Infinite dimensional Kolmogorov operators with time dependent drift coefficients

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Keywords: Kolmogorov operator, m-dissipativity, stochastic partial differential equation

The main object of this paper is the Kolmogorov operator in $[0,T]\times H$, where H is a separable Hilbert space with norm $|\cdot|$ and inner product $\langle\cdot,\cdot\rangle$ and $T>0$ is fixed, defined by

$$L_0u = D_tu + N(t)u,$$

$$N(t)u(t,x) = \frac{1}{2}\,\mathrm{Tr}\,[CD_x^2u(t,x)] + \langle x, A^*D_xu(t,x)\rangle + \langle F(t,x), D_xu(t,x)\rangle.$$

Here $A\colon D(A)\subset H\to H$ is the infinitesimal generator of a C_0 semigroup e^{tA} in H, A^* is the adjoint of A, C is a positive linear operator on H and the mapping $F\colon D(F)\subset [0,T]\times H\to H$ is such that $F(t,\cdot)$ is quasi dissipative for all $t\in[0,T]$ (see the definitions below). By D_t and D_x we denote the derivatives in t and x respectively. In order to define the domain of L_0 we introduce some functional spaces. We denote by $\mathcal{E}_A(H)$ the linear span of all real and imaginary parts of functions $e^{i\langle x,h\rangle}$ where $h\in D(A^*)$. For any $\varphi\in C^1([0,T])$ such that $\varphi(T)=0$ and any $h\in C^1([0,T];D(A^*))$ let us consider the function

$$u_{\varphi,h}(t,x) = \varphi(t)e^{i\langle x,h(t)\rangle},\quad t\in\mathbb{R},\ x\in H,$$

and denote by $\mathcal{E}_A([0,T]\times H)$ the linear span of all real and imaginary parts of such functions $u_{\varphi,h}$. The operator L_0 is defined on the space $D(L_0):=\mathcal{E}_A([0,T]\times H)$. Let $\mathcal{P}(H)$ be the set of all Borel probability measures on H and let $L(H)$ be the space of bounded linear operators on H.

First we shall assume that F is sufficiently regular and prove that, for any $\nu_0\in\mathcal{P}(H)$, there exists a unique family of probability measures $(\nu_t)_{t\in[0,T]}\subset\mathcal{P}(H)$ such that

$$\frac{d}{dt}\int_H u(t,x)\,\nu_t(dx) = \int_H L_0u(t,x)\,\nu_t(dx),\quad u\in D(L_0),\ t\in[0,T]. \tag{1}$$

This statement is an infinite dimensional analog of some results in [1]–[3]. Then we show that L_0 is essentially m-dissipative in the space $L^2([0,T]\times H;\nu)$ where ν is the measure in $[0,T]\times H$ defined by $\nu(dt,dx)=\nu_t(dx)dt$. In the case of irregular drifts we prove an analogous result under the assumption that there exists a suitable family of probability measures ν_t (see Condition (H2)). Finally, we apply the obtained results to reaction-diffusion equations with time-dependent coefficients.

Let us list the assumptions on the linear operator A which we will assume throughout.

Condition (H1):

(i) There is $\omega>0$ such that $\langle Ax,x\rangle\le -\omega|x|^2,\ x\in D(A)$;

(ii) $C\in L(H)$ is symmetric, nonnegative and such that the linear operator

$$Q_t := \int_0^t e^{sA}Ce^{sA^*}ds$$

is of trace class for all $t>0$;

(iii) $e^{tA}(H)\subset Q_t^{1/2}(H)$ for all $t>0$ and there is a bounded operator Λ_t such that $Q_t^{-1/2}\Lambda_t = e^{tA}$ and

$$\gamma_\lambda := \int_0^{+\infty} e^{-\lambda t}\|\Lambda_t\|dt < +\infty.$$

Let us note that assumption (iii) implies that the Ornstein–Uhlenbeck operator associated to L_0 (i.e., corresponding to $F = 0$) is strong Feller. This assumption is not essential but it allows to simplify several proofs. The Ornstein–Uhlenbeck operator is defined by

$$U\varphi(x) = \frac{1}{2}\,\mathrm{Tr}\,[CD_x^2\varphi(x)] + \langle x, A^*D_x\varphi(x)\rangle, \varphi \in \mathcal{E}_A(H).$$

In addition, we introduce the operator

$$V_0u(t,x) = D_tu(t,x) + Uu(t,x), \quad u \in \mathcal{E}_A([0,T]\times H),$$

and its maximal extension V.

