SELF-ORGANIZED CRITICALITY VIA STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT. Models of self-organized criticality which can be described as singular diffusions with or without (multiplicative) Wiener forcing term (as e.g. the Bak/Tang/Wiesenfeld- and Zhang-models) are analyzed. Existence and uniqueness of nonnegative strong solutions are proved. Previously numerically predicted transition to the critical state in 1-D is confirmed by a rigorous proof that this indeed happens in finite time with high probability.

1. INTRODUCTION

Within the past twenty years the notion of self-organized criticality (SOC) has become a new paradigm for the explanation of a huge variety of phenomena in nature and social sciences. Its origin lies in the attempt to explain the widespread appearance of power-law-statistics for characteristic events. In this paradigm an external perturbation may induce a chain reaction or avalanche in the system. Furthermore, a stationary state, the SOC-state, is reached where the average incoming flux is balanced by the average outgoing flux. This phenomenon was quite unexpected since attaining the critical state of a thermodynamic system usually requires a fine tuning of some control parameter, which is absent in the definition of the SOC models. Several models have been proposed to mimic this mechanism including the sand pile BTW-model [1, 2] and the Zhang-model [22]. The presence of thresholds in the definition of the dynamics implies that the energy can be accumulated locally, eventually generating a chain reaction which may transport energy on arbitrary large scales.

The literature on SOC is vast. We refer e.g. to [1, 2], [3], [7], [19], [21], [9, 10], [16], [18], [13], [8], [14], [17], [15], [22] for various studies. In [3] it was proposed to describe this phenomenon, e.g. in the case of the avalanche dynamics in the BTW- (see [1, 2]) and Zhang- (see [22]) models, by a singular diffusion. In the absense of noise the density $\varrho(t,\xi)$, $t \geq 0$, $\xi \in \mathbb{R}^d$, of this diffusion is formally described by the evolution equation

$$(1) \qquad \frac{\partial}{\partial t}\varrho(t,\xi) = \Delta\Psi(\varrho(t,\xi)),$$

where $\Psi(\varrho) := f(\varrho)H(\varrho-\varrho_c)$, $\varrho_c \geq 0$, H is the Heaviside function and $f(\varrho) = \text{const.}$ in the BTW-model and $f(\varrho) = \varrho$ in the Zhang-model. Applying the chain rule informally, the equation turns into

$$(2) \qquad \begin{aligned} \frac{\partial}{\partial t}\varrho(t,\xi) &= \nabla \cdot [\Psi'(\varrho(t,\xi))\,\nabla\varrho(t,\xi)] \\ &= \Psi''(\varrho(t,\xi))\,|\nabla\varrho(t,\xi)|^2 + \Psi'(\varrho(t,\xi))\,\Delta\varrho(t,\xi), \end{aligned}$$

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where

$$\Psi'(\varrho) = f'(\varrho)H(\varrho - \varrho_c) + f(\varrho)\delta(\varrho - \varrho_c).$$

To discuss the problem in a heuristic way, let us consider a smooth version of H, for example

$$H_\varepsilon(\varrho) = \frac{1}{2} + \frac{1}{\pi}\arctan(\frac{\varrho}{\varepsilon})$$

(or the mathematical more convenient one in (7) below). Since $H_\varepsilon(\varrho - \varrho_c)$ is convex left of ϱ_c and concave right of ϱ_c, we see that e.g. in the BTW-model $H_\varepsilon''(\varrho - \varrho_c)|\nabla\varrho|^2$ is positive if ϱ is below ϱ_c and negative if ϱ is above ϱ_c, i.e. according to (2), ϱ is "pushed" towards the critical value ϱ_c. This has been predicted numerically in 1-D by Bantay and Janosi in [3].

In [13] (see also [14]) Diaz–Guilera pointed out that it is more realistic to consider (1) perturbed by (an additive) noise to model a random amount of energy put into the system varying all over the underlying domain. So, (1) turns into a stochastic partial differential equation (SPDE). In [11] (see also the references therein) based on numerical tests, Carlson and Swindle observed that in the presence of such a noise the self-organized behaviour does not necesssarily occur, i.e. the system fails to converge to the critical value ϱ_c.

