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POSITIVE DENSITIES OF TRANSITION PROBABILITIES

OF DIFFUSION PROCESSES¹⁾

For diffusion processes in \mathbb{R}^d with locally unbounded drift coefficients we obtain a sufficient condition for the strict positivity of transition probabilities. To this end, we consider parabolic equations of the form $\mathcal{L}^*\mu = 0$ with respect to measures on $\mathbb{R}^d \times (0, 1)$ with the operator

$$\mathcal{L}u := \partial_t u + \partial_{x_i}(a^{ij}\partial_{x_j}u) + b^i\partial_{x_i}u.$$

It is shown that if the diffusion coefficient $A = (a^{ij})$ is sufficiently regular and the drift coefficient $b = (b^i)$ satisfies the condition $\exp(\kappa|b|^2) \in L^1_{loc}(\mu)$, where the measure μ is nonnegative, then μ has a continuous density $\varrho(x, t)$ which is strictly positive for $t > \tau$ provided that it is not identically zero for $t \leq \tau$. Applications are obtained to finite-dimensional projections of stationary distributions and transition probabilities of infinite-dimensional diffusions.

Keywords: density of transition probability, stationary distribution, parabolic equation, infinite-dimensional diffusion.

1. INTRODUCTION

This work is devoted to obtaining sufficient conditions for the strict positivity of densities of transition probabilities of finite-dimensional diffusion processes with singular drift coefficients and to the existence of strictly positive continuous densities of finite-dimensional projections of stationary distributions and transition probabilities of infinite-dimensional diffusions. To this end, we consider equations of the form

$$\mathcal{L}^*\mu = 0 \tag{1.1}$$

with respect to Borel measures μ on \mathbb{R}^d or on $\mathbb{R}^d \times (0, 1)$. Here \mathcal{L} is an elliptic or parabolic second order operator of the form

$$\mathcal{L}u(x) := \partial_{x_i}(a^{ij}\partial_{x_j}u(x)) + b^i(x)\partial_{x_i}u(x) \tag{1.2}$$

or

$$\mathcal{L}u(x, t) := \partial_t u(x, t) + \partial_{x_i}(a^{ij}(x, t)\partial_{x_j}u(x, t)) + b^i(x, t)\partial_{x_i}u(x, t), \tag{1.3}$$

where the summation over repeated indices is taken, and the interpretation of our equation is as follows.

We shall say that a Borel measure μ on \mathbb{R}^d satisfies the weak elliptic equation (1.1) if the functions a^{ij} belong to the local Sobolev class $W^{1,1}_{loc}(\mathbb{R}^d)$, the functions a^{ij} , $\partial_{x_i}a^{ij}$, and b^i are Borel measurable and locally integrable with respect to $|\mu|$, and, for every

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$u \in C_0^\infty(\mathbb{R}^d)$, we have the equality

$$\int_{\mathbb{R}^d} [a^{ij} \partial_{x_i} \partial_{x_j} u + \partial_{x_i} a^{ij} \partial_{x_j} u + b^i \partial_{x_i} u] d\mu = 0. \quad (1.4)$$

Similarly, a Borel measure μ on $\mathbb{R}^d \times (0, 1)$ satisfies the weak parabolic equation (1.1) if, for every compact interval $J \subset (0, 1)$ and every ball $U \subset \mathbb{R}^d$, the functions a^{ij} belong to the class $\mathbb{H}^{1,1}(U, J)$ defined below, the functions a^{ij} , $\partial_{x_i} a^{ij}$ and b^i are Borel measurable and locally integrable with respect to $|\mu|$, and, for every function $u \in C_0^\infty(\mathbb{R}^d \times (0, 1))$, we have the equality

$$\int_{\mathbb{R}^d \times (0, 1)} [\partial_t u + a^{ij} \partial_{x_i} \partial_{x_j} u + \partial_{x_i} a^{ij} \partial_{x_j} u + b^i \partial_{x_i} u] d\mu = 0. \quad (1.5)$$

Similarly one defines equation (1.1) for the operator \mathcal{L} with the extra term $\theta \cdot u$, where the function θ is locally μ -integrable.

Throughout we assume that $a^{ij} = a^{ji}$ and that the matrices $A(x, t) = (a^{ij}(x, t))$ are positive. The vector field $b = (b^1, \dots, b^d)$ is called the drift coefficient and A is called the diffusion coefficient.

Under very broad assumptions, such equations are satisfied for stationary distributions and transition probabilities of diffusion processes. If a diffusion process ξ_t in \mathbb{R}^d is defined by the stochastic differential equation in the Ito form

$$d\xi_t = \sqrt{A(\xi_t, t)} dW_t + \frac{1}{2} b(t, \xi_t) dt,$$

where W_t is a Wiener process in \mathbb{R}^d , then the transition probabilities $P(t, B) := P(\xi_t \in B)$ of the process ξ_t (or $P(s, z; t, B) := P(\xi_t \in B | \xi_s = z)$ with fixed $s \in \mathbb{R}^1$ $z \in \mathbb{R}^d$) generate the measure $\mu = P(t, dx) dt$ on $\mathbb{R}^d \times (0, 1)$, which satisfies the parabolic equation $L^* \mu = 0$ in the sense of the identity analogous to (1.5) but with the operator L in the *non-divergence form*

$$Lu(x, t) := \partial_t u(x, t) + a^{ij}(x, t) \partial_{x_i} \partial_{x_j} u(x, t) + b^i(x, t) \partial_{x_i} u(x, t), \quad (1.6)$$

and in the case where the coefficients are independent of time and there is a stationary distribution μ , this distribution satisfies the elliptic equation $L^* \mu = 0$ in the sense of the identity analogous to (1.4) also with the operator L in the non-divergence form

$$Lu(x) := a^{ij} \partial_{x_i} \partial_{x_j} u(x) + b^i(x) \partial_{x_i} u(x). \quad (1.7)$$

Equations with divergence-form operators of type (1.2) and (1.3) arise for stochastic equations in the Stratonovich form. However, under our assumptions on a^{ij} , the non-divergence form operator L can be obviously written as a divergence-form operator of type (1.2) or (1.3) with the new drift with the components $b^j - \partial_{x_i} a^{ij}$. For this reason, all results of this work are valid also for non-divergence form operators (1.6) and (1.7) and, thereby, are applicable to stochastic equations in the Ito form, provided that the corresponding conditions on the drift are fulfilled for the drift changed as indicated above. We consider divergence-form operators for the only reason that this leads to some technical convenience from the point of view of parabolic equations.

It is well-known that in the case of sufficiently regular coefficients A and b with non-degenerate A the transition probabilities and stationary distributions possess continuous strictly positive densities; the strict positivity is usually deduced from Harnack's inequality. It has been shown in [1] that if the coefficients b^i are locally Lebesgue integrable in a sufficiently high power (greater than d in the elliptic case and greater than $d + 2$ in the parabolic case) and the diffusion coefficient A is nondegenerate and sufficiently regular, then μ admits a continuous strictly positive density. However, there are important applications where the integrability of b is given only with respect to μ ; a typical application concerned with finite-dimensional projections of infinite-dimensional diffusions is

discussed below. Another typical example is a measure μ with a smooth density ϱ , which satisfies elliptic equation (1.1) with

$$A(x) = \text{Id} \quad \text{and} \quad b(x) = \frac{\nabla \varrho(x)}{\varrho(x)},$$

where $b(x) := 0$ if $\varrho(x) = 0$. Indeed, the corresponding integral relation is

$$\int_{\mathbb{R}^d} [\Delta u(x) \varrho(x) + (\nabla \varrho(x), \nabla u(x))] dx = 0,$$

which is fulfilled for all $u \in C_0^\infty(\mathbb{R}^d)$ due to the integration by parts formula. If ϱ vanishes (at some points or even on certain sets with nonempty interior), then $b = \nabla \varrho / \varrho$ may be very singular from the Lebesgue measure point of view, but is locally integrable with respect to μ . In this case the continuous density of μ may vanish even under very high integrability of b with respect to μ . For example, the measure μ on the real line with density ϱ such that $\varrho(x) = \exp(-x^{-2})$ if $x > 0$ and $\varrho(x) = 0$ if $x \leq 0$ satisfies equation (1.1) with $A = 1$ and $b(x) = 2x^{-3}$. Here $b \in L^p(\mu)$ for all $p \in [1, +\infty)$. However, in the elliptic case, it has been shown in [2] that the μ -integrability of $\exp(\kappa|b|)$ for some $\kappa > 0$ yields the strict positivity of the density of μ . Here we prove a parabolic analog of this result, which states that under reasonable assumptions on the coefficient A , the local μ -integrability of $\exp(\kappa|b|^2)$ with some $\kappa > 0$ ensures the existence of a continuous density ϱ of μ such that $\varrho(x, t) > 0$ for all $t > \tau$ if $\varrho(x, t)$ is not zero identically for $t < \tau$. In application to the densities of transition probabilities, this yields the strict positivity everywhere for all $t > 0$. Unlike the elliptic case, one cannot replace $|b|^2$ by $|b|$, which is demonstrated by the following simple example. Set $d = 1$,

$$b(x, t) = -\varepsilon(t - a)^{-2} e^{\varepsilon x} + 2\varepsilon^{-1}(t - a)^{-1},$$

where $\varepsilon, a \in (0, 1)$, and $\mathcal{L}f(x, t) = \partial_t f(x, t) + \partial_x^2 f(x, t) + b(x, t) \partial_x f(x, t)$. Then the measure

$$\mu = \exp(-(t - a)^{-2} e^{\varepsilon x}) dx dt$$

satisfies the equation $\mathcal{L}^* \mu = 0$ on $\mathbb{R}^1 \times (0, 1)$, but at $t = a$ its continuous density vanishes. It is easily seen that $\exp(\kappa|b|)$ is integrable with respect to μ if $\varepsilon < \kappa^{-1}$. As an application, in the last section we obtain sufficient conditions on the coefficients of an infinite dimensional diffusion ξ_t with a drift $b(x, t)$ that guarantee that the finite dimensional projections of stationary distributions and transition probabilities possess strictly positive densities. For example, if the diffusion coefficient of the process ξ_t in \mathbb{R}^∞ is constant and nondegenerate and $b = (b^i)$, then for the existence of strictly positive continuous densities of all finite-dimensional projections of the transition probabilities $P_t(x, \cdot)$ it suffices that the functions $\exp(\kappa|b^i|^2)$ be integrable with respect to the measure $P_t(x, \cdot) dt$. For the diffusion in a Banach space with the diffusion coefficient Id and a drift b it suffices to have the estimate $\|b(x, t)\| \leq C + C\|x\|$.

