Some results on stochastic porous media equations

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Abstract. Some recent results about nonnegative solutions of stochastic porous media equations in bounded open subsets of $\mathbb{R}^3$ are considered. The existence of an invariant measure is proved.

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1 Introduction

Let $\mathscr{O}$ be a non empty bounded open subset of $\mathbb{R}^3$ with smooth boundary $\partial\mathscr{O}$, of class C^2 for instance. We are concerned with the following *porous*

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media equation in $\mathscr{O}$ perturbed by noise

$$\begin{cases} dX(t) = \Delta(\beta(X(t))dt + \displaystyle\sum_{k=1}^{\infty} \sigma_k(X(t))d\gamma_k(t), \quad t \geq 0, \\ \beta(X(t)) = 0, \quad \text{on } \partial\mathscr{O}, \quad t \geq 0, \\ X(0) = x, \end{cases} \tag{1.1}$$

under the following assumptions,

Hypothesis 1.1

(i) $\beta(r) = \alpha r^m + \lambda r$ *where* m *is an odd integer strictly greater than* 1 *and* $\alpha > 0, \lambda \geq 0$.

(ii) $\sigma_k(x) = \mu_k\, x e_k,\ k \in \mathbb{N}$, *where* $\{\mu_k\}$ *is a sequence of positive numbers and* $\{e_k\}$ *is the complete orthonormal system in* $L^2(\mathscr{O})$ *consisting of eigenfunctions of the Dirichlet Laplacian problem in* $\mathscr{O}$.

(iii) $\{\gamma_k\}$ *is a sequence of (mutually) independent standard Brownian motions on a filtered probability space* $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$.

An additional assumption on the sequence $\{\mu_k\}$ will be made later.

When the $\{\sigma_k\}$ are independent of x we say that the noise is *additive* (see the paper [6]). It is well known that in this case the positivity of the solution to (1.1) for $x \geq 0$ does not hold. Since we are here interested in finding positive solutions of (1.1), we will consider the multiplicative noise (ii).

We note that the assumption on β covers many important models of dynamics of ideal gases in porous media and extends to functions β with polynomial growth which are coercive, i.e.,

$$\beta(r)r \geq \alpha_1 r^{m+1} + \alpha_2 r^2, \quad |\beta(r)| \leq \alpha_3(r^m + 1),$$

with $\alpha_i \geq 0, i = 1, 2, 3$ (see [4]).

Other important cases, with more general β have been studied, in [10] and [5].

In this paper we shall give a review of the main results in [4], trying to explain the main ideas which are involved and avoiding technicalities as much as possible. In addition we shall discuss invariant measures for equation (1.1).

2 Notations and setting of the problem

2.1 Some functional spaces

We shall use the following notations.

- $L^2(\mathscr{O})$ is the Hilbert space consisting of all (equivalence classes) of mappings $x : \mathscr{O} \to \mathbb{R}$ which are measurable and square integrable, endowed with the scalar product

$$\langle x, y\rangle = \int_{\mathscr{O}} x(\xi)y(\xi)d\xi, \quad x, y \in L^2(\mathscr{O}).$$

  We identify $L^2(\mathscr{O})$ with its topological dual.

  For $p > 2$ the space $L^p(\mathscr{O})$ is similarly defined. We note the norm in $L^p(\mathscr{O})$ by $|\cdot|_p$.

- $H^1(\mathscr{O})$ (resp. $H^2(\mathscr{O})$) is the space of all mappings $x \in L^2(\mathscr{O})$ whose first (resp. first and second) derivatives in the sense of distributions belong to $L^2(\mathscr{O})$. We set moreover

$$H_0^1(\mathscr{O}) = \{x \in H^1(\mathscr{O}) : \ x = 0 \text{ on } \partial\mathscr{O}\}.$$

- Δ is the realization of the Laplace operator with Dirichlet boundary conditions in $L^2(\mathscr{O})$,

$$\begin{cases} \Delta x = \displaystyle\sum_{k=1}^{3} \partial_k^2 x, \quad \forall\, x \in D(\Delta), \\ \\ D(\Delta) = H^2(\mathscr{O}) \cap H_0^1(\mathscr{O}). \end{cases}$$

  It is well known that $-\Delta$ is self-adjoint, positive and anti-compact operator. So, there exists a complete orthonormal system $\{e_k\}$ in $L^2(\mathscr{O})$ of eigenfunctions of $-\Delta$ [1]. We denote by $\{\lambda_k\}$ the corresponding sequence of eigenvalues,

$$\Delta e_k = -\lambda_k e_k, \quad k \in \mathbb{N}.$$

  By the Sobolev embedding theorem [2] it follows that

$$e_k \in C(\overline{\mathscr{O}}), \quad \forall\, k \in \mathbb{N};$$

[1] the system which is considered in Hypothesis 1.1.