The purpose of this note is to provide rigorous mathematical proofs for the above phenomena. First, we sketch the proofs for existence and uniqueness of solutions to (1) perturbed by noise (more precisely for multiplicative noise, so that positivity of initial data is preserved). Second, we prove that at least in 1-D we have convergence to ϱ_c in finite time in the deterministic case (confirming the numerical results in [3]) and convergence to ϱ_c with high probability in the stochastic case. In regard to [11] one can probably not achieve more, but so far we failed to prove that this probability is really not equal to 1.

Let us introduce our framework, where we switch to common notation in SPDE, i.e. replace $\varrho(t,\xi)$ by $X(t,x)(\xi)$ with x being the density at $t = 0$, $t \geq 0$, $\xi \in \mathcal{O}$, where $\mathcal{O}$ is an open bounded domain in $\mathbb{R}^d$, $d = 1, 2, 3$, with smooth boundary $\partial\mathcal{O}$. The appropriate class of SPDE is then of the form

$$(3) \quad \begin{cases} dX(t) - \Delta\Psi(X(t))dt \ni \sigma(X(t))dW(t), \\ \qquad\qquad\qquad\qquad\qquad \text{in } (0,\infty) \times \mathcal{O}, \\ \Psi(X(t)) \ni 0, \quad \text{on } (0,\infty) \times \partial\mathcal{O}, \\ X(0,x) = x \quad \text{on } \mathcal{O}, \end{cases}$$

where x is an initial datum, $\Psi : \mathbb{R} \to 2^{\mathbb{R}}$ a maximal monotone graph, i.e. Ψ is not strictly contained in another monotone graph, and

$$\sigma(X)dW = \sum_{k=1}^{\infty} \mu_k X d\beta_k \, e_k, \quad t \geq 0,$$

is a random forcing term, where $\{e_k\} \subset L^2(\mathcal{O})$ is the eigenbasis of the Laplacian $-\Delta$ on $\mathcal{O}$ with Dirichlet boundary conditions, μ_k are positive numbers and β_k independent standard Brownian motions on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq0}, \mathbb{P})$.

Throughout this note we make the following assumptions:

Hypothesis 1.   (i) $\Psi(r) = \rho \, \text{sign } r + \widetilde{\Psi}(r)$, *for $r \in \mathbb{R}$, where $0 \in \Psi(0)$, $\rho > 0$, $\widetilde{\Psi} : \mathbb{R} \to \mathbb{R}$ is Lipschitzian, $\widetilde{\Psi} \in C^1(\mathbb{R} \setminus \{0\})$ and for some $\delta > 0$ it satisfies $\widetilde{\Psi}'(r) \geq \delta$ for all $r \in \mathbb{R} \setminus \{0\}$.*

(ii) *The sequence $\{\mu_k\}$ is such that*

$$\sum_{k=1}^{\infty} \mu_k^2 \lambda_k^2 < +\infty,$$

where λ_k are the eigenvalues of $-\Delta$.

A typical example is given by

$$\Psi(r) = \psi_0(r) + c, \ r \in \mathbb{R},$$

where

$$\psi_0(r) := \begin{cases} \alpha_1 r, & r > 0 \\ [-\varrho, \varrho], & r = 0 \\ \alpha_2 r, & r < 0 \end{cases}$$

and $\alpha_1, \alpha_2 > 0$, $\varrho \geq 0$, $c \in [-\varrho, \varrho]$ are constants.

The following notations will be used. $L^p(\mathcal{O})$, $p \geq 1$, is the usual space of p-integrable functions on $\mathcal{O}$ with norm $|\cdot|_p$. The scalar product in $L^2(\mathcal{O})$ and the duality induced by the pivot space $L^2(\mathcal{O})$ will be denoted by $\langle \cdot, \cdot \rangle_2$. $H_0^1(\mathcal{O}) \subset L^2(\mathcal{O})$ is the first order Sobolev space on $\mathcal{O}$ with zero trace on the boundary. For a fixed measure space $(E, \mathcal{E}, m)$, a Banach space B and $p \in [1, \infty]$ we denote the space of all with respect to the measure m p-integrable maps from E to B by $L^p(E; B)$.