We note that in papers [3], [4], [5], and [6] upper and lower estimates of densities of solutions can be found. The problems of existence of solutions to the equations of the indicated type are considered in [7] and [8], and the uniqueness problems are addressed in papers [9], [10], and [11].

For an arbitrary domain $\Omega \subset \mathbb{R}^d$ let $W^{p,1}(\Omega)$ denote the Sobolev space of functions belonging to $L^p(\Omega)$ along with their generalized first order partial derivatives. This space is equipped with the standard norm

$$\|f\|_{W^{p,1}(\Omega)} := \|f\|_{L^p(\Omega)} + \|\nabla f\|_{L^p(\Omega)},$$

where $\|\cdot\|_{L^p(\Omega)}$ denotes the $L^p(\Omega)$ -norm of scalar or vector functions. The closure of $C_0^\infty(\Omega)$ in $W^{p,1}(\Omega)$ is denoted by the symbol $W_0^{p,1}(\Omega)$. Let $W_{loc}^{p,1}(\Omega)$ and $L_{loc}^p(\Omega)$ denote the spaces of functions belonging to $W^{p,1}(B)$ and $L^p(B)$, respectively, for every ball B with the closure in Ω .

If μ is a Borel measure (possibly, signed) on a domain Ω in \mathbb{R}^d or in $\mathbb{R}^d \times (0, 1)$ and $|\mu|$ is its variation, then $L^p(\mu)$ denotes the space $L^p(|\mu|)$ and $L_{loc}^p(\mu)$ denotes the set of all functions f on Ω such that $\zeta f \in L^p(\mu)$ for all $\zeta \in C_0^\infty(\Omega)$.

Let $J \subset \mathbb{R}^1$ be an interval and let U be an open set in \mathbb{R}^d . Let $\mathbb{H}^{p,1}(U, J)$ denote the space of all measurable functions u on $U \times J$ such that $u(\cdot, t) \in W^{p,1}(U)$ with finite norm

$$\|u\|_{\mathbb{H}^{p,1}(U, J)} = \left(\int_J \|u(\cdot, t)\|_{W^{p,1}(U)}^p dt \right)^{1/p}.$$

The space $\mathbb{H}_0^{p,1}(U, J)$ is defined similarly by replacing $W^{p,1}(U)$ with $W_0^{p,1}(U)$. The dual of the space $\mathbb{H}_0^{p,1}(U, J)$ is denoted by $\mathbb{H}^{p',-1}(U, J)$, where $p' := p(p-1)^{-1}$. In relation to parabolic equations, it is useful to introduce also the following spaces. Let $\mathcal{H}^{p,1}(U, J)$ be the space of all functions $u \in \mathbb{H}^{p,1}(U, J)$ with $\partial_t u \in \mathbb{H}^{p,-1}(U, J)$ and finite norm

$$\|u\|_{\mathcal{H}^{p,1}(U, J)} = \|\partial_t u\|_{\mathbb{H}^{p,-1}(U, J)} + \|u\|_{\mathbb{H}^{p,1}(U, J)}.$$

For functions u of (x, t) we write

$$\nabla u := \nabla_x u := (\partial_{x_1} u, \dots, \partial_{x_d} u).$$

Let $|\Omega|$ denote the Lebesgue volume of a set Ω . The norm of a vector v in \mathbb{R}^d is denoted by $|v|$.

An operator-valued mapping A on an open set is called locally uniformly nondegenerate if the mapping A^{-1} is locally bounded.

2. EXISTENCE OF DENSITIES

In this section, we give modifications of some results from [1] on the existence of continuous densities. The following result, which is Theorem 2.8 in [1], was obtained for equations with the differential operator in the non-divergence form and with $\beta_2 = 0$. For the reader's convenience, we indicate the necessary changes in the proof.

Let B_R be an open ball of radius R in \mathbb{R}^d .

Theorem 2.1. *Let $p > d$, $r \in (p', \infty)$, and let μ be a measure on B_R with a density $\varrho \in L_{loc}^r(B_R)$. Let $a^{ij} \in W_{loc}^{p,1}(B_R)$, $\beta_1 \in L_{loc}^p(B_R)$, and $\beta_2 \in L_{loc}^p(\mu)$, where A is locally uniformly nondegenerate on B_R . Assume that, for every $\varphi \in C_0^\infty(B_R)$, we have*

$$\left| \int_{B_R} a^{ij} \partial_{x_i} \partial_{x_j} \varphi d\mu \right| \leq \int_{B_R} (|\varphi| + |\nabla_x \varphi|) (|\beta_1 \varrho| + |\beta_2 \varrho|) dx.$$

Then $\varrho \in W_{loc}^{p,1}(B_R)$, hence ϱ has a locally Hölder continuous version.

Proof. Since $r > p'$ we have $pr > p + r$. Then

$$q = pr/(pr - p - r) > 1 \quad \text{and} \quad q' = pr/(p + r) > 1.$$

According to Hölder's inequality we have $\beta_1 \varrho \in L_{loc}^{q'}(B_R)$. The same is true for $\beta_2 \varrho$ since

$$|\beta_2|^{q'} |\varrho|^{q'} = |\beta_2|^{pr/(p+r)} |\varrho|^{r/(p+r)} |\mu|^{(pr-r)/(p+r)},$$

where $|\beta_2|^{pr/(p+r)} |\varrho|^{r/(p+r)} \in L_{loc}^s(B_R)$ and $|\varrho|^{(pr-r)/(p+r)} \in L_{loc}^{s'}(B_R)$ with $s = (p+r)/r$. Hence $|\beta_1 \varrho| + |\beta_2 \varrho| \in L_{loc}^{q'}(B_R)$. Now the same reasoning as in Theorem 2.8 in [1] completes the proof. \square

Corollary 2.1. *Let μ be a locally finite Borel measure on B_R . Let A be locally nondegenerate on B_R with $a^{ij} \in W_{loc}^{p,1}(B_R)$, where $p > d$ and continuous versions of a^{ij} are chosen, $\partial_{x_i} a^{ij} \in L_{loc}^p(\mu)$, and let $b^i, c \in L_{loc}^p(\mu)$. Suppose that*

$$\int_{B_R} \left[(a^{ij} \partial_{x_i} \partial_{x_j} \varphi + \partial_{x_i} a^{ij} \partial_{x_j} \varphi + b^i \partial_{x_i} \varphi + c \varphi) \right] d\mu = 0, \quad \forall \varphi \in C_0^\infty(B_R).$$

Then, μ has a density in $W_{loc}^{p,1}(B_R)$ that is locally Hölder continuous.

Proof. It suffices to take $\beta_1 = |\partial_{x_i} a^{ij}|$ and $\beta_2 = |b| + |c|$ and apply Theorem 2.1. \square

We extend these results to the parabolic case. Our strategy is essentially the same as in the elliptic case, but due to some additional technicalities we include complete proofs.

Let $\Omega_T = \Omega \times (0, T)$, $T > 0$, and let $A(\cdot, \cdot) = (a^{ij}(\cdot, \cdot))_{i,j=1}^d$ be a Borel mapping on Ω_T with values in the space of nonnegative symmetric operators in \mathbb{R}^d . Let $\omega_0 \in \Omega$ be a fixed point and let $B_R \subset \Omega$ be the ball of radius R centered at ω_0 . Set $B_{R,T} := B_R \times (0, T)$.

We shall assume that the functions $a^{ij}(x, t)$ are continuous in x uniformly in t . Note that a^{ij} has a modification with this property provided that

$$\sup_t \|a^{ij}(\cdot, t)\|_{W^{p,1}(B_R)} < \infty, \quad \text{where } p > d. \quad (2.1)$$

The following result is Theorem 3.7 in [1].

Theorem 2.2. *Let $d \geq 2$, $p > d$, $q \in [p', +\infty)$. Let A be uniformly bounded and uniformly nondegenerate and let (2.1) hold. Suppose that μ is a finite measure on $B_{R,T}$ such that, for some $N > 0$, one has*

$$\left| \int \left[\partial_t \varphi + a^{ij} \partial_{x_i} \partial_{x_j} \varphi \right] d\mu \right| \leq N \|\nabla_x \varphi\|_{L^q(B_{R,T})}, \quad \forall \varphi \in C_0^\infty(B_{R,T}). \quad (2.2)$$

Then $\mu \in \mathbb{H}^{q',1}(B_{R'}, [t_0, t_1])$ and $\mu \in \mathcal{H}^{q',1}(B_{R'}, [t_0, t_1])$ if $R' < R$ and $[t_0, t_1] \subset (0, T)$.

Now we prove a modification of Theorem 3.8 in [1], which differs in that the integrability condition on the coefficient β_2 (which was absent in the cited theorem) is expressed in terms of the measure μ and not in terms of Lebesgue measure.

Theorem 2.3. *Let A be locally bounded and locally uniformly nondegenerate on $B_{R,T}$ and let (2.1) be fulfilled, where we assume now that $p > d + 2$. Let μ be a finite Borel measure on $B_{R,T}$ with a density $\varrho \in L^r(B_{R,T})$ with some $r > p'$. Let $\beta_1 \in L_{loc}^p(B_{R,T})$ and $\beta_2 \in L_{loc}^p(\mu)$. Suppose that for all $\varphi \in C_0^\infty(B_{R,T})$ one has*

$$\left| \int_{B_{R,T}} \left[\partial_t \varphi + a^{ij} \partial_{x_i} \partial_{x_j} \varphi \right] d\mu \right| \leq \int_{B_{R,T}} (|\varphi| + |\nabla_x \varphi|) (|\beta_1 \varrho| + |\beta_2 \varrho|) dx dt.$$

Then ϱ has a version that is locally Hölder continuous on $B_R \times (0, T)$ and belongs to the classes $\mathbb{H}^{p,1}(B_{R'}, [T_0, T_1])$ and $\mathcal{H}^{p,1}(B_{R'}, [T_0, T_1])$ for all $R' < R$ and $[T_0, T_1] \subset (0, T)$.

Proof. We modify the proof of Theorem 3.8 in [1]. Let us fix a number $R' < R$ and a closed interval $[T_0, T_1] \subset (0, T)$. It follows by Hölder's inequality with the exponents θ/s and $\theta/(\theta - s)$ applied to the product

$$|\beta_2|^\theta |\varrho|^\theta = |\beta_2|^\theta |\varrho|^{\theta/p} |\varrho|^{\theta - \theta/p},$$

that $\beta_2 \varrho \in L_{loc}^\theta(B_{R,T})$ if $(p\theta - \theta)(p - \theta)^{-1} = r$, i.e. $\theta = pr(p + r - 1)^{-1}$. Set $s := pr(p + r)^{-1}$. Then $\theta > s$, hence $\beta_2 \varrho \in L_{loc}^s(B_{R,T})$. In addition, $|\beta_1 \varrho| \in L_{loc}^s(B_{R,T})$ also by Hölder's inequality and our assumptions on β_1 and ϱ . We have $q := s' > p'$. Hence (2.2) holds with $q = s' > p'$. This follows Hölder's inequality with the exponents q and $q' = s$ taking into account the fact that for any $\varphi \in C_0^\infty(B_{R,T})$ the integral of $(|\varphi| + |\nabla \varphi|)^q$ over $B_{R,T}$ is estimated by $C \|\nabla_x \varphi\|_{L^q(B_{R,T})}^q$ with some number C independent of φ .