[2] Since $\mathscr{O} \subset \mathbb{R}^3$ we have $H^2(\mathscr{O}) \subset C(\overline{\mathscr{O}})$ and $H^1(\mathscr{O}) \subset L^6(\mathscr{O})$.

however the sequence $\{e_k\}$ is not equibounded in $C(\overline{\mathscr{O}})$ in general. The following elementary estimate is useful

$$|e_k|_\infty \le c_0 |e_k|_{H^2} \le c_1 |\Delta e_k|_2 = c_1 \lambda_k, \quad k \in \mathbb{N}, \tag{2.1}$$

where c_0 and c_1 are suitable positive constants.

- $H^{-1}(\mathscr{O})$ is the topological dual of $H_0^1(\mathscr{O})$. It is well known that the Laplace operator Δ can be extended to an isomorphism of $H_0^1(\mathscr{O})$ onto $H^{-1}(\mathscr{O})$ (which we shall still denote by Δ).

  We denote again by $\langle \cdot, \cdot \rangle$ the duality between $H_0^1(\mathscr{O})$ and $H^{-1}(\mathscr{O})$.

  $H^{-1}(\mathscr{O})$ is endowed with the inner product

  $$\langle x, y \rangle_{-1} = -\langle \Delta^{-1} x, y \rangle, \quad x, y \in H^{-1}(\mathscr{O}).$$

  For further use we note that there exists a constant $c_2 > 0$ such that

  $$|x e_k|_{-1} \le c_2 \lambda_k |x|_{-1}, \quad \forall\, k \in \mathbb{N}. \tag{2.2}$$

  We have in fact

  $$|x e_k|_{-1}^2 = \sup\{|\langle x e_k, \phi \rangle|^2 : \ \phi \in H_0^1(\mathscr{O}),\ \|\phi\|_{H_0^1(\mathscr{O})} \le 1\}.$$

  Moreover,

  $$\begin{aligned} &|\langle x e_k, \phi \rangle|^2 \le |x|_{-1}^2 \ |e_k \phi|_{H_0^1}^2 \le 2|x|_{-1}^2 \ (|\phi \nabla e_k|_2^2 + |e_k \nabla \phi|_2^2) \\ &\le 2|x|_{-1}^2 \ (|\nabla e_k|_4^2 \ |\phi|_4^2 + |e_k|_\infty^2 \ |\phi|_{H_0^1}^2) \\ &\le C|x|_{-1}^2 \ |\phi|_{H_0^1}^2 (|e_k|_{H^2}^2 + |e_k|_\infty^2), \end{aligned}$$

  which implies (2.2).

Notice also that

$$\begin{aligned} \mathbb{E}\left|\sum_{k=1}^\infty \mu_k \int_0^t X(s) e_k d\gamma_k(s)\right|_{-1}^2 &= \sum_{k=1}^\infty \mu_k^2 \mathbb{E} \int_0^t |X(s) e_k|_{-1}^2 ds \\ &\le c_2^2 \sum_{k=1}^\infty \mu_k^2 \lambda_k^2 \ \mathbb{E} \int_0^t |X(s)|_{-1}^2 ds. \end{aligned}$$

In order that this quantity is finite (as we shall need later in several computations) we shall also assume that

Hypothesis 2.1 *We have*

$$\sum_{k=1}^{\infty} \mu_k^2 \lambda_k^2 := \kappa_1 < +\infty. \tag{2.3}$$

2.2 Abstract formulation of the problem

Let us we write equation (1.1) in an abstract form. For this purpose we introduce the following nonlinear operator in $H^{-1}(\mathscr{O})$.

$$\begin{cases} A(x) = -\Delta(\beta(x)), \quad x \in D(A), \\ \\ D(A) = \{x \in H^{-1}(\mathscr{O}) \cap L^1(\mathscr{O}) : \ \beta(x) \in H_0^1(\mathscr{O})\}. \end{cases} \tag{2.4}$$

It happens that the operator A is maximal monotone (see e.g. [2]) and this is the reason for studying equation (1.1) in the space $H^{-1}(\mathscr{O})$ which will denote by H in the following.

