On two-component contact model in continuum with one independent component*

Denis O. Filonenko

National University "Kyiv-Mohyla Academy"; 2 Skovoroda str., 03070, Kyiv, Ukraine

denfil@ukr.net

Dmitri L. Finkelshtein

Institute of Mathematics, Ukrainian National Academy of Sciences, Kiev, Ukraine

fdl@imath.kiev.ua

Yuri G. Kondratiev

Fakultät für Mathematik, Universität Bielefeld, D 33615 Bielefeld, Germany;

Department of Mathematics, University of Reading, UK

kondrat@mathematik.uni-bielefeld.de

Abstract

Properties of a contact process in continuum for a system of particles of two types, one which is independent of the other are considered. We study dynamics of the first and second order correlation functions, their asymptotics and dependence on parameters of the system.

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1 Preliminaries

The configuration space $\Gamma := \Gamma_{\mathbb{R}^d}$ over $\mathbb{R}^d$, $d \in \mathbb{N}$, is defined as the set of all locally finite subsets of $\mathbb{R}^d$,

$$\Gamma := \left\{ \gamma \subset \mathbb{R}^d : |\gamma_\Lambda| < \infty \text{ for every compact } \Lambda \subset \mathbb{R}^d \right\}, \tag{1.1}$$

where $|\cdot|$ denotes the cardinality of a set and $\gamma_\Lambda := \gamma \cap \Lambda$. As usual we identify each $\gamma \in \Gamma$ with the non-negative Radon measure $\sum_{x\in\gamma} \delta_x \in \mathcal{M}(\mathbb{R}^d)$, where δ_x is the Dirac measure with unit mass at x, $\sum_{x\in\varnothing} \delta_x$ is, by definition, the zero measure, and $\mathcal{M}(\mathbb{R}^d)$ denotes the space of all non-negative Radon measures on the Borel σ-algebra $\mathcal{B}(\mathbb{R}^d)$. This identification allows to endow Γ with the topology induced by the vague topology on $\mathcal{M}(\mathbb{R}^d)$, i.e., the weakest topology on Γ with respect to which all mappings

$$\Gamma \ni \gamma \longmapsto \langle f, \gamma \rangle := \int_{\mathbb{R}^d} f(x) d\gamma(x) = \sum_{x\in\gamma} f(x), \quad f \in C_0(\mathbb{R}^d),$$

are continuous. Here $C_0(\mathbb{R}^d)$ denotes the set of all continuous functions on $\mathbb{R}^d$ with compact support. We denote by $\mathcal{B}(\Gamma)$ the corresponding Borel σ-algebra on Γ.

Let us now consider the space of finite configurations

$$\Gamma_0 := \bigsqcup_{n=0}^{\infty} \Gamma^{(n)},$$

where $\Gamma^{(n)} := \Gamma^{(n)}_{\mathbb{R}^d} := \{\gamma \in \Gamma : |\gamma| = n\}$ for $n \in \mathbb{N}$ and $\Gamma^{(0)} := \{\varnothing\}$. For $n \in \mathbb{N}$, there is a natural bijection between the space $\Gamma^{(n)}$ and the symmetrization $\widetilde{(\mathbb{R}^d)^n}/S_n$ of the set $\widetilde{(\mathbb{R}^d)^n} := \{(x_1, ..., x_n) \in (\mathbb{R}^d)^n : x_i \neq x_j \text{ if } i \neq j\}$ under the permutation group S_n over $\{1, ..., n\}$ acting on $\widetilde{(\mathbb{R}^d)^n}$ by permuting the coordinate indexes. This bijection induces a metrizable topology on $\Gamma^{(n)}$, and we endow Γ_0 with the topology of disjoint union of topological spaces. By $\mathcal{B}(\Gamma^{(n)})$ and $\mathcal{B}(\Gamma_0)$ we denote the corresponding Borel σ-algebras on $\Gamma^{(n)}$ and Γ_0, respectively.

Given a constant $z > 0$, let λ_z be the Lebesgue-Poisson measure

$$\lambda_z := \sum_{n=0}^{\infty} \frac{z^n}{n!} m^{(n)},$$

where each $m^{(n)}$, $n \in \mathbb{N}$, is the image measure on $\Gamma^{(n)}$ of the product measure $dx_1...dx_n$ under the mapping $\widetilde{(\mathbb{R}^d)^n} \ni (x_1, ..., x_n) \mapsto \{x_1, ..., x_n\} \in \Gamma^{(n)}$. For $n = 0$ we set $m^{(0)}(\{\varnothing\}) := 1$.

We proceed to consider the K-transform [8], [9], [10], [4], that is, a mapping which maps functions defined on Γ_0 into functions defined on the space Γ. Let $\mathcal{B}_c(\mathbb{R}^d)$ denote the set of all bounded Borel sets in $\mathbb{R}^d$, and for any $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$

let $\Gamma_\Lambda := \{\eta \in \Gamma : \eta \subset \Lambda\}$. Evidently $\Gamma_\Lambda = \bigsqcup_{n=0}^{\infty} \Gamma_\Lambda^{(n)}$, where $\Gamma_\Lambda^{(n)} := \Gamma_\Lambda \cap \Gamma^{(n)}$ for each $n \in \mathbb{N}_0$, leading to a situation similar to the one for Γ_0, described above. We endow Γ_Λ with the topology of the disjoint union of topological spaces and with the corresponding Borel σ-algebra $\mathcal{B}(\Gamma_\Lambda)$.

Given a $\mathcal{B}(\Gamma_0)$-measurable function G with local support, that is, $G\!\upharpoonright_{\Gamma_0 \setminus \Gamma_\Lambda} \equiv 0$ for some $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$, the K-transform of G is a mapping $KG : \Gamma \to \mathbb{R}$ defined at each $\gamma \in \Gamma$ by

$$(KG)(\gamma) := \sum_{\eta \Subset \gamma} G(\eta), \tag{1.2}$$

where $\eta \Subset \gamma$ means that $\eta \subset \gamma$ and $|\eta| < \infty$. Note that for every such function G the sum in (1.2) has only a finite number of summands different from zero, and thus KG is a well-defined function on Γ. Moreover, if G has support described as before, then the restriction $(KG)\!\upharpoonright_{\Gamma_\Lambda}$ is a $\mathcal{B}(\Gamma_\Lambda)$-measurable function and $(KG)(\gamma) = (KG)\!\upharpoonright_{\Gamma_\Lambda}(\gamma_\Lambda)$ for all $\gamma \in \Gamma$, i.e., KG is a cylinder function.

Let now G be a bounded $\mathcal{B}(\Gamma_0)$-measurable function with bounded support, that is, $G\!\upharpoonright_{\Gamma_0 \setminus \bigsqcup_{n=0}^{N} \Gamma_\Lambda^{(n)}} \equiv 0$ for some $N \in \mathbb{N}_0, \Lambda \in \mathcal{B}_c(\mathbb{R}^d)$. In this situation, for each $C \geq |G|$ one finds $|(KG)(\gamma)| \leq C(1 + |\gamma_\Lambda|)^N$ for all $\gamma \in \Gamma$. As a result, besides the cylindricity property, KG is also polynomially bounded. In the sequel we denote the space of all bounded $\mathcal{B}(\Gamma_0)$-measurable functions with bounded support by $B_{bs}(\Gamma_0)$. It has been shown in [4] that the K-transform is a linear isomorphism whose inverse mapping is defined on cylinder functions by

$$\left(K^{-1}F\right)(\eta) := \sum_{\xi \subset \eta} (-1)^{|\eta \setminus \xi|} F(\xi), \quad \eta \in \Gamma_0. \tag{1.3}$$

Note that as mapping K^{-1} is well-defined on the set of measurable functions.

2 The description of problem and main results

2.1 Basic facts and notations

Two-component contact process in $\mathbb{R}^d$ describes a birth-and-death stochastic dynamics of a infinite system of two type particles. Such system may be interpreted as pair of configurations in $\mathbb{R}^d$ as well as one configuration of marked particles that means that each particle has mark (spin) $+1$ or -1. The first interpretation sometimes is more useful but we should additionally assume that these two configurations don't interact.

Let us give the rigorous definitions. Consider two copies of the space Γ: Γ^+ and Γ^-. Let

$$\Gamma^2 := \Big\{ (\gamma^+, \gamma^-) \in \Gamma^+ \times \Gamma^- : \gamma^+ \cap \gamma^- = \varnothing \Big\}. \tag{2.1}$$

Any configuration $\gamma := (\gamma^+, \gamma^-) \in \Gamma^2$ may be identified with marked configuration

$$\hat{\gamma} = \left\{ (x, \sigma_x) : x \in \gamma^+ \cup \gamma^-, \sigma_x = 1\!\!1_{x \in \gamma^+} - 1\!\!1_{x \in \gamma^-} \right\} \in \hat{\Gamma},$$

since $\gamma^+ \sqcup \gamma^- \in \Gamma$. Here $\hat{\Gamma}$ is the space of all marked configurations in $\mathbb{R}^d$ with marks equal to ± 1. One can induce topology on Γ^2 from the weakest topology on $\hat{\Gamma}$ such that all functions

$$\hat{\Gamma} \ni \hat{\gamma} \longmapsto \sum_{(x,\sigma_x)\in\hat{\gamma}} \hat{f}((x,\sigma_x)) \in \mathbb{R}$$

are continuous for all $\hat{f} \in C_0(\mathbb{R}^d \times \{-1;1\})$. Clearly, in this induced topology on Γ^2 all functions

$$\Gamma^2 \ni \gamma = (\gamma^+,\gamma^-) \longmapsto \sum_{x\in\gamma^+} f(x) + \sum_{y\in\gamma^-} g(y) \in \mathbb{R}$$

will be continuous for any $f, g \in C_0(\mathbb{R}^d)$.

On the other hand this topology may be induced from the topology on product $\Gamma^+ \times \Gamma^-$. Let $\mathcal{B}(\Gamma^2) := \mathcal{B}(\Gamma^+) \times \mathcal{B}(\Gamma^-)$ be the corresponding σ-algebra.

Let us now consider the space of finite configurations. Consider two copies of the space Γ_0: Γ_0^+ and Γ_0^-. Let

$$\Gamma_0^2 := \Big\{(\eta^+,\eta^-) \in \Gamma_0^+ \times \Gamma_0^- : \eta^+ \cap \eta^- = \varnothing\Big\}. \tag{2.2}$$

Again one can consider the topology on Γ_0^2 induced by the product-topology. By $\mathcal{B}(\Gamma_0^2) := \mathcal{B}(\Gamma_0^+) \times \mathcal{B}(\Gamma_0^-)$ we denote the corresponding σ-algebra.

