Parabolic equations for measures on infinite-dimensional spaces

V.I. Bogachev, G. Da Prato, M. Röckner

Keywords: parabolic equation for measures, Kolmogorov equation, stochastic partial differential equation

In a series of papers [1]–[4] we considered parabolic equations for measures on $\mathbb{R}^d$. Our motivation was a study of the Kolmogorov equations for transition probabilities of diffusion processes. Here we are concerned with similar problems in infinite dimensions. Applications are given to stochastic partial differential equations such as stochastic equations of Navier–Stokes and reaction-diffusion types. Some ideas of our works [5] and [6] on elliptic equations will be employed. First we explain our problem in the finite-dimensional case. Let us consider a second order parabolic operator

$$Lu(x,t) := \frac{\partial u(x,t)}{\partial t} + \sum_{i,j\le d} a^{ij}(x,t)\partial_{x_i}\partial_{x_j}u(x,t) + \sum_{i=1}^{d} b^i(x,t)\partial_{x_i}u(x,t),$$

where the matrices $A(x,t) := (a^{ij}(x,t))_{i,j\le d}$ are symmetric nonnegative. We assume that the functions a^{ij} and b^i are Borel measurable. Suppose we are given a family $\mu = (\mu_t)_{t\in(0,1)}$ of Borel probability measures on $\mathbb{R}^d$. By the same symbol μ we denote the probability measure $\mu = \mu_t\,dt$ on $\mathbb{R}^d \times (0,1)$ defined by

$$\int_{\mathbb{R}^d\times(0,1)} f(x,t)\,\mu(dx\,dt) := \int_0^1 \int_{\mathbb{R}^d} f(x,t)\,\mu_t(dx)\,dt.$$

We shall say that μ satisfies the weak parabolic equation

$$L^*\mu = 0 \tag{1}$$

if the functions a^{ij} and b^i are integrable on every compact set in $\mathbb{R}^d \times (0,1)$ with respect to the measure μ and, for every $u \in C_0^\infty(\mathbb{R}^d \times (0,1))$, one has

$$\int_0^1 \int_{\mathbb{R}^d} Lu(x,t)\mu_t(dx)\,dt = 0. \tag{2}$$

We shall say that μ satisfies the initial condition $\mu_0 := \nu$ at $t = 0$ (or has an initial distribution ν) if ν is a Borel probability measure on $\mathbb{R}^d$ and

$$\lim_{t\to 0} \int_{\mathbb{R}^d} \zeta(x)\,\mu_t(dx) = \int_{\mathbb{R}^d} \zeta(x)\,\nu(dx) \tag{3}$$

for all $\zeta \in C_0^\infty(\mathbb{R}^d)$. In this case we write $\mu = (\mu_t)_{t\in[0,1)}$. In the above cited papers we obtained a number of results on existence and uniqueness of solutions to such equations along with certain a priori estimates. For example, a solution μ of the indicated form exists for any initial distribution ν if the coefficients a^{ij} and b^i satisfy certain rather mild local conditions and one has $LV \le C - CV$, where $C \ge 0$ is a constant and $V \ge 0$ is a C^2-function on $\mathbb{R}^d$ such that the sets $\{V \le R\}$ are compact and there is a sequence $c_n \to \infty$ for which the sets $V^{-1}(c_n)$ are C^1-surfaces. Our considerations in the inifinite-dimensional case will be based on some results from [3], [5] and [6]. We need the following corollary of Lemma 2.2 in [3].