Let us write equation (1.1) in the following form.

$$\begin{cases} dX(t) + A(X(t))dt = \displaystyle\sum_{k=1}^{\infty} \mu_k X(t) e_k d\gamma_k(t), \quad t \geq 0, \\ \\ X(0) = x. \end{cases} \tag{2.5}$$

We note that, in view of Hypothesis 2.1, the series above is convergent provided $X(t) \in H^{-1}(\mathscr{O})$.

We are now going to define a concept of solution for (2.5). Since we have no hope to find a solution $X(t)$ belonging to $D(A)$, we shall give a weak concept of solution. For this we need some functional spaces.

For any $T > 0$ we shall denote by $L_W^2(0, T; L^2(\Omega, H))$ the set of all adapted processes $X(t)$ such that

$$\mathbb{E} \int_0^T \int_{\mathscr{O}} |X(t, \xi)|^2 dt d\xi < +\infty. \tag{2.6}$$

Moreover, by $C_W([0, T]; L^2(\Omega, H))$ we denote the subspace of $L_W^2(0, T; L^2(\Omega, H))$ of all mean square continuous processes.

Definition 2.2 *A solution of* (2.5) *is an H-valued continuous adapted process X such that*

$$X \in C_W([0, T]; L^2(\Omega, H)) \cap L^{m+1}(\Omega \times (0, T) \times \mathscr{O})$$

and for any $j \in \mathbb{N}$

$$\begin{aligned}(X(t), e_j)_2 &= (x, e_j)_2 - \lambda_j \int_0^t \int_{\mathscr{O}} \beta(X(s)) e_j d\xi ds \\ &\quad + \sum_{k=1}^{\infty} \mu_k \int_0^t (X(s)e_k, e_j)_2 d\gamma_j(s).\end{aligned} \tag{2.7}$$

Since

$$(X(t), e_j)_2 = \lambda_j \langle X(t), e_j \rangle_{-1}, \quad j \in \mathbb{N},$$

we may equivalently write (2.7) as follows

$$\begin{aligned}\langle X(t), e_j \rangle_{-1} &+ \int_0^t \int_{\mathscr{O}} \beta(X(s)) e_j d\xi ds = \langle x, e_j \rangle_{-1} \\ &\quad + \sum_{k=1}^{\infty} \mu_k \int_0^t \langle X(s)e_k, e_j \rangle_{-1} d\gamma_j(s).\end{aligned} \tag{2.8}$$

3 Existence and uniqueness

We shall first consider the equation

$$\begin{cases} dX^*(t) + A(X^*(t))dt = \displaystyle\sum_{k=1}^{\infty} \mu_k Z(t) e_k d\gamma_k(t), \quad t \geq 0, \\ \\ X^*(0) = x, \end{cases} \tag{3.1}$$

where $Z \in C_W([0,T]; L^2(\Omega, H))$ has been fixed. Then we shall solve (2.5) showing that the mapping

$$C_W([0,T]; L^2(\Omega, H)) \to C_W([0,T]; L^2(\Omega, H)), \ Z \to X^*$$

has a fixed point.

Also equation (3.1) will be solved in a weak sense, precised by the following definition.

Definition 3.1 *A solution of* (3.1) *is an* H*-valued continuous adapted process* X^* *such that*

$$X^* \in C_W([0,T]; L^2(\Omega, H)) \cap L^{m+1}(\Omega \times (0,T) \times \mathscr{O})$$

and for any $j \in \mathbb{N}$

$$
\begin{aligned}
\langle X^*(t), e_j \rangle_{-1} \quad + \quad & \int_0^t \int_{\mathscr{O}} \beta(X^*(s)) e_j d\xi ds = \langle x, e_j \rangle_{-1} \\
& + \sum_{k=1}^{\infty} \mu_k \int_0^t \langle Z(s) e_k, e_j \rangle_{-1} d\gamma_j(s).
\end{aligned}
\tag{3.2}
$$

3.1 The solution of (3.1)

Let us introduce the approximating equation,

$$
\begin{cases}
dX_\varepsilon(t) + A_\varepsilon(X_\varepsilon(t))dt = \displaystyle\sum_{k=1}^{\infty} \mu_k Z(t) e_k d\gamma_k(t), \quad t \geq 0, \\[2ex]
X_\varepsilon(0) = x,
\end{cases}
\tag{3.3}
$$

where A_ε are the Yosida approximations of the maximal monotone operator A,

$$
A_\varepsilon(x) = \frac{1}{\varepsilon}\,(x - J_\varepsilon(x)) = A(J_\varepsilon(x)), \quad \varepsilon > 0,\ x \in H,
$$

and $J_\varepsilon(x) = (1 + \varepsilon A)^{-1}(x)$.