We will say that a function $G : \Gamma_0^2 \to \mathbb{R}$ is a bounded function with bounded support if for any $(\eta^+,\eta^-) \in \Gamma_0^2$

$$G(\cdot,\eta^-) \in B_{bs}(\Gamma_0^+), \quad G(\eta^+,\cdot) \in B_{bs}(\Gamma_0^-).$$

Class of all such functions we denote by $B_{bs}(\Gamma_0^2)$.

For any $G \in B_{bs}(\Gamma_0^2)$ one can define the $\mathcal{K}$-transform of G as mapping $\mathcal{K}G : \Gamma^2 \to \mathbb{R}$ defined at each $\gamma = (\gamma^+,\gamma^-) \in \Gamma^2$ by

$$(\mathcal{K}G)(\gamma) = \sum_{\substack{\eta^+\Subset\gamma^+\\ \eta^-\Subset\gamma^-}} G(\eta^+,\eta^-). \tag{2.3}$$

On the other hand if $1\!\!1^\pm$ are unit operators on functions on $\Gamma_0^\pm$ and $K^+ := K \otimes 1\!\!1^-$, $K^- := 1\!\!1^+ \otimes K$ then

$$\mathcal{K} = K^+K^- = K^-K^+.$$

Hence, $\mathcal{K}G < \infty$ and $\mathcal{K}G$ is cylinder function on both variables.

Moreover, $\mathcal{K}G$ is polynomially bounded: for the proper $C > 0$, $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$, $N \in \mathbb{N}$

$$|(\mathcal{K}G)(\gamma)| \le C(1 + |\gamma_\Lambda^+|)^N (1 + |\gamma_\Lambda^-|)^N.$$

The inverse mapping is given on cylinder (on both variables) functions by

$$(\mathcal{K}^{-1}F)(\eta) := \sum_{\substack{\xi^+\subset\eta^+\\ \xi^-\subset\eta^-}} (-1)^{|\eta^+\setminus\xi^+|+|\eta^-\setminus\xi^-|} F(\xi^+,\xi^-), \quad \eta = (\eta^+,\eta^-) \in \Gamma_0^2 \tag{2.4}$$

and again this formula has sense for any measurable function F.

Let μ be a probability measure on $\left(\Gamma^2, \mathcal{B}(\Gamma^2)\right)$ (we denote class of the all such measures by $\mathcal{M}^1(\Gamma^2)$). The function $k_\mu : \Gamma_0^2 \to \mathbb{R}$ is called a *correlation function* of the measure μ if for any $G \in B_{bs}(\Gamma_0^2)$

$$\int_{\Gamma^2} (\mathcal{K}G)(\gamma)d\mu(\gamma) = \int_{\Gamma_0^2} G(\eta^+, \eta^-)k_\mu(\eta^+, \eta^-)d\lambda_1(\eta^+)d\lambda_1(\eta^-). \tag{2.5}$$

2.2 Description of model

Let us consider the generator L of two-component contact process with one independent component. The detailed explanation and interpretation of such processes can be found in [2]. This generator is well-defined at least on cylindric functions on Γ^2 and has the following form:

$$L = L^+_{\mathtt{CM}} + L^-_{\mathtt{CM}} + L^+_{\mathtt{int}}. \tag{2.6}$$

Here $L^+_{\mathtt{CM}}$ is the generator of the one-component contact model of (+)-system (see [7]), $L^-_{\mathtt{CM}}$ is the analogous generator of (−)-system, $L^+_{\mathtt{int}}$ is interaction term that describes birth of (+)-particles under influence of (−)-particles. Therefore, birth of (+)-particles is by the influence of two types of particles, birth of (−)-particles is by the influence of particles of the same type; the death of all particles is independent. Namely,

$$\begin{aligned}
\left(L^+_{\mathtt{CM}}F\right)(\gamma^+, \gamma^-) &= \sum_{x\in\gamma^+} \left[F\left(\gamma^+ \setminus x, \gamma^-\right) - F(\gamma^+, \gamma^-)\right] \\
&\quad + \lambda^+ \int_{\mathbb{R}^d} \left(\sum_{x'\in\gamma^+} a^+(x - x')\right) \left[F\left(\gamma^+ \cup x, \gamma^-\right) - F(\gamma^+, \gamma^-)\right] dx, \\
\left(L^-_{\mathtt{CM}}F\right)(\gamma^+, \gamma^-) &= \sum_{y\in\gamma^-} \left[F\left(\gamma^+, \gamma^- \setminus y\right) - F(\gamma^+, \gamma^-)\right] \\
&\quad + \lambda^- \int_{\mathbb{R}^d} \left(\sum_{y'\in\gamma^-} a^-(y - y')\right) \left[F\left(\gamma^+, \gamma^- \cup y\right) - F(\gamma^+, \gamma^-)\right] dy, \\
\left(L^+_{\mathtt{int}}F\right)(\gamma^+, \gamma^-) &= \lambda \int_{\mathbb{R}^d} \left(\sum_{y\in\gamma^-} a(x - y)\right) \left[F\left(\gamma^+ \cup x, \gamma^-\right) - F(\gamma^+, \gamma^-)\right] dx.
\end{aligned}$$

Constants $\lambda^+, \lambda^-, \lambda$ are positive, functions a^+, a^-, a are non-negative, even, integrable and normalized:

$$\langle a^+ \rangle = \langle a^- \rangle = \langle a \rangle = 1.$$

Here and in the sequel we use the following notation

$$\langle f \rangle := \int_{\mathbb{R}^d} f(x)dx, \quad f \in L^1(\mathbb{R}^d).$$

We also denote the Fourier transform of such f as $\hat{f}$:

$$\hat{f}(p) = \int_{\mathbb{R}^d} e^{-i(p,x)} f(x)dx,$$

where $(\cdot,\cdot)$ is a scalar product in $\mathbb{R}^d$.

Next theorem is the partial case of the results obtained in [2].

Theorem 2.1. *Let $d \geq 2$ and there exists constants $A > 0, \delta > 2d$ such that*

$$a^+(x) + a^-(x) + a(x) \leq \frac{A}{(1+|x|)^\delta}. \tag{2.7}$$

Then there exists a Markov process X_t on Γ^2 with generator L.

We will always suppose also that

$$\hat{a}, \hat{a}^+, \hat{a}^- \in L^1(\mathbb{R}^d). \tag{2.8}$$

Hence, one has stochastic dynamics of configurations that implies dynamics of measures, namely $\mathcal{M}^1\big(\Gamma^2\big) \ni \mu_0 \mapsto \mu_t \in \mathcal{M}^1\big(\Gamma^2\big)$ such that for any measurable bounded function $F : \Gamma^2 \to \mathbb{R}$

$$\int_{\Gamma^2} F(\gamma)d\mu_t(\gamma) := \mathbb{E}\left[\int_{\Gamma^2} F(X_t^\gamma)d\mu_0(\gamma)\right],$$

where process X_t^γ starts from $\gamma \in \Gamma^2$ (more precisely, γ belongs to proper support set, see [2]).

This dynamics of measures implies dynamics of corresponding correlation functions (if they exist). For obtain explicit differential equations for this dynamics we should calculate so-called descent operator $\hat{L}$ which defined on functions $G \in B_{bs}(\Gamma_0^2)$ by

$$\big(\hat{L}G\big)(\eta) = \big((\mathcal{K}^{-1}L\mathcal{K})G\big)(\eta), \quad \eta \in \Gamma_0^2. \tag{2.9}$$

Next we should obtain the adjoint operator $\hat{L}^*$ (with respect to measure $d\lambda_1 d\lambda_1$):

$$\int_{\Gamma_0^2} \hat{L}G(\eta^+,\eta^-)k(\eta^+,\eta^-)d\lambda_1(\eta^+)d\lambda_1(\eta^-)$$
$$= \int_{\Gamma_0^2} G(\eta^+,\eta^-)\hat{L}^*k(\eta^+,\eta^-)d\lambda_1(\eta^+)d\lambda_1(\eta^-). \tag{2.10}$$

Then equations for time evolution of correlation function will be following:

$$\frac{\partial k_t(\eta^+,\eta^-)}{\partial t} = \big(\hat{L}^*k_t\big)(\eta^+,\eta^-). \tag{2.11}$$

In the present article we concentrate our attention on the correlation functions of the first and second orders:

$$\begin{aligned}
k_t^{+}(x) &:= k_t(\{x\},\varnothing), && x\in\mathbb{R}^d;\\
k_t^{-}(y) &:= k_t(\varnothing,\{y\}), && y\in\mathbb{R}^d;\\
k_t^{++}(x_1,x_2) &:= k_t(\{x_1,x_2\},\varnothing), && x_1,x_2\in\mathbb{R}^d;\\
k_t^{+-}(x,y) &:= k_t(\{x\},\{y\}), && x,y\in\mathbb{R}^d;\\
k_t^{--}(y_1,y_2) &:= k_t(\varnothing,\{y_1,y_2\}), && y_1,y_2\in\mathbb{R}^d.
\end{aligned} \tag{2.12}$$

The main subject for our studying will be explicit expression for correlation functions of the first and second orders and their asymptotics at $t\to\infty$.

2.3 Problems and results

In this subsection we state main problems and formulate results. All proofs are presented in the next section.

First two results give explicit forms of the equation (2.11) for the first and second order correlation functions (2.12).