Lemma 1. *Let $\mu = (\mu_t)_{t\in[0,1)}$, where each μ_t is a Borel probability measure on $\mathbb{R}^d$, satisfy* (1) *and* (3)*, where ν is a Borel probability measure on $\mathbb{R}^d$. Suppose that there exist a nonnegative function $V \in C^2(\mathbb{R}^d)$ and a number $M \ge 0$ such that $V \in L^1(\nu)$, $\lim_{|x|\to\infty} V(x) = +\infty$, and*

$$LV(x,t) \le MV(x) - \Theta(x,t) \quad \mu\text{-a.e.}$$

for some Borel function Θ with the property that the function $\min(\Theta, n)$ is μ-integrable for every n (the latter is fulfilled if $\Theta \geq C$ for some constant C). Then, for a.e. $t \in [0,1)$, one has

$$\int_{\mathbb{R}^d} V\, d\mu_t + \int_0^t \int_{\mathbb{R}^d} \Theta\, d\mu_s\, ds \leq (Me^M + 1) \int_{\mathbb{R}^d} V\, d\nu + Me^M \int_0^t \int_0^s \int_{\mathbb{R}^d} \max(-\Theta, 0)\, d\mu_r\, dr\, ds. \quad (4)$$

If $\Theta \geq 0$, then we have

$$\int_{\mathbb{R}^d} V\, d\mu_t + \int_0^t \int_{\mathbb{R}^d} \Theta\, d\mu_s\, ds \leq (Me^M + 1) \int_{\mathbb{R}^d} V\, d\nu. \quad (5)$$

Furthermore, if the functions $t \mapsto \int_{\mathbb{R}^d} \zeta(x)\, \mu_t(dx)$ for all $\zeta \in C_0^\infty(\mathbb{R}^d)$ are continuous on $[0,1)$, then (4) is true for all $t \in [0,1)$.

Proof. We have $LV(x,t) \leq MV(x) - \min(n, \Theta(x,t))$ for μ-almost all (x,t). By the cited lemma, estimate (4) is true for $\min(n,\Theta)$ in place of Θ. By Fatou's theorem this estimate holds for Θ. □

Let us consider the infinite-dimensional case. Let H be a separable Hilbert space with inner product $(\cdot,\cdot)$ and an orthonormal basis $\{e_n\}$. Set $x_i := (x, e_i)$ and $P_n x := \sum_{i=1}^n x_i e_i$, $x \in H$. The linear span of $e_1, \ldots, e_n$ is denoted by H_n.

Let $a^{ij}\colon H \times [0,1] \to \mathbb{R}^1$ and $b^i\colon H \times [0,1] \to \mathbb{R}^1$ be continuous functions. Suppose that the matrices $(a^{ij})_{i,j \leq n}$ are symmetric nonnegative for all n. Let b denote the vector (b^i) in $\mathbb{R}^\infty$ (we do not assume that $b \in l^2$). Set

$$Lu(x,t) := \frac{\partial u(x,t)}{\partial t} + \sum_{i,j=1}^n a^{ij}(x,t) \partial_{x_i} \partial_{x_j} u(x,t) + \sum_{i=1}^n b^i(x,t) \partial_{x_i} u(x,t)$$

for functions u that are smooth functions of the variables $x_1, \ldots, x_n, t$.

We say that a Borel measure $\mu = \mu_t\, dt$, where each μ_t is a Borel probability measure on H, satisfies the parabolic equation

$$L^*\mu = 0$$

on $H \times (0,1)$ if the functions a^{ij} and b^i are μ-integrable and, for every function u of the form

$$u(x,t) = w(x_1, \ldots, x_n, t), \quad w \in C_0^\infty(\mathbb{R}^n \times (0,1)),$$

one has

$$\int_{H \times (0,1)} Lu(x,t)\, \mu(dxdt) = 0.$$

We shall say that μ has the initial distribution μ_0 if μ_0 is a Borel probability measure on H and one has

$$\lim_{t \to 0} \int_H \zeta(x)\, \mu_t(dx) = \int_H \zeta(x)\, \mu_0(dx)$$

for every function ζ on H of the form $\zeta(x) = \zeta_0(x_1, \ldots, x_n)$, $\zeta_0 \in C_0^\infty(\mathbb{R}^n)$. If we identify H with l^2 and regard l^2 as a subset of $\mathbb{R}^\infty$, then all these objects make sense on $\mathbb{R}^\infty$ as well.