As is well known (see e.g. [2]), A_ε is maximal monotone and Lipschitzian on H. Notice also that

$$
\begin{aligned}
\langle A_\varepsilon(x), x \rangle_{-1} \quad &= \quad \langle A J_\varepsilon(x), J_\varepsilon(x) \rangle_{-1} + \langle A J_\varepsilon(x), x - J_\varepsilon(x) \rangle_{-1} \\
&= \quad \langle A J_\varepsilon(x), J_\varepsilon(x) \rangle_{-1} + \varepsilon |A_\varepsilon(x)|^2,
\end{aligned}
$$

so that

$$
\langle A_\varepsilon x, x \rangle_{-1} = \langle A J_\varepsilon(x), J_\varepsilon(x) \rangle_{-1} + \frac{1}{\varepsilon}\,|x - J_\varepsilon(x)|_{-1}^2 \tag{3.4}
$$

By standard existence theory for stochastic equations in Hilbert spaces, equation (3.3) has a unique solution $X_\varepsilon := \Gamma_\varepsilon(Z) \in C_W([0,T]; L^2(\Omega; H))$ (see e.g. [7]).

Lemma 3.2 *Assume that Hypotheses* 1.1 *and* 2.1 *are fulfilled. Then for any* $x \in H^{-1}(\mathscr{O})$ *and any* $Z \in C_W([0,T]; L^2(\Omega, H))$ *there exists a unique solution* $X^* := \Gamma(Z)$ *of* (3.1) *such that*

$$
X^* \in C_W([0,T]; L^2(\Omega, H)) \cap L^{m+1}(\Omega \times (0,T) \times \mathscr{O}).
$$

Moreover, there exists a constant $C > 0$ such that for any $Z, Z_1 \in C_W([0,T]; L^2(\Omega, H))$ we have

$$\mathbb{E}|X^*(t) - X_1^*(t)|_{-1}^2 \leq CE \int_0^t |Z(s) - Z_1(s)|_{-1}^2 ds, \quad \forall\, t \in [0,T], \quad (3.5)$$

where $X_1^* = \Gamma(Z_1)$.

Proof. By Itô's formula we have

$$\frac{1}{2}\,\mathbb{E}|X_\varepsilon(t)|_{-1}^2 + \mathbb{E}\int_0^t \langle A_\varepsilon X_\varepsilon(s), X_\varepsilon(s)\rangle_{-1} ds$$

$$= \frac{1}{2}\,\mathbb{E}|x|_{-1}^2 + \sum_{k=1}^\infty \mu_k^2\,\mathbb{E}\int_0^t |Z(s)e_k|_{-1}^2 ds.$$

Now, setting $Y_\varepsilon = J_\varepsilon(X_\varepsilon)$ and taking into account (3.4) and Hypothesis 2.1, we obtain

$$\begin{aligned}
&\frac{1}{2}\,\mathbb{E}|X_\varepsilon(t)|_{-1}^2 + \mathbb{E}\int_0^t (\beta(Y_\varepsilon(s)), Y_\varepsilon(s))\, ds + \frac{1}{\varepsilon}\,\mathbb{E}\int_0^t |X_\varepsilon(s) - Y_\varepsilon(s)|_{-1}^2 ds \\
&= \frac{1}{2}\,\mathbb{E}|x|_{-1}^2 + \sum_{k=1}^\infty \mu_k^2\,\mathbb{E}\int_0^t |Z(s)e_k|_{-1}^2 ds \\
&\leq \frac{1}{2}\,\mathbb{E}|x|_{-1}^2 + \kappa_1\mathbb{E}\int_0^t |Z(s)|_{-1}^2 ds.
\end{aligned} \quad (3.6)$$

From (3.6) it follows that

$$\begin{cases} \{X_\varepsilon\} & \text{is bounded in } C_W([0,T]; L^2(\Omega, H)), \\ \{Y_\varepsilon\} & \text{is bounded in } L^{m+1}(\Omega \times (0,T) \times \mathscr{O}). \end{cases}$$

Therefore there exists a sequence $\varepsilon_k \downarrow 0$, and a pair of processes (X^*, η^*) such that

$$X^* \in L^{m+1}(\Omega \times (0,T) \times \mathscr{O}).$$

and

$$\eta^* \in L^{\frac{m+1}{m}}(\Omega \times (0,T) \times \mathscr{O})$$

such that

$$\begin{cases} \lim_{k\to\infty} X_{\varepsilon_k} = X^* \quad \text{weakly in } L^{m+1}(\Omega\times(0,T)\times\mathscr{O}), \\ \\ \lim_{k\to\infty} \beta(Y_{\varepsilon_k}(s)) = \eta^* \quad \text{weakly in } L^1(\Omega\times(0,T)\times\mathscr{O}). \end{cases}$$

