open sets in X, $U \subset W$, $x_0 \in \mathbb{R}^d$, $a \in (0, 1/2)$. For every $n \in \mathbb{N}$, let A_n denote the (finite) union of all balls $B(z, a/n)$ such that $z \in (1/n)(x_0 + \mathbb{Z}^d)$ and $B(z, 1/n) \subset U \cap B(0,n)$. Then, for every $q \in \mathcal{P}(X)$,*

$$\lim_{n\to\infty} R_q^{A_n \cup W^c} = R_q^{U\cup W^c}. \tag{5.1}$$

Proof. It suffices to consider a strictly positive $q \in \mathcal{P}(X)$. Let K be a compact set in U and $0 < \varepsilon < 1$. We intend to show that

$$R_q^{A_n\cup W^c} \geq (1-\varepsilon) R_q^{K\cup W^c}, \tag{5.2}$$

if n is sufficiently large. Since q is continuous and strictly positive, there exists $r \in (0,1)$ such that $r < \operatorname{dist}(K, \mathbb{R}^d \setminus U)$ and, for every $x \in K$, $q > (1-\varepsilon/2)q(x)$ on $B(x,r)$. By Lemma 5.1, there exists $n_0 \in \mathbb{N}$, $n_0 > \sqrt{d}$, such that $K \subset B(0, n_0)$ and

$$v_{n_0} > 1 - \frac{\varepsilon}{2} \quad \text{on } [-1,1]^d. \tag{5.3}$$

Now let $x \in K$ and $n \in \mathbb{N}$ such that $n_0 < nr/2$. Then $B(x,r) \subset B(0,n)$, since $x \in K \subset B(0,n_0)$ and $n_0 + r < n$. There exists a point $\tilde{x} \in x_0 + \mathbb{Z}^d$ such that $nx - \tilde{x} \in [-1,1]^d$. We define

$$\tilde{U} := \frac{1}{n}(\tilde{x} + U(0,n_0)), \quad \tilde{Z} := \frac{1}{n}(\tilde{x} + Z(n_0)), \quad \tilde{v} := \inf\{v \in \mathcal{S}^+(\tilde{U})\colon v \geq 1 \text{ on } \tilde{Z}\}.$$

By translation and scaling invariance, (5.3) implies that

$$\tilde{v} > 1 - \frac{\varepsilon}{2} \quad \text{on the set } \frac{1}{n}(\tilde{x} + [-1,1]^d) \tag{5.4}$$

containing the point x. Moreover, $|(1/n)\tilde{x} - x| \leq \sqrt{d}/n < r/2$ and $n_0/n < r/2$. Therefore $\tilde{U} \subset B(x,r)$ and hence $\tilde{Z} \subset B(x,r) \cap A_n$, since $\tilde{x} + \mathbb{Z}^d = x_0 + \mathbb{Z}^d$ and $B(x,r) \subset U \cap B(0,n)$. Defining $c := (1-\varepsilon/2)q(x)$ and knowing that $q > c$ on $B(x,r)$, we conclude that

$$R_q^{A_n} \geq c R_1^{A_n \cap B(x,r)} \geq c R_1^{\tilde{Z}} \geq c\tilde{v} \quad \text{on } \tilde{U}.$$

In particular, by (5.4),

$$R_q^{A_n}(x) \geq c\tilde{v}(x) > c(1 - \frac{\varepsilon}{2}) > (1-\varepsilon)q(x).$$

Since, of course, $R_q^{A_n\cup W^c} = q \geq (1-\varepsilon)q$ on W^c, we arrive at the inequality

$$R_q^{A_n\cup W^c} \geq (1-\varepsilon)R_q^{K\cup W^c}. \tag{5.5}$$

If (K_l) is a sequence of compact sets which is increasing to U, then $R_q^{K_l\cup W^c} \uparrow R_q^{U\cup W^c}$ by (2.2). Thus (5.1) follows, since trivially $R_q^{U\cup W^c} \geq R_q^{A_n\cup W^c}$. □

COROLLARY 5.3. *Let $U_1,\dots,U_k$ be open sets in W, $\nu \in \mathcal{M}(\mathcal{P}(X))$, $q \in \mathcal{P}_\nu(X)$, and $\delta > 0$.*

Then there exist compact sets L_j in U_j, $1 \leq j \leq k$, such that $L_1,\dots,L_k$ are pairwise disjoint, $L_1\cup\dots\cup L_k$ is the union of a $(1+\delta)$-Harnack family of closed balls $B_1,\dots,B_m$ of radius $r \leq \delta$ in W, the measure ν does not charge the centers of $B_1,\dots,B_m$, and

$$\left|\nu^{L_j\cup W^c}(q) - \nu^{U_j\cup W^c}(q)\right| < \delta. \tag{5.6}$$

Proof. Taking $c := 1+\delta$, we choose $a \in (0,1)$ according to Proposition 3.3. Moreover, we fix $x_0 \in \mathbb{R}^d$ such that ν does not charge any of the sets $(1/m)x_0 + \mathbb{Q}^d$, $m \in \mathbb{N}$, and define

$$x_j := x_0 + \Big(\frac{j}{k},0,\dots,0\Big), \quad 1 \leq j \leq k.$$

Let $M \in \mathbb{N}$ and, for every $1 \leq j \leq k$, let L_j be the (disjoint) union of all balls $B(x, a/(3kM))$, $x \in (1/M)(x_j + \mathbb{Z}^d)$, such that $B(x, 1/(3kM)) \subset U_j$. By Proposition 5.2, the inequalities (5.6) will hold and $r := a/(3kM)$ will be at most δ provided that M is sufficiently large.

By definition, the sets $L_1,\dots,L_k$ are pairwise disjoint. By Proposition 3.3, the set $L_1\cup\dots\cup L_k$ is the union of a $(1+\delta)$-Harnack family of closed balls $B_1,\dots,B_m$ in W. The measure ν does not charge the centers of $B_1,\dots,B_m$, since they are contained in the set $(1/M)x_0 + \mathbb{Q}^d$. □

6 Approximation of convex combinations of reduced measures

To prove Theorem 1.1 we shall first settle a special case.

THEOREM 6.1. *Let W be an open set in X, let $U_1,\dots,U_k$ be open sets in W, $\nu \in \mathcal{M}(\mathcal{P}(X))$, and $\lambda \in \Lambda_k$. Then there exist finite unions C_n, $n \in \mathbb{N}$, of pairwise disjoint closed balls in $U_1\cup\dots\cup U_k$ such that*

$$\lim_{n\to\infty} \nu^{C_n\cup W^c} = \sum_{j=1}^k \lambda_j \nu^{U_j\cup W^c}.$$

Proof. Let $\mathcal{Q}$ be a finite subset of $\mathcal{P}_\nu(X)$ and $\eta \in (0,1]$. By Lemma 2.1, it suffices to construct a finite union C of pairwise disjoint closed balls in $U_1\cup\dots\cup U_k$ such that, for every $q \in \mathcal{Q}$,

$$\left|\nu^{C\cup W^c}(q) - \sum_{j=1}^k \lambda_j \nu^{U_j\cup W^c}(q)\right| < \eta \tag{6.1}$$

(having chosen (q_m) according to Lemma 2.1, then, for every $n \in \mathbb{N}$, we may consider $\mathcal{Q} = \{q_1, \dots, q_n\}$ and $\eta = 1/n$).

1. Let p denote the sum of all $q \in \mathcal{Q}$. By Lemma 2.3, we may assume without loss of generality that $U := U_1 \cup \dots \cup U_k$ is relatively compact in W and that $p \geq 1$ on $\overline{U}$. Let $\varepsilon := (6\nu(p) + 1)^{-1}\eta$. There exists $0 < \delta \leq \varepsilon$ such that

$$|q(y) - q(z)| < \varepsilon, \qquad \text{whenever } q \in \mathcal{Q} \text{ and } y, z \in \overline{U},\ |y - z| < \delta. \tag{6.2}$$

By Corollary 5.3, there exist compact sets L_j in U_j, $1 \leq j \leq k$, such that $L_1, \dots, L_k$ are pairwise disjoint, $L_1 \cup \dots \cup L_k$ is the union of a $(1+\delta)$-Harnack family of closed balls $B_1, \dots, B_m$ of radius $r \leq \delta$ in W, the measure ν does not charge the centers of $B_1, \dots, B_m$, and

$$\left|\nu^{L_j \cup W^c}(p) - \nu^{U_j \cup W^c}(p)\right| < \delta.$$

Hence, for all $q \in \mathcal{Q}$ and $1 \leq j \leq k$,

$$0 \leq \nu^{U_j \cup W^c}(q) - \nu^{L_j \cup W^c}(q) \leq \nu^{U_j \cup W^c}(p) - \nu^{L_j \cup W^c}(p) < \delta. \tag{6.3}$$

Let

$$A := L_1 \cup \dots \cup L_k \quad \text{ and } \quad \mu := \sum_{j=1}^{k} \lambda_j \nu^{L_j \cup W^c}.$$

Obviously,

$$\mu(p) = \sum_{j=1}^{k} \lambda_j \nu^{L_j \cup W^c}(p) \leq \sum_{j=1}^{k} \lambda_j \nu(p) = \nu(p).$$

We intend to apply Theorem 4.4 to W in place of X. To that end we have to consider measures ${}^W\nu^E$ obtained by reducing the measure ν on $E \subset W$ *with respect to* W. By [6, VI.2.9]) and (2.1), ${}^W\nu^E = \nu^{E \cup W^c}|_W$ for every subset E of W. So, by Theorem 4.4, there exist $s_1, \dots, s_m \in [0,1]$ such that the union C of the scaled balls $B_1^{s_1}, \dots, B_m^{s_m} \subset A$ satisfies

$$\nu^{C \cup W^c}(B_i) = (1+\delta)^{-1}\mu(B_i) \qquad \text{for every } 1 \leq i \leq m. \tag{6.4}$$

2. We now fix $q \in \mathcal{Q}$ and consider $\varphi := \sum_{i=1}^{m} q(x_{B_i}) 1_{B_i}$. By (6.4),

$$\nu^{C \cup W^c}(\varphi) = (1+\delta)^{-1}\mu(\varphi). \tag{6.5}$$

By (6.2), $|\varphi - 1_A q| \leq \varepsilon 1_A \leq \varepsilon p$. Therefore

$$\left|\nu^{C \cup W^c}(\varphi) - \nu^{C \cup W^c}(1_A q)\right| \ \leq\ \varepsilon \nu^{C \cup W^c}(p) \leq \nu(p)\varepsilon, \tag{6.6}$$

$$|\mu(\varphi) - \mu(1_A q)| \ \leq\ \varepsilon \mu(p) \leq \nu(p)\varepsilon. \tag{6.7}$$

Combining (6.5), (6.6) and (6.7), we see that

$$\left|\nu^{C \cup W^c}(1_A q) - \mu(1_A q)\right| \leq \left|\nu^{C \cup W^c}(1_A q) - (1+\delta)^{-1}\mu(1_A q)\right| + \delta\mu(1_A q) \leq 3\nu(p)\varepsilon.$$

In fact, since $\nu^{C \cup W^c}$ and μ do not charge $W \setminus A$, we have shown that

$$\left|\nu^{C \cup W^c}(1_W q) - \mu(1_W q)\right| < 3\nu(p)\varepsilon. \tag{6.8}$$

3. It may be surprising that (6.8), which merely indicates that $\nu^{C\cup W^c}$ is a good approximation for μ on W, also implies that $\nu^{C\cup W^c}$ approximates μ as well on $X\setminus W$. We claim that

$$\rho := \nu^{C\cup W^c}|_{W^c} - \mu|_{W^c} \geq 0, \quad \text{and} \quad \rho(p) \leq 3\nu(p)\delta. \tag{6.9}$$

Indeed, for every compact subset K of W, $\nu^{W^c} = \nu^{K\cup W^c}|_{W^c} + (\nu^{K\cup W^c}|_W)^{W^c}$ by (2.7). Therefore

$$\mu|_{W^c} + (\mu|_W)^{W^c} = \nu^{W^c} = \nu^{C\cup W^c}|_{W^c} + (\nu^{C\cup W^c}|_W)^{W^c}.$$

Defining $\sigma := \mu|_W$ and $\tau := \nu^{C\cup W^c}|_W$ we hence see that

$$\rho = \sigma^{W^c} - \tau^{W^c}.$$

Let $B \in \{B_1, \dots, B_m\}$. By (6.4), $\tau(B) = (1+\delta)^{-1}\sigma(B)$. By (3.1), $\varepsilon_x^{W^c} \leq (1+\delta)\varepsilon_y^{W^c}$ for all $x, y \in B$. Hence, by Lemma 3.1,

$$(1_B\tau)^{W^c} \leq (1_B\sigma)^{W^c} \leq (1+\delta)^2(1_B\tau)^{W^c} \leq (1+3\delta)(1_B\tau)^{W^c}.$$

Taking the sum we obtain that $0 \leq \rho \leq 3\delta\tau^{W^c}$, where $\tau^{W^c}(p) \leq \mu^{W^c}(p) \leq \mu(p) \leq \nu(p)$. Thus (6.9) holds and

$$\left|\nu^{C\cup W^c}(1_{W^c}q) - \mu(1_{W^c}q)\right| = \rho(q) \leq \rho(p) \leq 3\nu(p)\delta. \tag{6.10}$$

4. Combining (6.8) and (6.10),

$$\left|\nu^{C\cup W^c}(q) - \sum_{j=1}^k \lambda_j \nu^{L_j\cup W^c}(q)\right| < 6\nu(p)\varepsilon.$$

Together with (6.3), this estimate finally yields

$$\left|\nu^{C\cup W^c}(q) - \sum_{j=1}^k \lambda_j \nu^{U_j\cup W^c}(q)\right| < 6\nu(p)\varepsilon + \delta \leq \eta,$$

that is, (6.1) holds. $\square$

As a consequence we now obtain our main theorem (see Theorem 1.1).