Assume first that, in addition to Condition (H1), the mapping $F\colon [0,T]\times H \to H$ is continuous along with D_xF and there exists $K > 0$ such that

$$|F(t,x) - F(t,y)| \le K|x-y|, \;\; x,y \in H, \; t \in [0,T].$$

It is known (see, e.g., [4]) that, under this assumption, for any $s \ge 0$, there exists a unique mild solution $X(\cdot,s,x)$ of the stochastic differential equation

$$dX = (AX + F(t,X))dt + \sqrt{C}\;dW(t), \;\; X(s) = x \in H, \tag{2}$$

where $W(t)$, $t \ge 0$, is a cylindrical Wiener process in H defined on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$. A mild solution $X(t,s,x)$ of (2) is an adapted stochastic process $X \in C([s,T]; L^2(\Omega,\mathcal{F},\mathbb{P}))$ such that

$$X(t,s,x) = e^{(t-s)A}x + \int_s^t e^{(t-r)A}F(r,X(r,s,x))dr + W_A(t,s), \quad t \ge s,$$

where $W_A(t,s)$ is the *stochastic convolution*:

$$W_A(t,s) = \int_s^t e^{(t-r)A}\sqrt{C}\;dW(r), \quad t \ge s.$$

In view of Condition (H1)-(ii), $W_A(t,s)$ is a Gaussian random element in H with mean 0 and covariance operator $Q_{s,t}$ given by

$$Q_{s,t}x = \int_s^t e^{sA}Ce^{sA^*}xds, \quad t \ge s, \; x \in H.$$

We define the transition evolution operator

$$P_{s,t}\varphi(x) = \mathbb{E}[\varphi(X(t,s,x))], \quad t \ge s, \; \varphi \in C_b(H),$$

where $C_u(H)$ is the Banach space of all uniformly continuous and bounded functions $\varphi\colon H \to \mathbb{R}$ endowed with the usual supremum norm

$$\|\varphi\|_0 = \sup_{x\in H}|\varphi(x)|.$$

The spaces $C_u^k(H)$, $k \in \mathbb{N}$, are defined similarly.

We denote by $C_u^*(H)$ the topological dual of $C_u(H)$. For any $0 \le s < t \le T$, let $P_{s,t}^*$ be the adjoint operator of $P_{s,t}$. It is easy to see that if $\nu \in \mathcal{P}(H)$ we have $P_{s,t}^*\nu \in \mathcal{P}(H)$ and

$$\int_H \varphi(x)(P_{s,t}^*\nu)(dx) = \int_H P_{s,t}\varphi(x)\nu(dx), \quad \forall\,\varphi \in C_u(H).$$

In the case under consideration we have the following existence and uniqueness results and moment estimates.

Proposition 1. *Let $\nu_0 \in \mathcal{P}(H)$. Then $\nu_t = P_{0,t}^*\nu_0$ satisfies* (1).

Proposition 2. *Let $(\zeta_t)_{t\in[0,T]}$ be a solution of* (1) *such that*

$$\sup_{t\in[0,T]}\int_H |x|^2\zeta_t(dx) < +\infty.$$

Then $\zeta_t = P_{0,t}^*\nu_0$ for all $t \in [0,T]$.

Proposition 3. *Let $\nu_0 \in \mathcal{P}(H)$ and let $\nu_t = P^*_{0,t}\nu_0$, $t \in [0,T]$. Then for any $m \in \mathbb{N}$ there exists $c_m > 0$ such that*

$$\int_H |x|^{2m}\nu_t(dx) \leq c_m \left(1 + \int_H |x|^{2m}\nu(dx)\right), \quad t \geq 0.$$

We recall that a mapping $T\colon D(T) \subset E \to E$, where E is a Hilbert space, is called dissipative if $\langle T(x) - T(y), x - y\rangle \leq 0$ for all $x, y \in D(T)$. If, in addition, $(I - T)(D(T)) = E$, then T is called m-dissipative. The mapping T is called quasi dissipative if $T - KI$ is dissipative for some $K > 0$. If $T - KI$ is m-dissipative, then T is called m-quasi dissipative.