Passing to the limit in equation (3.3) we see that X^* fulfills the identity

$$\begin{aligned} \langle X^*(t),\phi\rangle_{-1} &= \langle x,\phi\rangle_{-1} - \int_0^t \int_{\mathscr{O}} \eta(s)\phi d\xi ds \\ &\quad + \sum_{k=1}^{\infty} \mu_k\lambda_k \int_0^t (Z(s)e_k,e_j)_2 d\gamma_j(s). \end{aligned} \tag{3.7}$$

To conclude the proof of existence it suffices to show that

$$\eta = \beta(X^*) \quad \text{a.e. in } \ \Omega\times(0,T)\times\mathscr{O}. \tag{3.8}$$

Indeed, in such a case we may take in (3.7) $\phi = \Delta e_j$ for $j \in \mathbb{N}$.

To show (3.8) consider the lower semicontinuous convex function ossn $L^m(\Omega\times(0,T)\times\mathscr{O})$,

$$\Phi(x) = \frac{1}{m+1}\,\mathbb{E}\int_0^T\int_{\mathscr{O}} |x(t,\xi)|^{m+1}\,dtd\xi + \frac{\lambda}{2}\,\mathbb{E}\int_0^T\int_{\mathscr{O}} |x(t,\xi)|^2\,dtd\xi.$$

We claim that

$$\Phi(X^*)-\Phi(U) \le \mathbb{E}\int_0^T\int_{\mathscr{O}} \eta(X^*-U)dtd\xi, \quad \forall\, U \in L^{m+1}(\Omega\times(0,T)\times\mathscr{O}). \tag{3.9}$$

It is clear that (3.9) yields (3.8). We tray to deduce (3.9) letting $k\to\infty$ in the inequality

$$\Phi(Y_{\varepsilon_k})-\Phi(U) \le \mathbb{E}\int_0^T\int_{\mathscr{O}} \beta(Y_{\varepsilon_k})(Y_{\varepsilon_k}-U)dtd\xi \quad \forall\, U \in L^{m+1}(\Omega\times(0,T)\times\mathscr{O}). \tag{3.10}$$

We obtain by the lower semicontinuity of Φ and the fact that $\{\beta(Y_{\varepsilon_k})\}$ weakly converges to η, that

$$\Phi(X^*)-\Phi(U) \le \liminf_{k\to\infty}\mathbb{E}\int_0^T\int_{\mathscr{O}} \beta(Y_{\varepsilon_k})Y_{\varepsilon_k}dtd\xi - \mathbb{E}\int_0^T\int_{\mathscr{O}} \eta U dtd\xi. \tag{3.11}$$

So, in order to prove (3.9) it remains to show that

$$\liminf_{k\to\infty} \mathbb{E}\int_0^T \int_{\mathscr{O}} \beta(Y_{\varepsilon_k})Y_{\varepsilon_k} dt d\xi \le \mathbb{E}\int_0^T \int_{\mathscr{O}} \eta X^* dt d\xi. \tag{3.12}$$

For this we go back to the Itô formula (3.6) from which we deduce that

$$\begin{aligned}&\frac{1}{2}\,\mathbb{E}|X_\varepsilon(t)|_{-1}^2 + \mathbb{E}\int_0^T \int_{\mathscr{O}} \beta(Y_{\varepsilon_k})Y_{\varepsilon_k} dt d\xi\\ &\le \frac{1}{2}\,\mathbb{E}|x|_{-1}^2 + \sum_{k=1}^\infty \mu_k^2\,\mathbb{E}\int_0^t |Z(s)e_k|_{-1}^2 ds.\end{aligned} \tag{3.13}$$

Next we apply Itô formula to (3.6) and find that

$$\begin{aligned}&\frac{1}{2}\,\mathbb{E}|X_\varepsilon(t)|_{-1}^2 + \mathbb{E}\int_0^T \int_{\mathscr{O}} \eta(s)X^*(s) dt d\xi\\ &\le \frac{1}{2}\,\mathbb{E}|x|_{-1}^2 + \sum_{k=1}^\infty \mu_k^2\,\mathbb{E}\int_0^t |Z(s)e_k|_{-1}^2 ds.\end{aligned} \tag{3.14}$$

Comparing (3.13) and (3.14) yields (3.12). So, existence is proved.