By Theorem 2.2 we obtain the inclusions $\varrho \in \mathbb{H}^{s,1}(B_{R_1}, [t_0, t_1])$ and $\varrho \in \mathcal{H}^{s,1}(B_{R_1}, [t_0, t_1])$ for every interval $[t_0, t_1] \subset (0, T)$ and every $R_1 < R$. In particular, we can take any $t_0 \in (0, T_0)$, $t_1 \in (T_1, T)$, $R_1 \in (R', R)$.

Let us recall an embedding theorem for the spaces $\mathcal{H}^{p,1}(\mathbb{R}^d, [0, T])$. A proof can be found, e.g., in [12, Corollary 7.6] or in [13, Theorem 7.2] (where the restriction $p > 2$ was only needed in the case of stochastic Sobolev spaces). Let $q > p > 1$ and $(d+2)(1/p - 1/q) < 1$. Then there is a number $N(d, p, q, T)$ such that, for each $u \in \mathcal{H}^{p,1}(\mathbb{R}^d, [0, T])$, one has the inequality

$$\|u\|_{L^q(\mathbb{R}^d \times [0, T])} \leq N(d, p, q, T) \|u\|_{\mathcal{H}^{p,1}(\mathbb{R}^d, [0, T])}. \quad (2.3)$$

It is readily verified that the same is true for a ball instead of \mathbb{R}^d . According to (2.3), we obtain that $\varrho \in L^{r_1}(B_{R_1} \times [t_0, t_1])$ for each r_1 with

$$\frac{1}{r_1} > \frac{1}{s} - \frac{1}{d+2},$$

$$\frac{1}{r_1} > \frac{p+r}{pr} - \frac{1}{d+2} = \frac{(d+2)(p+r) - pr}{pr(d+2)},$$

which can be written as

$$r_1 < r \frac{p(d+2)}{(d+2)(p+r) - pr},$$

provided that $(d+2)(p+r) - pr > 0$; if $(d+2)(p+r) - pr \leq 0$, then we can take for r_1 any number $r_1 > 1$.

One can choose r_1 sufficiently close to $rp(d+2)((d+2)(p+r) - pr)^{-1}$ in such a way that

$$\begin{aligned} \frac{r_1}{r} &= \frac{p(d+2)}{(d+2)(p+r) - pr} = \frac{p}{p+r - \frac{pr}{d+2}} \\ &= \frac{p}{p - r(\frac{p}{d+2} - 1)} \geq \frac{p}{p - \frac{p}{d+2} + 1} > 1. \end{aligned}$$

Therefore, repeating the above procedure finitely many times we arrive at the situation with $(d+2)(p+r) - pr \leq 0$, when

$$\varrho \in L^r(B_{R_0} \times [\tau_0, \tau_1]) \quad \text{for all } r \in (1, +\infty),$$

$$\text{where } 0 < \tau_0 < T_0, T_1 < \tau_1 < T, R' < R_0 < R.$$

Hence $\varrho \in \mathbb{H}^{s,1}(B_{R_0}, [\tau_0, \tau_1])$ and $\varrho \in \mathcal{H}^{s,1}(B_{R_0}, [\tau_0, \tau_1])$ for each $s < p$. Let us choose $s > d+2$, $\alpha > 0$, and $\kappa > 0$ such that $\kappa > \alpha > 1/s$ and $s(1 - 2\kappa) > d$, which is possible since $p > d+2$. By [12, Theorem 7.2] combined with the Sobolev embedding theorem, for some $\gamma > 0$ the function ϱ belongs to the class $C^{\alpha-1/s}([\tau_0, \tau_1], C^\gamma(B_{R_0}))$ consisting of all $(\alpha - 1/s)$ -Hölder continuous mappings on $[\tau_0, \tau_1]$ with values in the space $C^\gamma(B_{R_0})$ of γ -Hölder functions on B_{R_0} . Thus, the function ϱ has a Hölder continuous version. \square

Corollary 2.2. *Let $p > d+2$ and let A be locally uniformly bounded and locally uniformly nondegenerate on Ω_T and let (2.1) be fulfilled with some $p > d+2$ for every ball B_R with compact closure in Ω . Assume that μ is a locally finite signed Borel measure on Ω_T such that $b^i, c \in L_{loc}^p(\mu)$ and*

$$\int_{\Omega_T} [\partial_t \varphi + a^{ij} \partial_{x_i} \partial_{x_j} \varphi + \partial_{x_i} a^{ij} \partial_{x_j} \varphi + b^i \partial_{x_i} \varphi + c \varphi] d\mu = 0, \quad \forall \varphi \in C_0^\infty(\Omega_T).$$

Then μ has a locally Hölder continuous density that belongs to the spaces $\mathbb{H}^{p,1}(U, J)$ and $\mathcal{H}^{p,1}(U, J)$ for every interval J and every open set U such that $U \times J$ has compact closure in Ω_T .

Proof. By [1, Corollary 3.2], the measure μ has a density in $L_{loc}^r(\Omega_T)$ with any $r < (d+2)'$, in particular, with some $r > p'$. Hence Theorem 2.3 applies with $\beta_1 = |\partial_{x_i} a^{ij}|$ and $\beta_2 = |b| + |c|$. The corollary is proven. \square

3. The strict positivity of densities

The previous results give existence and Sobolev regularity of densities of solutions of the weak elliptic and parabolic equations. The next theorems give sufficient conditions for the strict positivity of densities of solutions. Below we consider only nonnegative measures. To begin with, we recall a similar result in the elliptic case. It was obtained in [2].

Theorem 3.1. Let μ be a nonzero locally finite nonnegative Borel measure on \mathbb{R}^d . Let A be locally uniformly nondegenerate with $a^{ij} \in W_{loc}^{p,1}(\mathbb{R}^d)$, where $p > d$. Assume that, for every compact set K , there is a number $\kappa = \kappa(K) > 0$ such that the function $\exp(\kappa|b|)$ is integrable on K with respect to μ . Suppose that $\Theta \in L_{loc}^p(\mu)$ and that there exists a measurable function $\Psi \geq 0$ such that $\Theta \geq -\Psi$ and $\exp(\Psi) \in L_{loc}^1(\mu)$. Assume that μ satisfies the equation $\mathcal{L}^*\mu = 0$ on \mathbb{R}^d , where

$$\mathcal{L}u(x) := \partial_{x_i}(a^{ij}(x)\partial_{x_j}u(x)) + b^i(x)\partial_{x_i}u(x) + \Theta(x)u(x).$$

Then, the measure μ has a strictly positive continuous density ϱ .

Note that the existence of a continuous density follows immediately from Corollary 2.1.

Remark 3.1. If it is known that a nonzero nonnegative measure μ has a locally bounded density $\varrho \in W_{loc}^{2,1}(\mathbb{R}^d)$, then we can treat the inequality $\mathcal{L}^*\mu \leq 0$ in place of equation (1.1), where

$$\mathcal{L}u(x) := \partial_{x_i}(a^{ij}(x)\partial_{x_j}u(x)) + b^i(x)\partial_{x_i}u(x) - \Psi(x)u(x).$$

The inequality is understood as follows:

$$\int_{\mathbb{R}^d} \mathcal{L}\varphi(x) d\mu \leq 0$$

for every $\varphi \in C_0^\infty(\mathbb{R}^d)$ such that $\varphi \geq 0$. Assume that A is locally uniformly nondegenerate with $a^{ij} \in W_{loc}^{p,1}(\mathbb{R}^d)$, where $p > d$, $\Psi \geq 0$ and $\exp(\Psi) \in L_{loc}^1(\mu)$. Let b satisfy the same local exponential integrability condition as in the theorem. Then, ϱ has a version that is locally strictly separated from zero.

Let us consider the parabolic case. Let μ be a locally finite nonnegative Borel measure on $\mathbb{R}^d \times (0, 1)$ satisfying equation (1.1).

We assume that the mapping $A(x, t) = (a^{ij}(x, t))_{1 \leq i, j \leq d}$ with values in the spaces of positive symmetric matrices and the vector field $b(x, t) = (b^i(x, t))_{1 \leq i \leq d}$ satisfy the following conditions:

(C1) for some $p > d + 2$, the functions a^{ij} belong to $\mathbb{H}^{p,1}(U, J)$ for every ball U and every interval J with compact closure in $(0, 1)$;

(C2) for every compact set $K \subset \mathbb{R}^d \times (0, 1)$ there exist numbers $m(K), M(K) > 0$ such that for all $(x, t) \in K$ and all $y \in \mathbb{R}^d$ we have

$$m(K)|y|^2 \leq \sum_{1 \leq i, j \leq d} a^{ij}(x, t)y_i y_j \leq M(K)|y|^2;$$

(C3) for every compact set $K \subset \mathbb{R}^d \times (0, 1)$ there exists a number $\kappa(K) > 0$ such that the function $\exp(\kappa(K)|b|^2)$ is integrable with respect to the measure μ on K .

Let $K(r) = K(x, r)$ denote the cube of edge length r centered at x whose edges are parallel to the coordinate axes. First we obtain some a priori estimates.

For all $q \in (-\infty, 1]$ and $s > 0$ we set

$$h_q(s) := -\frac{1}{s|\ln s|^{2q-1}}.$$

Then $h_q(s) = H'_q(s)$, where $H_q(s) := (2 - 2q)^{-1}|\ln s|^{2-2q}$ if $q < 1$ and $H_1(s) := \ln|\ln s|$.

Lemma 3.1. For any $\tau \in (0, e^{-2})$ we have

- (i) $h_q^2(\tau)/h'_q(\tau) \leq |\ln \tau|^{2-2q}$,
- (ii) $\tau^2 h'_q(\tau)|\ln h'_q(\tau)| \leq 3(1 + |q|)|\ln \tau|^{2-2q}$.