It is readily verified that L_0 is dissipative in $L^2([0,T] \times H, \nu)$, hence it is closable. Let us denote its closure by L_2.

Theorem 1. *The operator L_2 is m-dissipative in the space $L^2([0,T] \times H; \nu)$.*

Let us turn to the case of irregular drifts. Suppose we are given a family $\{F(t,\cdot)\}_{t\in[0,T]}$ of m-quasi dissipative mappings $F(t,\cdot)\colon D(F(t,\cdot)) \subset H \to H$, where $D(F(t,\cdot))$ are Borel sets in H. For simplicity we shall assume that these mappings are m-dissipative. We are concerned with the Kolmogorov operator

$$L_0u(t,x) := D_tu(t,x) + Uu(t,x) + \langle F_0(t,\cdot), D_xu(t,x)\rangle, \ \ u \in D(L_0),$$

where $D(L_0) = \mathcal{E}_A([0,T] \times H)$ and U is the Ornstein–Uhlenbeck operator. Our goal is to prove that the closure of L_0 is m-dissipative in the space $L^2([0,T] \times H, \nu)$, where $\nu(dt, dx) = \nu_t(dx)dt$ and $(\nu_t)_{t\in[0,T]}$ is a given family of Borel measures on H such that

$$\frac{d}{dt}\int_H u(t,x)\nu_t(dx) = \int_H L_0u(t,x)\nu_t(dx), \quad \forall\, u \in D(L_0). \tag{3}$$

In addition to Condition (H1) we shall assume also

Condition (H2):

(i) There is a family $\{F(t,\cdot)\}_{t\in[0,T]}$ of m-quasi dissipative mappings in H such that $0 \in D(F(t,\cdot))$ and $F_0(t,0) = 0$ for all $t \in \mathbb{R}$;

(ii) there is a family $(\nu_t)_{t\in[0,T]}$ of Borel probability measures on H such that

$$\int_H |x|^2\, \nu_0(dx) < +\infty$$

and

$$\int_0^T \int_H (|x|^4 + |F_0(t,x)|^2 + |x|^4|F_0(t,x)|^2)\, \nu_t(dx)\, dt < +\infty;$$

(iii) for all $u \in D(L_0)$ we have $L_0u \in L^2([0,T] \times H, \nu)$ and (3) is fulfilled;

(iv) $\nu_t(D(F(t,\cdot)) = 1, \quad \forall\, t \in [0,T]$.

The problem of existence of measures ν_t as above is studied in our separate paper.

Let us assume that Conditions (H1) and (H2) hold.

Proposition 4. *For all $u \in D(L_0)$ we have*

$$\int_0^T \int_H L_0u(t,x)\; u(t,x)\, \nu_t(dx)\, dt = -\frac{1}{2}\int_0^T \int_H |C^{1/2}D_xu(t,x)|^2\, \nu_t(dx)\, dt - \int_H u^2(0,x)\, \nu_0(dx).$$

In particular, L_0 is dissipative in $L^2([0,T] \times H, \nu)$.

Since L_0 is dissipative, it is closable in $L^2([0,T] \times H, \nu)$. We shall denote its closure with domain $D(L_2)$ by L_2.

Now we can state the main result of this note.

Theorem 2. *Under Conditions* $(H1)$ *and* $(H2)$, *the operator* L_2 *is* m*-dissipative in the space* $L^2([0,+\infty)\times H,\nu)$.

Now we apply our result to the reaction-diffusion equation. Let us consider a stochastic heat equation perturbed by a time dependent polynomial drift of odd degree $d>1$ of the form $\lambda\xi - p(t,\xi)$, $\xi\in\mathbb{R}$, $t\in[0,T]$, where $\lambda\in\mathbb{R}$ is given and $\partial_\xi p(t,\xi)\ge 0$ for all $\xi\in\mathbb{R}$ and $t\in[0,T]$. We set $H=L^2(\mathcal{O})$, where $\mathcal{O}=(0,1)^n$, $n\in\mathbb{N}$, and denote by $\partial\mathcal{O}$ the boundary of $\mathcal{O}$. Let us consider the following stochastic partial differential equation on $\mathcal{O}$:

$$\begin{gathered} dX(t,s,\xi) = [\Delta_\xi X(t,s,\xi) + \lambda X(t,s,\xi) - p(t,X(t,s,\xi))]dt + BdW(t,\xi),\ t\ge s, \\ X(t,s,\xi)=0,\ t\ge s,\ \xi\in\partial\mathcal{O},\ X(s,s,\xi)=x(\xi),\ \xi\in\mathcal{O},\ x\in H, \end{gathered} \tag{4}$$

where Δ_ξ is the Laplace operator, $B\in L(H)$, and W is a cylindrical Wiener process in H defined on a probability space $(\Omega,\mathcal{F},\mathbb{P})$. We choose W of the form

$$W(t,\xi)=\sum_{k=1}^\infty e_k(\xi)\beta_k(t),\quad \xi\in\mathcal{O},\ t\ge 0,$$

where (e_k) is a complete orthonormal system in H and (β_k) is a sequence of independent standard Brownian motions on a probability space $(\Omega,\mathcal{F},\mathbb{P})$.

In order to write problem (4) as a stochastic differential equation in the Hilbert space H we denote by A the realization of the Laplace operator with Dirichlet boundary conditions, i.e.,

$$Ax=\Delta_\xi x,\quad x\in D(A),\ D(A)=H^2(\mathcal{O})\cap H_0^1(\mathcal{O}).$$

The operator A is self-adjoint and possesses a complete orthonormal system of eigenfunctions, namely

$$e_k(\xi)=(2/\pi)^{n/2}\ \sin(\pi k_1\xi_1)\cdots(\sin\pi k_n\xi_n),\quad \xi=(\xi_1,...,\xi_n)\in\mathbb{R}^n,$$

where $k=(k_1,\dots,k_n)$, $k_i\in\mathbb{N}$.

For any $x\in H$ we set $x_k=\langle x,e_k\rangle$, $\quad k\in\mathbb{N}^n$. Notice that

$$Ae_k=-\pi^2|k|^2e_k,\quad k\in\mathbb{N}^n,\ |k|^2=k_1^2+\cdots+k_n^2.$$

Therefore, we have

$$\|e^{tA}\|\le e^{-\pi^2t},\quad t\ge 0.$$

Concerning the operator B, we shall assume for simplicity that $B=(-A)^{-\gamma/2}$ with $n/2-1<\gamma<1$ (which implies $n<4$). Now it is easy to check that Condition (H1) is fulfilled. In fact we have

$$\begin{aligned} Q_tx &= \int_0^t e^{sA}BB^*e^{sA*}x\,ds = \int_0^t(-A)^{-\gamma}e^{2tA}x\,dt \\ &= (-A)^{-(1+\gamma)}(1-e^{2tA})x,\quad t\ge 0,\ x\in H. \end{aligned}$$

Then

$$\text{Tr}\,[(-A)^{-(1+\gamma)}]=\pi^{-2(1+\gamma)}\sum_{k\in\mathbb{N}^n}|k|^{-2(1+\gamma)}<+\infty,$$

since $\gamma>\frac{n}{2}-1$.

Now, setting $X(t,s)=X(t,s,\cdot)$ and $W(t)=W(t,\cdot)$, we can write problem (4) as

$$dX(t,s)=[AX(t,s)+F(t,X(t,s))]dt+(-A)^{-\gamma/2}dW(t),\ t\ge s,\ X(s,s)=x, \tag{5}$$

where F is the mapping

$$F\colon\ D(F)=[0,T]\times L^{2d}(\mathcal{O})\subset[0,T]\times H\to H,\ x(\xi)\mapsto\lambda\xi-p(t,x(\xi)).$$

It is convenient, following [4], to introduce two different notions of solution of (5). For this purpose, for any $s \in [0,T)$, we consider the space

$$C_W([s,T]; L^2(\Omega, \mathcal{F}, \mathbb{P}; H)) := C_W([s,T]; H))$$

consisting of all continuous mappings $F\colon [s,T] \to L^2(\Omega, \mathcal{F}, \mathbb{P}; H)$ adapted to W; endowed with the norm

$$\|F\|_{C_W([s,T];H))} = \left(\sup_{t\in[s,T]} \mathbb{E}\left(|F(t)|^2\right) \right)^{1/2}$$

$C_W([s,T]; H))$ is a Banach space.