In the following by H we shall denote the distribution space

$$H = H^{-1}(\mathcal{O}) = (H_0^1(\mathcal{O}))'$$

endowed with the scalar product and norm defined by

$$\langle u, v \rangle = \int_{\mathcal{O}} (-\Delta)^{-1} u(\xi) v(\xi) d\xi, \quad |u|_{-1} = \langle u, u \rangle^{1/2}.$$

We recall that the operator $x \to -\Delta\Psi(x)$ with the domain

$$\{x \in L^1(\mathcal{O}) \cap H : \exists \eta \in H_0^1(\mathcal{O}) \text{ s.th. } \eta \in \Psi(x) \text{ a.e. on } \mathcal{O}\},$$

is essentially maximal monotone in H (see e.g. [4]) and so the distribution space H is the natural functional setting for equation (3). However, the general existence theory of infinite dimensional stochastic equations in Hilbert space with nonlinear maximal monotone operators (see e.g. [12], [20]) is not directly applicable and so a direct approach must be used.

2. Existence, uniqueness and positivity

Definition 2. *Let $x \in H$. An H-valued continuous $\mathcal{F}_t$-adapted process $X = X(t, x)$ is called a solution to (3) on $[0, T]$ if for some $p \in [1, \infty[$*

$$X \in L^p(\Omega \times (0, T) \times \mathcal{O}) \cap L^2(0, T; L^2(\Omega, H)),$$

and there exists $\eta \in L^p(\Omega \times (0, T) \times \mathcal{O})$ such that $\mathbb{P}$-a.s.

(4)
$$\begin{aligned} &\langle X(t, x), e_j \rangle_2 \\ = \ &\langle x, e_j \rangle_2 + \int_0^t \int_{\mathcal{O}} \eta(s, \xi) \Delta e_j(\xi) d\xi ds \\ &+ \sum_{k=1}^{\infty} \mu_k \int_0^t \langle X(s, x) e_k, e_j \rangle_2 d\beta_k(s), \\ &\qquad \forall \ j \in \mathbb{N}, \ t \in [0, T], \end{aligned}$$

(5)
$$\eta \in \Psi(X) \quad \text{a.e. in } \Omega \times (0, T) \times \mathcal{O}.$$

Below for simplicity we often write $X(t)$ instead of $X(t,x)$.

Theorem 3. *Under Hypothesis 1 for each $x \in L^4(\mathcal{O})$, there is a unique solution X to (3). Moreover, if x is nonnegative a.e. in $\mathcal{O}$ then $\mathbb{P}$-a.s.*

$$X(t,x)(\xi) \geq 0, \quad \text{for a.e. } (t,\xi) \in (0,\infty) \times \mathcal{O}.$$

Sketch of Proof. Consider the approximating equation

(6)
$$\begin{cases} dX_\lambda(t) - \Delta(\Psi_\lambda(X_\lambda(t))dt = \sigma(X_\lambda(t))dW(t), \\ X_\lambda(0,x) = x, \end{cases}$$

where $\lambda > 0$,

(7)
$$\Psi_\lambda(r) := \rho\,(\text{sign})_\lambda(r) + \widetilde{\Psi}(r), \quad r \in \mathbb{R},$$

$$(\text{sign})_\lambda(r) := \begin{cases} 1 & \text{if } r > \lambda \\ \frac{r}{\lambda} & \text{if } r \in [-\lambda, \lambda] \\ -1 & \text{if } r < -\lambda. \end{cases}$$

By [6, Theorem 2.2] (applied with $m = 1$) equation (6) has a unique solution

$$X_\lambda \in L^2(\Omega \times (0,T) \times \mathcal{O}) \cap L^2(\Omega, C([0,T]; H))$$

in the sense of Definition 2 which is nonnegative, if so is x. Here as usual the space of continuous H-valued paths $C([0,T]; H)$ is equipped with the supremum norm.