Proof. According to the equality $h'_q(s) = (|\ln s| + 1 - 2q)s^{-2}|\ln s|^{-2q}$ we have

$$\frac{h_q^2(\tau)}{h'_q(\tau)} = \frac{\tau^2 |\ln \tau|^{2q}}{\tau^2 |\ln \tau|^{4q-2} (|\ln \tau| - 2q + 1)} = \frac{|\ln \tau|^{2-2q}}{|\ln \tau| - 2q + 1}.$$

Note that if $\tau \in (0, e^{-2})$ and $q \leq 1$ then $|\ln \tau| - 2q + 1 \geq 1$. This yields (i). Let us verify inequality (ii). One has

$$\tau^2 h'_q(\tau) |\ln h'_q(\tau)| \leq \frac{\tau^2 (|\ln \tau| - 2q + 1) (|\ln \tau| + 2q \ln |\ln \tau| + |\ln ||\ln \tau| + 2q - 1|)}{\tau^2 |\ln \tau|^{2q}}.$$

Hence we obtain

$$\tau^2 h'_q(\tau) |\ln h'_q(\tau)| \leq 3(1 + |q|) |\ln \tau|^{2-2q},$$

which completes the proof. \square

Let Ω be an arbitrary domain in \mathbb{R}^d and let $[T_0, T_1] \subset (0, 1)$. According to Corollary 2.2, the measure μ has a density $\varrho \in L^\infty(\Omega \times [T_0, T_1])$. Multiplying ϱ by $(2\|\varrho\|_{L^\infty})^{-1}e^{-2}$ we shall assume that

$$\|\varrho\|_{L^\infty} \leq 2^{-1}e^{-2}.$$

Set $\varrho_k := \varrho + k^{-1}$, where $k > 2e^2$. We observe that $k^{-1} \leq \varrho_k \leq e^{-2}$.

Lemma 3.2. *Let $\eta \in C_0^1(\Omega \times [T_0, T_1])$, $T_0 < \tau_1 < \tau_2 < T_1$, and $0 < \gamma < \kappa$, where $\kappa = \kappa(\Omega \times [T_0, T_1])$ is the number from Condition (C3). Then, the following estimates hold.*

If $q = 1$ and $\eta(t, x) = \eta_0(x)$ whenever $t \in [\tau_1, \tau_2]$, then

$$\begin{aligned} & \int_{\Omega} [\ln |\ln \varrho_k(x, \tau_2)| \eta_0^2(x) - \ln |\ln \varrho_k(x, \tau_1)| \eta_0^2(x)] dx + \frac{m}{3} \int_{\tau_1}^{\tau_2} \int_{\Omega} |\nabla \ln |\ln \varrho_k||^2 \eta_0^2 dx dt \\ & \leq C \left[(1 + \gamma^{-1}) \int_{\tau_1}^{\tau_2} \int_{\Omega} (\eta_0^2 + |\nabla \eta_0|^2) dx dt + \gamma^{-1} \|\varrho\|_{L^\infty} \int_{\tau_1}^{\tau_2} \int_{\Omega} e^{\gamma|b|^2} \varrho \eta_0^2 dx dt \right]. \end{aligned} \quad (3.1)$$

If $q < 1$, then

$$\begin{aligned} & \frac{1}{2-2q} \int_{\Omega} [|\ln \varrho_k(x, \tau_2)|^{2-2q} \eta^2(x, \tau_2) - |\ln \varrho_k(x, \tau_1)|^{2-2q} \eta^2(x, \tau_1)] dx \\ & + \frac{m}{3(1-q)^2} \int_{\tau_1}^{\tau_2} \int_{\Omega} |\nabla |\ln \varrho_k|^{1-q}|^2 \eta^2 dx dt \leq 2 \int_{\tau_1}^{\tau_2} \int_{\Omega} |\eta| |\eta_t| |\ln \varrho_k|^{2-2q} dx dt \\ & + C \left[(1 + \gamma^{-1})(1 + |q|) \int_{\tau_1}^{\tau_2} \int_{\Omega} (\eta^2 + |\nabla \eta|^2) |\ln \varrho_k|^{2-2q} dx dt \right. \\ & \left. + \gamma^{-1} \|\varrho\|_{L^\infty} \int_{\tau_1}^{\tau_2} \int_{\Omega} e^{\gamma|b|^2} \varrho \eta^2 dx dt \right]. \end{aligned} \quad (3.2)$$

In these estimates $m = m(\Omega \times [T_0, T_1])$ is the number from Condition (C2) and C is a number which depends only on d , $m(\Omega \times [T_0, T_1])$, and $M(\Omega \times [T_0, T_1])$.

Proof. For every function $\varphi \in \mathbb{H}_0^{2,1}(\Omega, (0, 1))$ one has the following identity:

$$\int_0^1 \int_{\Omega} [-\partial_t \varphi \varrho + (A \nabla \varrho, \nabla \varphi)] dx dt = \int_0^1 \int_B (b, \nabla \varphi) \varrho dx dt. \quad (3.3)$$

Let $\omega_r(s) := r^{-1} \omega(sr^{-1})$, where $0 \leq \omega \leq 1$, $\omega \in C_0^1(\mathbb{R}^1)$, $\text{supp } \omega \subset [1/2, 3/4]$ and

$$\int_0^1 \omega(s) ds = 1.$$

Set also

$$\varrho_r(x, t) := \int_0^1 \omega_r(t-s) \varrho(x, s) ds, \quad t > r.$$

Then the function ϱ_r is continuously differentiable in t . According to (3.3), the function ϱ_r satisfies the integral equality

$$\int_0^1 \int_{\Omega} \psi \partial_t \varrho_r + ((A \nabla \varrho) * \omega_r, \nabla \psi) dx dt = \int_0^1 \int_{\Omega} ((b \varrho) * \omega_r, \nabla \psi) dx dt \quad (3.4)$$

for every function $\psi \in \mathbb{H}_0^{2,1}(\Omega, (0, 1))$ such that $\psi = 0$ if $t \in (0, r]$. Note that the convolution $(b\varrho) * \omega_r$ is well-defined because by Fubini's theorem $b(x, \cdot)\varrho(x, \cdot) \in L^1[\delta, 1 - \delta]$ for almost all x and all $\delta > 0$.

Let $k > 0$ be fixed and let $\alpha_0 > 0$ be chosen in such a way that $[T_0, T_1] \subset [\alpha_0, 1 - \alpha_0]$ and $\eta(x, t) = 0$ whenever $t \notin [\alpha_0, 1 - \alpha_0]$. Further we consider $r < \alpha_0$. Set

$$\varrho_{r,k} := \varrho_r + k^{-1}, \quad \psi := I_{[\tau_1, \tau_2]} h_q(\varrho_{r,k}) \eta^2,$$

where $I_{[\tau_1, \tau_2]}$ denotes the indicator function of the interval $[\tau_1, \tau_2]$. Substituting such ψ in (3.4) and using the Newton–Leibniz formula, we find

$$\begin{aligned} & \int_{\Omega} [H_q(\varrho_{r,k}(x, \tau_2)) \eta^2(x, \tau_2) - H_q(\varrho_{r,k}(x, \tau_1)) \eta^2(x, \tau_1)] dx \\ & + \int_{\tau_1}^{\tau_2} \int_{\Omega} ((A\nabla\varrho) * \omega_r, \nabla\varrho_{r,k}) h'_q(\varrho_{r,k}) \eta^2 dx dt = I(r, k) + J(r, k) + L(r, k) + N(r, k), \end{aligned}$$

where

$$\begin{aligned} I(r, k) &= -2 \int_{\tau_1}^{\tau_2} \int_{\Omega} ((A\nabla\varrho) * \omega_r, \nabla\eta) h_q(\varrho_{r,k}) \eta dx dt, \\ J(r, k) &= \int_{\tau_1}^{\tau_2} \int_{\Omega} ((b\varrho) * \omega_r, \nabla\varrho_{r,k}) h'_q(\varrho_{r,k}) \eta^2 dx dt, \\ L(r, k) &= 2 \int_{\tau_1}^{\tau_2} \int_{\Omega} ((b\varrho) * \omega_r, \nabla\eta) h_q(\varrho_{r,k}) \eta dx dt, \\ N(r, k) &= 2 \int_{\tau_1}^{\tau_2} \int_{\Omega} H_q(\varrho_{r,k}) \eta \partial_t \eta dx dt. \end{aligned}$$

The function $|H(\varrho_{r,k})|$ on the support of η is bounded by some number uniformly in $r \leq \alpha_0$. Hence, according to the Lebesgue dominated convergence theorem, as $r \rightarrow 0$ we have

$$\begin{aligned} & \int_{\Omega} H_q(\varrho_{r,k}(x, \tau_i)) \eta^2(x, \tau_i) dx \rightarrow \int_{\Omega} H_q(\varrho_k(x, \tau_i)) \eta^2(x, \tau_i) dx, \quad i = 1, 2, \\ & \int_{\tau_1}^{\tau_2} \int_{\Omega} H_q(\varrho_{r,k}) \eta \partial_t \eta dx dt \rightarrow \int_{\tau_1}^{\tau_2} \int_{\Omega} H_q(\varrho_k) \eta \partial_t \eta dx dt. \end{aligned}$$

Note that $|h'_q(\varrho_{r,k})| \leq C(k)$ for some number $C(k)$ and all $r \in (0, 1)$. The functions $A\nabla\varrho$, $b\varrho$, and $\nabla\varrho_{r,k}$ belong to $L^2(\Omega \times [T_0, T_1])$ if $r < \alpha_0$. Hence, as $r \rightarrow 0$, we obtain

$$\begin{aligned} \|(A\nabla\varrho) * \omega_r - A\nabla\varrho\|_{L^2(\Omega \times [T_0, T_1])} &\rightarrow 0, \quad \|(b\varrho) * \omega_r - b\varrho\|_{L^2(\Omega \times [T_0, T_1])} \rightarrow 0, \\ \|\nabla\varrho_{r,k} - \nabla\varrho_k\|_{L^2(\Omega \times [T_0, T_1])} &\rightarrow 0. \end{aligned}$$

The terms

$$\int_{\tau_1}^{\tau_2} \int_{\Omega} ((A\nabla\varrho) * \omega_r, \nabla\varrho_{r,k}) h'_q(\varrho_{r,k}) \eta^2 dx dt$$

and $I(r, k), J(r, k), L(r, k)$ converge as $r \rightarrow 0$. Thus we have

$$\begin{aligned} & \int_{\Omega} [H_q(\varrho_k(x, \tau_2)) \eta^2(x, \tau_2) - H_q(\varrho_k(x, \tau_1)) \eta^2(x, \tau_1)] dx \\ & + \int_{\tau_1}^{\tau_2} \int_{\Omega} (A\nabla\varrho, \nabla\varrho) h'_q(\varrho_k) \eta^2 dx dt = I(k) + J(k) + L(k) + N(k), \quad (3.5) \end{aligned}$$

where

$$\begin{aligned} I(k) &= -2 \int_{\tau_1}^{\tau_2} \int_{\Omega} (A\nabla\varrho, \nabla\eta) h_q(\varrho_k) \eta dx dt, \quad J(k) = \int_{\tau_1}^{\tau_2} \int_{\Omega} (b\varrho, \nabla\varrho) h'_q(\varrho_k) \eta^2 dx dt, \\ L(k) &= 2 \int_{\tau_1}^{\tau_2} \int_{\Omega} (b\varrho, \nabla\eta) h_q(\varrho_k) \eta dx dt, \quad N(k) = 2 \int_{\tau_1}^{\tau_2} \int_{\Omega} H_q(\varrho_k) \eta \eta_t dx dt. \end{aligned}$$