COROLLARY 6.2. *Let $\nu \in \mathcal{M}(\mathcal{P}(X))$, let $A_1, \dots, A_k \in \mathcal{B}(X)$, and $\lambda \in \Lambda_k$. Moreover, let A_0 be a Borel subset of $A_1 \cap \cdots \cap A_k$ and let (V_n) be a sequence of open neighborhoods of $(A_1 \cup \cdots \cup A_k) \setminus A_0$ in X. Then there exist finite unions C_n of pairwise disjoint closed balls in V_n, $n \in \mathbb{N}$, such that*

$$\lim_{n\to\infty} \nu^{C_n\cup A_0} = \sum_{j=1}^k \lambda_j \nu^{A_j}.$$

Proof. Again, let $\eta \in (0,1]$, $\mathcal{Q}$ be a finite subset of $\mathcal{P}_\nu(X)$, and let p denote the sum of all $q \in \mathcal{Q}$. Moreover, let V be an open neighborhood of $(A_1 \cup \cdots \cup A_k) \setminus A_0$. By Lemma 2.3, there exists a closed set F in A_0 such that

$$\nu^{A_0}(p) - \nu^F(p) < \eta \tag{6.11}$$

and there exist open neighborhoods U_j of $A_j \setminus A_0$ in $V \setminus F$ such that

$$\nu^{U_j}(p) - \nu^{A_j \setminus A_0}(p) < \eta \qquad (1 \le j \le k). \tag{6.12}$$

By Lemma 2.4 and (6.11), for every $E \in \mathcal{B}(X)$, $\nu^{A_0 \cup E}(p) - \nu^{F \cup E}(p) < \eta$. Since trivially $\nu^{A_0 \cup E}(q) - \nu^{F \cup E}(q) \ge 0$ for every $q \in \mathcal{Q}$, we hence obtain that, for all $q \in \mathcal{Q}$ and $E \in \mathcal{B}(X)$,

$$0 \le \nu^{A_0 \cup E}(q) - \nu^{F \cup E}(q) < \eta. \tag{6.13}$$

In particular, for all $q \in \mathcal{Q}$ and $1 \le j \le k$,

$$0 \le \nu^{A_j}(q) - \nu^{F \cup (A_j \setminus A_0)}(q) < \eta.$$

Similarly, by Lemma 2.4 and (6.12), for all $q \in \mathcal{Q}$ and $1 \le j \le k$,

$$0 \le \nu^{F \cup U_j}(q) - \nu^{F \cup (A_j \setminus A_0)}(q) < \eta$$

and hence

$$\left|\nu^{A_j}(q) - \nu^{F \cup U_i}(q)\right| < \eta. \tag{6.14}$$

Applying Theorem 6.1 with $W := F^c$ (and using Lemma 2.1), we obtain a finite union C of pairwise disjoint closed balls in $U_1 \cup \cdots \cup U_k$ such that, for every $q \in \mathcal{Q}$,

$$\left|\nu^{C \cup F}(q) - \sum_{j=1}^{k} \lambda_j \nu^{U_j \cup F}(q)\right| < \eta. \tag{6.15}$$

In particular, C is contained in $V \setminus F$. Let us now fix $q \in \mathcal{Q}$. By (6.13),

$$\left|\nu^{C \cup A_0}(q) - \nu^{C \cup F}(q)\right| < \eta.$$

So we conclude, by (6.14) and (6.15), that

$$\left|\nu^{C \cup A_0}(q) - \sum_{j=1}^{k} \lambda_j \nu^{A_j}(q)\right| < 3\eta. \tag{6.16}$$

As before the proof is finished by Lemma 2.1. $\square$

To obtain Corollary 1.2 from Corollary 6.2, we take $k = 2$, $A_0 := (U \cup V)^c$, $A_1 := A_0 \cup (\partial U \cap V)$, and $A_2 := A_0 \cup (\partial V \cap U)$. Indeed, then $\nu^{A_1} = \nu^{U^c}$ by Lemma 2.5, since $A_1 \subset U^c$ and ν^{U^c} is supported by the subset ∂U of A_1. Similarly, $\nu^{A_2} = \nu^{V^c}$. Moreover, $(A_1 \cup A_2) \setminus A_0 = (\partial U \cap V) \cup (\partial V \cap U)$. Finally, having taken C_n according to Corollary 6.2, it suffices to observe that $C_n \cup A_0 = ((U \cup V) \setminus C_n)^c$.

7 Continuous growth of balayage on compact sets

In this section we shall see that the choice of *balls* in the dilation result is not as essential as it might seem. All we really needed was that, starting with a finite union A of closed balls $B_1, \dots, B_m$ which are pairwise disjoint, there is an increasing family $(A_t)_{t\in[0,1]^m}$ of compact sets such that, for the given measure $\nu \in \mathcal{M}(\mathcal{P}(X))$, the mapping $t \mapsto \nu^{A_t}$ is continuous, $\nu^{A_{(0,\dots,0)}} = 0$, and $\nu^{A_{(1,\dots,1)}} = \nu^A$.

We intend to prove that this can be achieved for finite unions A of *arbitrary compact sets* $K_1, \dots, K_m$ which are pairwise disjoint. The proof will show that such a result holds in the general context of balayage spaces provided points are polar.

PROPOSITION 7.1. *Let K be a compact set in X such that $\nu \in \mathcal{M}(\mathcal{P}(X))$ does not charge points in K. Then there exist compact sets K^t in K, $0 \le t \le 1$, such that $K^1 = K$ and the following holds:*

(i) *The family $(K^t)_{0\le t\le 1}$ is increasing and right continuous, that is, $K^s \subset K^t$ if $s \le t$, and each K^t, $t \in [0,1)$, is the intersection of all K^s, $s > t$.*

(ii) *The mapping $t \mapsto \nu^{K^t}$ is continuous on $[0,1]$ and $\nu^{K^0} = 0$.*

Proof. Let $p \in \mathcal{P}(X)$ be a strict potential such that $\nu(p) \le 1$ (see [6, p. 321]).

1. Firstly, we intend to show the following. Given any two compact sets L^0 and L^1 in K with $L^0 \subset L^1$, there exists a compact set L such that $L^0 \subset L \subset L^1$ and

$$\nu^L(p) = \frac{1}{2}\nu^{L^0}(p) + \frac{1}{2}\nu^{L^1}(p) =: \gamma. \tag{7.1}$$

To that end we shall recursively construct an increasing sequence (L_n^0) and a decreasing sequence (L_n^1) of compact sets in L^1 such that $L^0 \subset L_n^0 \subset L_n^1 \subset L^1$ and

$$\nu^{L_n^0}(p) \le \gamma \le \nu^{L_n^1}(p) \le \nu^{L_n^0}(p) + 2^{-(n-1)}, \qquad n = 1, 2, \dots. \tag{7.2}$$

Defining $L_1^0 := L^0$ and $L_1^1 := L^1$, (7.2) trivially holds for $n = 1$, since $\nu(p) \le 1$. Suppose that $n \in \mathbb{N}$ and that compact sets L_n^0, L_n^1 satisfying $L_n^0 \subset L_n^1 \subset K$ and $\nu^{L_n^0}(p) \le \gamma \le \nu^{L_n^1}(p)$ have been constructed.

Let us consider $y \in L_n^1$. Since points are polar and $\nu(\{y\}) = 0$, we know that $\nu^{\{y\}} = 0$. So, by Lemma 2.3, there exists $r_y > 0$ such that

$$\nu^{B(y,r_y)\cap L_n^1}(p) < 2^{-n}. \tag{7.3}$$

There exist $y_1, \dots, y_m \in L_n^1$ such that L_n^1 is covered by the sets $A_j := B(y_j, r_{y_j}) \cap L_n^1$, $1 \le j \le m$. We define

$$C_k := L_n^0 \cup \bigcup_{j=1}^k A_j \qquad (0 \le k \le m).$$

Then $L_n^0 = C_0 \subset C_1 \subset \dots \subset C_m = L_n^1$. Since $\nu(L_n^0) \le \gamma \le \nu(L_n^1)$ and, for every $0 \le k \le m$,

$$\nu^{C_k}(p) \le \nu^{C_{k+1}}(p) \le \nu^{C_k}(p) + \nu^{A_k}(p) < \nu^{C_k}(p) + 2^{-n}$$

by ([6, VI.9.3]), there exists $l \in \{0, 1, \ldots, m\}$ such that

$$\nu^{C_l}(p) \leq \gamma \leq \nu^{C_{l+1}}(p) \leq \nu^{C_l}(p) + 2^{-n}.$$

The induction step is finished defining $L^0_{n+1} := C_l$ and $L^1_{n+1} := C_{l+1}$.

By (7.2), the set $L := \bigcap_{n=1}^{\infty} L^1_n$ has the desired properties.

2. We now begin our construction of the family $(K^t)_{0 \leq t \leq 1}$ taking $L^0 := \emptyset$ and $L^1 := K$. By part one, we obtain a compact set $L^{1/2}$ such that $L^{1/2} \subset K$ and $\nu^{L^{1/2}}(p) = (1/2)\nu^K(p)$. Continuing in an obvious way, we obtain compact sets L^s in K, $s \in D := \{k2^{-n} \colon n \in \mathbb{N},\ k = 0, 1, 2, \ldots, 2^n\}$, such that $L^s \subset L^{\tilde{s}}$ if $s \leq \tilde{s}$, and $\nu^{L^s}(p) = s \cdot \nu^K(p)$ for all $s \in D$. We finish the construction defining $K^1 := K$ and

$$K^t := \bigcap_{s \in D, s > t} L^s \qquad (0 \leq t < 1).$$

Then $(K^t)_{0 \leq t \leq 1}$ is an increasing and right continuous family of compact sets and

$$\nu^{K^t}(p) = t \cdot \nu^K(p) \qquad \text{for all } 0 \leq t \leq 1. \tag{7.4}$$

In particular, $t \mapsto \nu^{K^t}(p)$ is continuous on $[0, 1]$. Thus, by Lemma 2.2, the mapping $t \mapsto \nu^{K^t}$ is continuous on $[0, 1]$. □

COROLLARY 7.2. *Let ν and $(K^t)_{0 \leq t \leq 1}$ be as in Proposition 7.1. If, in addition, $\nu(U) > 0$ for every non-empty open subset U of K^c, then, for every $\mu \in \mathcal{M}(\mathcal{P}(X))$ not charging K, the mapping $t \mapsto \mu^{K^t}$ is continuous on $[0, 1]$.*

Proof. Let us fix $\mu \in \mathcal{M}(\mathcal{P}(X))$ not charging K and let $p \in \mathcal{P}(X)$ be strict such that $(\nu + \mu)(p) \leq 1$. Let us fix $0 \leq t \leq 1$ and let $s_n, \tilde{s}_n \in [0, 1]$ such that $s_n \downarrow t$ and $\tilde{s}_n \uparrow t$. Of course, $\varepsilon_y^{K^{\tilde{s}_n}}(p) \leq \varepsilon_y^{K^t}(p) \leq \varepsilon_y^{K^{s_n}}(p)$ for all $n \in \mathbb{N}$ and $y \in K^c$. We define

$$h_n := R_p^{K^{s_n}} - R_p^{K^{\tilde{s}_n}}.$$

Then $h_n \geq 0$, h_n is harmonic on K^c (see [6, VI.2.6]), and $\nu(h_n) = \nu^{K^{s_n}}(p) - \nu^{K^{\tilde{s}_n}}(p)$. The sequence (h_n) is decreasing to a harmonic function $h \geq 0$ on K^c (see [6, III.3.1]) satisfying $\nu(h) = 0$. Therefore $h = 0$ on K^c. This implies that, for every $y \in K^c$, $\lim_{s \to t} \varepsilon_y^{K^s}(p) = \varepsilon_y^{K^t}(p)$, whence $\lim_{s \to t} \mu^{K^s}(p) = \mu^{K^t}(p)$, since μ is supported by K^c. So, by Lemma 2.2, $\lim_{s \to t} \mu^{K^s} = \mu^{K^t}$. □

Now let $K_1, K_2, \ldots, K_m$ be disjoint compact subsets of X such that ν does not charge points of $K_1 \cup \cdots \cup K_m$. For each $i \in \{1, \ldots, m\}$, we choose an increasing right continuous family $(K_i^{t_i})_{0 \leq t_i \leq 1}$ of compact sets in K_i such that $K_i^1 = K_i$, $\nu^{K_i^0} = 0$, and $t_i \mapsto \nu^{K_i^{t_i}}$ is continuous on $[0, 1]$.

As we did earlier with finite unions of balls, we then define

$$A_t := K_1^{t_1} \cup K_2^{t_2} \cup \cdots \cup K_m^{t_m}, \qquad t = (t_1, \ldots, t_m) \in [0, 1]^m. \tag{7.5}$$

The continuity of $t \mapsto \nu^{A_t}$ will be an easy consequence of the following general result.