Proposition 2.1. *For any $x,y\in\mathbb{R}^d$*

$$\begin{aligned}
\frac{\partial k_t^{-}(y)}{\partial t} &= -k_t^{-}(y)+\lambda^{-}\int_{\mathbb{R}^d}a^{-}(y-y')k_t^{-}(y')dy',\\
\frac{\partial k_t^{+}(x)}{\partial t} &= -k_t^{+}(x)+\lambda^{+}\int_{\mathbb{R}^d}a^{+}(x-x')k_t^{+}(x')dx'+\lambda\int_{\mathbb{R}^d}a(x-y)k_t^{-}(y)dy
\end{aligned}$$

Proposition 2.2. *For any $x,y,x_1,x_2,y_1,y_2\in\mathbb{R}^d$*

$$\begin{aligned}
\frac{\partial k_t^{--}(y_1,y_2)}{\partial t} &= \lambda^{-}\int_{\mathbb{R}^d}a^{-}(y_2-y')k_t^{--}(y_1,y')dy'+\lambda^{-}\int_{\mathbb{R}^d}a^{-}(y_1-y')k_t^{--}(y_2,y')dy'\\
&\quad-2k_t^{--}(y_1,y_2)+\lambda^{-}a^{-}(y_1-y_2)[k_t^{-}(y_1)+k_t^{-}(y_2)],\\
\frac{\partial k_t^{+-}(x,y)}{\partial t} &= \lambda^{+}\int_{\mathbb{R}^d}a^{+}(x-x')k_t^{+-}(x',y)dx'+\lambda^{-}\int_{\mathbb{R}^d}a^{-}(y-y')k_t^{+-}(x,y')dy'\\
&\quad-2k_t^{+-}(x,y)+\lambda a(x-y)k_t^{-}(y)+\lambda\int_{\mathbb{R}^d}a(x-y')k_t^{--}(y,y')dy',\\
\frac{\partial k_t^{++}(x_1,x_2)}{\partial t} &= \lambda^{+}\int_{\mathbb{R}^d}a^{+}(x_1-x')k_t^{++}(x_2,x')dx'+\lambda^{+}\int_{\mathbb{R}^d}a^{+}(x_2-x')k_t^{++}(x_1,x')dx'\\
&\quad-2k_t^{++}(x_1,x_2)+\lambda^{+}a^{+}(x_1-x_2)[k_t^{+}(x_1)+k_t^{+}(x_2)]\\
&\quad+\lambda\int_{\mathbb{R}^d}a(x_1-y)k_t^{+-}(x_2,y)dy+\lambda\int_{\mathbb{R}^d}a(x_2-y)k_t^{+-}(x_1,y)dy.
\end{aligned}$$

Obviously, equations for $(-)$-system are independent. Recall that such equations were studied in [5].

Let us formulate the main problem for the first order correlation functions.

Problem 1. We should to study the asymptotic properties of the solutions of equations from Proposition 2.1 under following initial conditions:

$$k_0^+(x) = c^+ + \psi^+(x) \geq 0, \qquad k_0^-(y) = c^- + \psi^-(y) \geq \alpha^- > 0, \tag{2.13}$$

where constants c^+, c^- are positive, functions ψ^+, ψ^- and their Fourier transforms $\hat{\psi}^+, \hat{\psi}^-$ are integrable on $\mathbb{R}^d$.

Explicit expressions for solutions are in the next section. The answer of the Problem 1 may be found in the next theorem.

Theorem 2.2. *Let $d \geq 3$ and (2.7), (2.8) hold. The first correlation functions have the following asymptotics at $t \to \infty$:*

1) *for any $y \in \mathbb{R}^d$*

$$k_t^-(y) \to \begin{cases} 0, & \textit{if } \lambda^- < 1 \\ \infty, & \textit{if } \lambda^- > 1 \end{cases},$$

and in the case $\lambda^- = 1$

$$k_t^-(y) \to c^-;$$

2) *for any $x \in \mathbb{R}^d$*

$$k_t^+(x) \to \begin{cases} 0, & \textit{if } \max\{\lambda^+, \lambda^-\} < 1 \\ \infty, & \textit{if } \min\{\lambda^+, \lambda^-\} \geq 1 \end{cases},$$

next, in the case $1 = \lambda^+ > \lambda^-$

$$k_t^+(x) \to c^+ + \frac{\lambda c^-}{1 - \lambda^-},$$

and in the case $\lambda^+ < \lambda^- = 1$

$$k_t^+(x) \to \frac{\lambda c^-}{1 - \lambda^+}.$$

Let us discuss this result. Of course, first part about the independent $(-)$-system is the same as in [5, 7]. It state that $\lambda^- = 1$ is critical value; below of this value $(-)$-system will degenerate at infinity, above of this value $(-)$-system will grow (exponentially, see next section for details). At this critical value $(-)$-system continues to be stable.

$(+)$-system consists of two parts: independent contact and influence from the side of $(-)$-system. If $\max\{\lambda^+, \lambda^-\} < 1$ it means that independent part of $(+)$-system is sub-critical (and should disappear at infinity) and additionally it has influence of disappearing $(-)$-system; naturally, such $(+)$-system will disappear. If $\min\{\lambda^+, \lambda^-\} \geq 1$ it means that growing or stable independent part of $(+)$-system has influence by stable or growing $(-)$-system, hence, $(+)$-system will grow.

Let us concentrate our attention on two other cases. If $\lambda^+ = 1, \lambda^- < 1$ it means that independent part of $(+)$-system is stable and has influence by degenerating $(-)$-system. As a result, $(+)$-system will keep stability property but the limiting value will have the initial value of $(-)$-system which will disappearing at infinity. Hence, $(+)$-system will have memory about vanished $(-)$-system.

If $\lambda^+ < 1, \lambda^- = 1$ it means that degenerating independent part of $(+)$-system has influence by stable $(-)$-system. In result, $(+)$-system will stop disappearing and become stable. But "fare" for this will be absence of the initial value of $(+)$-system in limit. Therefore, $(+)$-system "will lost memory" about its origin and "remember" only about origin of "donor".

In studying asymptotics of the second correlation functions we concentrate our attention only on this two cases when $(+)$-system will be stable. For simplicity of computations we consider translation invariant case only:

$$\psi^+ = \psi^- \equiv 0. \tag{2.14}$$

Problem 2. We should to study the asymptotic properties of the solutions of equations from Proposition 2.2 under following initial conditions:

$$\begin{aligned} k_0^{++}(x_1, x_2) &= c^{++} + \varphi^{++}(x_1 - x_2) \geq 0, \\ k_0^{+-}(x, y) &= c^{+-} + \varphi^{+-}(x - y) \geq 0, \\ k_0^{--}(y_1, y_2) &= c^{--} + \varphi^{--}(y_1 - y_2) \geq 0, \end{aligned} \tag{2.15}$$

where $c^{--},\ c^{+-},\ c^{++}$ are positive constants and and functions $\varphi^{--},\ \varphi^{+-},\ \varphi^{++}$ are even functions which are integrable on $\mathbb{R}^d$ together with their Fourier transforms $\hat{\varphi}^{--},\ \hat{\varphi}^{+-},\ \hat{\varphi}^{++}$.

Explicit expressions for solutions are also in the next section. The answer of the Problem 2 may be found in the next theorem.

Theorem 2.3. *Let $d \geq 3$ and (2.7), (2.8), (2.14) hold. The second correlation functions have the following asymptotics at $t \to \infty$:*

1) *let $\lambda^+ = 1,\ 0 < \lambda^- < 1$, then for any $x, y, x_1, x_2, y_1, y_2 \in \mathbb{R}^d$*

$$\begin{cases} k_t^{--}(y_1, y_2) \to 0, \\ k_t^{+-}(x, y) \to 0, \\ k_t^{++}(x_1, x_2) \to \left(c^{++} - \dfrac{2\lambda c^{+-}}{\lambda^- - 1} + \dfrac{\lambda^2 c^{--}}{(\lambda^- - 1)^2} \right) + \Omega^{++}(x_1 - x_2) < \infty; \end{cases}$$

2) *let $\lambda^- = 1,\ 0 < \lambda^+ < 1$, then for any $x, y, x_1, x_2, y_1, y_2 \in \mathbb{R}^d$*

$$\begin{cases} k_t^{--}(y_1, y_2) \to c^{--} + \Xi^{--}(y_1 - y_2) < \infty, \\ k_t^{+-}(x, y) \to \dfrac{\lambda c^{--}}{1 - \lambda^+} + \Xi^{+-}(x - y) < \infty, \\ k_t^{++}(x_1, x_2) \to \dfrac{\lambda^2 c^{--}}{(1 - \lambda^+)^2} + \Xi^{++}(x_1 - x_2) < \infty; \end{cases}$$

here functions $\Xi^{--}, \Xi^{+-}, \Xi^{++}$ depend on initial value c^- only and function Ω^{++} depends on initial value c^+ only (of course, they also depend on $\lambda, \lambda^\pm, a, a^\pm$).

The explicit expressions for limits will be presented in the next section.

As we see, the situation with "memory" which we had for the first correlation functions is the same for the second one: in the first case $(+)$-system will obtain additional memory about vanished $(-)$-system; in the second case $(+)$-system will have memory about $(-)$-system only.

Remark 2.1. Note that if $c^{++} = (c^+)^2$, $c^{+-} = c^+ c^-$, $c^{--} = (c^-)^2$ then the previous theorems show, in fact, that there exist finite limits of so-called second order Ursell functions $k_t^{++} - (k_t^+)^2$, $k_t^{+-} - k_t^+ k_t^-$, $k_t^{--} - (k_t^-)^2$.

3 Proofs

In this section we present proofs of all our results.

3.1 Equations for time evolution of the correlation functions

First of all we show how to obtain the equations from the Propositions 2.1 and 2.2. We start from the explicit form of the descent operator $\hat{L}$.