Let us estimate every term on the right-hand side in (3.5) separately. Let $\varepsilon > 0$. Then

$$I(k) \leq \varepsilon \int_{\tau_1}^{\tau_2} \int_{\Omega} |\nabla \varrho|^2 h'_q(\varrho_k) \eta^2 dx dt + \varepsilon^{-1} M^2 \int_{\tau_1}^{\tau_2} \int_{\Omega} |\nabla \eta|^2 \frac{h_q^2(\varrho_k)}{h'_q(\varrho_k)} dx dt.$$

According to the first assertion of Lemma 3.1 one has

$$\frac{h_q^2(\varrho_k)}{h'_q(\varrho_k)} \leq |\ln \varrho_k|^{2-2q}.$$

Hence

$$I(k) \leq \varepsilon \int_{\tau_1}^{\tau_2} \int_{\Omega} |\nabla \varrho|^2 h'_q(\varrho) \eta^2 dx dt + \varepsilon^{-1} M^2 \int_{\tau_1}^{\tau_2} \int_{\Omega} |\nabla \eta|^2 |\ln \varrho_k|^{2-2q} dx.$$

Let us estimate $J(k)$. One has

$$J(k) \leq \int_{\tau_1}^{\tau_2} \int_{\Omega} |\nabla \eta|^2 \frac{h_q^2(\varrho_k)}{h'_q(\varrho_k)} dx dt + \int_{\tau_1}^{\tau_2} \int_{\Omega} |b|^2 \varrho^2 h'_q(\varrho_k) \eta^2 dx dt.$$

The first term on the right is estimated as above. Let us consider the second term. Note that for all $\alpha > 0, \beta > 0, \gamma > 0$ the inequality

$$\alpha\beta \leq \gamma^{-1}(e^{\gamma\beta} + \alpha \ln \alpha).$$

holds. Applying this inequality with $\alpha = h'_q(\varrho_k)$ and $\beta = |b|^2$ and taking into account that $\varrho < \varrho_k$, we obtain

$$|b|^2 \varrho^2 h'_q(\varrho_k) \leq \gamma^{-1} e^{\gamma|b|^2} \varrho^2 + \gamma^{-1} \varrho_k^2 h'_q(\varrho_k) |\ln h'_q(\varrho_k)|.$$

According to the second assertion of Lemma 3.1, one has

$$\varrho_k^2 h'_q(\varrho_k) |\ln h'_q(\varrho_k)| \leq 3(1 + |q|) |\ln \varrho_k|^{2-2q}.$$

Therefore,

$$\begin{aligned} \int_{\tau_1}^{\tau_2} \int_{\Omega} |b|^2 \varrho^2 h'_q(\varrho_k) \eta^2 dx dt &\leq \gamma^{-1} \|\varrho\|_{L^\infty} \int_{\tau_1}^{\tau_2} \int_{\Omega} e^{\gamma|b|^2} \varrho \eta^2 dx dt \\ &\quad + 3\gamma^{-1}(1 + |q|) \int_{\tau_1}^{\tau_2} \int_{\Omega} |\ln \varrho_k|^{2-2q} \eta^2 dx dt. \end{aligned}$$

Thus

$$\begin{aligned} J(k) &\leq 3(1 + \gamma^{-1})(1 + |q|) \int_{\tau_1}^{\tau_2} \int_{\Omega} (\eta^2 + |\nabla \eta|^2) |\ln \varrho_k|^{2-2q} dx dt \\ &\quad + \gamma^{-1} \|\varrho\|_{L^\infty} \int_{\tau_1}^{\tau_2} \int_{\Omega} e^{\gamma|b|^2} \varrho \eta^2 dx dt. \end{aligned}$$

Let us estimate L . One has

$$L(k) \leq \varepsilon \int_{\tau_1}^{\tau_2} \int_{\Omega} |\nabla \varrho|^2 h'_q(\varrho_k) \eta^2 dx dt + 4\varepsilon^{-1} \int_{\Omega} |b|^2 \varrho^2 h'_q(\varrho_k) \eta^2 dx dt.$$

Estimating the second term on the right as above we obtain

$$\begin{aligned} L(k) &\leq \varepsilon \int_{\tau_1}^{\tau_2} \int_{\Omega} |\nabla \varrho|^2 h'_q(\varrho_k) \eta^2 dx dt + 4\varepsilon^{-1} \gamma^{-1} \|\varrho\|_{L^\infty} \int_{\tau_1}^{\tau_2} \int_{\Omega} e^{\gamma|b|^2} \varrho \eta^2 dx dt \\ &\quad + 12\varepsilon^{-1} \gamma^{-1} (1 + |q|) \int_{\tau_1}^{\tau_2} \int_{\Omega} |\ln \varrho_k|^{2-2q} \eta^2 dx dt. \end{aligned}$$

Note that

$$\int_{\tau_1}^{\tau_2} \int_{\Omega} (A \nabla \varrho, \nabla \varrho) h'_q(\varrho_k) \eta^2 dx \geq m \int_{\tau_1}^{\tau_2} \int_{\Omega} |\nabla \varrho|^2 h'_q(\varrho_k) \eta^2 dx.$$

Since $h'_q(\varrho_k) = (|\ln \varrho_k| - 2q + 1)(\varrho_k^2 |\ln \varrho_k|^{2q})^{-1}$, $0 < \varrho_k < e^{-2}$ and $q \leq 1$, we have the inequality

$$h'_q(\varrho_k) \geq \frac{1}{\varrho_k^2 |\ln \varrho_k|^{2q}}.$$

Hence

$$\int_{\tau_1}^{\tau_2} \int_{\Omega} (A \nabla \varrho, \nabla \varrho) h'_q(\varrho_k) \eta^2 dx dt \geq m \int_{\tau_1}^{\tau_2} \int_{\Omega} \frac{|\nabla \varrho|^2}{\varrho_k^2 |\ln \varrho_k|^{2q}} \eta^2 dx dt.$$

Summing the above estimates and letting $\varepsilon := m/3$, we arrive at the estimate

$$\begin{aligned} & \int_{\Omega} (H_q(\varrho_k) \eta^2) \Big|_{\tau_1}^{\tau_2} dx + 3^{-1} m \int_{\tau_1}^{\tau_2} \int_{\Omega} \frac{|\nabla \varrho|^2}{\varrho_k^2 |\ln \varrho_k|^{2q}} \eta^2 dx dt \\ & \leq 2 \int_{\tau_1}^{\tau_2} \int_{\Omega} H_q(\varrho_k) \eta \eta_t dx dt \\ & + C \left[(1 + \gamma^{-1})(1 + |q|) \int_{\tau_1}^{\tau_2} \int_{\Omega} (\eta^2 + |\nabla \eta|^2) |\ln \varrho_k|^{2-2q} dx dt \right. \\ & \quad \left. + \gamma^{-1} \|\varrho\|_{L^\infty} \int_{\tau_1}^{\tau_2} \int_{\Omega} e^{\gamma|b|^2} \varrho \eta^2 dx dt \right] \end{aligned}$$

In the case $q = 1$, when $\eta(x, t) = \eta_0(x)$ for all $t \in [\tau_1, \tau_2]$, one has $\partial_t \eta = 0$ on $[\tau_1, \tau_2]$ and $N(k) = 0$. The proof is complete. \square

The following two lemmas can be found in [14], [15] or [16].

Let $V^2(U \times J)$ be the space of functions $u \in \mathbb{H}^{2,1}(U \times J)$ with finite norm

$$\|u\|_{V^2} = \|u\|_{\mathbb{H}^{2,1}(U \times J)} + \sup_{t \in J} \|u(\cdot, t)\|_{L^2(U)}.$$

Lemma 3.3. *Let $d > 2$. Suppose that $v \in V^2(\Omega \times J)$ and that for almost all $t \in J$ the function $x \rightarrow v(x, t)$ has compact support in Ω . Then*

$$\|v\|_{L^{2(d+2)/d}(\Omega \times J)} \leq C \|v\|_{V^2(\Omega \times J)},$$

where C depends only on d and the volume $|\Omega|$ of the set Ω .

Let us fix a cube $K(r) = K(y, r)$ and define the function ψ_K as follows:

$$\psi_K(x) = \prod_{i=1}^d \chi_i(x_i), \quad x = (x_1, \dots, x_d), \quad (3.6)$$

where $\chi_i \in C^1([y_i - 2r, y_i + 2r])$, $\chi_i(x_i) = 1$ if $|x_i - y_i| \leq r$ and $\chi_i(x_i) = 0$ if $|x_i - y_i| \geq 2r$. Let also $0 \leq \psi_K \leq 1$ and $|\nabla \psi_K| \leq cr^{-1}$ for some positive number c .

Lemma 3.4. *Assume that $v \in V^2(K(R) \times (T_0, T_1))$ and that for every function $\psi \in C_0^1(K(R))$ the function*

$$t \mapsto \int_{K(R)} v(x, t) \psi^2(x) dx$$

is absolutely continuous. Suppose that there exist a nonnegative function $g \in L^1(T_0, T_1)$ and constants $A_1, A_2 > 0$ such that for each cube $K(r) \subset K(R)$ and almost all $t \in (T_0, T_1)$ we have

$$\frac{d}{dt} \int_{K(2r)} v(x, t) \psi_K^2(x) dx + A_1 \int_{K(2r)} |\nabla v|^2 \psi_K^2 dx dt \leq A_2 g(t) |K(r)|^{1-2/d}.$$

Then, whenever $T_0 < s_1 < s_2 < t_1 < t_2 < T_1$, there exist numbers $\lambda > 0$ and $\sigma > 0$ depending on $s_1, s_2, t_1, t_2, d, A_1, A_2$, and $\|g\|_{L^1(T_0, T_1)}$ such that

$$\int_{s_1}^{s_2} \int_{K(R)} e^{-\lambda v} dx dt \int_{t_1}^{t_2} \int_{K(R)} e^{\lambda v} dx dt \leq \sigma.$$

Let us fix a cube $K(R) = K(y, R)$ and numbers $s_1 < s_2 < t_1 < t_2$ such that $s_1, s_2, t_1, t_2 \in [T_0, T_1]$, where $[T_0, T_1] \subset (0, 1)$.