LEMMA 7.3. *Let $(B^t)_{0\le t\le 1}$ be an increasing family in $\mathcal{B}(X)$ such that the mapping $t\mapsto \nu^{B^t}$ is continuous on $[0,1]$. Then, for every $B\in\mathcal{B}(X)$, the mapping $t\mapsto \nu^{B^t\cup B}$ is continuous on $[0,1]$.*

Proof. Let us fix $q\in\mathcal{P}_\nu(X)$. By Lemma 2.1, the function $t\mapsto \nu^{B^t}(q)$ is continuous on $[0,1]$ and we only have to show that the function $t\mapsto \nu^{B^t\cup B}(q)$ is continuous as well. So let $t\in[0,1]$ and $s,\tilde s\in[0,1]$ such that $\tilde s\le t\le s$. Then trivially $\nu^{B^{\tilde s}\cup B}(q)\le \nu^{B^t\cup B}(q)\le \nu^{B^s\cup B}(q)$. By Lemma 2.4,

$$\nu^{B^s\cup B}(q)-\nu^{B^{\tilde s}\cup B}(q)\le \nu^{B^s}(q)-\nu^{B^{\tilde s}}(q),$$

where the right side converges to 0 as $s-\tilde s\to 0$. Thus $\lim_{\tau\to t}\nu^{B^\tau\cup B}(q)=\nu^{B^t\cup B}(q)$. □

PROPOSITION 7.4. *The mapping $t\mapsto \nu^{A_t}$ is continuous on $[0,1]^m$.*

Proof. Let $q\in\mathcal{P}_\nu(X)$. By Lemma 7.3, the function $\varphi\colon t\mapsto \nu^{A_t}(q)$ is separately continuous on $[0,1]^m$. Moreover, φ is obviously increasing. Therefore φ is continuous on $[0,1]^m$. □

Proceeding almost word by word as in Section 4 we now obtain the following.

THEOREM 7.5. *Let $\delta>0$ and let $L_1,\dots,L_k$ be pairwise disjoint sets such that $L:=L_1\cup\dots L_k$ is the union of a $(1+\delta)$-Harnack family of compact sets $K_1,\dots,K_m$ in X and $\nu\in\mathcal{M}(\mathcal{P}(X))$ does not charge points in L. Moreover, let $\beta_1,\dots,\beta_m\in[0,(1+\delta)^{-1}]$ and $\lambda\in\Lambda_k$. Then there exists a compact subset K of L such that*

$$\nu^K(K_i)=\beta_i\sum_{j=1}^k\lambda_j\nu^{L_j}(K_i)\qquad \textit{for every } 1\le i\le m.$$

8 A Faraday cage result

The following result is inspired by the proof of [13, Théorème 1]. It immediately yields an alternative proof for Proposition 5.2, a proof which shows that a similar approximation by balayage on disjoint compact pieces which are small with respect to their mutual distances can be established under very general assumptions on the potential theoretic setting.

Moreover, it will allow us to strengthen Corollary 1.2 provided the boundaries of the open sets U and V have the weak capacity doubling property (see Section 9).

PROPOSITION 8.1. *Let K be a compact set in X and let q be a continuous potential on X which is harmonic outside K. Moreover, let $\varepsilon,\eta\in(0,1)$, and $M>1$. Then there exists $\rho_0>0$ such that, for every $0<\rho\le\rho_0$, the following holds:*

If $x_1,\dots,x_N\in K$ such that the balls $B(x_i,\rho/M)$ are pairwise disjoint, the set K is covered by the balls $B(x_i,\rho)$, and $A\in B(X)$ (see Figure 5) such that

$$\operatorname{cap}\big(A\cap B(x_i,M\rho)\big)\ge\eta\operatorname{cap}\big(K\cap B(x_i,\rho)\big)\qquad \textit{for every } 1\le i\le N, \tag{8.1}$$

then

$$R_q^A\ge(1-\varepsilon)\,q.$$

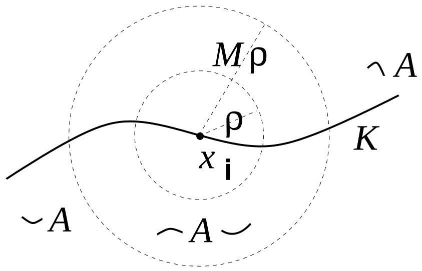

Figure 5. Illustration of (8.1)

Proof. We may assume without loss of generality that $q \geq 1$ on a compact neighborhood L of K in X. Let μ denote the Riesz measure for q, that is,

$$G^\mu = q,$$

where G denotes the Green function for X. Let $\delta := \varepsilon/4$ and let $a \in (0,1)$ such that $(1+a)^{d-\frac{\alpha}{2}} \leq (1+\delta)(1-a)^{d+\frac{\alpha}{2}}$. We define

$$c := \big((1+a^{-1})M^2\big)^d \quad \text{and} \quad \beta := \frac{\eta\delta}{c}. \tag{8.2}$$

There exists $0 < \rho_0 < \operatorname{dist}(K, L^c)/M$ such that

$$G^{1_{B(x,\rho_0)}\mu} \leq \beta \qquad \text{for every } x \in K$$

(cf. [11, Proposition 7.1]).

Let us fix $0 < \rho \leq \rho_0$ and consider $A \in \mathcal{B}(X)$ and $x_1, \dots, x_N \in K$ such that the assumptions of the Proposition are satisfied.

There exist measures μ_i, $1 \leq i \leq N$, such that $\sum_{i=1}^N \mu_i = \mu$ and each measure μ_i is supported by $B(x_i, \rho)$, $1 \leq i \leq N$. Then certainly

$$G^{\mu_i} \leq \beta \qquad \text{for every } 1 \leq i \leq N. \tag{8.3}$$

Let J denote the set of all $1 \leq i \leq N$ such that $\mu_i \neq 0$ and let $i \in J$. By (8.3),

$$\operatorname{cap}\big(K \cap B(x_i, \rho)\big) \geq \frac{\|\mu_i\|}{\beta}.$$

Hence (8.1) implies that

$$\operatorname{cap}\big(A \cap B(x_i, M\rho)\big) \geq \frac{\eta\|\mu_i\|}{\beta}.$$

So there exists a compact set L_i in $A \cap B(x_i, M\rho)$ such that

$$\operatorname{cap}(L_i) > \frac{\eta\|\mu_i\|}{2\beta}. \tag{8.4}$$

Let ν_i denote the equilibrium measure for L_i, that is, the Riesz measure for $\hat{R}_1^{L_i}$, and

$$\tilde{\mu}_i := \frac{\|\mu_i\|}{\operatorname{cap}(L_i)}\,\nu_i = \frac{\|\mu_i\|}{\|\nu_i\|}\,\nu_i.$$

Then, by (8.4),

$$G^{\tilde{\mu}_i} = \frac{\|\mu_i\|}{\operatorname{cap}(L_i)}\, G^{\nu_i} \le \frac{\|\mu_i\|}{\operatorname{cap}(L_i)} < \frac{2\beta}{\eta}. \tag{8.5}$$

By Proposition 3.3, for all $x, y \in B(x_i, M\rho)$ and $z \notin V_i := B(x_i, M\rho/a)$,

$$G(x,z) = \varepsilon_x^{V_i^c}(G(\cdot, z)) \le (1+\delta)\varepsilon_y^{V_i^c}(G(\cdot, z)) = (1+\delta)G(y,z).$$

Since $\|\tilde{\mu}_i\| = \|\mu_i\|$ and the measures $\tilde{\mu}_i$, μ_i are supported by $B(x_i, M\rho)$, we conclude that, outside $B(x_i, M\rho/a)$,

$$G^{\tilde{\mu}_i} \le (1+\delta)G^{\mu_i} \quad \text{and} \quad G^{\mu_i} \le (1+\delta)G^{\tilde{\mu}_i}.$$

Defining

$$J_i := \{j \in J \colon B(x_j, M\rho/a) \cap B(x_i, M\rho) \neq \emptyset\}$$

we hence know that, for every $j \in J \setminus J_i$ and for all $x \in B(x_i, M\rho)$,

$$G^{\tilde{\mu}_j}(x) \le (1+\delta)G^{\mu_j}(x) \quad \text{and} \quad G^{\mu_j}(x) \le (1+\delta)G^{\tilde{\mu}_j}(x). \tag{8.6}$$

It is easily verified that $B(x_j, \rho/M) \subset B(x_i, (1+a^{-1})M\rho)$ for every $j \in J_i$. Since the balls $B(x_1, \rho/M), \dots, B(x_N, \rho/M)$ are disjoint by assumption, the sum of the volumes of the balls $B(x_j, \rho/M)$, $j \in J_i$, is certainly bounded by the volume of the ball $B(x_i, (1+a^{-1})M\rho)$. Therefore J_i has less than c elements (see (8.2)).

Let us define

$$\tilde{\mu} := \sum_{j \in J} \tilde{\mu}_j.$$

Of course, $\mu = \sum_{j\in J} \mu_j$. So, by (8.5), (8.3), and (8.6), the inequalities

$$G^{\tilde{\mu}} \le (1+\delta)G^{\mu} + \frac{2\beta}{\eta}\, c \le (1+2\delta)q, \tag{8.7}$$

$$G^{\mu} \le (1+\delta)G^{\tilde{\mu}} + \beta c \le (1+\delta)G^{\tilde{\mu}} + \delta G^{\mu} \tag{8.8}$$

hold on each $B(x_i, M\rho)$, $i \in J$, and hence on X, by the minimum principle. By definition, $\tilde{\mu}$ is supported by a compact set in A. So, by (8.7) and the domination principle,

$$R_q^A \ge (1+2\delta)^{-1} G^{\tilde{\mu}}.$$

Since

$$G^{\tilde{\mu}} \ge \frac{1-\delta}{1+\delta}\, q \ge (1-2\delta)q$$

by (8.8) and $(1-2\delta)/(1+2\delta) \ge 1-4\delta \ge 1-\varepsilon$, we finally see that $R_q^A \ge (1-\varepsilon)q$. $\square$

9 Approximation using a capacity doubling property

Given $\gamma \in (0,1)$ and an open set U in X, let us say that a set A is a γ-*ball set in* U, if A is compact and if there exist pairwise disjoint closed balls $B_1, \dots, B_m$ contained in U such that $A \subset B_1^\gamma \cup \dots \cup B_m^\gamma$. We know that, for every $\delta > 0$, there exists $\gamma > 0$ such that every γ-ball set is the union of a $(1+\delta)$-Harnack family (see Proposition 3.3).

We recall that the *base* $b(A)$ of a subset A of X is the set of all points $x \in X$ such that $\hat{R}_p^A(x) = p(x)$ for every $p \in \mathcal{P}(X)$. It is a G_δ-set, the fine closure of A is $A \cup b(A)$, and the set $A \setminus b(A)$ is polar whence $b(A \setminus b(A)) = \emptyset$. Moreover, the mapping $b\colon A \mapsto b(A)$ is additive (see [6, Section VI] for details). Therefore

$$b(A) = b(A \cap b(A)) \cup b(A \setminus b(A)) = b(A \cap b(A)).$$

This shows that $b(A)$ is the fine closure of $A \cap b(A)$ and hence, assuming that $A \in \mathcal{B}(X)$, $\nu^{b(A)} = \nu^{A \cap b(A)}$ for every $\nu \in \mathcal{M}(\mathcal{P}(X))$. In particular, if ν does not charge the (polar) set $A \setminus b(A)$, then

$$\nu^{b(A)} = \nu^A, \tag{9.1}$$

since, for every $p \in \mathcal{P}(X)$,

$$\nu(R_p^A) \le \nu(R_p^{A\setminus b(A)}) + \nu(R_p^{A\cap b(A)}) = \nu(R_p^{A\cap b(A)}) \le \nu(R_p^A).$$

Finally, let us recollect that a set $A \subset X$ is called *subbasic*, if $A \subset b(A)$, that is, if $b(A)$ is the fine closure of A.

For every $A \in \mathcal{B}(X)$, let $D(A)$ denote the set of all points $x \in A \cap b(A)$ such that, for some $c > 0$ and $r_0 > 0$,

$$\operatorname{cap}(A \cap U(x,2r)) \le c \operatorname{cap}(A \cap U(x,r)) \qquad \text{for every } 0 < r \le r_0. \tag{9.2}$$

Let us note that then, for every $\gamma \in (0,1)$, there exists $\eta > 0$ such that

$$\operatorname{cap}(A \cap B(x,\gamma r)) \ge \eta \operatorname{cap}(A \cap B(x,r)) \qquad \text{for every } 0 < r \le r_0. \tag{9.3}$$

Indeed, assume that (9.2) holds and let $0 < r \le r_0$. Taking $k \in \mathbb{N}$ such that $2^{-k} < \gamma$, and then $\rho \in (r, 2r_0)$ such that $2^{-k}\rho < \gamma r$, we obtain that

$$\begin{aligned} \operatorname{cap}(A \cap B(x,\gamma r)) &\ge \operatorname{cap}(A \cap U(x,2^{-k}\rho)) \\ \ge c^{-k} \operatorname{cap}(A \cap U(x,\rho)) &\ge c^{-k} \operatorname{cap}(A \cap B(x,r)). \end{aligned}$$

We shall say that $A \in \mathcal{B}(X)$ has the *weak capacity doubling property* if $D(A)$ is finely dense in $b(A)$. In applications of this property, we shall use the subsets $D_n(A)$, $n \in \mathbb{N}$, consisting of all $x \in A \cap b(A)$ such that

$$\operatorname{cap}(A \cap U(x,2r)) \le n \operatorname{cap}(A \cap U(x,r)) \qquad \text{for every } 0 < r \le 1/n. \tag{9.4}$$

Clearly, the sequence $(D_n(A))$ is increasing to $D(A)$.