Proposition 3.1. *Let $G \in B_{bs}(\Gamma_0^2)$. Then for any $\eta = (\eta^+, \eta^-) \in \Gamma_0^2$*

$$
\begin{aligned}
\left(\hat{L}G\right)(\eta^+, \eta^-) = &- \left(|\eta^+| + |\eta^-|\right) G(\eta^+, \eta^-) \\
&+ \lambda^+ \int_{\mathbb{R}^d} G\left(\eta^+ \cup x, \eta^-\right) \left(\sum_{x' \in \eta^+} a^+ (x - x') \right) dx \\
&+ \lambda^+ \int_{\mathbb{R}^d} \sum_{x' \in \eta^+} G\left(\eta^+ \setminus x' \cup x, \eta^-\right) a^+ (x - x')\, dx \\
&+ \lambda^- \int_{\mathbb{R}^d} G\left(\eta^+, \eta^- \cup y\right) \left(\sum_{y' \in \eta^-} a^- (y - y') \right) dy \\
&+ \lambda^- \int_{\mathbb{R}^d} \sum_{y' \in \eta^-} G(\eta^+, \eta^- \setminus y' \cup y) a^- (y - y')\, dy \\
&+ \lambda \int_{\mathbb{R}^d} G\left(\eta^+ \cup x, \eta^-\right) \left(\sum_{y' \in \eta^-} a\,(x - y') \right) dx \\
&+ \lambda \int_{\mathbb{R}^d} \sum_{y' \in \eta^-} G\left(\eta^+ \cup x, \eta^- \setminus y'\right) a\,(x - y')\, dx
\end{aligned}
$$

Proof. Let us denote death and birth parts of the operator $L^{+}_{\mathtt{CM}}$ by

$$(L^{+}_{d}F)(\gamma^{+},\gamma^{-}) := \sum_{x\in\gamma^{+}} \left[F\left(\gamma^{+}\setminus x,\gamma^{-}\right) - F(\gamma^{+},\gamma^{-})\right],$$

$$(L^{+}_{b}F)(\gamma^{+},\gamma^{-}) := \lambda^{+}\int_{\mathbb{R}^{d}} \left(\sum_{x'\in\gamma^{+}} a^{+}\left(x-x'\right)\right)\left[F\left(\gamma^{+}\cup x,\gamma^{-}\right) - F(\gamma^{+},\gamma^{-})\right]dx.$$

In the same way we denote death and birth parts of the operator $L^{-}_{\mathtt{CM}}$: $L^{-}_{\mathtt{CM}} = L^{-}_{d} + L^{-}_{b}$. As a result,

$$L = L^{+}_{d} + L^{+}_{b} + L^{-}_{d} + L^{-}_{b} + L^{+}_{\mathtt{int}}.$$

Now we calculate pre-image under $\mathcal{K}$-transform of all this operators. One has for any $\eta = (\eta^{+},\eta^{-}) \in \Gamma^{2}_{0}$

$$\begin{aligned}
\left(\hat{L}^{+}_{b}G\right)(\eta) &= \left(\mathcal{K}^{-1}L^{+}_{b}\mathcal{K}G\right)(\eta)\\
&= \sum_{\xi^{+}\subset\eta^{+}} (-1)^{|\eta^{+}\setminus\xi^{+}|} \sum_{\xi^{-}\subset\eta^{-}} (-1)^{|\eta^{-}\setminus\xi^{-}|}\lambda^{+}\int_{\mathbb{R}^{d}}\sum_{x'\in\xi^{+}} a^{+}(x-x')\\
&\quad\times\left(\sum_{\zeta^{+}\subset\xi^{+}\cup x}\sum_{\zeta^{-}\subset\xi^{-}} G(\zeta^{+},\zeta^{-}) - \sum_{\zeta^{+}\subset\xi^{+}}\sum_{\zeta^{-}\subset\xi^{-}} G(\zeta^{+},\zeta^{-})\right)dx\\
&= \lambda^{+}\int_{\mathbb{R}^{d}}\sum_{x'\in\eta^{+}} G\left(\eta^{+}\cup x,\eta^{-}\right)a^{+}\left(x-x'\right)dx\\
&\quad + \lambda^{+}\int_{\mathbb{R}^{d}}\sum_{x'\in\eta^{+}} G\left(\eta^{+}\setminus x'\cup x,\eta^{-}\right)a^{+}\left(x-x'\right)dx,
\end{aligned}$$

analogously, we have that

$$\begin{aligned}
\left(\hat{L}^{-}_{b}G\right)(\eta^{+},\eta^{-}) &= \lambda^{-}\int_{\mathbb{R}^{d}}\sum_{y'\in\eta^{-}} G\left(\eta^{+},\eta^{-}\cup y\right)a^{-}\left(y-y'\right)dy\\
&\quad + \lambda^{-}\int_{\mathbb{R}^{d}}\sum_{y'\in\eta^{-}} G\left(\eta^{+},\eta^{-}\setminus y'\cup y\right)a^{-}\left(y-y'\right)dy.
\end{aligned}$$

Next,

$$\begin{aligned}
\left(\hat{L}^{+}_{\mathtt{int}}G\right)(\eta) &= \left(\mathcal{K}^{-1}L^{+}_{\mathtt{int}}\mathcal{K}G\right)(\eta)\\
&= \sum_{\xi^{+}\subset\eta^{+}} (-1)^{|\eta^{+}\setminus\xi^{+}|} \sum_{\xi^{-}\subset\eta^{-}} (-1)^{|\eta^{-}\setminus\xi^{-}|}\lambda\int_{\mathbb{R}^{d}}\sum_{y\in\xi^{-}} a(x-y)\\
&\quad\times\left(\sum_{\zeta^{+}\subset\xi^{+}\cup x}\sum_{\zeta^{-}\subset\xi^{-}} G(\zeta^{+},\zeta^{-}) - \sum_{\zeta^{+}\subset\xi^{+}}\sum_{\zeta^{-}\subset\xi^{-}} G(\zeta^{+},\zeta^{-})\right)dx
\end{aligned}$$

$$= \lambda \int_{\mathbb{R}^d} \sum_{y' \in \eta^-} G\left(\eta^+ \cup x, \eta^-\right) a\left(x - y'\right) dx$$

$$+ \lambda \int_{\mathbb{R}^d} \sum_{y' \in \eta^-} G\left(\eta^+ \cup x, \eta^- \setminus y'\right) a\left(x - y'\right) dx.$$

Finally,

$$\begin{aligned}
\left(\hat{L}_d^- G\right)(\eta) &= \left(\mathcal{K}^{-1} L_d^- \mathcal{K} G\right)(\eta) \\
&= \sum_{\xi^+ \subset \eta^+} (-1)^{|\eta^+ \setminus \xi^+|} \sum_{\xi^- \subset \eta^-} (-1)^{|\eta^- \setminus \xi^-|} \\
&\quad \times \sum_{y \in \xi^-} \left(\sum_{\zeta^+ \subset \xi^+} \sum_{\zeta^- \subset \xi^- \setminus y} G(\zeta^+, \zeta^-) - \sum_{\zeta^+ \subset \xi^+} \sum_{\zeta^- \subset \xi^-} G(\zeta^+, \zeta^-) \right) \\
&= - \left|\eta^-\right| G(\eta^+, \eta^-),
\end{aligned}$$

and, analogously,

$$\left(\hat{L}_d^+ G\right)(\eta^+, \eta^-) = - \left|\eta^+\right| G(\eta^+, \eta^-).$$

The statement is proved. □

Now we should calculate the adjoint operator $\hat{L}^*$.

Proposition 3.2. *The adjoint operator $\hat{L}^*$ has the following form:*

$$\begin{aligned}
\left(\hat{L}^* k\right)(\eta^+, \eta^-) &= - \left(\left|\eta^+\right| + \left|\eta^-\right|\right) k(\eta^+, \eta^-) \\
&\quad + \lambda^+ \sum_{x \in \eta^+} \sum_{x' \in \eta^+ \setminus x} a^+(x - x') k\left(\eta^+ \setminus x, \eta^-\right) \\
&\quad + \lambda^+ \sum_{x \in \eta^+} \int_{\mathbb{R}^d} a^+(x - x') k\left(\eta^+ \setminus x \cup x', \eta^-\right) dx' \\
&\quad + \lambda^- \sum_{y \in \eta^-} \sum_{y' \in \eta^- \setminus y} a^-(y - y') k\left(\eta^+, \eta^- \setminus y\right) \\
&\quad + \lambda^- \sum_{y \in \eta^-} \int_{\mathbb{R}^d} a^-(y - y') k\left(\eta^+, \eta^- \setminus y \cup y'\right) dy' \\
&\quad + \lambda \sum_{x \in \eta^+} \sum_{y \in \eta^-} a(x - y) k(\eta^+ \setminus x, \eta^-) \\
&\quad + \lambda \sum_{x \in \eta^+} \int_{\mathbb{R}^d} a(x - y) k\left(\eta^+ \setminus x, \eta^- \cup y\right) dy
\end{aligned}$$

Proof. We may use the following corollaries of the classical Mecke formula (see, e.g., [11]):

$$\int_{\Gamma_0^2} \sum_{x\in\eta^+} h_+(x,\eta^+,\eta^-)d\lambda_1(\eta^+)d\lambda_1(\eta^-)$$
$$= \int_{\Gamma_0^2}\int_{\mathbb{R}^d} h_+(x,\eta^+\cup x,\eta^-)dxd\lambda_1(\eta^+)d\lambda_1(\eta^-),$$
$$\int_{\Gamma_0^2} \sum_{y\in\eta^-} h_-(y,\eta^+,\eta^-)d\lambda_1(\eta^+)d\lambda_1(\eta^-)$$
$$= \int_{\Gamma_0^2}\int_{\mathbb{R}^d} h_-(y,\eta^+,\eta^-\cup y)dyd\lambda_1(\eta^+)d\lambda_1(\eta^-),$$
$$\int_{\Gamma_0^2} \sum_{x\in\eta^+}\sum_{y\in\eta^-} h(x,\eta^+,\eta^-)d\lambda_1(\eta^+)d\lambda_1(\eta^-)$$
$$= \int_{\Gamma_0^2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} h(x,\eta^+\cup x,\eta^-\cup y)dxdyd\lambda_1(\eta^+)d\lambda_1(\eta^-).$$

Then one can obtain the explicit formula for the operator $\hat{L}^*$ directly from definition (2.10). □

As a result, the statements of the Propositions 2.1 and 2.2 are directly follow from the Proposition 3.2 and (2.11)–(2.12).