Now (3.5) follows from Itô's formula and therefore uniqueness follows from (3.5) and the Gronwall lemma. □

3.2 Existence and uniquenesss for (2.8)

Theorem 3.3 *Assume that Hypotheses* 1.1 *and* 2.1 *are fulfilled. Then for any $x \in H^{-1}(\mathscr{O})$ there exists a unique solution X of* (2.8) *such that*

$$X \in C_W([0,T]; L^2(\Omega, H)) \cap L^{m+1}(\Omega \times (0,T) \times \mathscr{O}).$$

Proof. By (3.5) it follows that

$$|\Gamma(Z) - \Gamma(Z_1)|_{C_W([0,T];L^2(\Omega,H))} \le CT|Z - Z_1|_{C_W([0,T];L^2(\Omega,H))},$$

for all $Z, Z_1 \in C_W([0,T]; L^2(\Omega, H))$. Thus the operator Γ is a contraction in $C_W([0,T_1]; L^2(\Omega, H))$, where $T_1 = \frac{1}{2C}$. Therefore there exists a unique solution of (2.8) in the interval $[0,T_1]$. In a similar way we can prove existence and uniqueness of a solution in the interval $[T_1, 2T_1]$ and so on. The conclusion follows now in a finite numbers of steps. □

In fact, one can prove that X has continuous sample paths in H (see [10])

4 Regularity

By Theorem 3.3 it follows that there exists a unique solution

$$X \in C_W([0,T]; L^2(\Omega, H)) \cap L^{m+1}(\Omega \times (0,T) \times \mathscr{O}).$$

of (2.8) provided $x \in H^{-1}(\mathscr{O})$. Our aim is to show that if $x \geq 0$ (in the sense of distributions) then $X(t) \geq 0$ for all $t \in [0,T]$.

Let us introduce the approximating equation,

$$\begin{cases} dX_\varepsilon(t) + A_\varepsilon(X_\varepsilon(t))dt = \displaystyle\sum_{k=1}^{\infty} \mu_k X_\varepsilon(t) e_k d\gamma_k(t), \quad t \geq 0, \\ \\ X_\varepsilon(0) = x, \end{cases} \tag{4.1}$$

We are going to find a unique solution X_ε of equation (4.1) in $C_W([0,T]; L^2(\Omega \times \mathscr{O}))$ and prove that $X_\varepsilon \to X$ in $C_W([0,T]; L^2(\Omega; H))$ as $\varepsilon \to 0$.

It is easier to discuss positivity in the space $L^2(\mathscr{O})$ instead of in $H^{-1}(\mathscr{O})$. For this we shall prove some regularity results for the solution of equation (4.1), namely that if $x \in L^p(\mathscr{O})$ then $X_\varepsilon(t) \in L^p(\mathscr{O})$ for all $t \in [0,T]$ (with estimates independent of ε. These regularity results are also needed in order to prove that $X_\varepsilon \to 0$ in $C_W([0,T]; L^2(\Omega; H)$.

To solve equation (4.1) in $L^p(\mathscr{O})$ we need some additional properties of the operators J_ε in $L^p(\mathscr{O})$ which are gathered in Lemma 4.1 below. However, the proof of this lemma requires that $\beta(r) = r^m + \lambda r$ with $\lambda > 0$. So, we will make this assumption in this section. Finally, in Section 5 we shall show how to remove this condition and prove the positivity of the solution of (2.8) for all $x \in H^{-1}(\mathscr{O})$.

Lemma 4.1 *For any $p \geq m+1$, $\varepsilon > 0$ and any $x \in L^p(\mathscr{O})$ there is a unique $y = J_\varepsilon(x) \in L^p(\mathscr{O})$ such that*

$$y - \varepsilon \Delta \beta(y) = x. \tag{4.2}$$

Moreover,

$$|J_\varepsilon(x)|_p \leq |x|_p, \quad \forall p \geq 2. \tag{4.3}$$

Finally, J_ε is Lipschitz continuous in $L^2(\mathscr{O})$.