LEMMA 9.1. *For every $A \in \mathcal{B}(X)$, the sets $D_n(A)$, $n \in \mathbb{N}$, are Borel sets. In particular, $D(A) \in \mathcal{B}(X)$.*

Proof. Let us define

$$f(x,r) := \operatorname{cap}(A \cap U(x,r)) \qquad (x \in X,\, r > 0).$$

For every $x \in X$, the function $r \mapsto f(x,r)$ is left continuous, since $U(x,s) \uparrow U(x,r)$ as $s \uparrow r$. Therefore

$$D_n(A) = \bigcap_{0<r\le 1/n,\, r\in\mathbb{Q}} \{x \in A \cap b(A) \colon f(x,2r) \le nf(x,r)\} \qquad (n \in \mathbb{N}).$$

We know that $A \cap b(A) \in \mathcal{B}(X)$. So the proof will be finished, if we show that the functions $x \mapsto f(x,r)$, $r > 0$, are lower semicontinuous. To that end let us fix $r > 0$ and $a \in \mathbb{R}$ such that $f(x,r) > a$. By the left continuity of $s \mapsto f(x,s)$, there exists $0 < s < r$ such that $f(x,s) > a$. If $y \in U(x, r-s)$, then $U(x,s) \subset U(y,r)$ and hence $f(y,r) \ge f(x,s) > a$. $\square$

LEMMA 9.2. *Let $A \in \mathcal{B}(X)$ have the weak capacity doubling property. Then $D(A)$ is subbasic. In particular, $b(D(A)) = b(A)$.*

Proof. By definition of the weak capacity doubling property, $b(A) = D(A) \cup b(D(A))$. Since $bb = b$ and b is additive, we hence obtain that

$$D(A) \subset b(A) = b(b(A)) = b(D(A)) \cup b(b(D(A))) = b(D(A)).$$

$\square$

PROPOSITION 9.3. *Let $A \in \mathcal{B}(X)$ have the weak capacity doubling property, let L be a compact subset of $D(A)$, and let V be an open neighborhood of L. Moreover, let $\varepsilon \in (0,1)$, $\nu \in \mathcal{M}(\mathcal{P}(X))$, and $p \in \mathcal{P}(X)$ such that $\nu(p) \le 1$.*

Then there exists an ε-ball set $\tilde{A}$ in V such that $\tilde{A} \subset A$, the measure ν does not charge points of $\tilde{A}$, and $\nu^{\tilde{A}}(p) > \nu^L(p) - 2\varepsilon$.

Proof. Knowing that $D(A) \cap V$ is subbasic, we conclude from [6, VI.6.12] that there exists a compact set $\tilde{L}$ in $D(A) \cap V$ with $L \subset b(\tilde{L})$. By [6, VI.4.16], there exists $\tilde{q} \in \mathcal{P}(X)$ such that $R_p^L \le \tilde{q} \le R_p^{b(\tilde{L})}$ and $R_{\tilde{q}}^{b(\tilde{L})} = \tilde{q}$. In particular, $\tilde{q}$ is harmonic outside $\tilde{L}$. Since $\tilde{L} \cap D_n(A) \uparrow \tilde{L}$ as $n \to \infty$, there exist $n \in \mathbb{N}$ and $q, q' \in \mathcal{P}(X)$ such that $q + q' = \tilde{q}$, q is harmonic outside a compact subset K of $\tilde{L} \cap D_n(A)$, and $\nu(q') < \varepsilon$ (see [6, II.6.17]).

By (9.3), there exists $\eta > 0$ such that, for every $x \in K$ and $r \in (0, 1/n]$,

$$\operatorname{cap}(A \cap B(x, \frac{\varepsilon r}{3})) \ge 2\eta \operatorname{cap}(A \cap B(x,r)). \tag{9.5}$$

Taking $M := 3$ we choose $\rho_0 \le (1/n) \wedge \operatorname{dist}(K, \mathbb{R}^d \setminus V)$ according to Proposition 8.1 and fix $\rho \in (0, \rho_0)$. There exist points $x_1, \dots, x_N$ in K such that the balls

$B(x_1,\rho/3),\ldots,B(x_N,\rho/3)$ are pairwise disjoint and the balls $B(x_1,\rho),\ldots,B(x_N,\rho)$ cover K (see [34, Lemma 7.3]).

Let $1\le i\le N$. Since countable sets in X have zero capacity and since there are at most countably many points $y\in X$ such that $\nu(\{y\})>0$, we may choose a compact subset A_i in $A\cap B(x_i,\varepsilon\rho/3)$ such that ν does not charge points of A_i, the capacity of A_i is at least $(1/2)\operatorname{cap}(A\cap B(x_i,\varepsilon\rho/3))$, and hence, by (9.5),

$$\operatorname{cap}(A_i)\ge\eta\operatorname{cap}(A\cap B(x_i,\rho))\ge\eta\operatorname{cap}(K\cap B(x_i,\rho)).$$

Let $\tilde A:=A_1\cup\cdots\cup A_N$. By construction, $\tilde A$ is a ε-ball set in V, $\tilde A\subset A$, the measure ν does not charge points of $\tilde A$, and

$$\operatorname{cap}(\tilde A\cap B(x_i,3\rho))\ge\operatorname{cap}(A_i)\ge\eta\operatorname{cap}(K\cap B(x_i,\rho))\qquad\text{for all } 1\le i\le N.$$

Thus $R_q^{\tilde A}\ge(1-\varepsilon)q$ by Proposition 8.1 and hence

$$\nu^{\tilde A}(p)\ge\nu^{\tilde A}(q)\ge(1-\varepsilon)\nu(q)>\nu(\tilde q)-2\varepsilon\ge\nu^L(p)-2\varepsilon.$$

□

COROLLARY 9.4. *Let $A_1,A_2\in\mathcal B(X)$ such that $A_1':=A_1\setminus A_2$ and $A_2':=A_2\setminus A_1$ have the capacity doubling property. Moreover, let us suppose that $\nu\in\mathcal M(\mathcal P(X))$ does not charge the sets $A_j'\setminus b(A_j')$, $j=1,2$, and let V be an open neighborhood of $A_1'\cup A_2'$.*

Then there exist compact sets $A_{1,n}$ in A_1' and $A_{2,n}$ in A_2', respectively, such that each union $A_{1,n}\cup A_{2,n}$, $n\in\mathbb N$, is a $(1/n)$-ball set in V, the measure ν does not charge points in $A_{1,n}\cup A_{2,n}$, and

$$\lim_{n\to\infty}\nu^{A_{j,n}\cup(A_1\cap A_2)}=\nu^{A_j},\qquad j\in\{1,2\}.$$

Proof. Let us fix a strict potential $p\in\mathcal P(X)$ such that $\nu(p)\le1$, and let $\varepsilon\in(0,1)$. By (9.1) and the weak capacity doubling property, $\nu^{D(A_j')}=\nu^{A_j'}$ and hence, by Lemma 2.3, there exists compact sets L_j in $D(A_j')$ such that

$$\nu^{L_j}(p)>\nu^{A_j'}(p)-\varepsilon,\qquad j\in\{1,2\}.\tag{9.6}$$

Let V_1 and V_2 be disjoint open neighborhoods of L_1 and L_2 in V, respectively. For the moment, let us fix $j\in\{1,2\}$. By Proposition 9.3, there exists an ε-ball set $\tilde A_j$ in V_j such that $\tilde A_j\subset A_j$, the measure ν does not charge points in $\tilde A_j$, and

$$\nu^{\tilde A_j}(p)>\nu^{L_j}(p)-2\varepsilon.\tag{9.7}$$

Let $A_0:=A_1\cap A_2$. Then $A_j=A_j'\cup A_0$ and therefore, by (9.6), (9.7), and Lemma 2.4,

$$0\le\nu^{A_j}(p)-\nu^{\tilde A_j\cup A_0}(p)\le\nu^{A_j'}(p)-\nu^{\tilde A_j}(p)<3\varepsilon.$$

Obviously, $\tilde A_1\cup\tilde A_2$ is an ε-ball set in V, since V_1 and V_2 are disjoint subsets of V.

Taking $\varepsilon=1/n$, $n\in\mathbb N$, we obtain sets $A_{j,n}$ in $A_j\setminus A_0$ such that

$$0\le\nu^{A_j}(p)-\nu^{A_{j,n}\cup A_0}(p)<\frac{3}{n},\qquad j\in\{1,2\},$$

the unions $A_{1,n}\cup A_{2,n}$ are $(1/n)$-ball sets in V, and the measure ν does not charge points of $A_{1,n}\cup A_{2,n}$. Lemma 2.2 finishes the proof. □

Using Corollary 9.4, Theorem 7.5, and proceeding as in the proof of Theorem 6.1 and Corollary 6.2, we obtain the following result.

THEOREM 9.5. *Let $A_1, A_2 \in \mathcal{B}(X)$ such that $A_1' := A_1 \backslash A_2$ and $A_2' := A_2 \backslash A_1$ have the weak capacity doubling property. Moreover, let us suppose that $\nu \in \mathcal{M}(\mathcal{P}(X))$ does not charge the sets $A_j' \setminus b(A_j')$, $j = 1, 2$.*

Then, for every $\lambda \in (0,1)$, there exist $(1/n)$-ball sets $C_n \subset A_1' \cup A_2'$ and closed sets $F_n \subset A_1 \cap A_2$, $n \in \mathbb{N}$, such that

$$\lim_{n\to\infty} \nu^{C_n \cup F_n} = \lambda \nu^{A_1} + (1-\lambda)\nu^{A_2}. \tag{9.8}$$

If $A_0 := A_1 \cap A_2$ is closed, then the sets F_n can be replaced by A_0.

COROLLARY 9.6. *Let U, V be open sets in X such that $U \setminus V$ and $V \setminus U$ have the weak capacity doubling property. Moreover, let us suppose that ν is supported by $U \cap V$, let $W := U \cup V$, and $\lambda \in (0,1)$.*

Then there exist $(1/n)$-ball sets C_n in W, $n \in \mathbb{N}$, such that $C_n \subset (U \setminus V) \cup (V \setminus U)$ and

$$\lim_{n\to\infty} \nu^{(W\setminus C_n)^c} = \lambda \nu^{U^c} + (1-\lambda)\nu^{V^c}.$$

COROLLARY 9.7. *Let $\alpha = 2$ (classical case) and let U, V be open sets in X such that $\partial U \cap V$ and $\partial V \cap U$ have the weak capacity doubling property. Moreover, let us suppose that ν is supported by $U \cap V$, let $W := U \cup V$, and $\lambda \in (0,1)$.*

Then there exist $(1/n)$-ball sets C_n in W, $n \in \mathbb{N}$, such that each C_n is contained in $(\partial U \cap V) \cup (\partial V \cap U)$ and

$$\lim_{n\to\infty} \nu^{(W\setminus C_n)^c} = \lambda \nu^{U^c} + (1-\lambda)\nu^{V^c}.$$

Proof. Let $A_1 := (\partial U \cap V) \cup W^c$ and $A_2 := (\partial V \cap U) \cup W^c$. Then $\nu^{A_1} = \nu^{U^c}$ and $\nu^{A_2} = \nu^{V^c}$ (see the end of Section 6). Moreover, $A_1' := A_1 \setminus A_2 = \partial U \cap V$, $A_2' := A_2 \setminus A_1 = \partial V \cap U$, and $A_1 \cap A_2 = W^c$. Thus the result follows immediately from Theorem 9.5. □

10 Intrinsic metric on Brelot spaces

Let X be a locally compact space with countable base which is not compact. Moreover, we assume that X is connected and locally connected.

Given a harmonic sheaf $\mathcal{H}$ on X such that $(X, \mathcal{H})$ is a $\mathcal{P}$-harmonic space, let us say that a Borel measurable function $G\colon X \times X \to [0,\infty]$ is a *Green function for* $(X,\mathcal{H})$ provided the following conditions are satisfied:

(i) For every $y \in X$, $G(\cdot, y)$ is a potential on X which is harmonic on $X \setminus \{y\}$.

(ii) For every continuous real potential p on X which is harmonic outside a compact set, there exists a measure μ on X such that $p = \int G(\cdot, y)\, d\mu(y)$.

We observe that G determines the harmonic sheaf $\mathcal{H}$ uniquely, since continuous real potentials determine the harmonic kernels and hence harmonic functions.

In the following let $\mathcal{H}$ be a harmonic sheaf on X and $G\colon X \times X \to [0,\infty]$ such that $(X,\mathcal{H})$ is a $\mathcal{P}$-harmonic Brelot space and G is a Green function for $(X,\mathcal{H})$, $G > 0$, and $G(x,x) = \infty$ for all $x \in X$. We define the adjoint ${}^*G$ of G by

$$ {}^*G(x,y) := G(y,x) \qquad (x,y \in X) $$

and suppose that ${}^*G$ is a Green function for some Brelot space $(X, {}^*\mathcal{H})$. Notions related to $(X, {}^*\mathcal{H})$ will be distinguished from those related to $(X,\mathcal{H})$ by adding an asterisk (for example, $*$-harmonic function and μ^{*A}).