By Ito's formula for $\alpha > 0$ large enough it follows that for all $\lambda, \mu \in (0,1)$ and $t \in [0,T]$

(8)
$$\begin{aligned} &\tfrac{1}{2}\,|X_\lambda(t) - X_\mu(t))|_{-1}^2 e^{-\alpha t} \\ &\leq C \max\{\lambda, \mu\} \int_0^t \int_{\mathcal{O}} \Big(|\Psi_\lambda(X_\lambda(s))|^2 + |X_\lambda(s)|^2 \\ &\quad + |\Psi_\mu(X_\mu(s))|^2 + |X_\mu(s)|^2\Big)e^{-\alpha s}d\xi\,ds \\ &\quad + \int_0^t e^{-\alpha s}\langle X_\lambda(s) - X_\mu(s), \sigma(X_\lambda(s) - X_\mu(s))dW(s)\rangle, \end{aligned}$$

(where $|\cdot|$ denotes the absolute value in $\mathbb{R}$). Hence by the Burkholder-Davis-Gundy inequality $\{X_\lambda\}$ is a Cauchy net in $L^2(\Omega; C([0,T], H))$ and by a standard technique from stochatic partial differential equations one shows that the limit X is the desired solution to (3) (cf. [5]). $\square$

Remark 4. *One can also show (see [5, Prop. 3.4]) that $X, X_\lambda \in L^2(0,T; L^2(\Omega, H_0^1(\mathcal{O})))$, that*

$$\lim_{\lambda \to 0} \mathbb{E} \int_0^T |X_\lambda - X|_{L^2(\mathcal{O})}^2 dt = 0,$$

where $\mathbb{E}$ denotes expectation with respect to $\mathbb{P}$, and that both X and X_λ have continuous paths in $L^2(\mathcal{O})$. Theorem 3 is true for more general not linear growing Ψ (see [5]).

3. Extinction in finite time and self-organized criticality

In this section we assume $N < \infty$. Let τ be the stopping time

$$\tau = \inf\{t \geq 0 : \ |X(t,x)|_{-1} = 0\},$$

where $X(t,x), t \geq 0$, is the solution from Theorem 3.

Proposition 5. *Assume there exists $N \in \mathbb{N}$ such that $\mu_k = 0 \quad \forall k \geq N + 1$. Then*

$$X(t,x) = 0 \quad \text{for } t \geq \tau, \ \mathbb{P}\text{-a.s.}.$$

Sketch of Proof. For simplicity we consider the case with $\varrho \equiv 1$. Define

$$\mu(t) := -\sum_{k=1}^{N} \mu_k e_k \beta_k(t), \quad t \in [0, T], \quad \tilde{\mu} := \sum_{k=1}^{N} \mu_k^2 e_k^2$$

and

$$Y(t) := e^{\mu(t)} X(t), \quad t \geq 0.$$

Then by Ito's product rule Y satisfies $\mathbb{P}$-a.s. the following ordinary PDE

$$\frac{dY(t)}{dt} = e^{\mu(t)} \Delta \eta(t) - \frac{1}{2} \tilde{\mu} Y(t), \quad t \geq 0,$$

with $\eta(t) \in \Psi(X(t))$. Setting $Y_\lambda := e^\mu X_\lambda$ we consider the approximating equation

$$(9) \qquad \frac{dY_\lambda(t)}{dt} = e^{\mu(t)} \Delta \eta_\lambda(t) - \frac{1}{2} \tilde{\mu} Y_\lambda(t), \quad t \geq 0,$$

where

$$(10) \qquad \eta_\lambda(t) = \Psi_\lambda(X_\lambda(t)) \in H_0^1(\mathcal{O}).$$