Proof. For existence of y one uses the assumption $\lambda > 0$ which implies that β^{-1} is Lipschitz continuous. Estimate (4.3) follows multiplying both sides of equation (4.2) by $|x|^{p-2}x$ and then integrating on $\mathscr{O}$. To prove the last statement one considers another element $x_1 \in L^p(\mathscr{O})$ and the corresponding element y_1 such that $y_1 - \varepsilon\Delta\beta(y_1) = x_1$. Then one multiplies both sides of the last identity by $\beta(y) - \beta(y_1)$ and integrates on $\mathscr{O}$ [3] (For details see [4]). $\Box$

Proposition 4.2 *Assume that Hypotheses* 1.1 *and* 2.1 *are fulfilled and that* $\lambda > 0$. *Then equation* (4.1) *has a unique solution* $X_\varepsilon \in C_W([0,T]; L^2(\Omega\times\mathscr{O}))$. *Moreover, if* $x \in L^p(\mathscr{O})$, $p \ge m+1$, *there exists* $C > 0$ *such that*

$$\mathbb{E}|X_\varepsilon(t)|_p^p \le C(|x|_p). \tag{4.4}$$

Finally

$$\lim_{\varepsilon\to 0} X_\varepsilon = X, \quad \text{in } C_W([0,T]; L^p(\Omega\times\mathscr{O})),$$

where X *is the solution to* (2.5).

Proof. Let us prove (4.4). We start from the case $p = 2$. By the Itô formula we have,

$$\mathbb{E}|X_\varepsilon(t)|_2^2 + 2\mathbb{E}\int_0^t (A_\varepsilon(s), X_\varepsilon(s))_2 ds$$

$$= |x|_2^2 + \sum_{k=1}^\infty \mu_k^2 \mathbb{E}\int_0^t |X_\varepsilon(s)e_k|_2^2 ds.$$

Since $(A_\varepsilon(s), X_\varepsilon(s))_2 \ge 0$ we have, recalling (2.1)

$$\mathbb{E}|X_\varepsilon(t)|_2^2 \le |x|_2^2 + c_3\int_0^t \mathbb{E}|X_\varepsilon(s)|_2^2 ds,$$

where c_3 is a suitable constant. So, (4.4) follows for $p = 2$.

Let now p be arbitrary. Applying (formally) the Itô formula to the function

$$\Phi(x) = \int_{\mathscr{O}} |x(\xi)|^p d\xi,$$

(4.4) follows. To make rigorous the argument we have to apply the Itô formula to the function

$$\Phi_\rho(x) = \int_{\mathscr{O}} \frac{|x(\xi)|^p}{1+\rho|x(\xi)|^p} d\xi,$$

and let $\rho \to 0$.

Finally, the last statement follows from the monotonicity of β and the L^p estimate for X_ε, see [4] for details. $\Box$

[3] A similar argument does not work on $L^p(\mathscr{O})$ for $p \ne 2$. So, we are able to show Lipschitzianity of J_ε in $L^2(\mathscr{O})$ only.

5 Positivity

Theorem 5.1 *Assume that Hypotheses* 1.1 *and* 2.1 *are fulfilled. Let* $x \in L^p(\mathscr{O})$ *be nonnegative a.e. on* $\mathscr{O}$ *where* $p \geq m+1$ *is a natural number. Then the solution* X *to* (2.5) *is such that* $X \in L^\infty_W(0,T;L^p(\Omega;L^p(\mathscr{O})))$ *and* $X \geq 0$ *a.e. on* $\Omega \times (0,\infty) \times \mathscr{O}$.

Proof. First assume that $\lambda > 0$. Then in view of Proposition 4.2 to prove positivity of the solution X of (2.5) it is enough to prove positivity of the solution X_ε of (4.1). Let us consider the modified equation

$$\begin{cases} dZ_\varepsilon(t) + A_\varepsilon(Z_\varepsilon^+(t))dt = \displaystyle\sum_{k=1}^\infty \mu_k Z_\varepsilon^+(t) e_k d\gamma_k(t), \quad t \geq 0, \\ \\ Z_\varepsilon(0) = x, \end{cases} \tag{5.1}$$

where $Z_\varepsilon^+(t) = \max\{Z_\varepsilon^+(t), 0\}$ which can be solved as equation (4.1). If we show that $Z_\varepsilon(t) \geq 0$ it follows clearly that

$$X_\varepsilon(t) = Z_\varepsilon(t) \geq 0.$$

To show positivity of Z_ε^+ we use Itô's formula for the function $(Z_\varepsilon^-)^4$. Formally we obtain

$$\mathbb{E}(Z_\varepsilon^-)^4 \leq 0$$

(for details see [4]). This implies that $Z_\varepsilon^+(t) \geq 0$.

Finally, denote by X_λ the solution of (2.5) for a fixed $\lambda > 0$. Then it is easy, using the monotonicity of β, to show that there exists the limit X of X_λ as $\lambda \to 0$ and to show that X is the solution of (2.5). $\square$

6 The invariant measure

We assume here that $\beta(r) = r^m$.