It may be of interest to note that, in view of the axiom of proportionality (cf. [10, Satz 3.2]), G is almost uniquely determined by the harmonic sheaves $\mathcal{H}$ and ${}^*\mathcal{H}$ (see [22, Remarks 2.1]). Indeed, suppose that $\tilde{G}$ has the same properties as G and let $x_0 \in X$. Then there exists a function $\varphi : X \to (0,\infty)$ such that $\tilde{G}(\cdot,y) = \varphi(y)G(\cdot,y)$ for all $y \in X$. Moreover, ${}^*\tilde{G}(\cdot,x_0) = a\,{}^*G(\cdot,x_0)$ for some $a > 0$, that is, $\tilde{G}(x_0,\cdot) = a\,G(x_0,\cdot)$. Therefore $\varphi(y) = a$ for all $y \in X$ and hence $\tilde{G} = a\,G$.

Let
$$ \rho := G^{-1} + {}^*G^{-1}. $$

We assume, in addition, that G and ${}^*G$ are locally comparable and that the triangle property holds locally, that is, X can be covered by open sets V having the following property (see [22, p. 102]). There exists $c > 0$ such that, for all points $x,y,z \in V$,

$$ G(x,y) \le c\,{}^*G(x,y) \quad \text{and} \quad \min(G(x,y), G(y,z)) \le c\,G(x,z). \tag{10.1} $$

or, equivalently, there exists $c > 0$ such that, for all $x,y,z \in V$,

$$ \rho(x,y) \le c\,G(x,y)^{-1} \quad \text{and} \quad \rho(x,y) \le c\,(\rho(x,z) + \rho(z,y)). \tag{10.2} $$

Let L be an arbitrary compact subset of X. By [22, Lemma 2.2], there exists $c > 0$ such that (10.2) holds for all $x,y,z \in L$. By [23, Proposition 14.5], there exists a metric d on L and $\gamma > 0$ such that $\rho \approx d^\gamma$ on $L \times L$, that is, there exists $c > 0$ such that

$$ c^{-1}d(x,y)^\gamma \le \rho(x,y) \le c\,d(x,y)^\gamma \qquad (x,y \in L). $$

Consequently,

$$ G \approx d^{-\gamma} \quad \text{on } L \times L. \tag{10.3} $$

Let us note that the topology induced by such an *intrinsic metric* d is the original topology of L. Indeed, for every $y \in L$, the sets $L \cap \{G(\cdot,y) > a\}$, $a > 0$, form a fundamental system of neighborhoods of y in L and $c^{-1}d^{-\gamma} \le G \le c\,d^{-\gamma}$ on $L \times L$ implies that, for every $r > 0$,

$$ L \cap \{G(\cdot,y) > c\,r^{-\gamma}\} \subset \{d(\cdot,y) < r\} \subset L \cap \{G(\cdot,y) > c^{-1}r^{-\gamma}\}. $$

We recall from [24, Theorem 31.1] that, for all $x, y \in X$ and $A \subset X$,

$$R^A_{G(\cdot,y)}(x) = R^{*A}_{G(x,\cdot)}(y) = \int G(x,z)\, d\varepsilon_y^{*A} = G^{\varepsilon_y^{*A}}(x). \tag{10.4}$$

Moreover, let us note that the fine topologies for $(X, \mathcal{H})$ and $(X, {}^*\mathcal{H})$ coincide (see [22, p. 103]) and hence, by [27, Theorem 2.4], the axiom of domination is satisfied for both $(X, \mathcal{H})$ and $(X, {}^*\mathcal{H})$ (cf. [15, Section 9.2] for the definition). Hence all semipolar sets are polar (see [15, Corollary 9.2.3]). In particular, for every set A in X, the set $A \setminus b(A)$ is polar.

EXAMPLES 10.1. Various classes of linear partial differential operators of second order on open subsets X of $\mathbb{R}^d$ lead to Green functions and Brelot spaces satisfying our assumptions:

1. If
$$\mathcal{L} = \sum_{i,j=1}^d a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^d b_i \frac{\partial}{\partial x_i} + c$$
such that the functions a_{ij}, b_i, c are Hölder continuous and the quadratic forms $\xi \mapsto \sum a_{ij}(x)\xi_i\xi_j$, $x \in X$, are positive definite, then
$$\mathcal{H}(U) := \{u \in \mathcal{C}^2(U) : \mathcal{L}u = 0\}$$
yields a Brelot space $(X, \mathcal{H})$ ([24, 7]). See [28] for the case where the coefficients are only assumed to be continuous.

2. If
$$\mathcal{L} = \sum_{j=1}^d \frac{\partial}{\partial x_j}\Big(\sum_{i=1}^d a_{ij}\frac{\partial}{\partial x_i} + d_i\Big) + \sum_{i=1}^d b_i \frac{\partial}{\partial x_i} + c$$
such that the functions a_{ij} are measurable, bounded and the matrix $(a_{ij}(x))$ is uniformly elliptic, then (under mild restrictions on the functions b_i, d_i, c, see [25]) we obtain a Brelot space defining a harmonic function u on an open subset U of X to be (a continuous version of) a weak solution of $\mathcal{L}u = 0$, that is, such that $u \in H^1_{\mathrm{loc}}(U)$ and, for all $\varphi \in \mathcal{D}(U)$,
$$\int_U \Big[\sum_j \Big(\sum_i a_{ij}\frac{\partial u}{\partial x_i} + d_j u\Big)\frac{\partial \varphi}{\partial x_j} + \Big(\sum_i b_i \frac{\partial u}{\partial x_i} + cu\Big)\varphi\Big]\, d\lambda = 0.$$

3. If
$$\mathcal{L} = \sum_{j=1}^r X_j^2 + Y$$
with smooth vector fields $X_1, \dots, X_r, Y$ such that Hörmander's condition for hypoellipticity (full rank of the Lie algebra generated by $X_1, \dots, X_r$) is satisfied, then we get a Brelot space (see [9, 8, 26, 6]) defining
$$\mathcal{H}(U) := \{u \in \mathcal{C}^2(U) : \mathcal{L}u = 0\}.$$

In these examples, we have Green functions G which are (at least locally) equivalent to the classical Green function (cases (1) and (2)) or rather different, but still equivalent to some negative power of a metric (case (3)). In particular, in all these examples, G and its adjoint ${}^*G$ are locally comparable and satisfy locally the triangle property (see (10.1)). Details may be found in [21, 22].

11 Scaling invariant Harnack's inequalities

In this section, we shall see that, even in our general setting of Brelot spaces, Harnack's inequalities hold which locally are scaling invariant with respect to an intrinsic metric and which will help us to construct suitable $(1+\delta)$-Harnack families for any given $\delta \in (0,1)$.

Let V be a relatively compact open subset of X and let d be a metric on $\overline{V}$ and $\gamma > 0$ such that $G \approx d^{-\gamma}$ on $\overline{V} \times \overline{V}$. For all $x \in \overline{V}$ and $r \geq 0$, let $U(x,r)$, $B(x,r)$ denote the set of all $y \in \overline{V}$ with $d(x,y) < r$, $d(x,y) \leq r$, respectively.

PROPOSITION 11.1. *There exist $\beta \in (0,1/3)$ and $c > 0$ such that, for all points $y_1, y_2 \in U(x,\beta r)$ and $z \in U(x,3\beta r) \setminus U(x,2\beta r)$,*

$$G_{U(x,r)}(z,y_1) \leq c\, G_{U(x,r)}(z,y_2), \tag{11.1}$$

whenever $x \in V$ and $r > 0$ with $B(x,r) \subset V$.

Proof. Let $c_1 \geq 1$ such that $c_1^{-1} d^{-\gamma} \leq G \leq c_1 d^{-\gamma}$ on $V \times V$ and let h be a harmonic function on V which is bounded and bounded away from 0 (for example, $h = 1$, if constants are harmonic, or $h := G(\cdot, x_1)|_V$, $x_1 \in X \setminus \overline{V}$). Let $c_2 \geq 1$ such that $c_2^{-1} \leq h \leq c_2$. We define

$$\beta := \frac{1}{6(c_1c_2)^{2/\gamma}} \quad \text{and} \quad c := \frac{c_1^2}{4^{-\gamma} - 5^{-\gamma}}.$$

For later use, let us note that $5(c_1c_2)^{2/\gamma}\beta = 1 - (c_1c_2)^{2/\gamma}\beta \leq 1 - \beta$ and hence

$$(1-\beta)^{-\gamma} \leq (c_1c_2)^{-2}(5\beta)^{-\gamma}. \tag{11.2}$$

Let us fix $x \in V$ and $r > 0$ such that $B(x,r) \subset V$. Obviously,

$$H_{U(x,r)}1 \leq c_2 H_{U(x,r)}h = c_2 h \leq c_2^2. \tag{11.3}$$

If $0 < t < s \leq 1$, $y \in U(x,tr)$, and $z \in \partial U(x,sr)$, then $(s-t)r \leq d(y,z) \leq (s+t)r$ and therefore

$$c_1^{-1}(s+t)^{-\gamma} r^{-\gamma} \leq G(z,y) \leq c_1 (s-t)^{-\gamma} r^{-\gamma}. \tag{11.4}$$

Let us now fix $y \in U(x,\beta r)$. By (11.4), $G(\cdot,y) \leq c_1(1-\beta)^{-\gamma} r^{-\gamma}$ on $\partial U(x,r)$ and hence, by (11.3) and (11.2),

$$H_{U(x,r)} G(\cdot,y) \leq c_2^2 c_1 (1-\beta)^{-\gamma} r^{-\gamma} \leq c_1^{-1}(5\beta r)^{-\gamma} \quad \text{on } U(x,r).$$

Finally, let $z \in U(x, 3\beta r) \setminus U(x, 2\beta r)$. Applying (11.4) with $t = \beta$ and $2\beta \le s < 3\beta$, we obtain that

$$c_1^{-1}(4\beta r)^{-\gamma} \le G(z, y) \le c_1(\beta r)^{-\gamma}.$$

Since $G_{U(x,r)}(z, y) = G(z, y) - \big(H_{U(x,r)}G(\cdot, y)\big)(z)$, we see that

$$(4^{-\gamma} - 5^{-\gamma})c_1^{-1}(\beta r)^{-\gamma} \le G_{U(x,r)}(z, y) \le c_1(\beta r)^{-\gamma}.$$

Thus (11.1) follows by our definition of c. □

PROPOSITION 11.2. *There exist $\beta \in (0, 1/3)$ and $c > 0$ such that, for all $x \in V$ and $r > 0$ with $B(x, r) \subset V$ and all harmonic functions $h \ge 0$ on $U(x, r)$,*

$$h(y_1) \le c\, h(y_2) \qquad \text{for all } y_1, y_2 \in U(x, \beta r).$$

Proof. Applying Proposition 11.1 to ${}^*G$, we obtain $\beta \in (0, 1/3)$ and $c > 0$ such that, for all $y_1, y_2 \in U(x, \beta r)$ and $z \in U(x, 3\beta r) \setminus U(x, 2\beta r)$,

$$G_{U(x,r)}(y_1, z) \le c\, G_{U(x,r)}(y_2, z), \tag{11.5}$$

whenever $x \in V$ and $r > 0$ with $B(x, r) \subset V$.

Let us fix $x \in V$ and $r > 0$ such that $B(x, r) \subset V$ and let $h \ge 0$ be a harmonic function on $U(x, r)$. We may choose a continuous function $0 \le \varphi \le 1$ on $U(x, r)$ such that $\varphi = 1$ on $U(x, 2\beta r)$ and the support L of φ is contained in $U(x, 3\beta r)$. Let p denote the smallest superharmonic function on $U(x, r)$ majorizing φh. Then p is a continuous potential on $U(x, r)$ and $p = h$ on $U(x, 2\beta r)$. Moreover, p is harmonic on $U(x, r) \setminus L$. So there exists a measure μ on $L \setminus U(x, 2\beta r)$ such that

$$p := \int G_{U(x,r)}(\cdot, z)\, d\mu(z).$$

By integration with respect to μ, we see by (11.5) that, for all $y_1, y_2 \in U(x, \beta r)$,

$$h(y_1) = p(y_1) \le c\, p(y_2) = c\, h(y_2).$$

□

By Moser's trick, leading from scaling invariant Harnack's inequalities to Hölder continuity, Proposition 11.2 can be improved considerably.

PROPOSITION 11.3. *For every $\delta > 0$, there exists $\gamma \in (0, 1)$ such that, for all $x \in V$ and $r > 0$ with $B(x, r) \subset V$ and all harmonic functions $h \ge 0$ on $U(x, r)$,*

$$h(y_1) \le (1 + \delta)h(y_2) \qquad \text{for all } y_1, y_2 \in U(x, \gamma r). \tag{11.6}$$

Proof. We choose $\beta \in (0, 1/3)$ and $c > 0$ according to Proposition 11.2, and fix $\delta \in (0, 1)$.

1. Let us first suppose that $1 \in \mathcal{H}(V)$. We define $C := \frac{c^2}{c-1}$, $\eta := \ln \frac{c}{c-1} / \ln \frac{1}{\beta}$ and choose $\gamma > 0$ such that $C\gamma^\eta \le \delta/3$.