3.2 Solution of the equations for time evolution of the correlation functions

To solve the equations from the Propositions 2.1 and 2.2 using classical perturbation method we rewrite these equations in the following forms:

$$\frac{\partial k_t^-(y)}{\partial t} = (\lambda^- - 1)k_t^-(y) + \lambda^-(L^-k_t^-)(y), \tag{3.1}$$
$$\frac{\partial k_t^+(x)}{\partial t} = (\lambda^+ - 1)k_t^+(x) + \lambda^+(L^+k_t^+)(x) + \lambda\int_{\mathbb{R}^d} a(x-y)k_t^-(y)dy, \tag{3.2}$$

where Markov-type generators $L^\pm$ are defined on functions on $\mathbb{R}^d$ by

$$(L^-f)(y) = \int_{\mathbb{R}^d} a^-(y-y')[f(y')-f(y)]dy',$$
$$(L^+f)(x) = \int_{\mathbb{R}^d} a^+(x-x')[f(x')-f(x)]dx';$$

and for the second order correlation functions:

$$\frac{\partial k_t^{--}(y_1,y_2)}{\partial t} = 2k_t^{--}(y_1,y_2)(\lambda^- - 1) + \lambda^-(L_1^{--}k_t^{--})(y_1,y_2) + \lambda^-(L_2^{--}k_t^{--})(y_1,y_2) + \lambda^- a^-(y_1-y_2)[k_t^-(y_1)+k_t^-(y_2)], \tag{3.3}$$

$$\frac{\partial k_t^{+-}(x,y)}{\partial t} = (\lambda^+ + \lambda^- - 2)k_t^{+-}(x,y) + \lambda^+ L_1^{+-}k_t^{+-}(x,y) + \lambda^- L_2^{+-}k_t^{+-}(x,y) + \lambda a(x-y)k_t^-(y) + \lambda\int_{\mathbb{R}^d} a(x-y')k_t^{--}(y,y')dy', \tag{3.4}$$

$$\begin{aligned}\frac{\partial k_t^{++}(x_1,x_2)}{\partial t} &= 2k_t^{++}(x_1,x_2)(\lambda^+ - 1) + \lambda^+ L_1^{++}k_t^{++}(x_1,x_2) + \lambda^+ L_2^{++}k_t^{++}(x_1,x_2)\\ &+ \{\lambda^+ a^+(x_1-x_2)[k_t^+(x_1)+k_t^+(x_2)]\\ &+ \lambda\int_{\mathbb{R}^d} a(x_1-y)k_t^{+-}(x_2,y)dy + \lambda\int_{\mathbb{R}^d} a(x_2-y)k_t^{+-}(x_1,y)dy\},\end{aligned} \tag{3.5}$$

where Markov-type generators $L_i^{\pm\pm}$, $i=1,2$ are defined on functions on $\mathbb{R}^d\times\mathbb{R}^d$ by

$$\begin{aligned}(L_1^{--}f)(y_1,y_2) &= \int_{\mathbb{R}^d} a^-(y_1-y')[f(y_2,y')-f(y_2,y_1)]dy',\\ (L_2^{--}f)(y_1,y_2) &= \int_{\mathbb{R}^d} a^-(y_2-y')[f(y_1,y')-f(y_1,y_2)]dy',\\ (L_1^{+-}f)(x,y) &= \int_{\mathbb{R}^d} a^+(x-x')[f(x',y)-f(x,y)]dx',\\ (L_2^{+-}f)(x,y) &= \int_{\mathbb{R}^d} a^-(y-y')[f(x,y')-f(x,y)]dy',\\ (L_1^{++}f)(x_1,x_2) &= \int_{\mathbb{R}^d} a^+(x_1-x')[f(x_2,x')-f(x_2,x_1)]dx',\\ (L_2^{++}f)(x_1,x_2) &= \int_{\mathbb{R}^d} a^+(x_2-x')[f(x_1,x')-f(x_1,x_2)]dx'.\end{aligned}$$

Next propositions are direct corollaries of the perturbation method (note also that any Markov semigroup preserves constants).

Proposition 3.3. *The solutions of* (3.1)–(3.2) *with initial values* (2.13) *have the following forms:*

$$k_t^-(y) = c^- e^{t(\lambda^- -1)} + e^{t(\lambda^- -1)}e^{t\lambda^- L^-}\psi^-(y), \tag{3.6}$$

$$k_t^+(x) = c^+ e^{t(\lambda^+ -1)} + e^{t(\lambda^+ -1)}e^{t\lambda^+ L^+}\psi^+(x) + \lambda c^- e^{t(\lambda^+ -1)}\int_0^t e^{\tau(\lambda^- -\lambda^+)}d\tau \tag{3.7}$$

$$+ \lambda e^{t(\lambda^+ - 1)} \int_0^t e^{\tau(\lambda^- - \lambda^+)} e^{(t-\tau)\lambda^+ L^+} (a * (e^{\tau\lambda^- L^-} \psi^-))(x) d\tau.$$

Proposition 3.4. *Let* (2.14) *holds. Then the solutions of* (3.3)–(3.5) *with initial values* (2.15) *have the following forms:*

$$k_t^{--}(y_1, y_2) = e^{t2(\lambda^- - 1)} e^{t\lambda^- L_1^{--}} e^{t\lambda^- L_2^{--}} (c^{--} + \varphi^{--}(y_1 - y_2))$$
$$+ \int_0^t e^{(t-\tau)2(\lambda^- - 1)} e^{(t-\tau)\lambda^- L_1^{--}} e^{(t-\tau)\lambda^- L_2^{--}} \lambda^- a^-(y_1 - y_2)[k_\tau^-(y_1) + k_\tau^-(y_2)] d\tau, \tag{3.8}$$

$$k_t^{+-}(x, y) = e^{t(\lambda^+ + \lambda^- - 2)} e^{t\lambda^+ L_1^{+-}} e^{t\lambda^- L_2^{+-}} (c^{+-} + \varphi^{+-}(x - y))$$
$$+ \int_0^t e^{(t-\tau)(\lambda^+ + \lambda^- - 2)} e^{(t-\tau)\lambda^+ L_1^{+-}} e^{(t-\tau)\lambda^- L_2^{+-}}$$
$$\times \{\lambda a(x - y) k_\tau^-(y) + \lambda \int_{\mathbb{R}^d} a(x - y') k_\tau^{--}(y, y') dy'\} d\tau, \tag{3.9}$$

$$k_t^{++}(x_1, x_2) = e^{t2(\lambda^+ - 1)} e^{t\lambda^+ L_1^{++}} e^{t\lambda^+ L_2^{++}} (c^{++} + \varphi^{++}(x_1 - x_2))$$
$$+ \int_0^t e^{(t-\tau)2(\lambda^+ - 1)} e^{(t-\tau)\lambda^+ L_1^{++}} e^{(t-\tau)\lambda^+ L_2^{++}} \{\lambda^+ a^+(x_1 - x_2)[k_\tau^+(x_1) + k_\tau^+(x_2)]$$
$$+ \lambda \int_{\mathbb{R}^d} a(x_1 - y) k_\tau^{+-}(x_2, y) dy + \lambda \int_{\mathbb{R}^d} a(x_2 - y) k_\tau^{+-}(x_1, y) dy\} d\tau \tag{3.10}$$

3.3 Technical lemmas

In this subsection we present several useful notations and notes and prove technical lemmas needed in the sequel. Let us define

$$\mu^+ := \lambda^+ - 1, \qquad \mu^- := \lambda^- - 1, \tag{3.11}$$
$$f^+(p) := \lambda^+ \hat{a}^+(p) - 1, \qquad f^-(p) := \lambda^- \hat{a}^-(p) - 1. \tag{3.12}$$

Note that conditions $0 < \lambda^\pm \le 1$ equivalent to $-1 < \mu^\pm \le 0$ and $\mu^\pm = 0$ only if $\lambda^\pm = 1$. Recall that $a^\pm$ are positive, even and normalized. Then

$$\hat{a}^\pm(p) = \int_{\mathbb{R}^d} \cos(p, x) a^\pm(x) dx, \qquad |\hat{a}^\pm(p)| \le 1, \tag{3.13}$$

and $\hat{a}^\pm(p) = 1$ only at $p = 0$. Hence, the conditions $0 < \lambda^\pm \le 1$ imply

$$-\lambda^\pm - 1 \le f^\pm(p) \le \mu^\pm \le 0, \tag{3.14}$$

and $f^\pm(p) = \mu^\pm$ only at point $p = 0$.

Let $C^-(\mathbb{R}^d)$ be the set of non-positive continuous functions on $\mathbb{R}^d$ which equal to 0 only on countable sets. Since Fourier image of integrable function is continuous one has $f^\pm \in C^-(\mathbb{R}^d)$. For any $f \in C^-(\mathbb{R}^d)$ define two closed sets

$$\mathfrak{D}_f^\pm := \{x \in \mathbb{R}^d : f(x) = f^\pm(x)\}. \tag{3.15}$$

Note that that set $\mathbb{R}^d \setminus \mathfrak{D}^+_{f^-} = \mathbb{R}^d \setminus \mathfrak{D}^-_{f^+}$ has zero Lebesgue measure only if $\lambda^+\hat{a}^+ \equiv \lambda^-\hat{a}^-$ and, hence, $\lambda^+ = \lambda^-$.

Lemma 3.1. *Let $d \geq 3$ and $b \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. Then*

$$c^\pm(p) = \frac{b(p)}{\hat{a}^\pm(p) - 1}$$

are integrable functions on $\mathbb{R}^d$.

Proof. By (3.13), $\hat{a}^\pm(0) = 1$. Due to (2.7), $a^\pm$ has at least first and second finite moments. Then using (3.13) one has in some neighbourhood of the origin

$$\hat{a}^\pm(p) - 1 = \int_{\mathbb{R}^d} [\cos(p,x) - 1]a^\pm(x)dx \sim -\frac{1}{2}\int_{\mathbb{R}^d} (p,x)^2 a^\pm(x)dx \sim -\frac{1}{2}|p|^2$$

and outside of this neighbourhood $|\hat{a}^\pm(p) - 1|$ are bounded from below.

Hence, $c^\pm$ are integrable in this neighbourhood since b is bounded and $\dfrac{1}{|p|^2} \in L^1(\mathbb{R}^d)$ for $d \geq 3$; and $c^\pm$ are integrable outside of this neighbourhood since b is integrable. $\square$

Lemma 3.2. *Let $d \geq 3$, $0 < \lambda^\pm \leq 1$, and $b \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. Then for any $f \in C^-(\mathbb{R}^d)$*

$$d^\pm(p) = b(p) \sup_{t \geq 0} \frac{e^{tf(p)} - e^{tf^\pm(p)}}{f(p) - f^\pm(p)}$$

are integrable functions on $\mathbb{R}^d \setminus \mathfrak{D}^\pm_f$.