Our motivation to consider this equation is that, under broad assumptions, it is fulfilled for the transition probabilities of the diffusion process governed by the stochastic differential equation

$$d\xi_t = \sqrt{2A(\xi_t, t)}dW_t + b(\xi_t, t)dt$$

associated with $A = (a^{ij})$ and $b = (b^i)$. However, we do not assume that such a diffusion exists. Our results show that the Kolmogorov equation for transition probabilities can have a solution under much weaker assumptions that are usually imposed in order to construct a diffusion process.

Theorem 1. *Suppose that the functions b^i are weakly sequentially continuous and there exists a number $C>0$ such that for all $x\in H_n$, $n\ge 1$, we have*

$$\sum_{i=1}^{\infty} a^{ii}(x,t)\le C+C\|x\|^2, \quad (b(x,t),x):=\sum_{i=1}^{n} b^i(x,t)x_i\le C+C\|x\|^2.$$

Assume also that there exist constants C_i and k_i such that

$$|b^i(x,t)|\le C_i+C_i\|x\|^{k_i}.$$

Then, for every Borel probability measure μ_0 on H with finite moments of each order the equation $L^*\mu=0$ with the initial distribution μ_0 has a solution of the form $\mu=\mu_t\,dt$ with Borel probability measures μ_t on H.

Proof. For every fixed n let a_n^{ij} denote the restriction of a^{ij} to $H_n\times(0,1)$. Let $b_n=(b^1,\ldots,b^n)$ and $A_n:=(a_n^{ij})_{i,j\le n}$. Finally, let $\mu_{0,n}$ be the projection of μ_0 on H_n. We show that there exists Borel probability measures $\mu_{t,n}$ on H_n such that the measure $\mu_n:=\mu_{t,n}\,dt$ satisfies equation (1) with the coefficients A_n and b_n on $H_n\times(0,1)$ and the initial distributions $\mu_{0,n}$. To this end we consider a Lyapunov function $V_m(x)=\|x\|^{2m}$ on H_n, where $m\ge 2$. By using that

$$(A(x,t)z,z)\le \mathrm{Trace}\,A(x,t)\|z\|^2\le C\|z\|^2+C\|x\|^2\|z\|^2,$$

we obtain

$$\begin{aligned}LV_m(x)&=2m\|x\|^{2m-2}\sum_{i=1}^{n} a^{ii}(x,t)+4m(m-1)\|x\|^{2m-4}(A_n(x,t)x,x)\\&\quad+2m\|x\|^{2m-2}(b(x,t),x)\le 2mC\|x\|^{2m-2}+2mC\|x\|^{2m}+4m(m-1)[C\|x\|^{2m-2}\\&\qquad+C\|x\|^{2m}]+2mC\|x\|^{2m-2}+2mC\|x\|^{2m}\le C_m+C_m\|x\|^{2m}=C_m+C_mV_m(x).\end{aligned}$$

Since the function V_m is $\mu_{0,n}$-integrable, we can apply our result from [3] and obtain the desired probability measures $\mu_{t,n}$ on H_n such that the function

$$t\mapsto \int_{H_n}\zeta(x)\,\mu_{t,n}(dx)$$

is continuous on $[0,1)$ for every $\zeta\in C_0^\infty(H_n)$. Moreover, the following estimate holds for all $m\ge 2$ and almost all $t\in(0,1)$:

$$\int_{H_n} V_m(x)\,\mu_{t,n}(dx)\le M_m+M_m\int_{H_n} V_m(x)\,\mu_{0,n}(dx) \tag{6}$$

with some $M_m>0$ independent of n. Therefore, by Fatou's theorem and the above stated continuity of $t\mapsto\mu_{t,n}$ it follows that (6) holds for all $t\in[0,1)$. Suppose now that $\zeta\in C_0^\infty(\mathbb{R}^d)$. Let us identify H_n with $\mathbb{R}^n$. If $n\ge d$, then ζ regarded as a function on $\mathbb{R}^n$ belongs to the class $C_b^\infty(\mathbb{R}^n)$. Let $m=\max(k_1,\ldots,k_d)$. Then we have the estimate