Hence

$$(11) \qquad \begin{aligned} &\left\langle \frac{dY_\lambda(t)}{dt}, Y_\lambda(t) \right\rangle_2 \\ &= \left\langle \eta_\lambda(t), \Delta(e^{\mu(t)} Y_\lambda(t)) \right\rangle_2 - \frac{1}{2} \langle \tilde{\mu} Y_\lambda(t), Y_\lambda(t) \rangle_2, \end{aligned}$$

where by (10), (7) and integrating by parts we have

$$\begin{aligned} &\langle \eta_\lambda(t), \Delta(e^{\mu(t)} Y_\lambda(t)) \rangle_2 \\ &= -\frac{1}{\lambda} \int_{\{e^{-\mu(t)}|Y_\lambda(t)|<\lambda\}} (|\nabla Y_\lambda(t)|^2 \\ &\qquad\qquad -|Y_\lambda(t)|^2 \, |\nabla \mu(t)|^2) d\xi \\ &\quad - \int_{\mathcal{O}} \widetilde{\Psi}'(e^{-\mu(t)} Y_\lambda(t)) \\ &\qquad\qquad (|\nabla Y_\lambda(t)|^2 - |Y_\lambda(t)|^2 \, |\nabla \mu(t)|^2) d\xi. \end{aligned}$$

This yields

$$(12) \qquad \langle \eta_\lambda(t), \Delta(e^{\mu(t)} Y_\lambda(t)) \rangle_2 \leq C \left(|Y_\lambda(t)|_2^2 + \lambda \right).$$

Hence (11) and Gronwall's lemma imply

$$|Y_\lambda(t)|_2^2 \leq e^{C(t-s)} \left(|Y_\lambda(s)|_2^2 + C\lambda T \right), \quad t \geq s.$$

Now letting $\lambda \to 0$ we get

$$(13) \qquad |Y(t)|_2^2 \leq e^{C(t-s)} |Y(s)|_2^2, \quad t \geq s.$$

Taking in (13) $s = \tau$ we get $Y(t) = 0$ for all $t \geq \tau$ as claimed. $\qquad\square$

For proving the extinction result we need $\mathcal{O} \subset \mathbb{R}$, i.e. $d = 1$. In this case we can easily prove the assertion of Proposition 5 by a supermartingale argument without having to assume that $\mu_k = 0 \; \forall k \geq N+1$ for some $N \in \mathbb{N}$ (see the proof of Theorem 6 below). To be more specific let $\mathcal{O} = (0, \pi)$. Then $e_k(\xi) = \sqrt{\frac{2}{\pi}} \sin k\xi, \quad \xi \in [0, \pi]$, $\lambda_k = k^2$ and $L^1(0, \pi) \subset H$ continuously, so

$$(14) \qquad \gamma = \inf \left\{ \frac{|x|_{L^1}}{|x|_{-1}} : \; x \in L^1(0, \pi) \right\} > 0.$$

Theorem 6. *Assume* $0 \in \mathrm{int}\Psi(0)$ *and that* $x \in L^4(\mathcal{O})$. *Consider the equation*

(15)
$$
\begin{cases}
dX(t) - \Delta(\rho\,\mathrm{sign}\,(X(t) - x_c) + \widetilde{\Psi}(X(t) - x_c))dt \\
\qquad\qquad \ni \sigma(X(t) - x_c)\sum_{k=1}^{\infty}\mu_k e_k d\beta_k, \quad t \geq 0, \\
\rho\,\mathrm{sign}\,(X(t) - x_c) + \widetilde{\Psi}(X(t) - x_c) \ni 0, \text{ on } \partial(0,\pi), \\
X(0, x) = x.
\end{cases}
$$

where $x_c \in \mathbb{R}$.

Define $\tilde{\rho} := \rho - |\tilde{\Psi}(0)|$ (> 0 *since* $0 \in \mathrm{int}\Psi(0)$) *and assume that*

$$
|x - x_c|_{-1} < \tilde{\rho}\gamma C_\infty^{-1},
$$

where $C_\infty := \frac{\pi}{4}\sum_{k=1}^{\infty}(1 + k)^2\mu_k^2$ *and* γ *is as in* (14). *Then for all* $t > 0$