Proof. Let $p \in \mathbb{R}^d \setminus \mathfrak{D}^+_f$ for example. Without loss of generality assume that $p \neq 0$ and $f(p) \neq 0$. Set $a = f(p), b = f^+(p)$. Then $a < 0$, $b < 0$, $a \neq b$. Let us define

$$h(t) := \frac{e^{ta} - e^{tb}}{a - b}, \quad t \geq 0.$$

Clearly, $h(t) \geq 0$ and $h(t) = 0$ only at $t = 0$. One has

$$h'(t) := \frac{be^{ta}\left(\dfrac{a}{b} - e^{t(b-a)}\right)}{a - b}.$$

Set $t_0 = \dfrac{1}{b-a} \ln \dfrac{a}{b}$. If $0 > a > b$ then $t_0 > 0$ and for $0 < t < t_0$ we have $e^{t(b-a)} > \dfrac{a}{b}$, hence, $h'(t) > 0$; for $t > t_0$ one has $h'(t) < 0$. If $0 > b > a$ then $t_0 > 0$ also and for $0 < t < t_0$ we obtain $e^{t(b-a)} < \dfrac{a}{b}$, therefore, $h'(t) > 0$; for $t > t_0$ again $h'(t) < 0$. As a result,

$$\max_{[0;\infty)} h(t) = h(t_0) = \frac{e^{t_0 a}(1 - e^{t_0(b-a)})}{a - b} = \frac{e^{t_0 a}\left(1 - \dfrac{a}{b}\right)}{a - b} = -\frac{1}{b}e^{t_0 a} < -\frac{1}{b},$$

since $-b > 0$, $a < 0$.

Hence, for any $p \in \mathbb{R}^d \setminus \mathfrak{D}_f^+$, $t \geq 0$

$$0 \leq \frac{e^{tf(p)} - e^{tf^+(p)}}{f(p) - f^+(p)} < -\frac{1}{f^+(p)}.$$

Then using (3.14), (3.11) for $\lambda^+ < 1$ one has $\mu^+ < 0$ and $d^+(p) < \dfrac{b(p)}{-\mu^+}$ that imply the statement of this Lemma. For $\lambda^+ = 1$ the result is followed from Lemma 3.1. □

3.4 Asymptotic behaviour of the first order correlation functions

In this subsection we prove the Theorem 2.2.

1) We should use (3.6). Note that $\psi^- \in L^1(\mathbb{R}^d)$ and Markov semigroup maps $L^1(\mathbb{R}^d)$ into $L^1(\mathbb{R}^d)$. Then using inverse Fourier transform one has

$$\left(e^{t\lambda^- L^-}\psi^-\right)(y) = c_d \int_{\mathbb{R}^d} e^{i(p,y)} e^{t\lambda^-(\hat{a}^-(p)-1)} \hat{\psi}^-(p)\, dp, \tag{3.16}$$

where $c_d := \dfrac{1}{(2\pi n)^d}$. Using (3.13), the expression in the integral in (3.16) goes to 0 for any y and a.a. p. Since $\hat{\psi}^- \in L^1(\mathbb{R}^d)$ and $\left|e^{i(p,y)} e^{t\lambda^-(\hat{a}^-(p)-1)}\right| \leq 1$ one has that the integral also goes to 0 for any y. Then the statement is directly followed from (3.6).

2) We will use (3.7). Note that similarly to the first step $e^{t\lambda^+ L^+}\psi^+ \to 0$ point-wisely.

2.1) If $\lambda^+ > 1$ then for any $\lambda^- > 0$

$$k_t^+(x) \to \infty,$$

since $\psi^- \geq \alpha^- - c^- > -c^-$, hence, the last term in (3.7) is bigger than

$$-\lambda c^- e^{t(\lambda^+ - 1)} \int_0^t e^{\tau(\lambda^- - \lambda^+)} d\tau$$

and, therefore,

$$k_t^+(x) > c^+ e^{t(\lambda^+-1)} + e^{t(\lambda^+-1)} e^{t\lambda^+ L^+}\psi^+(x) \to \infty$$

2.2) Let now $\lambda^+ \leq 1$. Divide proof on several sub-steps.

2.2.1) Suppose $\lambda^+ = \lambda^- = \nu$ then using (3.7) one has

$$k_t^+(x) = e^{t(\nu-1)}c^+ + e^{t(\nu-1)} e^{t\nu L^+}\psi^+(x) + \lambda e^{t(\nu-1)} c^- t + u_t(x) \tag{3.17}$$

where

$$u_t(x) = \lambda e^{t(\nu-1)} \int_0^t e^{(t-\tau)\nu L^+} \Big(a * (e^{\tau\nu L^-}\psi^-)\Big)(x)\, d\tau.$$

Let us find $\lim\limits_{t\to\infty} u_t(x)$, for $\nu \le 1$. Note that $u_t \in L^1(\mathbb{R}^d)$ since semigroup and convolution preserve integrability. Hence, we may compute the Fourier transform of u_t:

$$\hat{u}_t(p) = \begin{cases} \lambda\hat{a}(p)\hat{\psi}^-(p)e^{tf^+(p)}t, & p \in \mathfrak{D}^+_{f^-}, \\ \lambda\hat{a}(p)\hat{\psi}^-(p)\dfrac{e^{tf^-(p)} - e^{tf^+(p)}}{f^-(p) - f^+(p)}, & p \in \mathbb{R}^d \setminus \mathfrak{D}^+_{f^-}. \end{cases} \tag{3.18}$$

Since $\hat{\psi}^-$ is bounded and $\hat{a}$ is bounded and integrable due to (2.7) one can apply Lemma 3.2, hence, $\hat{u}_t(p)$ has integrable majorant on $\mathbb{R}^d \setminus \mathfrak{D}^+_{f^-}$. Since $e^{ta}t < -\dfrac{e^{-1}}{a}$ for any $t \ge 0$, $a < 0$ one has for any $p \in \mathfrak{D}^+_{f^-} \setminus \{0\}$

$$\big|\hat{u}_t(p)\big| \le c_1 \left|\frac{\hat{a}(p)}{f^+(p)}\right|.$$

Again if $\nu < 1$ then denominator is separated from zero, otherwise one can apply Lemma 3.1. As a result, $\hat{u}_t(p)$ has integrable majorant on whole $\mathbb{R}^d$ and pointwisely goes to 0 as $t \to \infty$ (except case $\nu = 1$, $p = 0$). Therefore, using dominated convergence theorem the inverse Fourier transform of $\hat{u}_t(p)$ converges to zero, i.e. pointwisely $u_t(x) \to 0$ as $t \to \infty$.

Thus, using (3.17) one has that $k_t^+ \to \infty$ if $\nu = 1$ and $k_t^+ \to 0$ if $\nu < 1$.

2.2.2) Let now $\lambda^+ \ne \lambda^-$. Using (3.7) obtain

$$\begin{aligned} k_t^+(x) &= c^+ e^{t\left(\lambda^+-1\right)} + e^{t\left(\lambda^+-1\right)} e^{t\lambda^+ L^+}\psi^+(x) \\ &\quad + \lambda c^- \frac{1}{\lambda^- - \lambda^+}\left(e^{t\left(\lambda^--1\right)} - e^{t\left(\lambda^+-1\right)}\right) \\ &\quad + \lambda e^{t\left(\lambda^+-1\right)} \int_0^t e^{\tau\left(\lambda^- - \lambda^+\right)} e^{(t-\tau)\lambda^+ L^+}\left(a * e^{\tau\lambda^- L^-}\psi^-\right)(x)\, d\tau. \end{aligned} \tag{3.19}$$

2.2.2.1) Suppose that $\lambda^- > 1$. Then since $\lambda^+ \le 1$ and $\psi^- \ge \alpha^- - c^- > 0$ we obtain that

$$k_t^+(x) \to \infty, \quad t \to \infty$$

2.2.2.2) Next, let $\lambda^- < 1$, $\lambda^+ < 1$. Since $\hat{\psi}^-$ is bounded one has for $M = \sup_{\mathbb{R}^d}|\hat{\psi}^-|$ that the last term in (3.19) is not bigger (by absolute value) than

$$\frac{M}{\lambda^- - \lambda^+}\left(e^{t(\lambda^--1)} - e^{t(\lambda^+-1)}\right) \to 0.$$

Then due to (3.19) $k_t^+(x) \to 0$.

2.2.2.3) Finally, let $\lambda^- < 1,\ \lambda^+ = 1$ or $\lambda^- = 1,\ \lambda^+ < 1$. The last term in (3.19) is integrable function since semigroup and convolution preserve integrability. By direct computation its Fourier transform has form (3.18). Hence, this last term pointwisely goes to 0.

As a result, by (3.19) we obtain that if $\lambda^+ = 1$, $\lambda^- < 1$

$$k_t^+(x) \to c^+ + \frac{\lambda c^-}{1-\lambda^-}, \quad t \to \infty;$$

and if $\lambda^+ < 1$, $\lambda^- = 1$

$$k_t^+(x) \to \frac{\lambda c^-}{1-\lambda^+}, \quad t \to \infty.$$

Theorem 2.2 is proved.

3.5 Asymptotic behaviour of the second order correlation functions

In this subsection we prove the Theorem 2.3.

First of all we present explicit expressions for Ω^{++}, Ξ^{--}, Ξ^{+-}, Ξ^{++}, and after that we prove the Theorem. These functions are inverse Fourier transforms of the following

$$\omega^{++}(p) = \frac{\lambda^- + \lambda - 1}{\lambda^- - 1} \cdot \frac{c^+ \hat{a}^+(p)}{1 - \hat{a}^+(p)}, \tag{3.20}$$

$$\xi^{--}(p) = \frac{c^- \hat{a}^-(p)}{1 - \hat{a}^-(p)}, \tag{3.21}$$

$$\xi^{+-}(p) = \frac{1}{2} \cdot \frac{\mu^- + 2}{2 - \lambda^+ \hat{a}^+(p) - \hat{a}^-(p)} \cdot \frac{c^- \lambda \hat{a}(p)}{1 - \hat{a}^-(p)}, \tag{3.22}$$

$$\xi^{++}(p) = \frac{\lambda}{1 - \lambda^+ \hat{a}^+(p)} \left(\frac{\lambda^+ c^- \hat{a}^+(p)}{1 - \lambda^+} + \frac{\lambda c^-}{2 - \lambda^+ \hat{a}^+(p) - \hat{a}^+(p)} \cdot \frac{\hat{a}^2(p)}{1 - \hat{a}^-(p)} \right), \tag{3.23}$$

correspondingly.