$$|L\zeta(x,t)|=|\mathrm{trace}\,(A_n\zeta'')+(b_n,\nabla\zeta)|\le K+K\|x\|^2+K\|x\|^m,$$

where K is some number which depends on ζ (but is independent of n since ζ is a function of $x_1,\ldots,x_d$). Therefore, one has the identity

$$\int_{H_n}\zeta(x)\,\mu_{t,n}(dx)=\int_0^t\int_{H_n} L\zeta(x,s)\,\mu_{s,n}(dx)\,ds+\int_{H_n}\zeta(x)\,\mu_{0,n}(dx).$$

According to [3], this identity holds for all $\zeta\in C_0^\infty(\mathbb{R}^d)$, hence in our situation it remains valid also for all $\zeta\in C_b^\infty(\mathbb{R}^d)$. Letting

$$\varphi_n(t):=\int_{H_n}\zeta(x)\,\mu_{t,n}(dx),$$

we see that the function φ_n is Lipschitzian and (7) yields the estimate

$$|\varphi_n'(t)| \le \int_{H_n} |L\zeta(x,t)|\,\mu_{t,n}(dx) \\ \le K + K\int_{H_n} \|x\|^2\,\mu_{t,n}(dx) + K\int_{H_n} \|x\|^m\,\mu_{t,n}(dx) \le \Lambda \quad (7)$$

with some Λ independent of n. It follows from (6) that, for every fixed $t \in (0,1)$, the sequence of measures $\mu_{t,n}$ is uniformly tight on H with the weak topology. Hence we can find a subsequence, denoted for simplicity by the same indices n, such that it converges weakly on H with the weak topology for every rational $t \in (0,1)$. However, (6) and (7) yield that this convergence holds for every t. The family of measures μ_t obtained in this way is the desired solution. Indeed, by using (6) and (7) it is readily verified that $\mu = \mu_t\,dt$ satisfies our parabolic equation with the initial distribution μ_0. □

The same reasoning along with a priori estimates from [2], [3] yields the following result.

Corollary 1. *Suppose that in Theorem* 1 *we have the following stronger estimates*:

$$\sum_{i=1}^{\infty} a^{ij}(x,t) \le C \quad \textit{and} \quad (b(x,t),x) \le M - M\|x\|^2 \quad \textit{whenever } x \in \textstyle\bigcup_n H_n,$$

where $C, M > 0$. Let $\kappa \in (0, 2M/C)$. Then, for every Borel probability measure μ_0 on H such that $\exp(\kappa\|x\|^2) \in L^1(\mu_0)$ the equation $L^*\mu = 0$ with the initial distribution μ_0 has a solution of the form $\mu = \mu_t\,dt$ with Borel probability measures μ_t on H such that

$$\sup_{t\in[0,1)} \int_H \exp(\kappa\|x\|^2)\,\mu_t(dx) < \infty.$$

Under the assumptions of this corollary we have $\exp(\varepsilon_i|b^i|) \in L^1(\mu)$ for any $\varepsilon_i \in (0, \kappa/k_i)$. Now the next result follows from [7].

Corollary 2. *Suppose that in the previous corollary the functions a^{ij} are constant and the matrices $(a^{ij})_{i,j\le n}$ are invertible. Assume additionally that $k_i = 1$ for all i. Then the projections on $H_n \times (0,1)$ of the measure μ constructed above possess strictly positive continuous densities.*

The assumption of the sequential weak continuity is rather stringent. However, there are important examples where it is satisfied (see the elliptic case in [6]). In order to drop this assumption one can require a stronger bound in terms of a Lyapunov function.

Let $\Theta\colon X \to [0, +\infty]$ be a Borel function such that the sets $\{\Theta \le R\}$ are compact and Θ is finite on each H_n. For example, one can take numbers $\alpha_i > 0$ tending to $+\infty$ and set $\Theta(x) = \sum_{i=1}^{\infty} \alpha_i^2 x_i^2$.