Set $Q^- := K(R) \times [s_1, s_2]$ and $Q^+ := K(R) \times [t_1, t_2]$.

The next theorem is the main result of this work.

Theorem 3.2. *Let $\mu = \varrho dx dt$ be a solution of equation (1.1), where the coefficients a^{ij}, b^i satisfy conditions (C1), (C2), and (C3). Suppose that $\text{ess sup}_{(x,t) \in Q^-} \varrho(x, t) > 0$. Then $|\ln \varrho| \in L^\infty(Q^+)$.*

Proof. Let $K_n := K(R + 2^{-n}R), \tau_n = t_1 - (t_1 - s_2)^{-1}2^{-n-1}$, where $n \geq 0$. Set $\kappa := \kappa(K(R) \times [T_0, T_1]), m := m(K(R) \times [T_0, T_1])$.

1. We show that for some $\lambda > 0$ the functions $|\ln \varrho_k|^\lambda$, where $\varrho_k = \varrho + k^{-1}$, have integrals uniformly bounded in k . Set $f(\varrho_k) = \ln |\ln \varrho_k|$. According to Lemma 3.2 with $q = 1$, whenever $T_0 < s < t < T_1$ we have the estimate

$$\begin{aligned} & \int_{K_0} [f(\varrho_k(x, t))\eta^2(x) - f(\varrho_k(x, s))\eta^2(x)] dx + \frac{m}{3} \int_s^t \int_{K_0} |\nabla f(\varrho_k)|^2 \eta^2 dx d\tau \\ & \leq C \left[(1 + \gamma^{-1}) \int_s^t \int_{K_0} (\eta^2 + |\nabla \eta|^2) dx d\tau + \gamma^{-1} \|\varrho\|_{L^\infty} \int_s^t \int_{K_0} e^{\gamma|b|^2} \varrho \eta^2 dx d\tau \right] \end{aligned} \quad (3.7)$$

whenever $\eta \in C_0^\infty(K_0 \times (T_0, T_1))$ and $\eta(x, \tau) = \eta_0(x)$ if $\tau \in [s, t]$. Let $d\gamma = \kappa$. According to Hölder's inequality

$$\int_{K_0} e^{\gamma|b|} \varrho \eta^2 dx \leq \left(\int_{K_0} e^{\kappa|b|} \varrho \eta^2 dx \right)^{1/d} \left(\int_{K_0} \varrho \eta^2 dx \right)^{(d-1)/d}.$$

Set

$$g(t) := 1 + \left(\int_{K_0} e^{\kappa|b(x,t)|} \varrho(x, t) dx \right)^{1/d}.$$

Applying Hölder's inequality it is easily seen that $g \in L^1(T_0, T_1)$. According to equality (3.5) the function

$$t \mapsto \int_{K_0} f(\varrho_k(x, t))\eta_0^2(x) dx$$

is absolutely continuous on every inner subinterval in $[T_0, T_1]$. For an arbitrary cube $K(r)$ such that $K(2r) \subset K_0$, we take η such that for all $\tau \in [s, t]$ one has $\eta(x, \tau) = \eta_0(x) := \psi_K(x)$, where the function ψ_K is defined by (3.6). By (3.7) there exist numbers A_1, A_2 such that

$$\frac{d}{dt} \int_{K(2r)} f(\varrho_k(x, t))\psi_K^2(x) dx + A_1 \int_{K(2r)} |\nabla f(\varrho_k(x, t))|\psi_K^2(x) dx \leq A_2 g(t) |K(r)|^{1-2/d}.$$

This follows by the fact that (3.7) holds for all pairs s, t with $s < t$. Thus the assumptions of Lemma 3.4 are fulfilled, hence there exist numbers $\lambda, \sigma > 0$ independent of k such that

$$\int_{s_1}^{s_2} \int_{K_0} e^{-\lambda f(\varrho_k)} dx dt \int_{\tau_0}^{t_2} \int_{K_0} e^{\lambda f(\varrho_k)} dx dt \leq \sigma.$$

Since $\text{ess sup}_{(x,t) \in Q^-} \varrho(x, t) > 0$, the first multiplier on the left does not vanish. Taking into account the estimate $0 < f(\varrho_k) < f(\varrho)$, we obtain that there exists a number $C_1 > 0$ independent of k such that

$$\int_{\tau_0}^{t_2} \int_{K_0} e^{\lambda f(\varrho_k)} dx dt \leq \sigma \left(\int_{s_1}^{s_2} \int_{K_0} e^{-\lambda f(\varrho)} dx dt \right)^{-1} \leq C_1.$$

Since $f(\varrho_k) = \ln |\ln \varrho_k|$, we have

$$\int_{\tau_0}^{t_2} \int_{K_0} |\ln \varrho_k|^\lambda dx dt \leq C_1.$$

2. Let us show that the norms $\|\ln \varrho\|_{L^p}$ are uniformly bounded in $p \in [1, +\infty)$. Set $f(\varrho_k) := |\ln \varrho_k|^{1-q}$. Let $\eta \in C_0^1(K_0 \times (T_0, T_1))$. By Lemma 3.2 with $\gamma = \kappa$, whenever $\tau_0 < s < t < t_2$ we have

$$\begin{aligned} & \frac{1}{2-2q} \int_{K_0} [f(\varrho_k)^2 \eta^2(x, t) - f(\varrho_k)^2 \eta^2(x, s)] dx \\ & + \frac{m}{3(1-q)^2} \int_s^t \int_{K_0} |\nabla(f(\varrho_k))|^2 \eta^2 dx d\tau \leq 2 \int_s^t \int_{K_0} |\eta| |\partial_t \eta| f(\varrho_k)^2 dx d\tau \\ & + C \left[(1 + \delta^{-1})(1 + |q|) \int_s^t \int_{K_0} (\eta^2 + |\nabla \eta|^2) f(\varrho_k)^2 dx d\tau \right. \\ & \left. + \delta^{-1} \|\varrho\|_{L^\infty} \int_s^t \int_{K_0} e^{\kappa|b|^2} \varrho \eta^2 dx d\tau \right]. \quad (3.8) \end{aligned}$$

Set

$$\Lambda := \int_{T_0}^{T_1} \int_{K_0} e^{\kappa|b|^2} \varrho dx dt.$$

Let η_n be a continuously differentiable function such that

$$\eta_n(x, t) = 1 \quad \text{if } (x, t) \in K_{n+1} \times (\tau_{n+1}, t_2), \quad \eta_n(x, t) = 0 \quad \text{if } (x, t) \notin K_n \times (\tau_n, T_1),$$

and $|\eta(x, t)| \leq 1$, $|\partial_t \eta_n(x, t)| \leq c_1 2^n$, $|\nabla_x \eta_n(x, t)| \leq c_2 2^n$, where c_1, c_2 are some positive numbers. Then, according to (3.8) we have

$$\begin{aligned} \|\nabla(f(\varrho_k))\|_{L^2(K_{n+1} \times (\tau_{n+1}, t_2))}^2 & \leq C_2(1 + |q|)^3 4^n \left(\|f(\varrho_k)\|_{L^2(K_n \times (\tau_n, t_2))}^2 + \Lambda \right), \\ \sup_{t \in (\tau_{n+1}, t_2)} \|f(\varrho_k(\cdot, t))\|_{L^2(K_{n+1})}^2 & \leq C_2(1 + |q|)^2 4^n \left(\|f(\varrho_k)\|_{L^2(K_n \times (\tau_n, t_2))}^2 + \Lambda \right), \end{aligned}$$

where the number C_2 is independent of n and k . Hence by Lemma 3.3 there exists a number C_3 independent of n and k such that

$$\|f(\varrho_k)\|_{L^{2(d+2)/d}(K_{n+1} \times (\tau_{n+1}, t_2))}^2 \leq C_3(1 + |q|)^3 4^n \left(\|f(\varrho_k)\|_{L^2(K_n \times (\tau_n, t_2))}^2 + \Lambda \right).$$

Let $p_0 = \lambda$, $s = (d+2)d^{-1} > 1$, $p_n = s^n \lambda$. Then $p_{n+1} = s p_n$ and $p_n \rightarrow \infty$. Substituting the numbers $q_n = 1 - p_n/2$ in place of q in $f(\varrho_k) = |\ln \varrho_k|^{1-q}$, we find

$$\|\ln \varrho_k\|_{L^{p_{n+1}}(K_{n+1} \times (\tau_{n+1}, t_2))}^{p_n} \leq C_3(1 + p_n)^3 4^n \left(\|\ln \varrho_k\|_{L^{p_n}(K_n \times (\tau_n, t_2))}^{p_n} + \Lambda \right).$$

For every fixed n , beginning with $n = 0$, we apply Fatou's theorem as $k \rightarrow \infty$. This yields a number $C_4 > 1$ independent of n such that

$$\|\ln \varrho\|_{L^{p_{n+1}}(K_{n+1} \times (\tau_{n+1}, t_2))}^{p_n} \leq C_4^n \left(\|\ln \varrho\|_{L^{p_n}(K_n \times (\tau_n, t_2))}^{p_n} + 1 \right).$$

Let $Z_n = \max\{\|\ln \varrho\|_{L^{p_n}(K_n \times (\tau_n, t_2))}, 1\}$. Then $Z_{n+1} \leq (2C_4)^{\lambda^{-1} n s^{-n}} Z_n$, whence it follows that

$$Z_n \leq (2C_4)^{\lambda^{-1} \sum_n n s^{-n}} Z_0.$$

Thus, the sequence $\{Z_n\}$ is bounded and there exists a number C_5 independent of n such that $\|\ln \varrho\|_{L^{p_n}(K_n \times (\tau_n, t_2))} \leq C_5$. Hence $|\ln \varrho| \in L^\infty(K_0 \times (t_1, t_2))$. The theorem is proven. \square

Corollary 3.1. *Let $\mu = \varrho dx dt$ be a solution of equation (1.1), where the coefficients a^{ij} and b^i satisfy conditions (C1), (C2), and (C3). Let $0 < \tau < 1$ and*

$$\text{esssup}_{(x,t) \in \mathbb{R}^d \times (0,\tau)} \varrho(x, t) > 0.$$

Then, the measure μ has a continuous density ϱ that is strictly positive on $\mathbb{R}^d \times (\tau, 1)$.