Let us introduce the following denotations for the Markov semigroups

$$T_t^{11} = e^{t\lambda^+ L_1^{++}}, \quad T_t^{12} = e^{t\lambda^+ L_2^{++}}, \quad T_t^{13} = e^{t\lambda^+ L_1^{+-}},$$
$$T_t^{21} = e^{t\lambda^- L_1^{--}}, \quad T_t^{22} = e^{t\lambda^- L_2^{--}}, \quad T_t^{23} = e^{t\lambda^- L_1^{+-}}.$$

We start with trivial remark that for any even functions $c, g \in L^1(\mathbb{R}^d)$

$$(L_1 g)(x_1 - x_2) = (L_2 g)(x_1 - x_2),$$

where

$$(L_1 f)(x_1, x_2) := \int_{\mathbb{R}^d} c(x_1 - x')[f(x_2, x') - f(x_2, x_1)]dx',$$
$$(L_2 f)(x_1, x_2) := \int_{\mathbb{R}^d} c(x_2 - x')[f(x_1, x') - f(x_1, x_2)]dx'.$$

After transformations, substitutions and simplifying we obtain for (3.8)–(3.10) the following representations:

$$
\begin{aligned}
k_t^{--}(y_1,y_2) &= c^{--}e^{2\mu^-t} + e^{2\mu^-t}T_t^{21}T_t^{22}\varphi^{--}(y_1-y_2) + U_t^{--}(y_1-y_2),\\
k_t^{+-}(x,y) &= \left(c^{+-} - \frac{\lambda c^{--}}{\mu^- - \mu^+}\right)e^{(\mu^+ + \mu^-)t} + \frac{\lambda c^{--}}{\mu^- - \mu^+}e^{2\mu^-t}\\
&\quad + e^{(\mu^+ + \mu^-)t}T_t^{13}T_t^{23}\varphi^{+-}(x-y) + U_t^{+-}(x-y),\\
k_t^{++}(x_1,x_2) &= \left(c^{++} - \frac{2\lambda c^{+-}}{\mu^- - \mu^+} + \frac{\lambda^2 c^{--}}{(\mu^- - \mu^+)^2}\right)e^{2\mu^+t}\\
&\quad + \left(\frac{2\lambda c^{+-}}{\mu^- - \mu^+} - \frac{2\lambda^2 c^{--}}{(\mu^- - \mu^+)^2}\right)e^{(\mu^+ + \mu^-)t}\\
&\quad + \frac{\lambda^2 c^{--}}{(\mu^- - \mu^+)^2}e^{2\mu^-t}\\
&\quad + e^{2\mu^+t}T_t^{11}T_t^{12}\varphi^{++}(x_1-x_2) + U_t^{++}(x_1-x_2).
\end{aligned}
$$

Here

$$
U_t^{--}(y_1-y_2) = 2\lambda^- c^- \int_0^t e^{\mu^-\tau}e^{2\mu^-(t-\tau)}T_{t-\tau}^{21}T_{t-\tau}^{22}a^-(y_1-y_2)d\tau,
$$

$$
\begin{aligned}
&U_t^{+-}(x-y)\\
&= \lambda c^- \int_0^t e^{\mu^-\tau}e^{(\mu^+ + \mu^-)(t-\tau)}T_{t-\tau}^{13}T_{t-\tau}^{23}a(x-y)d\tau\\
&\quad + \lambda\int_0^t e^{2\mu^-\tau}e^{(\mu^+ + \mu^-)(t-\tau)}T_{t-\tau}^{13}T_{t-\tau}^{23}\int_{\mathbb{R}^d} a(x-y')T_\tau^{21}T_\tau^{22}\varphi^{--}(y-y')dy'd\tau\\
&\quad + 2c^-\lambda\lambda^- \int_0^t e^{(\mu^+ + \mu^-)(t-\tau)}T_{t-\tau}^{13}T_{t-\tau}^{23}\\
&\qquad \times \int_{\mathbb{R}^d} a(x-y')\int_0^\tau e^{\mu^-s}e^{2\mu^-(\tau-s)}T_{\tau-s}^{21}T_{\tau-s}^{22}a^-(y-y')dsdy'd\tau,
\end{aligned}
$$

$$
\begin{aligned}
&U_t^{++}(x_1-x_2)\\
&= 2\lambda^+c^+\int_0^t e^{\mu^+\tau}e^{2\mu^+(t-\tau)}T_{t-\tau}^{11}T_{t-\tau}^{12}a^+(x_1-x_2)d\tau\\
&\quad + 2\lambda\lambda^+c^-\int_0^t e^{2\mu^+(t-\tau)}T_{t-\tau}^{11}T_{t-\tau}^{12}a^+(x_1-x_2)\int_0^\tau e^{\mu^-s}e^{\mu^+(\tau-s)}dsd\tau\\
&\quad + 2\lambda\int_0^t e^{(\mu^+ + \mu^-)\tau}e^{2\mu^+(t-\tau)}T_{t-\tau}^{11}T_{t-\tau}^{12}\int_{\mathbb{R}^d} a(x_1-y)T_\tau^{13}T_\tau^{23}\varphi^{+-}(x_2-y)dyd\tau\\
&\quad + 2\lambda^2c^-\int_0^t e^{2\mu^+(t-\tau)}T_{t-\tau}^{11}T_{t-\tau}^{12}\int_{\mathbb{R}^d} a(x_1-y)\\
&\qquad \times \int_0^\tau e^{\mu^-s}e^{(\mu^+ + \mu^-)(\tau-s)}T_{\tau-s}^{13}T_{\tau-s}^{23}a(x_2-y)dydsd\tau
\end{aligned}
$$

$$
\begin{aligned}
&+2\lambda^2\int_0^t e^{2\mu^+(t-\tau)}T^{11}_{t-\tau}T^{12}_{t-\tau}\int_{\mathbb{R}^d}a(x_1-y)\int_0^\tau e^{2\mu^- s}e^{(\mu^++\mu^-)(\tau-s)}T^{13}_{\tau-s}T^{23}_{\tau-s}\\
&\quad\times\int_{\mathbb{R}^d}a(x_2-y')T^{21}_sT^{22}_s\varphi^{--}(y-y')dy'dsdyd\tau\\
&+4\lambda^-c^-\lambda^2\int_0^t e^{2\mu^+(t-\tau)}T^{11}_{t-\tau}T^{12}_{t-\tau}\int_{\mathbb{R}^d}a(x_1-y)\\
&\quad\times\int_0^\tau e^{(\mu^++\mu^-)(\tau-s)}T^{13}_{\tau-s}T^{23}_{\tau-s}\int_{\mathbb{R}^d}a(x_2-y')\\
&\quad\times\int_0^s e^{\mu^-\theta}e^{2\mu^-(s-\theta)}T^{21}_{s-\theta}T^{22}_{s-\theta}a^-(y-y')d\theta dy'dsdyd\tau.
\end{aligned}
$$

Since semigroups and convolutions preserve integrability we have that $T^{21}_tT^{22}_t\varphi^{--}$, $T^{13}_tT^{23}_t\varphi^{+-}$, $T^{11}_tT^{12}_t\varphi^{++}$ as well as U^{--}_t, U^{+-}_t and U^{++}_t are integrable on $\mathbb{R}^d$ functions. So, to find their limits as $t\to\infty$ we may use the Fourier transforms.

Namely,

$$
\begin{aligned}
T^{21}_tT^{22}_t\varphi^{--}(y_1-y_2)&=c_d\int_{\mathbb{R}^d}e^{ip(y_1-y_2)}e^{2(f^-(p)-\mu^-)t}\hat\varphi^{--}(p)dp,\\
T^{13}_tT^{23}_t\varphi^{+-}(x-y)&=c_d\int_{\mathbb{R}^d}e^{ip(x-y)}e^{(f^+(p)-\mu^+)t}e^{(f^-(p)-\mu^-)t}\hat\varphi^{+-}(p)dp,\\
T^{11}_tT^{12}_t\varphi^{++}(x_1-x_2)&=c_d\int_{\mathbb{R}^d}e^{ip(x_1-x_2)}e^{2(f^+(p)-\mu^+)t}\hat\varphi^{++}(p)dp.
\end{aligned}
$$

Since $\hat\varphi^{--},\hat\varphi^{+-},\hat\varphi^{++}$ are integrable we have using (3.14) and dominated convergence theorem that these three terms go to 0.

Let us introduce for further simplicity of notations the following functions

$$
\begin{aligned}
h_1(p)&:=\mu^+-2f^+(p)\ge 0,\\
h_2(p)&:=\mu^--2f^-(p)\ge 0,\\
h_3(p)&:=f^+(p)+f^-(p)<0,\\
h_4(p)&:=\mu^--f^+(p)-f^-(p)\ge 0.
\end{aligned}
$$

These inequalities are followed from (3.11), (3.12) and (3.14) as well as the fact that equalities are possible only at $p=0$.

Consider also the following two functions g_1 and g_2

$$
\begin{aligned}
g_1(p)&=f^-(p)-f^+(p),\\
g_2(p)&=\mu^--2f^+(p).
\end{aligned}
$$

They can be equal zero on a set of non-zero measure.

We have in the new notations:

$$\widehat{U_t}^{--}(p) = 2c^-\lambda^-\hat{a}^-(p)e^{2f^-(p)t}\int\limits_0^t e^{h_2(p)\tau}d\tau,$$

$$\begin{aligned}\widehat{U_t}^{+-}(p) = {} & c^-\lambda\hat{a}(p)e^{h_3(p)t}\int\limits_0^t e^{h_4(p)\tau}d\tau \\ & + \lambda\hat{a}(p)\hat{\varphi}^{--}(p)e^{h_3(p)t}\int_0^t e^{g_1(p)\tau}d\tau \\ & + 2c^-\lambda\hat{a}(p)\lambda^-\hat{a}^-(p)e^{h_3(p)t}\int\limits_0^t e^{g_1(p)\tau}\int_0^\tau e^{h_2(p)s}dsd\tau,\end{aligned}$$

$$\begin{aligned}\widehat{U_t}^{++}(p) = {} & 2c^+\lambda^+\hat{a}^+(p)e^{2f^+(p)t}\int\limits_0^t e^{h_1(p)\tau}d\tau \\ & + \frac{2\lambda c^-\lambda^+\hat{a}^+(p)}{\mu^- - \mu^+}e^{2f^+(p)t}\left(\int_0^t e^{g_2(p)\tau}d\tau - \int_0^t e^{h_1(p)\tau}d\tau\right) \\ & + 2\lambda\hat{a}(p)\hat{\varphi}^{+-}(p)e^{2f^+(p)t}\int\limits_0^t e^{g_1(p)\tau}d\tau \\ & + 2c^-\lambda^2\hat{a}^2(p)e^{2f^+(p)t}\int_0^t e^{g_1(p)\tau}\int\limits_0^\tau e^{h_4(p)s}dsd\tau \\ & + 2\lambda^2\hat{a}^2(p)\hat{\varphi}^{+-}(p)e^{2f^+(p)t}\int\limits_0^t e^{g_1(p)\tau}\int_0^\tau e^{g_1(p)s}dsd\tau \\ & + 4c^-\lambda^-\hat{a}^-(p)\lambda^2\hat{a}^2(p)e^{2f^+(p)t}\int\limits_0^t e^{g_1(p)\tau}\int_0^\tau e^{g_1(p)s}\int\limits_0^s e^{h_2(p)\theta}d\theta dsd\tau.\end{aligned}$$

Let us consider the following closed set $\mathfrak{D} = \mathfrak{D}_1 \cup \mathfrak{D}_2$, where $\mathfrak{D}_1 := \{p : g_1(p) = 0\} = \mathfrak{D}^+_{f^-}$, $\mathfrak{D}_2 = \{p : \ g_2(p) = 0\}$. It's easy to see that $\mathfrak{D}_1 \cap \mathfrak{D}_2 = \varnothing$. Indeed, by (3.14) for any $p \in \mathfrak{D}_1 \cap \mathfrak{D}_2$

$$\mu^- = 2f^+(p) = 2f^-(p) \le 2\mu^-.$$

But $\mu^- \le 0$, hence, it should be equality that implies $f^-(p) = \mu^-$, and with necessity $p = 0$. But if $0 \in \mathfrak{D}_1 \cap \mathfrak{D}_2$, then $f^+(0) = f^-(0)$, i.e., $\mu^+ = \mu^-$, that contradicts to the condition of the theorem.