Definition 1. (i) *Let $x \in L^{2d}(\mathcal{O})$. We say that a mapping $X(\,\cdot, s, x) \in C_W([s,T]; H)$ is a mild solution of problem (5) if $X(t,s,x) \in L^{2d}(\mathcal{O})$ for all $t \in [s,T]$ and the following integral equality holds:*

$$X(t,s,x) = e^{(t-s)A}x + \int_s^t e^{(t-r)A} F(r, X(r,s,x))\, dr + W_A(s,t), \quad t \ge 0,$$

where $W_A(s,t)$ is the stochastic convolution

$$W_A(s,t) = \int_s^t e^{(t-r)A}(-A)^{-\gamma/2}\, dW(s), \quad t \ge 0.$$

(ii) *Let $x \in H$ and $s \in [0,T]$. We say that $X(\,\cdot, s, x) \in C_W([s,T]; H)$ is a generalized solution of problem (5) if there exists a sequence $(x_n) \subset L^{2d}(\mathcal{O})$ such that*

$$\lim_{n\to\infty} x_n = x \quad \text{in} \;\; H,$$

mild solutions $X(\,\cdot, s, x_n)$ exist and

$$\lim_{n\to\infty} X(\,\cdot, s, x_n) = X(\,\cdot, s, x) \quad \text{in} \;\; C_W([s,T]; H).$$

We shall denote by $X(t,s,x)$ mild and generalized solutions of (5).

Theorem 3. *The following statements are true.*

(i) *If $x \in L^{2d}(\mathcal{O})$, problem (5) has a unique mild solution $X(\cdot, s, x)$. Moreover for any $m \in \mathbb{N}$, there is $c_{m,p,T} > 0$ such that*

$$\mathbb{E}\left(|X(t,s,x)|^{2m}_{L^{2d}(\mathcal{O})}\right) \le c_{m,p,T}\left(1 + |x|^{2m}_{L^{2d}(\mathcal{O})}\right), \quad 0 \le s \le t \le T.$$

(ii) *If $x \in H$, problem (4) has a unique generalized solution $X(\cdot, s, x)$.*

For any $0 \le s \le t \le T$, let us consider the transition evolution operator

$$P_{s,t}\varphi(x) = \mathbb{E}[\varphi(X(t,s,x))], \quad \varphi \in C_u(H),$$

where $X(t,s,x)$ is a generalized solution of (5). Then, given $\nu_0 \in \mathcal{P}(H)$, set

$$\nu_t = P^*_{0,t}\nu_0, \quad t \in [0,T]. \tag{6}$$

Corollary 1. *Let $m \in \mathbb{N}$ and assume that $\nu_0 \in \mathcal{P}(H)$ satisfies*

$$\int_H |x|^{2m}_{L^{2d}(\mathcal{O})} \nu_0(dx) < +\infty.$$

Then we have

$$\int_H |x|^{2m}_{L^{2d}(\mathcal{O})} \nu_t(dx) \le c_{m,p,T} \int_H |x|^{2m}_{L^{2d}(\mathcal{O})} \nu_0(dx).$$

Theorem 4. *Assume that $\nu_0 \in \mathcal{P}(H)$ satisfies*

$$\int_H \left(|x|^4 + |F(0,x)|^2 + |x|^4 |F_0(0,x)|^4\right)\, \nu_0(dx) < +\infty.$$

Then the operator L_0 associated with (5) is closable and its closure is m-dissipative in $L^2([0,+\infty) \times H, \nu)$, where $\nu(dt,dx) = \nu_t(dx)dt$ and ν_t is defined by (6).

On the dissipativity of Kolmogorov operators see also [5]–[7].

This work has been supported by the RFBR projects 07-01-00536, 05-01-02941-JF, 06-01-39003 GFEN, the DFG Grant 436 RUS 113/343/0(R), the ARC Discovery Grant DP0663153 (Sydney), the programme SFB 701 at the University of Bielefeld, and the research programme "Equazioni di Kolmogorov" of the Italian "Ministero della Ricerca Scientifica e Tecnologica".

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