Let us note that the condition $\text{esssup}_{(x,t) \in \mathbb{R}^d \times (0,\tau)} \varrho(x,t) > 0$ is trivially satisfied if, for every $t > 0$, the function $x \mapsto \varrho(x,t)$ is a probability density, which is the case in applications to transition probabilities. However, in the general case, this condition cannot be replaced by the hypothesis that the measure μ is nonnegative and not identically zero on $\mathbb{R}^d \times (0,\tau)$ (as in the elliptic case). This is seen from the following simple example. Let $d = 1$ and $a \in (0,1)$. Let $\varrho(x,t) = \exp((a-t)^{-1} \exp x)$ if $t > a$ and $\varrho(x,t) = 0$ if $t \leq a$. Then the measure μ with density ϱ on $\mathbb{R}^1 \times (0,1)$ satisfies the equation $\mathcal{L}^* \mu = 0$ with $a^{11}(x) = e^{-x}$ and $b = 0$ (which has the form $\partial_t \varrho = \partial_x(a^{11} \partial_x \varrho)$ in terms of ϱ), but this solution is zero when $t \leq a$.

Remark 3.2. Theorem 3.1 and Corollary 3.1 remain true with similar proofs if the operator L has the form

$$\mathcal{L}u(x,t) := \partial_t u(x,t) + \partial_{x_i}(a^{ij}(x,t) \partial_{x_j} u(x,t)) + b^i(x,t) \partial_{x_i} u(x,t) + \Theta(x,t)u(x,t),$$

where a^{ij} and b^i satisfy conditions (C1), (C2), and (C3), $\Theta \in L^p_{loc}(\mu)$ and there exists a measurable function $\Psi \geq 0$ such that $\Theta \geq -\Psi$ and $\exp(\Psi) \in L^1_{loc}(\mu)$.

Remark 3.3. If it is known that the measure μ has a locally bounded density $\varrho \in \mathbb{H}^{2,1}_{loc}(\mathbb{R}^d, (0,1))$ then we can treat the inequality $\mathcal{L}^* \mu \leq 0$ in place of equation (1.1), where

$$\mathcal{L}u(x,t) := \partial_t u(x,t) + \partial_{x_i}(a^{ij}(x,t) \partial_{x_j} u(x,t)) + b^i(x,t) \partial_{x_i} u(x,t) - \Psi(x,t)u(x,t).$$

The inequality is understood as follows:

$$\int_0^1 \int_{\mathbb{R}^d} \mathcal{L}\varphi(x,t) d\mu \leq 0$$

for every nonnegative function $\varphi \in C^\infty_0(\mathbb{R}^d \times (0,1))$. Assume that a^{ij} and b^i satisfy conditions (C1), (C2), and (C3), $\Psi \geq 0$ and $\exp(\Psi) \in L^1_{loc}(\mu)$. Then ϱ has a version that is strictly positive whenever $t > \tau$, where $\text{esssup}_{(x,t) \in \mathbb{R}^d \times (0,\tau)} \varrho(x,t) > 0$.

We emphasize once again that the obtained results are applicable to equations with non-divergence form operators, one just needs to impose the corresponding exponential integrability assumptions on the new drift with the components $b^j - \partial_{x_i} a^{ij}$.

Let us observe that our results show that the local exponential integrability of the drift (with the square in the parabolic case) with respect to the solution yields that in fact the drift is locally exponentially integrable with respect to Lebesgue measure (due to the continuity and strict positivity of the density of the solution). As we have already noted, this phenomenon does happen in the case of power integrability. Although the exponential integrability is not necessary, it cannot be substantially weakened even in the one-dimensional case. In paper [17], the following question was investigated. Suppose we are given a probability measure μ on $[0, +\infty)$ with an absolutely continuous density ϱ and let ψ be a positive convex function on $[0, +\infty)$. When does the integrability of $\psi(|\varrho'/\varrho|)$ with respect to μ yield that $\varrho(x) > 0$ almost everywhere on $[0, +\infty)$? This question is directly related to our problem in the elliptic case since the measure μ satisfies equation (1.1) with $A(x) = 1$ and $b(x) = -\varrho'(x)/\varrho(x)$, i.e., one is concerned precisely with the μ -integrability of $\psi(|b|)$ in this very special case. It turns out that a sufficient condition for the validity of the aforementioned implication is the non-integrability at the infinity of the function $x^{-2} \ln \psi(x)$. Moreover, this non-integrability is also necessary in order that for every absolutely continuous probability density ϱ the μ -integrability of $\psi(|\varrho'/\varrho|)$ imply the positivity of ϱ almost everywhere. For example, one can take for ψ the functions $\exp(\kappa x)$ with $\kappa > 0$ (as we did) and even $\exp(x/|\ln x|^2)$.

4. APPLICATIONS

The obtained results enable us to establish the existence of strictly positive continuous densities of finite-dimensional projections of stationary distributions and transition probabilities of infinite-dimensional diffusions. We begin with the elliptic case. Let \mathbb{R}^∞ be the

space of infinite real sequences with the product topology. Suppose we are given Borel functions B^i on \mathbb{R}^∞ and numbers $\alpha_i > 0$. We consider the infinite-dimensional elliptic operator

$$L\varphi := \sum_{i=1}^{\infty} [\alpha_i \partial_{x_i}^2 \varphi + B^i \partial_{x_i} \varphi]$$

defined on smooth functions of finitely many variables. We shall say that a Borel probability measure μ on \mathbb{R}^∞ satisfies the weak elliptic equation

$$L^* \mu = 0 \tag{4.1}$$

if for each $i \geq 1$ one has $B^i \in L^1(\mu)$ and

$$\int_{\mathbb{R}^\infty} \sum_{i=1}^{\infty} [\alpha_i \partial_{x_i}^2 \varphi + B^i \partial_{x_i} \varphi] d\mu = 0 \tag{4.2}$$

for all functions φ on \mathbb{R}^∞ of the form $\varphi(x) = \varphi_0(x_1, \dots, x_n)$, $\varphi_0 \in C_0^\infty(\mathbb{R}^n)$. The series above is in fact a finite sum for such φ . Measures satisfying equation (4.1) are called infinitesimally invariant for the operator L because, under broad assumptions, this equation is fulfilled for true invariant measures of the diffusion generated by L (certainly, provided that this diffusion exists and has stationary distributions). An advantage of consideration of infinitesimally invariant measures is that the equation may make sense and possess solutions under assumptions much weaker than those needed for the existence of the associated diffusion. This concerns even the finite-dimensional case. Moreover, the existence of solutions of the elliptic equation often enables one to construct the diffusion (these matters are studied in papers [19], [20], and [21]).

Let us fix d and consider the projection $P: \mathbb{R}^\infty \rightarrow \mathbb{R}^d$, $P(x) = (x_1, x_2, \dots, x_d)$. Let $\mu_P = \mu \circ P^{-1}$ be the corresponding finite-dimensional projection of the measure μ . Since $B^i \in L^1(\mu)$ there exists the conditional expectation B_P of the mapping $P \circ B$ with respect to the measure μ and the σ -field σ_P generated by P . Then one has $B_P(x) = b(Px)$ for some Borel mapping $b: \mathbb{R}^d \rightarrow \mathbb{R}^d$. Clearly, $|b| \in L^1(\mu_P)$ since $|B_P| \in L^1(\mu)$. The measure μ_P satisfies the equation $L_P^* \mu = 0$, where the operator L_P has the form

$$L_P u := \sum_{i=1}^d \alpha_i \partial_{x_i}^2 u + (b, \nabla u).$$

Theorem 4.1. *Suppose that a Borel probability measure μ satisfies equation (4.1) and there exist numbers $\kappa_i > 0$ such that $\exp(\kappa_i |B^i|) \in L^1(\mu)$ for each $i \geq 1$. Then, for any projection $P: \mathbb{R}^\infty \rightarrow \mathbb{R}^d$, the measure μ_P on \mathbb{R}^d has a continuous strictly positive density.*

Proof. Let $\kappa = \min(\kappa_1, \dots, \kappa_d)$. Due to Jensen's inequality $\exp(\kappa |b|) \in L^1(\mu_P)$ since $\exp(\kappa |B^i|) \in L^1(\mu)$. Hence we can apply Theorem 3.1. \square

In [18], by using the method of Lyapunov functions sufficient conditions were obtained ensuring the exponential integrability with respect to the measure μ satisfying equation (4.1). These conditions employ certain coercivity of the drift. Let us give typical examples.

Example 4.1. Let $q_n > 0$, $\sum_{n=1}^{\infty} q_n < \infty$ and let X be the Hilbert space of sequences $x = (x_n)$ with $\|x\|_0^2 := \sum_{n=1}^{\infty} q_n x_n^2 < \infty$. Suppose that we are given functions B^n on X , $n \in \mathbb{N}$, continuous on balls in the weak topology, and that for some numbers $C_n > 0$ one has the estimates

$$|B^n(x)| \leq C_n + C_n \|x\|_0^2.$$

Assume that there is a number $C > 0$ for which

$$\sum_{n=1}^{\infty} q_n x_n B^n(x) \leq C - C \|x\|_0^2$$

for all x of the form $x = (x_1, \dots, x_n, 0, \dots)$. By Theorem 5.3 of [18], there exists a Borel probability measure μ on X which satisfies equation (4.1) with $\alpha_i \equiv 1$, and the function $\exp(\varepsilon\|x\|_0^2)$ is integrable with respect to μ for some $\varepsilon > 0$. Then $\exp(\varepsilon_n|B^n|) \in L^1(\mu)$ with $\varepsilon_n < \varepsilon C_n^{-1}$, which yields the existence of continuous strictly positive densities of finite dimensional projections of the measure μ generated by the projections $x \mapsto (x_1, \dots, x_d)$.

If for some $r > 1$ we have the estimates

$$|B^n(x)| \leq C_n + C_n\|x\|_0^r,$$

then it suffices to require the inequality

$$\sum_{n=1}^{\infty} q_n x_n B^n(x) \leq C - C\|x\|_0^r$$

on finite sequences.