Theorem 2. *Suppose that the functions a^{ij} and b^i are continuous on the sets $\{\Theta \le R\}$ and there exists a number $C > 0$ such that for all $x \in H_n$, $n \ge 1$, one has*

$$\sum_{i=1}^{\infty} \alpha_i^2 a^{ii}(x,t) \le C + C\|x\|^2, (b(x,t),x) := \sum_{i=1}^{n} b^i(x,t)x_i \le C + C\|x\|^2 - \Theta(x).$$

Assume also that there exist constants C_i and k_i such that

$$|b^i(x,t)| \le C_i + C_i\|x\|^{k_i}(1 + \delta(\Theta(x))\Theta(x)),$$

where δ is a nonnegative Borel function on $[0, +\infty)$ with $\lim\limits_{s\to 0} \delta(s) = 0$. Then, for every Borel probability measure μ_0 on H such that all functions $\Theta(x)\|x\|^k$ are μ_0-integrable, the equation $L^*\mu = 0$ with the initial distribution μ_0 has a solution of the form $\mu = \mu_t\,dt$ with Borel probability measures μ_t on H such that $\Theta(x)\|x\|^k \in L^1(\mu)$.

The proof is similar to the proof of Theorem 1, but in place of the uniform tightness with respect to the weak topology we establish the uniform tightness of finite-dimensional solutions μ_n in the norm topology by using estimate (6) from Lemma 1 and compactness of the sets $\{\Theta \le R\}$. The justification of (1) employs (6) and the method of proof of Theorem 5.1 in [6]. Details will be presented in a separate paper, where we also prove parabolic versions of Theorem 5.1 and Theorem 5.2 in [6], which involve more general Lyapunov functions. Let us consider examples related to stochastic partial differential equations of reaction-diffusion type and Navier–Stokes type.

Example 1. Let $\Omega = \{\omega \in \mathbb{R}^d\colon\ |\omega| \le 1\}$, let $H = L^2(\Omega)$, and let $\{e_i\}$ be the eigenbasis of the Laplacian in H with zero boundary condition, $\Delta e_i = -\alpha_i^2 e_i$. We consider the mapping

$$b(x) = \Delta x + \Phi(x),$$

where Φ is a continuous function on the real line with $|\Phi(s)| \le C + C|s|^\alpha$, where $\alpha = 2 + 4/d$. Although b is not defined on all of H we have the following well-defined components on $W^{2,1}(\Omega)$ (and even on $L^\alpha(\Omega)$):

$$b^i(x) = -\alpha_i^2 x_i + (\Phi(x), e_i)_{L^2}.$$

By using the bound on Φ and the multiplicative Sobolev inequality one estimates $\|\Phi(x)\|_{L^1}$ via $C_1 + C_1\|x\|_{L^2}^2\|x\|_{W^{2,1}}^2$. In addition, the Sobolev embedding $W^{2,1}(\Omega) \subset L^q(\Omega)$ with $q = 2d/(d-2)$ in the case $d > 2$ and any $q > 1$ in the case $d = 2$ yields that the functions b^i are continuous on all balls in the Sobolev space $W^{2,1}(\Omega)$ with respect to the topology from $L^2(\Omega)$. Therefore, the previous theorem applies if we take $a^{ij} = 0$ whenever $i \ne j$ and constant $a^{ii} > 0$ such that $\sum_{i=1}^\infty \alpha_i^2 a^{ii} < \infty$. For Θ we take $\Theta(x) = \sum_{i=1}^\infty \alpha_i^2 x_i^2$. Clearly, one has $c_0\|x\|_{W^{2,1}}^2 \le \Theta(x) \le \|x\|_{W^{2,1}}^2$ with some $c_0 > 0$. Our restriction on Φ shows that only in the case $d \le 4$ one can take for Φ a polynomial of degree 3; in higher dimensions it allows only slightly more than quadratic growth.