Let $x \in V$, $r > 0$ with $B(x,r) \subset V$, and let $h \geq 0$ be a harmonic function on $U(x,r)$. Then, by [22, Proposition 7.1],

$$(11.7) \qquad |h(y) - h(x)| \leq C \left(\frac{d(x,y)}{r} \right)^{\eta} h(x) \qquad \text{for every } y \in U(x, \beta r).$$

In particular,

$$(1 - \tfrac{\delta}{3}) h(x) \leq h(y) \leq (1 + \tfrac{\delta}{3}) h(x) \qquad \text{for every } y \in U(x, \gamma r)$$

and hence

$$h(y_1) \leq (1+\delta) h(y_2) \qquad \text{for all } y_1, y_2 \in U(x, \gamma r),$$

since $(1+\frac{\delta}{3})/(1-\frac{\delta}{3}) \leq 1+\delta$.

2. Let us now consider the general case. Let h_0 be a strictly positive harmonic function on a neighborhood of $\overline{V}$ and let $r_0 > 0$ such that $d(x,y) \leq r_0$ for all $x, y \in \overline{V}$. There exists $\varepsilon > 0$ such that

$$(11.8) \quad h_0(y_1) \leq (1 + \tfrac{\delta}{3}) h_0(y_2), \qquad \text{whenever } y_1, y_2 \in \overline{V} \text{ such that } d(y_1, y_2) < \varepsilon r_0.$$

By Proposition 11.2, we know that

$$\frac{h(x)}{h_0(x)} \leq c^2 \frac{h(y)}{h_0(y)} \qquad \text{for all } x, y \in U(x, \beta r).$$

Using [22, Proposition 7.1] and proceeding similarly as in part one, we may now choose $\gamma \in (0, \varepsilon/2)$ such that, for all $y_1, y_2 \in B(x, \gamma r)$,

$$\frac{h(y_1)}{h_0(y_1)} \leq (1 + \frac{\delta}{3}) \frac{h(y_2)}{h_0(y_2)},$$

where $h_0(y_1) \leq (1+\frac{\delta}{3}) h_0(y_2)$ by (11.8), and hence

$$h(y_1) \leq (1 + \tfrac{\delta}{3})^2 h(y_2) \leq (1+\delta) h(y_2).$$

□

COROLLARY 11.4. *For every $\delta > 0$, there exists $a \in (0,1)$ such that the following holds. If $x_1, \dots, x_m \in V$ and $r_1, \dots, r_m \in (0, \infty)$ such that $B(x_1, r_1), \dots, B(x_m, r_m)$ are pairwise disjoint subsets of V, then $(B(x_i, a r_i))_{1 \leq i \leq m}$ is a $(1+\delta)$-Harnack family in V.*

Proof. Proposition 11.3 and Lemma 3.2. □

12 General convexity properties of reduced measures

In addition to the hypotheses made at the beginning of Section 10, let us assume that the following doubling property related to the Green function G holds:

(DG) For every compact set K in X, there exist $m_0 \in \mathbb{N}$ and $a_0 > 0$ such that, for all $a \ge a_0$ and $x \in K$, the set $\{G(\cdot, x) > a\}$ contains at most m_0 pairwise disjoint sets of the form $\{G(\cdot, y) > 2a\}$, $y \in K$.

Let K be a compact set in X and V a relatively compact open neighborhood of K. Let d be a metric on $\overline{V}$ and $c, \gamma \in (0, \infty)$ such that $c^{-1}d^{-\gamma} \le G \le cd^{-\gamma}$ on $\overline{V} \times \overline{V}$. It is easily verified that (DG) is equivalent to the following property:

(DB) There exist $m_0 \in \mathbb{N}$ and $r_0 > 0$ such that, for every $0 < r \le r_0$ and for every $x \in K$, the set $B(x, 2r)$ contains at most m_0 pairwise disjoint sets of the form $B(y, r)$, $y \in K$.

Clearly, (DB) holds if there exist $\tilde{c} > 0$ and a finite measure μ on V such that K is contained in the support of μ and $\mu(B(x, 2r)) \le \tilde{c}\mu(B(x, r))$ for all $x \in K$ and $0 < r \le r_0$. Indeed, then, for every $y \in K$ such that $B(y, r) \subset B(x, 2r)$, we have $B(x, 2r) \subset B(y, 4r)$, hence $\mu(B(x, 2r)) \le \tilde{c}^2\mu(B(y, r))$, and therefore (DB) holds with some $m_0 \le \tilde{c}^2$.

In particular, (DB) and (DG) are satisfied in the examples considered at the end of Section 10.

For simplicity, let us assume in the following that the constant 1 is both superharmonic and $*$-superharmonic. Let cap denote the capacity associated with G, that is, for every $A \in \mathcal{B}(X)$,

$$\operatorname{cap}(A) = \sup\{\mu(A)\colon \mu \in \mathcal{M}(X),\ G^\mu \le 1\}.$$

Of course, we expect the following estimates.

LEMMA 12.1. *There exist $c_1 > 0$ and $r_0 > 0$ such that, for all $x \in K$ and $0 < r \le r_0$,*

$$c_1^{-1} r^\gamma \le \operatorname{cap}(B(x, r)) \le c_1 r^\gamma. \tag{12.1}$$

Proof. Let us fix functions $g \ge 1$, $h \ge 1$ which are harmonic, $*$-harmonic, respectively, on a neighborhood of $\overline{V}$. Let $b \in \mathbb{R}$ be an upper bound for g and h on $\overline{V}$, and let $c_1 := b^2 c$. Finally, we choose $r_0 > 0$ such that, for every $y \in K$, both $G(\cdot, y)$ and $G(y, \cdot)$ are strictly smaller than $(bc)^{-1} r_0^{-\gamma}$ on ∂V and hence on V^c by the minimum principle.

Now let us fix $y \in K$, $0 < r \le r_0$. To obtain the first inequality in (12.1), we consider $U := \{G(\cdot, y) > cr^{-\gamma} g\}$. Then $\overline{U} \subset B(x, r)$, the set U is regular, and $R_{G(\cdot,y)}^{U^c} = G^{\varepsilon_y^{*U^c}}$ by (10.4). Of course, $R_{G(\cdot,y)}^{U^c} \le cr^{-\gamma} g \le bcr^{-\gamma}$ on V, and hence on X by the minimum principle. So the measure $\nu := (bc)^{-1} r^\gamma \varepsilon_y^{*U^c}$ satisfies $G^\nu \le 1$. Moreover, ν is supported by $\partial U \subset B(x, r)$ and

$$\nu(B(x, r)) \ge (bc)^{-1} r^\gamma \varepsilon_y^{*U^c}(b^{-1} h) = c_1^{-1} r^\gamma h(y) \ge c_1^{-1} r^\gamma.$$

Therefore $\operatorname{cap}(B(x, r)) \ge c_1^{-1} r^\gamma$.

Next let $\tilde{U} := \{G(y,\cdot) > (bc)^{-1} r^{-\gamma} h\}$. Then $B(x,r) \subset \overline{\tilde{U}}$, $\tilde{U}$ is $*$-regular, and $R^{*\tilde{U}^c}_{G(y,\cdot)} = {}^*G^{\varepsilon_y^{\tilde{U}^c}}$. Let $\sigma := bcr^\gamma \varepsilon_y^{\tilde{U}^c}$. Then ${}^*G^\sigma \geq h \geq 1$ on $\overline{\tilde{U}}$ and $\sigma(X) \leq \sigma(g) = bcr^\gamma g(y) \leq c_1 r^\gamma$. If μ is any measure on $B(x,r)$ such that $G^\mu \leq 1$, then

$$\mu(B(x,r)) \leq \int {}^*G^\sigma \, d\mu = \int G^\mu \, d\sigma \leq \sigma(X) \leq c_1 r^\gamma.$$

Thus $\operatorname{cap}(B(x,r)) \leq c_1 r^\gamma$ finishing the proof.

□

PROPOSITION 12.2. *Let $a, \varepsilon \in (0,1)$, $M > 1$, and let q be a continuous potential on X which is harmonic outside K. Then there exists $\rho_0 > 0$ such that, for every $0 < \rho \leq \rho_0$, the following holds:*

If $A := B(x_1, a\rho) \cup \cdots \cup B(x_N, a\rho)$, where $x_1, \dots, x_N \in K$ such that the d-balls $B(x_i, \rho)$ are pairwise disjoint and the d-balls $B(x_i, 3\rho)$ cover K, then

$$R_q^A \geq (1-\varepsilon)\, q.$$

Proof. It suffices to note that, by Lemma 12.1,

$$\operatorname{cap}(A \cap B(x_i, 9\rho)) \geq \operatorname{cap}(B(x_i, a\rho)) \geq c_1^{-1}(a\rho)^{-\gamma} \geq c_1^{-2}(3a)^{-\gamma} \operatorname{cap}(K \cap B(x_i, 3\rho))$$

and to proceed as in the proof of Proposition 8.1 (with $M = 3$ and 3ρ in place of ρ).

□

PROPOSITION 12.3. *Let U_1, U_2 be open sets in X, $q \in \mathcal{P}(X)$ with $\nu(q) \leq 1$, and $\delta > 0$. Then there exist $m_1, m_2 \in \mathbb{N}$ and a $(1+\delta)$-Harnack family $(K_i)_{1 \leq i \leq m_1+m_2}$ in the union $W := U_1 \cup U_2$ such that the compact sets $L_1 := K_1 \cup \cdots \cup K_{m_1}$ and $L_2 := K_{m_1+1} \cup \cdots \cup K_{m_1+m_2}$ satisfy*

$$\nu^{L_j \cup W^c}(q) > \nu^{U_j^c}(q) - \delta, \qquad j \in \{1,2\}.$$

Proof. Each of the sets $U_2 \setminus U_1$ and $U_1 \setminus U_2$ is a countable union of compact sets. So, by (2.2), there exist compact sets $L_1^0 \subset U_2 \setminus U_1$ and $L_2^0 \subset U_1 \setminus U_2$ such that

$$\nu^{L_j^0 \cup W^c}(q) > \nu^{U_j^c}(q) - \frac{\delta}{2}, \qquad j \in \{1,2\}. \tag{12.2}$$

Let V_1 and V_2 be disjoint relatively compact open neighborhoods of L_1^0 and L_2^0 in W, respectively. Applying the previous considerations to $V := V_1 \cup V_2$ and the compact subsets L_1^0, L_2^0, respectively, and arguing as in the proofs of Proposition 9.3 and Corollary 9.4 as well as Corollary 11.4, we obtain $m_1, m_2 \in \mathbb{N}$ and a family $(K_i)_{1 \leq i \leq m_1+m_2}$ having the desired properties. □

Using Proposition 12.3, Theorem 7.5, and proceeding as in the proofs of Proposition 6.1 and Corollary 6.2, we obtain the following result.

THEOREM 12.4. *Let $\nu \in \mathcal{M}(\mathcal{P}(X))$, $A_1, A_2 \in \mathcal{B}(X)$, and $\lambda \in (0,1)$. Moreover, let (V_m) be a sequence of open neighborhoods of $(A_1 \cup A_2) \setminus (A_1 \cap A_2)$. Then there exist compact sets C_m in V_m, $m \in \mathbb{N}$, such that*

$$\lim_{m \to \infty} \nu^{(A_1 \cap A_2) \cup C_m} = \lambda \nu^{A_1} + (1-\lambda)\nu^{A_2}.$$

In particular, Question 1 raised in the Introduction has a positive answer in this general setting as well. Moreover, we obtain analogues of Corollary 1.4 and Corollary 1.6 (see the Section 13 for a discussion of $\mathcal{M}_\nu(\mathcal{P}(X))$ in the context of balayage spaces). Generalizing the definition of the weak capacity doubling property, we finally see the following.

THEOREM 12.5. *Let U, V be open sets in X such that $\partial U \cap V$ and $\partial V \cap U$ have the weak capacity doubling property. Moreover, let us suppose that ν is supported by $U \cap V$, let $W := U \cup V$, and $\lambda \in (0,1)$.*

Then there exist compact sets C_n in $(\partial U \cap V) \cup (\partial V \cap U)$, $n \in \mathbb{N}$, such that

$$\lim_{n\to\infty} \nu^{(W\setminus C_n)^c} \;=\; \lambda \nu^{U^c} + (1-\lambda)\nu^{V^c}.$$

13 Appendix: Representing measures

To cover all situations discussed before (classical case, Riesz potentials, Brelot spaces) let us assume that $(X, \mathcal{W})$ is a balayage space satisfying the axiom of polarity (see [6]). Let W be an open set in X or, more generally, let W be a finely open Borel set in X. Let $S(W)$ denote the set of all continuous functions on X which are $\mathcal{P}(X)$-bounded (that is, bounded in modulus by some $p \in \mathcal{P}(X)$) and finely superharmonic on W. Moreover, let $H(W)$ be the set of all continuous $\mathcal{P}(X)$-bounded functions on X which are finely harmonic on W, that is, $H(W) = S(W) \cap (-S(W))$. If W is open, then $S(W)$, $H(W)$ is simply the set of all continuous $\mathcal{P}(X)$-bounded functions on X which are superharmonic on W, harmonic on W, respectively (see [20, Theorem 9.8]). We fix $\nu \in \mathcal{M}(\mathcal{P}(X))$ such that $\nu(p) < \infty$ for every $p \in \mathcal{P}(X)$, and define

$$\mathcal{M}_\nu(S(W)) := \{\mu \in \mathcal{M}(\mathcal{P}(X))\colon \mu(s) \le \nu(s) \text{ for every } s \in S(W)\}.$$

Given $\sigma, \tau \in \mathcal{M}(X)$, let us write $\sigma \prec \tau$ if $\sigma(p) \le \tau(p)$ for every $p \in \mathcal{P}(X)$.