Similar assertions are valid for finite dimensional projections of infinitesimally invariant measures for stochastic equations of Burgers and Navier–Stokes types, constructed in Section 7 of paper [18] under the assumption of at most quadratic growth of nonlinearities. For example, let us consider the stochastic Burgers-type equation

$$du(t, x) = \sqrt{2}dW(t, x) + \left[\mathcal{H}u(t, x) - \psi(u(t, x))\partial_x u(t, x) + f(x) \right] dt$$

in the space $X = L^2[0, 1]$, where \mathcal{H} is the Laplace operator with zero boundary conditions and the eigenbasis $\{\eta_n\}$ with the eigenvalues λ_n , W is the Wiener process in X of the form $W(t) = \sum_{n=1}^{\infty} \alpha_n w_n(t)\eta_n$, $\{w_n\}$ is a sequence of independent real Wiener processes, $\alpha_n > 0$, $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$, ψ is a Borel function such that $|\psi(s)| \leq C + C|s|$, $f \in L^\infty[0, 1]$. Set

$$\Psi(y) := \int_0^y \psi(s) ds,$$

$$B^n(u) = \lambda_n u_n + (\Psi(u), \eta_n')_2 + (f, \eta_n)_2, \quad u_n := (u, \eta_n)_2.$$

Then stationary distributions of the indicated stochastic process satisfy the equation $L^*\mu = 0$ with the operator

$$L\varphi = \sum_{n=1}^{\infty} [\alpha_n \partial_{\eta_n}^2 \varphi + B^n \partial_{\eta_n} \varphi]$$

defined on the functions of the form $\varphi(u) = \varphi_0(u_1, \dots, u_n)$, $\varphi_0 \in C_0^\infty(\mathbb{R}^n)$, where ∂_η denotes the partial derivative along a vector η . As shown in [18], there exists a Borel probability measure μ on X satisfying the equation $\mathcal{L}^*\mu = 0$ in the above sense. Moreover, the function $\exp(\varepsilon\|x\|_0^2)$ is integrable with respect to μ for some $\varepsilon > 0$ (this follows from the results in [18]). Therefore, here the condition $\exp(\varepsilon_n|B^n|) \in L^1(\mu)$ is fulfilled as well. Similarly one considers the stochastic Navier–Stokes type equation

$$du(t, x) = \sqrt{2}dW(t, x) + \left[\mathcal{H}u(t, x) - (u(t, x) \cdot \nabla)u(t, x) + F(x, u(t, x)) + \nabla p(t, x) \right] dt$$

with the Laplace operator \mathcal{H} on a domain in \mathbb{R}^d and a bounded continuous mapping F (in the classical case F does not depend on u). Here the functions B^n have the form

$$B^n(u) = (\mathcal{H}u, \eta_n)_2 + \sum_{j=1}^d (\partial_j u, u^j \eta_n)_2 + (F(\cdot, u), \eta_n)_2.$$

The corresponding measure μ is constructed on the space $L^2(D, \mathbb{R}^d)$ of vector-functions. We note that in papers [23], [24], in the absence of the nonlinearity F , a number of fine results was obtained about densities of finite-dimensional projections of stationary distributions of the indicated infinite-dimensional process.

Finally, yet another close example is related to the stochastic reaction-diffusion equation

$$du(t, x) = \left[\partial_x^2 u(t, x) + F(u(t, x)) \right] dt + \sqrt{2} dW(t),$$

where the function F on the real line is such that $|F(s)| \leq C + Cs^2$, $sF(s) \leq C + \varepsilon s^2$ and $\varepsilon > 0$ is sufficiently small (see Section 7 in [18]).

Let us consider the parabolic case. The analog of (4.2) for a measure μ on $\mathbb{R}^\infty \times (0, 1)$ is the identity

$$\int_{\mathbb{R}^\infty \times (0, 1)} \left(\partial_t \varphi + \sum_{i=1}^{\infty} [\partial_{x_i}^2 \varphi + B^i \partial_{x_i} \varphi] \right) d\mu = 0 \quad (4.3)$$

for all functions φ of the form $\varphi(x, t) = \varphi_0(x_1, \dots, x_n, t)$, $\varphi_0 \in C_0^\infty(\mathbb{R}^n \times (0, 1))$, where we also require the μ -integrability of the coefficients $B^i: \mathbb{R}^\infty \times (0, 1) \rightarrow \mathbb{R}^1$. Equation (4.1) is understood in the sense of this identity.

Theorem 4.2. *Let a Borel probability measure μ on $\mathbb{R}^\infty \times (0, 1)$ satisfy equation (4.1) and let there exist numbers $\kappa_i > 0$ such that $\exp(\kappa_i |B^i|^2) \in L^1(\mu)$ for each $i \geq 1$. Then, for every projection $P: (x, t) \mapsto (x_1, \dots, x_d, t)$, $\mathbb{R}^\infty \times (0, 1) \rightarrow \mathbb{R}^d \times (0, 1)$, the measure μ_P on $\mathbb{R}^d \times (0, 1)$ has a continuous strictly positive density.*

The proof is similar to the justification of the previous theorem. The same considerations apply to the following more general situation. Suppose that X is a locally convex space and, for each n , we are given a Borel function B^n on $X \times (0, 1)$. Let a^{ij} be real numbers such that the matrices $(a^{ij})_{i,j \leq d}$ are positive. Let l_i be continuous linear functionals on X and let h_j be vectors in X such that $l_i(h_j) = \delta_{ij}$. Let us consider the operator

$$L\varphi := \partial_t \varphi + \sum_{i,j \geq 1} a^{ij} \partial_{h_i} \partial_{h_j} \varphi + \sum_{i \geq 1} B^i \partial_{h_i} \varphi$$

defined on functions of the form $\varphi(x, t) = \varphi_0(l_1(x), \dots, l_n(x), t)$, $\varphi_0 \in C_0^\infty(\mathbb{R}^n \times (0, 1))$. Then, for a Borel probability measure μ on $X \times (0, 1)$ such that $B^i \in L^1(\mu)$ for all i , the equation $L^* \mu = 0$ is defined in the same sense as above. For a fixed number d , let b^i denote the conditional expectation of B^i with respect to the measure μ and the σ -algebra generated by l_1, \dots, l_d . Let $\pi(x, t) := (l_1(x), \dots, l_d(x), t)$, $(x, t) \in X \times (0, 1)$, and let $\mu_\pi := \mu \circ \pi^{-1}$. Then, the measure μ_π on $\mathbb{R}^d \times (0, 1)$ satisfies the parabolic equation $L_\pi^* \mu_\pi = 0$ with the operator L_π having the diffusion matrix $(a^{ij})_{i,j \leq d}$ and drift $b_\pi = (b^i)$. Therefore, we arrive at the following assertion.

Theorem 4.3. *Suppose that for each i there exists a number $\kappa_i > 0$ such that one has $\exp(\kappa_i |B^i|^2) \in L^1(\mu)$. Then the measure μ_π has a continuous strictly positive density on $\mathbb{R}^d \times (0, 1)$.*

It should be noted that in the previous results we dealt with finite-dimensional projections determined by the functionals of the form mentioned in the hypotheses (yet, in the case of \mathbb{R}^∞ , any continuous linear functional is a finite linear combination of coordinate functionals). In some cases, as in the next example, one can pass from special functionals to general ones.

Example 4.2. Let a diffusion process ξ_t in a separable Banach space X be defined by the stochastic differential equation

$$d\xi_t = dW_t + b(\xi_t, t)dt,$$

where W_t is some Wiener process in X (see §7.2 in [22]) such that the distribution of W_1 is positive on all balls and $b: X \times [0, 1] \rightarrow X$ is a Borel mapping such that $\|b(x, t) - b(y, t)\| \leq C\|x - y\|$ and $\|b(x, t)\| \leq C + C\|x\|$, where C is a constant. This equation is understood as the integral one

$$\xi_t(\omega) = \xi_0(\omega) + W_t(\omega) + \int_0^t b(\xi_s(\omega), s) ds.$$

It has a unique solution, which is constructed by means of the contracting mapping theorem. Suppose that for some $\varepsilon_0 > 0$ the function $\exp(\varepsilon_0\|x\|^2)$ is integrable with respect to the distribution of ξ_0 (which is fulfilled, e.g., if ξ_0 is a non-random point). By using the Gronwall lemma, we obtain

$$\|\xi_t(\omega)\| \leq [\|\xi_0(\omega)\| + \|W_t(\omega)\| + C]e^{Ct}, \quad t \in [0, 1].$$

Due to Fernique's theorem (see §2.6 in [22]), for some $\varepsilon_1 > 0$ the random variables $\exp(\varepsilon_1\|W_t\|^2)$ with $t \in [0, 1]$ have uniformly bounded expectations. Hence there exists $\varepsilon_2 > 0$ such that the random variables $\exp(\varepsilon_2\|\xi_t\|^2)$ with $t \in [0, 1]$ have uniformly bounded expectations as well. This means that the function $\exp(\varepsilon_2\|x\|^2)$ is integrable with respect to the measure $P(t, \cdot) dt$ on $X \times [0, 1]$, which yields also the integrability of $\exp(C^{-2}\varepsilon_2\|b\|^2)$ with respect to that measure. Suppose that $l_1, \dots, l_d \in X^*$ and the mapping $\pi = (l_1, \dots, l_d): X \rightarrow \mathbb{R}^d$ is surjective. Set $P_t(B) := P(\xi_t \in B)$ and consider the measure $\mu := P_t dt$ on $X \times (0, 1)$. As above, there exist Borel functions b^i on $\mathbb{R}^d \times (0, 1)$ such that $b^i \circ \pi$ coincides with the conditional expectation of $l_i \circ b$ with respect to the measure μ and the σ -algebra generated by π . It is readily verified that the measure $\mu_\pi := \mu \circ \pi^{-1}$ on $\mathbb{R}^d \times (0, 1)$ satisfies equation (1.1) with the drift $b_\pi := (b^1, \dots, b^d)$ and certain constant coefficients a^{ij} . By using that the distribution of W_1 is not concentrated on a proper closed subspace one can show that the matrix (a^{ij}) is strictly positive. Therefore, according to Theorem 3.2, the measure μ_π has a strictly positive continuous density. The same can be obtained directly from Theorem 4.3. Indeed, one can show that there exist functionals $f_i \in X^*$ such that $f_i(W_t)$ are independent Wiener processes and the functionals l_1, \dots, l_d are linear combinations of f_1, \dots, f_d . To this end we denote by γ the distribution of W_1 in X , apply the standard orthogonalization procedure to the elements l_1, \dots, l_d of $L^2(\gamma)$ and complement the obtained functionals f_1, \dots, f_d to an orthonormal basis of the Euclidean space X^* equipped with the inner product from $L^2(\gamma)$ (see §7.2 in [22] on infinite-dimensional Wiener processes). Actually, we construct another Wiener process \widetilde{W}_t of the form $\widetilde{W}_t = \sum_{i=1}^{\infty} w_t^i e_i$, where $\{e_i\}$ is an orthonormal basis in the Cameron–Martin space of γ and w_t^i are independent real Wiener processes, such that the process \widetilde{W}_t has the same finite-dimensional distributions as W_t and $f_i(e_j) = \delta_{ij}$. Then Theorem 4.3 applies to the projections of μ generated by the functionals f_i . It remains to observe that there is an invertible operator S on \mathbb{R}^d such that $S \circ (f_1, \dots, f_d) = (l_1, \dots, l_d)$.

It would be interesting to study finite-dimensional projections of the measure μ in Theorem 4.3 corresponding to arbitrary continuous finite-dimensional operators, not necessarily generated by the functionals l_i .

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