(16)
$$
\mathbb{P}(\tau_c \leq t) \geq 1 - \frac{|x - x_c|_{-1}}{\tilde{\rho}\gamma\int_0^t e^{-C_\infty s}ds},
$$

where

$$
\tau_c = \inf\{t \geq 0 : \ |X(t) - x_c|_{-1} = 0\}
$$
$$
= \sup\{t \geq 0 : \ |X(t) - x_c|_{-1} > 0\}
$$

and $X - x_c$ *is the solution from Theorem 3.*

Sketch of Proof. For simplicity we assume $x_c = 0$. An application of Ito's formula for $\varphi_\varepsilon(|X|_{-1}^2) = (|X|_{-1}^2 + \varepsilon^2)^{1/2}$ yields

$$
\varphi_\varepsilon(|X(t)|_{-1}^2) + \tilde{\rho}\int_r^t \frac{|X(s)|_{L^1((0,\pi))}}{(|X(s)|_{-1}^2 + \varepsilon^2)^{1/2}}ds
$$
$$
\leq \varphi_\varepsilon(|X(r)|_{-1}^2) + C_\infty\int_r^t \frac{|X(s)|_{-1}^2}{(|X(s)|_{-1}^2 + \varepsilon^2)^{1/2}}ds
$$
$$
+ 2\int_r^t \langle\sigma(X(s))dW(s), \varphi_\varepsilon'(|X(t)|_{-1}^2)X(s)\rangle \quad \mathbb{P}\text{-a.s., } r < t
$$

Hence by (14)

$$
\varphi_\varepsilon(|X(t)|_{-1}^2) + \gamma\tilde{\rho}\int_r^t \frac{|X(s)|_{-1}}{(|X(s)|_{-1}^2 + \varepsilon^2)^{1/2}}ds
$$
$$
\leq \varphi_\varepsilon(|X(r)|_{-1}^2) + C_\infty\int_r^t \frac{|X(s)|_{-1}^2}{(|X(s)|_{-1}^2 + \varepsilon^2)^{1/2}}ds
$$
$$
+ 2\int_r^t \langle\sigma(X(s))dW(s), \varphi_\varepsilon'(|X(s)|_{-1}^2)X(s)\rangle. \quad \mathbb{P}\text{-a.s., } r < t.
$$

Now, letting ε tend to zero we get

$$
|X(t)|_{-1} + \gamma\tilde{\rho}\int_r^t 1_{\{|X(s)|_{-1}>0\}}ds
$$
$$
\leq |X(r)|_{-1} + C_\infty\int_r^t |X(s)|_{-1}ds
$$
$$
+ \int_r^t 1_{\{|X(s)|_{-1}>0\}}\langle\sigma(X(s))dW(s), X(s)|X(s)|_{-1}^{-1}\rangle \quad \mathbb{P}\text{-a.s., } r < t.
$$

Hence by Itô's product rule

$$e^{-C_\infty t}|X(t)|_{-1} + \gamma\tilde\rho \int_r^t e^{-C_\infty s}1_{\{|X(s)|_{-1}>0\}}ds$$

$$(17)\quad \leq e^{-C_\infty r}|X(r)|_{-1}$$

$$+ \int_r^t e^{-C_\infty s}1_{\{|X(s)|_{-1}>0\}}\langle\sigma(X(s))dW(s), X(s)|X(s)|_{-1}^{-1}\rangle \quad \mathbb{P}\text{-a.s.},\ r < t.$$

In particular, $e^{-C_\infty t}|X(t)|_{-1}$, $t \geq 0$, is an $(\mathcal{F}_t)$-supermartingale, hence $|X(t)|_{-1} = 0\ \forall t \geq \tau$. Since $\tilde\rho > 0$, taking expectation in (17) with $r = 0$ yields

$$\int_0^t e^{-C_\infty s}\mathbb{P}(\tau > s)ds \leq \frac{|x|_{-1}}{\gamma\tilde\rho}.$$

and (16) follows.

$\square$

Corollary 7. *If in the situation of the above Theorem 6, the noise is zero, i.e. $C_\infty = 0$, then*

$$\tau_c \leq \frac{|x - x_c|_{-1}}{\tilde\varrho\gamma}.$$

Remark 8. *We note that there is a small error in the corresponding result in [5], namely [5, Theorem 4.2, Corollary 4.3 and Remark 4.4]. There we forgot to replace ρ by $\tilde\rho$ and to assume that $0 \in \mathrm{int}\Psi(0)$, rather than just $0 \in \Psi(0)$.*

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