THEOREM 13.1.

$$\begin{aligned}
\mathcal{M}_\nu(S(W)) \;&=\; \{\mu \in \mathcal{M}_\nu(\mathcal{P}(X))\colon \mu(h) = \nu(h) \textit{ for every } h \in H(W)\}\\
&=\; \{\mu \in \mathcal{M}_\nu(\mathcal{P}(X))\colon \mu^{W^c} = \nu^{W^c}\}\\
&=\; \{\mu \in \mathcal{M}(\mathcal{P}(X))\colon \nu^{W^c} \prec \mu \prec \nu\}.
\end{aligned}$$

Moreover, $\mathcal{M}_\nu(S(W))$ is a closed face of $\mathcal{M}_\nu(\mathcal{P}(X))$ and

$$\big(\mathcal{M}_\nu(S(W))\big)_e = \{\nu^A\colon A \in \mathcal{B}(X),\, W^c \subset A\}.$$

Proof. We replace the measure ε_x in the proof of [6, VII.9.5] by ν. Since semi-polar sets are polar, we know that $\beta(W^c) = b(W^c)$ and $\mu^{\beta(W^c)} = \mu^{b(W^c)} = \mu^{W^c}$ for every $\mu \in \mathcal{M}(\mathcal{P}(X))$ (see [6, VI.6.1, VI.6.6]). Therefore we obtain the first three identities, the fact that $\mathcal{M}_\nu(S(W))$ is a closed face of $\mathcal{M}_\nu(\mathcal{P}(X))$, and that every measure ν^A, where $A \in \mathcal{B}(X)$ and $W^c \subset A$, is contained in $\big(\mathcal{M}_\nu(S(W))\big)_e$.

Conversely, let $\mu \in \big(\mathcal{M}_\nu(S(W))\big)_e$. Of course, $\mu \in \big(\mathcal{M}_\nu(\mathcal{P}(X))\big)_e$, since the set $\mathcal{M}_\nu(S(W))$ is a closed face of $\mathcal{M}_\nu(\mathcal{P}(X))$. So there exists $A \in \mathcal{B}(X)$ such that $\mu = \nu^A$. We intend to show that $\mu = \nu^{A\cup W^c}$. This will finish the proof, since $A \cup W^c \in \mathcal{B}(X)$.

By the characterization of $\mathcal{M}_\nu(S(W))$ given above, $\nu^{W^c} \prec \nu^A$. By Lemma 2.3, this implies that, for every $p \in \mathcal{P}(X)$,

$$\nu^{W^c}(p) = \nu^{W^c}(R_p^{W^c}) = \inf_{U \text{open} \supset W^c} \nu^{W^c}(R_p^U) \le \inf_{U \text{open} \supset W^c} \nu^A(R_p^U) = \nu^A(R_p^{W^c}),$$

that is,

$$\nu^{W^c} \prec \nu^A|_{W^c} + (\nu^A|_W)^{W^c} \tag{13.1}$$

(see the proof of [6, VI.9.9]). In addition,

$$\nu^{A\cup W^c} + \nu^A|_{W^c} + (\nu^A|_W)^{W^c} \prec \nu^A + \nu^{W^c}. \tag{13.2}$$

Indeed, if $\nu(A) = 0$, this follows from [6, VI.9.8]. And if $\nu(A^c) = 0$, then $\nu^{A\cup W^c} = \nu^A = \nu$ and (13.2) reduces to the trivial statement $\nu + \nu^{W^c} \prec \nu + \nu^{W^c}$. The general case follows decomposing ν into $1_{A^c}\nu$ and $1_A\nu$.

Combining (13.1) and (13.2), we see that $\nu^{A\cup W^c} \prec \nu^A$. Since $\nu^A \prec \nu^{A\cup W^c}$ holds trivially, we conclude that $\mu = \nu^A = \nu^{A\cup W^c}$ as claimed above, and the proof is finished. $\square$

14 Appendix: Weak capacity density condition

Let us consider again the classical case or the case of Riesz potentials as described in the Introduction. For every open subset U of $\mathbb{R}^d$, let cap_U denote the Green capacity with respect to U where, as before, we simply write cap instead of cap_X.

For every $A \in \mathcal{B}(X)$, let $C(A)$ denote the set of all points $x \in A \cap b(A)$ for which there exist $\varepsilon > 0$ and $r_0 > 0$ such that $U(x, 2r_0) \subset X$ and

$$\mathrm{cap}_{U(x,2r)}(A \cap B(x,r)) \ge \varepsilon\, \mathrm{cap}_{U(x,2r)}(B(x,r)) \quad \text{for every } 0 < r \le r_0. \tag{14.1}$$

Of course, $C(A)$ contains the interior of A.

Generalizing the usual definition of the capacity density condition, given for the complement or the boundary of an open set (see, for example, [1, 2, 3, 35, 12, 18, 30, 29]), we say that $A \in \mathcal{B}(X)$ satisfies the *capacity density condition*, if there exist $\varepsilon > 0$ and $0 < r_0 \le (1/2)\mathrm{dist}(A, \mathbb{R}^d \setminus X)$ such that, for every $x \in A$, (14.1) holds.

Let us say that A satisfies the *weak capacity density condition*, if the set $C(A)$ is finely dense in $b(A)$. We shall prove that $C(A)$ is a subset of $D(A)$ defined in Section 9 and hence any $A \in \mathcal{B}(X)$ satisfying the weak capacity density condition has the weak capacity doubling property.

To that end let us discuss the relation between $\mathrm{cap}_{U(x,2r)}$ and cap. Trivially, $\mathrm{cap}_{U(x,2r)} \ge \mathrm{cap}$, since, given any two open sets U, V in X (or in $\mathbb{R}^d$, if $d - \alpha > 0$) such that $U \subset V$, then $G_U \le G_V$ on $U \times U$ and hence

$$\mathrm{cap}_U(B) \ge \mathrm{cap}_V(B) \qquad \text{for every Borel set } B \text{ in } U. \tag{14.2}$$

If $d-\alpha>0$, there exists a constant $\kappa>0$ such that, for all $x\in X$ and $r>0$ with $U(x,2r)\subset X$, $G_{U(x,2r)}\ge \kappa G_{\mathbb{R}^d}$ on $B(x,r)\times B(x,r)$ and hence

$$\text{(14.3)} \qquad \operatorname{cap}_{\mathbb{R}^d}(B)\ge \kappa \operatorname{cap}_{U(x,2r)}(B) \qquad \text{for every Borel set } B \text{ in } B(x,r).$$

Let us now consider the case $d=\alpha=2$. Then

$$\operatorname{cap}_{U(x,R)}(B(x,r))=(\ln(R/r))^{-1} \qquad (x\in\mathbb{R}^2, 0<r<R<\infty).$$

In particular,

$$\text{(14.4)} \qquad \operatorname{cap}_{U(x,2r)}(B(x,r))=(\ln 2)^{-1}$$

and, if for example $X=U(0,1)$, there is no constant $c>0$ such that $\operatorname{cap}(B(0,r))\ge c\operatorname{cap}_{U(0,2r)}(B(0,r))$ for every $0<r<1/2$.

Nevertheless the following result holds also in the case $d=\alpha=2$.

LEMMA 14.1. *There exists $\kappa>0$ such that the following holds. If $x\in X$ and $r>0$ such that $U(x,2r)\subset X$, and if $\varepsilon>0$ and $A\in\mathcal{B}(X)$ such that*

$$\text{(14.5)} \qquad \operatorname{cap}_{U(x,2r)}(A\cap B(x,r))\ge \varepsilon \operatorname{cap}_{U(x,2r)}(B(x,r)),$$

then $\operatorname{cap}(A\cap B(x,r))\ge \varepsilon\kappa \operatorname{cap}(B(x,r))$.

Proof. Because of (14.2) and (14.3) it remains to consider the case $d=\alpha=2$. In this case, we define

$$\kappa:=\inf\{G_{U(0,1)}(x,y)\colon x,y\in B(0,1/2)\}.$$

Let $\varepsilon>0$, $x\in X$, and $r>0$ such that $B(x,2r)\subset X$ and (14.5) holds. Let ν denote the equilibrium measure for $A\cap B(x,r)$ with respect to $U(x,2r)$, that is, ${}^{U(x,2r)}\hat R_1^{A\cap B(x,r)}=G^\nu_{U(x,2r)}$. Then, by translation and scaling invariance,

$${}^{U(x,2r)}\hat R_1^{A\cap B(x,r)}\ge \kappa\|\nu\| \quad \text{on } B(x,r),$$

where $\|\nu\|\ge \varepsilon/\ln 2\ge\varepsilon$ by (14.5) and (14.4). So $\hat R_1^{A\cap B(x,r)}\ge\kappa\varepsilon$ on $B(x,r)$ and therefore $\operatorname{cap}(A\cap B(x,r))\ge \varepsilon\kappa\operatorname{cap}(B(x,r))$. $\square$

PROPOSITION 14.2. *For every $A\in\mathcal{B}(X)$, the set $C(A)$ is contained in $D(A)$. In particular, $A\in\mathcal{B}(X)$ has the weak capacity doubling property, if A satisfies the weak capacity density condition.*

Proof. Let $\eta:=\operatorname{cap}_{U(0,8)}(B(0,1))/\operatorname{cap}_{U(0,8)}(B(0,4))$. Let $x\in X$ and $r>0$ such that $U(x,8r)\subset X$. Then $\operatorname{cap}_{U(x,8r)}(B(x,r))=\eta\operatorname{cap}_{U(x,8r)}(B(x,4r))$. Finally let $A\in\mathcal{B}(X)$ and $\varepsilon>0$ such that

$$\operatorname{cap}_{U(x,2r)}(A\cap B(x,r))\ge\varepsilon\operatorname{cap}_{U(x,2r)}(B(x,r)).$$

Then, by Lemma 14.1,

$$\operatorname{cap}(B(x,r))\ge\eta\kappa\operatorname{cap}(B(x,4r)) \quad\text{and}\quad \operatorname{cap}(A\cap B(x,r))\ge\varepsilon\kappa\operatorname{cap}(B(x,r)).$$

Therefore

$$\begin{aligned} & \operatorname{cap}(A \cap U(x, 4r)) \le \operatorname{cap}(B(x, 4r)) \le (\eta\kappa)^{-1} \operatorname{cap}(B(x, r)) \\ \le \quad & (\varepsilon\eta\kappa^2)^{-1} \operatorname{cap}(A \cap B(x, r)) \le (\varepsilon\eta\kappa^2)^{-1} \operatorname{cap}(A \cap U(x, 2r)). \end{aligned}$$

□

The following proposition shows that the weak capacity doubling is much weaker than the capacity density condition.

PROPOSITION 14.3. *Let $X = \mathbb{R}^d$ and $d - \alpha > 0$. Then there exists a Cantor set K and $c > 0$ such that, for all $x \in K$ and $0 < r < 1/e$,*

$$c^{-1} r^{d-\alpha}/|\ln r| \le \operatorname{cap}(K \cap B(x, r)) \le c r^{d-\alpha}/|\ln r|. \tag{14.6}$$

In particular, $b(K) = K$, $C(K) = \emptyset$, and $D(K) = K$.

Proof. The first part is a special case of [30, Corollary 1] taking $N = d$, $s = d - \alpha$, and $q = 1$.

Let us fix $x \in K$. Then, in particular,

$$\int_0^{1/e} \frac{\operatorname{cap}(K \cap B(x, r))}{r^{d-\alpha}} \frac{dr}{r} \ge c^{-1} \int_0^{1/e} \frac{1}{\ln(1/r)} \frac{dr}{r} = \infty$$

and therefore $x \in b(K)$ by Wiener's criterion. Moreover, for every $0 < r < 1/e$,

$$\frac{\operatorname{cap}(K \cap B(x, r))}{\operatorname{cap}(B(x, r))} \le \frac{c}{\ln(1/r)}$$

and hence $x \notin C(K)$, since $\lim_{r \to 0} (\ln(1/r))^{-1} = 0$. Finally, for every $0 < r < 1/e$,

$$\frac{\operatorname{cap}(K \cap B(x, r))}{\operatorname{cap}(K \cap B(x, r/2))} \le c^2 2^{d-\alpha} \frac{\ln(2/r)}{\ln(1/r)} \le c^2 2^{d+1-\alpha},$$

since $\ln 2 \le 1 \le \ln(1/r)$. Thus $x \in D(K)$. In fact, this shows that even $K = D_n(K)$, if $n \in \mathbb{N}$ is large enough. □

Let us observe that, for $d - \alpha > 0$ and relatively compact open sets V in X, the capacity density property for V^c is equivalent to a uniform regularity of V, which for the Laplacian has been studied in [3]. As shown there, it is closely related to the existence of strong barriers. The proof for the equivalence of the capacity density condition and the uniform regularity given in [3] can be adapted to our more general situation including Riesz potentials.

PROPOSITION 14.4. *Let $X = \mathbb{R}^d$, $d - \alpha > 0$, and let V be a non-empty relatively compact open set in $\mathbb{R}^d$. Then the following properties are equivalent:*

1. *V^c satisfies the capacity density condition.*

2. *There exists* $\gamma > 0$ *such that, for all* $z \in \partial V$ *and* $r > 0$,

$$\operatorname{cap}(V^c \cap B(z,r)) \ge \gamma\, r^{d-\alpha}. \tag{14.7}$$

3. *There exists* $\gamma > 0$ *such that* (14.7) *holds for all* $z \in V^c$ *and* $r > 0$.