Next we note that the functions $\widehat{U_t}^{+-}(p)$ and $\widehat{U_t}^{++}(p)$ have different explicit expressions for $p \in \mathfrak{D}$ and for $p \in \mathfrak{D}^c := \mathbb{R}^d \setminus \mathfrak{D}$. Note also that these functions are continuous functions of p as compositions of the integrals of the continuous functions of t with continuous dependence on a parameter p. Hence, for calculate these expressions for $p\in\mathfrak{D}$ we may calculate their for $p\in\mathfrak{D}^c$ and take limits as $\operatorname{dist}(p,\mathfrak{D})\to 0$.

By direct calculations for any $p\in\mathfrak{D}^c\setminus\{0\}$ we obtain

$$
\begin{aligned}
\widehat{U}_t^{--}(p) &= 2\lambda^- c^- \hat{a}^-(p)\frac{e^{\mu^- t}-e^{2f^-(p)t}}{\mu^- - 2f^-(p)},\\
\widehat{U}_t^{+-}(p) &= \lambda c^- \hat{a}(p)\cdot\frac{\mu^-+2}{\mu^- - 2f^-(p)}\cdot\frac{e^{\mu^- t}-e^{[f^+(p)+f^-(p)]t}}{\mu^- - [f^+(p)+f^-(p)]}\\
&\qquad + \left(\lambda\hat{a}(p)\hat{\varphi}^{--}(p) - \frac{2c^-\lambda\lambda^-\hat{a}(p)\hat{a}^-(p)}{\mu^- - 2f^-(p)}\right) G_t^{(1)}(p)e^{2f^-(p)t},\\
\widehat{U}_t^{++}(p) &= \left(\frac{2\lambda c^-\lambda^+\hat{a}^+(p)}{\mu^- - \mu^+} + \frac{2c^-\lambda^2\hat{a}^2(p)}{\mu^- - f^+(p)-f^-(p)}\cdot\frac{\mu^-+2}{\mu^- - 2f^-(p)}\right) G_t^{(2)}(p)e^{2f^+(p)t}\\
&\qquad + 2c^+\lambda^+\hat{a}^+(p)\cdot\frac{\mu^- - \mu^+ + \lambda}{\mu^- - \mu^+}\cdot\frac{e^{\mu^+ t}-e^{2f^+(p)t}}{\mu^+ - 2f^+(p)}\\
&\qquad + \left(\lambda^2\hat{a}^2(p)\hat{\varphi}^{--}(p) - \frac{2c^-\lambda^-\hat{a}^-(p)\lambda^2\hat{a}^2(p)}{\mu^- - 2f^-(p)}\right)\left(G_t^{(1)}(p)\right)^2 e^{2f^-(p)t}\\
&\qquad + \left(2\lambda\hat{a}(p)\hat{\varphi}^{+-}(p) - \frac{2c^-\lambda^2\hat{a}^2(p)}{\mu^- - f^+(p)-f^-(p)}\cdot\frac{\mu^-+2}{\mu^- - 2f^-(p)}\right)\\
&\qquad\times G_t^{(1)}(p)e^{[f^+(p)+f^-(p)]t},
\end{aligned}
$$

where we denote objects which are not defined for $p\in\mathfrak{D}$ by

$$
\begin{aligned}
G_t^{(1)}(p) &= \frac{e^{[f^+(p)-f^-(p)]t}-1}{f^+(p)-f^-(p)}, && p\in\mathfrak{D}_1^c := \mathbb{R}^d\setminus\mathfrak{D}_1,\\
G_t^{(2)}(p) &= \frac{e^{[\mu^- - 2f^+(p)]t}-1}{\mu^- - 2f^+(p)}, && p\in\mathfrak{D}_2^c := \mathbb{R}^d\setminus\mathfrak{D}_2.
\end{aligned}
$$

Obviously $\operatorname{dist}(p,\mathfrak{D}_1)\to 0$ implies $g_1(p)\to 0$ and, hence, $G_t^{(1)}(p)\to t$. In the same manner $\operatorname{dist}(p,\mathfrak{D}_2)\to 0$ provides $G_t^{(2)}(p)\to t$. Therefore, for obtain the explicit expressions for $\widehat{U_t}^{+-}(p)$ and $\widehat{U_t}^{++}(p)$ on $\mathfrak{D}\setminus\{0\}$ it's enough to define

$$
G_t^{(1)}(p) := t,\quad p\in\mathfrak{D}_1;\qquad G_t^{(2)}(p) := t,\quad p\in\mathfrak{D}_2.
$$

Then we have for any $b\in L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$

$$
|b(p)|G_t^{(1)}(p)e^{f^-(p)t} \le \begin{cases} |b(p)|\dfrac{e^{f^+(p)t}-e^{f^-(p)t}}{f^+(p)-f^-(p)}, & p\in\mathfrak{D}_1^c\setminus\{0\},\\[2mm] |b(p)|\dfrac{e^{-1}}{-2f^-(p)}, & p\in\mathfrak{D}_1. \end{cases}
$$

And by result and proof of Lemma 3.2 this function has integrable majorante (which doesn't depend on t) on whole $\mathbb{R}^d$. Note also that $e^{f^\pm(p)t}\le 1$, hence, all terms with $G_t^{(1)}$ have this property.

Next,

$$|b(p)|G_t^{(2)}(p)e^{2f^+(p)t} \le \begin{cases} |b(p)|\dfrac{e^{\mu^- t} - e^{2f^+(p)t}}{\mu^- - 2f^+(p)}, & p \in \mathfrak{D}_2^c \setminus \{0\}, \\ |b(p)|\dfrac{e^{-1}}{-2f^+(p)}, & p \in \mathfrak{D}_2. \end{cases}$$

If $\mu^- < 0$ then may apply the previous considerations ($\mu^- \in C^-$). Otherwise, we may use that a function $u(t) = \dfrac{1 - e^{at}}{-a}$ ($a < 0$) is increasing and, hence, bounded by $u(+\infty) = -\dfrac{1}{a}$.

Note also that other numerators depended on t in the expressions for $\widehat{U}_t^{--}$, $\widehat{U}_t^{+-}$, $\widehat{U}_t^{++}$ may be estimated by 2 (recall that corresponding denominators are not equal to 0 if $p \neq 0$).

Therefore, for prove that functions $\widehat{U}_t^{--}$, $\widehat{U}_t^{+-}$, $\widehat{U}_t^{++}$ have integrable majorants it's enough to show that all terms which independent on t are integrable. Recall that $\hat{\varphi}^{--}$, $\hat{\varphi}^{+-}$ and $\hat{\varphi}^{++}$ are bounded, $\hat{a}$, $\hat{a}^+$ and $\hat{a}^-$ are bounded and integrable. Thus, we should prove integrability of two terms:

$$\frac{b(p)}{\mu^\pm - 2f^\pm(p)} \quad \text{and} \quad \frac{b(p)}{\mu^- - f^-(p) - f^+(p)} \cdot \frac{1}{\mu^- - 2f^-(p)}, \tag{3.24}$$

where $b \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$.

If $\mu^\pm = 0$ then we have

$$\frac{b(p)}{\mu^\pm - 2f^\pm(p)} = -\frac{1}{2}\frac{b(p)}{\hat{a}^\pm(p) - 1}$$

and due to Lemma 3.1 these functions are integrable. If $\mu^\pm < 0$ then using (3.14) we obtain

$$0 < -\mu^\pm \le \mu^\pm - 2f^\pm(p),$$

that implies

$$\frac{|b(p)|}{\mu^\pm - 2f^\pm(p)} \le \frac{|b(p)|}{-\mu^\pm}$$

which are also integrable functions.

Next, if $\mu^- = 0$ then $\mu^+ < 0$ and using (3.14)

$$(\mu^- - f^-(p) - f^+(p))(\mu^- - 2f^-(p)) \ge -2\mu^+(1 - \hat{a}^-(p)),$$

and we again may use Lemma 3.1. Finally, if $\mu^- < 0$ then $\mu^+ = 0$ and

$$\big((\mu^- - f^-(p)) + (-f^+(p))\big) \cdot \big(\mu^- - 2f^-(p)\big) \ge -\mu^-(1 - \hat{a}^+(p)),$$

and we also may use Lemma 3.1.

As a result, the functions $\widehat{U}_t^{--}$, $\widehat{U}_t^{+-}$, $\widehat{U}_t^{++}$ have integrable majorants and by dominated convergence theorem for obtain limits of U_t^{--}, U_t^{+-}, U_t^{++} as $t \to \infty$ we may calculate limits of the Fourier transforms and after apply the inverse Fourier transforms. Hence, taking $t \to \infty$ in the expressions for $\widehat{U}_t^{--}$, $\widehat{U}_t^{+-}$, $\widehat{U}_t^{++}$ we immediately obtain the statement of the Theorem 2.3 with functions Ω^{++}, Ξ^{--}, Ξ^{+-}, Ξ^{++} which are inverse Fourier transforms of (3.20)–(3.23).

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