Example 2. Now we turn to a more complicated equation with the drift b formally given by

$$b(x,t)(\omega) = \Delta_\omega[\Psi(x(\omega),t)] + \Phi(x(\omega),t),$$

where Ψ and Φ are real functions on $\mathbb{R}^1 \times [0,1]$. Let

$$b^i(x,t) := \int_\Omega \Big[\Psi(x(\omega),t)\Delta e_i(\omega)\, d\omega + \Phi(x(\omega),t)e_i(\omega)\Big]\, d\omega, \quad x \in L^r(\Omega).$$

The corresponding operator L is given by

$$Lu = \partial_t u + \sum_{i=1}^\infty q_i \partial_{e_i}^2 u + \sum_{i=1}^\infty b^i \partial_{e_i} u,$$

where $q_i > 0$ and $\sum_{i=1}^\infty q_i < \infty$. We assume that Ψ and Φ are continuous functions and Ψ has a continuous partial derivative $\partial_s \Psi(s,t)$ such that

$$\kappa_0|s|^{r-1} \le \partial_s\Psi(s,t) \le C_1 + \kappa_1|s|^{r-1},\ |\Phi(s,t)| \le C_2 + \kappa_2|s|^r,$$

where $\kappa_0, \kappa_1, \kappa_2, C_1, C_2 \in (0,+\infty)$ are some constants and $r > 1$. Set $\zeta_r(s) := |s|^r \operatorname{sgn} s$, $s \in \mathbb{R}^1$. If r is an odd number, then $\zeta_r(s) = s^r$. Let us employ the function

$$\Theta(x) := \int_\Omega |\nabla(\zeta_{(r+1)/2} \circ x)(\omega)|^2\, d\omega,$$

which has compact sets $\{\Theta \le R\}$ in the topology of $L^2(\Omega)$ (this fact is verified in [5]). Similarly to [5] one obtains the estimates

$$|b^i(x)| \le C + C\Theta(x),\ LV(x) \le C_1 - C_2\Theta(x)$$

with some positive numbers C, C_1, C_2, where $V(x) = (x,x)$.

Example 3. The stochastic equation of Navier–Stokes type is considered in the space X_0 of $\mathbb{R}^d$-valued mappings $\xi = (\xi^1, \ldots, \xi^d)$ such that $\xi^j \in W_0^{2,1}(\Omega)$ and $\operatorname{div} \xi = 0$, where $\Omega \subset \mathbb{R}^d$ is a bounded domain with a smooth boundary. The space X_0 is equipped with its natural Hilbert norm $\|\xi\|_0$ defined by $\|\xi\|_0^2 := \sum_{j=1}^d \|\nabla \xi^j\|_{L^2}^2$. The equation is formally written as

$$d\xi(x,t) = \sqrt{2}dW(x,t) + \Big[\Delta_x \xi(x,t) - \sum_{j=1}^d \xi^j(x,t)\partial_{x_j}\xi(x,t) + F(x,\xi(x,t),t)\Big]dt,$$

where W is a suitable Wiener process in X_0 and $F\colon\ \Omega\times\mathbb{R}^d\times(0,1) \to \mathbb{R}^d$ is a bounded continuous mapping. Since the Laplacian Δ is not defined on all of X_0, this equation requires some interpretation. Our approach suggests the following procedure. Let $\{\eta_n\}$ be an orthonormal basis in the closure of X_0 in $L^2(\Omega, \mathbb{R}^d)$ formed by eigenfunctions of Δ such that $\eta_n \in X_0$. We introduce functions

$$b^n(\xi,t) := (\xi, \Delta\eta_n)_2 - \sum_{j=1}^d (\partial_{x_j}\xi, \xi^j \eta_n)_2 + (F(\,\cdot\,, \xi(\,\cdot\,), t), \eta_n)_2,$$

where $(a,b)_2 := (a,b)_{L^2(\Omega)}$. These functions are defined on all of X_0. It is readily verified that they are continuous on balls from X_0 with respect to the topology of $L^2(\Omega, \mathbb{R}^d)$ (see [7], p. 39). Choosing a suitable Wiener process, we arrive at the operator

$$Lu(\xi,t) = \partial_t u(\xi,t) + \sum_{n=1}^\infty \alpha_n \partial_{\eta_n}^2 u(\xi,t) + \sum_{n=1}^\infty b^n(\xi,t)\partial_{\eta_n} u(\xi,t).$$