4. *The set* V *is* uniformly regular (*with respect to* $-(-\Delta)^{\alpha/2}$), *that is, there exists* $\delta > 0$ *such that, for all* $z \in \partial V$ *and* $r > 0$,

$$\varepsilon_x^{(V\cap U(z,r))^c}\big(V^c \cap U(z,r)\big) \ge \delta \qquad \textit{for every } x \in V \cap B(z,r/2). \tag{14.8}$$

If $\alpha = 2$, the minimum principle shows that (14.8) holds if and only if

$$\varepsilon_x^{(V\cap U(z,r))^c}\big(V^c \cap U(z,r)\big) \ge \delta \qquad \text{for every } x \in V \cap \partial B(z,r/2).$$

Proof of Proposition 14.4. (1) $\Leftrightarrow$ (2): By (14.3), (2) implies (1) and (1) implies that (14.7) holds for all $z \in \partial V$ and $0 < r \le r_0$. Let $z \in \partial V$ and let R be the diameter of V. Assuming that $\operatorname{cap}(V^c \cap B(z,r_0)) \ge \gamma r_0^{d-\alpha}$ we obtain that, for every $r_0 \le r \le R$,

$$\operatorname{cap}(V^c \cap B(z,r)) \ge \operatorname{cap}(V^c \cap B(z,r_0)) \ge \gamma r_0^{d-\alpha} \ge \gamma \Big(\frac{r_0}{R}\Big)^{d-\alpha} r^{d-\alpha}.$$

Finally, for every $r > R$, $\operatorname{cap}(V^c \cap B(z,r)) \ge \operatorname{cap}(\partial B(z,r)) = r^{d-\alpha}$.

(2) $\Leftrightarrow$ (3): Trivially, (3) implies (2). So let us assume that (2) holds and let us fix $z \in U^c$. If $B(z,r/2) \subset U^c$, then obviously

$$\operatorname{cap}(U^c \cap B(z,r)) \ge \operatorname{cap}(B(z,r/2)) = 2^{\alpha-d} r^{d-\alpha}.$$

So let us assume that $B(z,r/2) \cap U \ne \emptyset$. Then there exists a point $z' \in \partial U \cap B(z,r/2)$ and $B(z',r/2)$ is contained in $B(z,r)$. So, by (14.7),

$$\operatorname{cap}(U^c \cap B(z,r)) \ge \operatorname{cap}(U^c \cap B(z',r/2)) \ge \gamma 2^{\alpha-d} r^{d-\alpha}.$$

(4) $\Rightarrow$ (2): Let V be uniformly regular, $z \in \partial V$, $r > 0$, and $E := V^c \cap B(z,r)$. Then, for every $x \in V \cap B(z,r/2)$,

$$R_1^E(x) \ge \varepsilon_x^{(V\cap U(z,r))^c}(V^c \cap U(z,r)) \ge \delta$$

Of course, $R_1^E = 1 \ge \delta$ on the subset $B(z,r/2) \setminus V$ of E. So we see that $R_1^E \ge \delta R_1^{B(z,r/2)}$, hence $\hat R_1^E \ge \delta \hat R_1^{B(z,r/2)}$ and

$$\operatorname{cap}(E) \ge \delta \operatorname{cap}(B(z,r/2)) = \delta 2^{\alpha-d} r^{d-\alpha}.$$

(2) $\Rightarrow$ (4): Suppose that (14.7) holds, let $\tilde\gamma := 3^{\alpha-d}\gamma$, and $M \ge 4$ such that $M^{\alpha-d} < \tilde\gamma/2$. By Harnack's inequalities, there exists $c > 0$ such that $h \ge c$ on $B(0,M/2) \setminus U(0,2)$ for every function $h \in \mathcal{H}^+(U(0,M) \setminus B(0,1))$ satisfying $h \ge 1$ on $\partial U(0,2)$.

Let us now fix $z \in \partial V$ and $r > 0$. We define $W := U(z,r)$, $\rho := r/M$, and $F := V^c \cap B(z,\rho)$. Let μ denote the equilibrium measure of F. Then $\|\mu\| \geq \gamma\rho^{d-\alpha}$ and therefore

$$\hat{R}_1^F = G^\mu \geq 3^{\alpha-d}\gamma = \tilde{\gamma} \quad \text{ on } B(z, 2\rho).$$

Since $F \subset B(z,\rho)$, we obviously have

$$(14.9) \qquad R_1^F(y) \leq \frac{\rho^{d-\alpha}}{|y-z|^{d-\alpha}} \leq \frac{1}{M^{d-\alpha}} < \frac{\tilde{\gamma}}{2}, \qquad \text{whenever } |y-z| \geq M\rho = r.$$

Defining $\nu_y := \varepsilon_y^{F\cup W^c}$, $y \in X$, we obtain that, for every $y \in B(z,2\rho)\setminus F$,

$$\tilde{\gamma} \leq R_1^F(y) = \nu_y(R_1^F) \leq \nu_y(F) + \frac{\tilde{\gamma}}{2},$$

since ν_y is supported by $F \cup W^c$, $R_1^F \leq 1$, $\|\nu_y\| \leq 1$, and $R_1^F \leq \tilde{\gamma}/2$ on W^c. Therefore $\nu_y(F) \geq \tilde{\gamma}/2$ for every $y \in B(z,2\rho)\setminus F$.

Since the function $y \mapsto \varepsilon_y^{F\cup W^c}(F)$ is harmonic on $U(z,M\rho)\setminus B(z,\rho)$, we conclude by scaling invariance and Harnack's inequalities that

$$\varepsilon_y^{F\cup W^c}(F) \geq c\,\frac{\tilde{\gamma}}{2} =: \delta \qquad \text{for all } y \in B(z,r/2)\setminus F.$$

Fixing $x \in V \cap B(z,r/2)$ and defining $\tilde{\nu}_x := \varepsilon_x^{(V\cap W)^c} = \varepsilon_x^{\tilde{F}\cup W^c}$, where $\tilde{F} := W \setminus V$, we know by (2.7) that $\nu_x(W^c) \geq \tilde{\nu}_x(W^c)$ and thus finally

$$\tilde{\nu}_x(\tilde{F}) = 1 - \tilde{\nu}_x(W^c) \geq 1 - \nu_x(W^c) = \nu_x(F) \geq \delta.$$

□

Finally, let us restrict our attention to classical case $\alpha = 2$ and let V be a bounded domain in $\mathbb{R}^d$, $d \geq 2$. If V is regular, then the boundary ∂V satisfies the capacity density condition, if and only if, for some $\beta > 0$, the Dirichlet solution to any β-Hölder continuous boundary function is β-Hölder continuous on V and the corresponding operator is bounded. In the plane case, ∂V satisfies the capacity density condition, if and only if V is *uniformly perfect*, that is, if there exist $r_0 > 0$ and $c \in (0,1)$ such that, for all $z \in \partial V$ and $0 < r \leq r_0$,

$$\partial V \cap \big(B(z,r)\setminus U(z,cr)\big) \neq \emptyset$$

(see [2] for details).

References

[1] D.R. Adams and L.I. Hedberg. *Function spaces and potential theory*, volume 314 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1996.

[2] H. Aikawa. Hölder continuity of the Dirichlet solution for a general domain. *Bull. London Math. Soc.*, 34(6):691–702, 2002.

[3] A. Ancona. On strong barriers and an inequality of Hardy for domains in $\mathbf{R}^n$. *J. London Math. Soc. (2)*, 34(2):274–290, 1986.

[4] D.H. Armitage and S.J. Gardiner. *Classical potential theory.* Springer Monographs in Mathematics. Springer-Verlag London Ltd., London, 2001.

[5] J.R. Baxter and R.V. Chacon. Compactness of stopping times. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 40(3):169–181, 1977.

[6] J. Bliedtner and W. Hansen. *Potential Theory – An Analytic and Probabilistic Approach to Balayage.* Universitext. Springer, Berlin-Heidelberg-New York-Tokyo, 1986.

[7] N. Boboc and P. Mustaţă. *Espaces harmoniques associés aux opérateurs différentiels linéaires du second ordre de type elliptique.* Lecture Notes in Mathematics, No. 68. Springer-Verlag, Berlin, 1968.

[8] J.M. Bony. Principe du maximum, inégalité de Harnack et unicité du problème de Cauchy pour les opérateurs elliptiques dégénérés. *Ann. Inst. Fourier*, 19,1:277–304, 1969.

[9] J.M. Bony. Opérateurs elliptiques dégénérés associés aux axiomatiques de la théorie du potentiel. *"Potential Theory" (CIME, 1^0 Ciclo, Stresa 1969)*, pages 69–119, 1970.

[10] A. Boukricha. Das Picard-Prinzip und verwandte Fragen bei Störung von harmonischen Räumen. *Math. Ann.*, 239:247–270, 1979.

[11] A. Boukricha, W. Hansen, and H. Hueber. Continuous solutions of the generalized Schrödinger equation and perturbation of harmonic spaces. *Exposition. Math.*, 5:97–135, 1987.

[12] J. Brossard and R. Carmona. Can one hear the dimension of a fractal? *Comm. Math. Phys.*, 104(1):103–122, 1986.

[13] G. Choquet. Convergence vague et suites de potentiels newtoniens. *Bull. Sci. Math. (2)*, 99(3):157–164, 1975.

[14] B.J. Cole and T.J. Ransford. Jensen measures and harmonic measures. *J. Reine Angew. Math.*, 541:29–53, 2001.

[15] C. Constantinescu and A. Cornea. *Potential Theory on Harmonic Spaces.* Grundlehren d. math. Wiss. Springer, Berlin - Heidelberg - New York, 1972.

[16] N. Falkner. Stopped distributions for Markov processes in duality. *Z. Wahrsch. Verw. Gebiete*, 62(1):43–51, 1983.

[17] N. Falkner and P.J. Fitzsimmons. Stopping distributions for right processes. *Probab. Theory Related Fields*, 89(3):301–318, 1991.

[18] C. Fernström. On the instability of capacity. *Ark. Mat.*, 15(2):241–252, 1977.

[19] P.J. Fitzsimmons. Skorokhod embedding by randomized hitting times. In *Seminar on Stochastic Processes, 1990 (Vancouver, BC, 1990)*, volume 24 of *Progr. Probab.*, pages 183–191. Birkhäuser Boston, Boston, MA, 1991.

[20] B. Fuglede. *Finely harmonic functions*. Springer-Verlag, Berlin, 1972. Lecture Notes in Mathematics, Vol. 289.

[21] W. Hansen. Harnack inequalities for Schrödinger operators. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.*, 28:413–470, 1999.

[22] W. Hansen. Normalized solutions of Schrödinger equations with potentially bounded measures. *Potential Analysis*, 21:99–135, 2004.

[23] J. Heinonen. *Lectures on analysis on metric spaces*. Springer-Verlag, New York, 2001.

[24] R.-M. Hervé. Recherches axiomatiques sur la théorie des fonctions surharmoniques et du potentiel. *Ann. Inst. Fourier*, 12:415–517, 1962.

[25] R.-M. and M. Hervé. Les fonctions surharmoniques associées à un opérateur elliptique du second ordre à coefficients discontinus. *Ann. Inst. Fourier*, 19,1:305–359, 1968.

[26] R.-M. and M. Hervé. Les fonctions surharmoniques dans l'axiomatique de M. Brelot associées à un opérateur elliptique dégénéré. *Ann. Inst. Fourier*, 22,2:131–145, 1972.

[27] K. Janssen. A co-fine domination principle for harmonic spaces. *Math. Z.*, 141:185–191, 1975.

[28] P. Kröger. Harmonic spaces associated with parabolic and elliptic differential operators. *Math. Ann.*, 285:393–403, 1988.

[29] P. Mattila and P.V. Paramonov. On density properties of capacities associated to general kernels. In *Papers on analysis*, volume 83 of *Rep. Univ. Jyväskylä Dep. Math. Stat.*, pages 281–286. Univ. Jyväskylä, Jyväskylä, 2001.

[30] P. Mattila and P.V. Paramonov. On density properties of the Riesz capacities and the analytic capacity γ_+. *Tr. Mat. Inst. Steklova*, 235(Anal. i Geom. Vopr. Kompleks. Analiza):143–156, 2001.

[31] G. Mokobodzki. Éléments extrémaux pour le balayage. In *Séminaire de Théorie du Potentiel, dirigé par M. Brelot, G. Choquet et J. Deny (1969/70), Exp. 5*, page 14. Secrétariat Math., Paris, 1971.

[32] T.J. Ransford. Jensen measures. In *Approximation, complex analysis, and potential theory (Montreal, QC, 2000)*, volume 37 of *NATO Sci. Ser. II Math. Phys. Chem.*, pages 221–237. Kluwer Acad. Publ., Dordrecht, 2001.

[33] H. Rost. The stopping distributions of a Markov process. *Invent. Math.*, 14:1–16, 1971.

[34] W. Rudin. *Real and complex analysis.* McGraw-Hill Book Co., New York, third edition, 1987.

[35] M. van den Berg and E. Bolthausen. Estimates for Dirichlet eigenfunctions. *J. London Math. Soc. (2)*, 59(2):607–619, 1999.

Wolfhard Hansen
Fakultät für Mathematik
Universität Bielefeld
33501 Bielefeld
Germany

hansen@math.uni-bielefeld.de

Ivan Netuka
Charles University
Faculty of Mathematics and Physics
Mathematical Institute
Sokolovská 86
186 75 Praha 8
Czech Republic

netuka@karlin.mff.cuni.cz