Since for every ξ from the linear span of $\{\eta_n\}$ one has

$$\sum_{n=1}^\infty \sum_{j=1}^d (\xi,\eta_n)_2 (\partial_{x_j}\xi, \xi^j\eta_n)_2 = \sum_{j=1}^d (\xi, \xi^j \partial_{x_j}\xi)_2 = -\frac{1}{2}\int_\Omega |\xi|^2 \operatorname{div}\xi \, dx = 0$$

and $(\Delta\xi, \xi)_2 = -\|\xi\|_0^2$, we arrive at the estimate

$$\sum_{n=1}^N (\xi, \eta_n)_2 b^n(\xi,t) \le C_1 - C_1\|\xi\|_0^2$$

for all ξ in the linear span of $\eta_1, \ldots, \eta_N$, where C_1 is a constant independent of N. Clearly, we have also the estimate $|b^n(\xi,t)| \le C_2 + C_2\|\xi\|_0^2$. Therefore, there is a probability measure μ on $X_0 \times [0,1)$ satisfying the equation $L^*\mu = 0$ with any initial distribution μ_0 for which $\|\xi\|_0^2\|\xi\|_2^k \in L^1(\mu_0)$ for all k.

Similar results are valid in the case of non-constant a^{ij}.

Although the idea to use a priori estimates based on Lyapunov functions is standard in this area (see, e.g., [8]–[10] for the case of stochastic equations), the described method of constructing solutions to infinite-dimensional Kolmogorov equations for measures leads to considerably weaker assumptions on the coefficients of the considered equations.

This work has been supported by the RFBR projects 07-01-00536, 06-01-39003 GFEN, the DFG Grant 436 RUS 113/343/0(R), the ARC Discovery Grant DP0663153, the programme SFB 701 at the University of Bielefeld, and the programme "Equazioni di Kolmogorov" of the Italian "Ministero della Ricerca Scientifica e Tecnologica".

References

[1] Bogachev V.I., Da Prato G., Röckner M. On weak parabolic equations for probability measures. Dokl. Math. 2002. V. 66, n 2. P. 192–196.

[2] Bogachev V.I., Da Prato G., Röckner M. Existence of solutions to weak parabolic equations for measures. Proc. London Math. Soc. 2004. V. 88, n 3. P. 753–774.

[3] Bogachev V.I., Da Prato D., Röckner M. On parabolic equations for measures. Comm. Partial Diff. Equat. 2008.

[4] Bogachev V.I., Da Prato D., Röckner M., Stannat W. Uniqueness of solutions to weak parabolic equations for measures. Bull. London Math. Soc. 2007. V. 39. P. 631–640.

[5] Bogachev V.I., Da Prato G., Röckner M. Invariant measures of generalized stochastic porous medium equations. Dokl. Math. 2004. V. 69, n 3. P. 321–325.

[6] Bogachev V.I., Röckner M. Elliptic equations for measures on infinite dimensional spaces and applications. Probab. Theory Relat. Fields. 2001. V. 120. P. 445–496.

[7] Bogachev V.I., Röckner M., Shaposhnikov S.V. Positive densities of transition probabilities of diffusion processes. Theory Probab. Appl. 2008. V. 53.

[8] Vishik M.I., Fursikov A.V. Mathematical problems of statistical hydromechanics. Kluwer, Dordrecht, 1980.

[9] Da Prato G., Zabczyk J. Stochastic equations in infinite dimensions. Cambridge Univ. Press, 1992.

[10] Da Prato G., Zabczyk J. Ergodicity for infinite-dimensional systems. Cambridge, Cambridge University Press, 1996.