Let $X(t,x)$ be the solution of (2.5) for $x \in H$. Define the *transition semigroup*

$$P_t\varphi(x) = \mathbb{E}[\varphi(X(t,x))], \quad t \geq 0, \varphi \in B_b(H),$$

where $B_b(H)$ is the space of all real Borel functions on H. It is easy to check that P_t is Feller, that is $P_t\varphi \in C_b(H)$ for all $\varphi \in C_b(H)$, where $C_b(H)$ is the space of all real continuous and bounded functions on H.

For any $t \geq 0$ and $x \in H$ we denote by $\pi_t(x,\cdot)$ the law of $X(t,x)$, so that we have

$$P_t\varphi(x) = \int_H \varphi(y)\pi_t(x,dy), \quad \varphi \in B_b(H). \tag{6.1}$$

We recall that a Borel probability measure ν on H is said to be *invariant* for the transition semigroup P_t if

$$\int_H P_t\varphi d\nu = \int_H \varphi d\nu, \quad \forall\, \varphi \in C_b(H).$$

It is clear that δ_0 is an invariant measure for P_t. For this it is convenient to consider a more general problem

$$\begin{cases} dX(t) + A(X(t))dt = \displaystyle\sum_{k=1}^{\infty} \mu_k X(t) e_k d\gamma_k(t) + g, \quad t \geq 0, \\ \\ X(0) = x. \end{cases} \tag{6.2}$$

where $g \in L^2(\mathscr{O})$ is a constant exterior force. We notice that all results established for problem (2.5) extend trivially to problem (6.2).

Theorem 6.1 *There exists an invariant measure for P_t.*

Proof. Let $x \in H$ and let $X(t,x)$ be the solution of (2.5). From Itô's formula we have

$$\begin{aligned} &\frac{1}{2}\, \mathbb{E}|X(t)|_{-1}^2 + \mathbb{E}\int_0^t |X(s)|_{m+1}^{m+1} ds \\ &= \frac{1}{2}\, \mathbb{E}|x|_{-1}^2 + \sum_{k=1}^{\infty} \mu_k^2 \mathbb{E}\int_0^t |X(s)e_k|_{-1}^2 ds \\ &\leq \frac{1}{2}\, \mathbb{E}|x|_{-1}^2 + \kappa_1 \mathbb{E}\int_0^t |X(s)|_{-1}^2 ds. \end{aligned} \tag{6.3}$$

It follows that

$$\mathbb{E}\int_0^t |X(s)|_{m+1}^{m+1} ds \leq \frac{1}{2}\, \mathbb{E}|x|_{-1}^2 + \kappa_1 \mathbb{E}\int_0^t |X(s)|_{-1}^2 ds. \tag{6.4}$$

By the Sobolev embedding theorem we have

$$H_0^1(\mathscr{O}) \subset L^{\frac{m+1}{m}}(\mathscr{O}),$$

the inclusion being compact. Therefore, the dual inclusion,

$$L^{m+1}(\mathscr{O}) \subset H^{-1}(\mathscr{O}),$$

holds and it is compact.

Consequently, there exists a positive constant κ_2 such that

$$|x|_{-1} \leq \kappa_2 |x|_{m+1}, \tag{6.5}$$

and from (6.4) we obtain

$$\mathbb{E}\int_0^t |X(s)|_{m+1}^{m+1} ds \leq \frac{1}{2}\, \mathbb{E}|x|_{-1}^2 + \kappa_1\kappa_2^2 \mathbb{E}\int_0^t |X(s)|_{m+1}^2 ds. \tag{6.6}$$

Now let κ_3 be a positive constant such that

$$\kappa_1\kappa_1^2 r^2 \leq \frac{1}{2}\, r^{m+1} + \kappa_3, \quad \forall r \in \mathbb{R}.$$

Then by (6.6) we deduce that

$$\frac{1}{t}\, \mathbb{E}\int_0^t |X(s)|_{m+1}^{m+1} ds \leq \mathbb{E}|x|_{-1}^2 + 2\kappa_1\kappa_1^2, \quad \forall\, t \geq 1. \tag{6.7}$$

Set now

$$\mu_t = \frac{1}{t}\, \int_0^t \pi_s(x, \cdot) ds, \quad t > 0.$$

We claim that the family of probability measures $\{\mu_t\}_{t \geq 1}$ on H is tight. Then the Krylov-Bogoliubov theorem will yields the existence of an invariant measure for P_t. To prove the claim consider for any $R > 0$ the ball B_R in $L^{m+1}(\mathscr{O})$ of center 0 and radius R, which is compact in H by the compactness of the embedding of $L^{m+1}(\mathscr{O})$ into $H^{-1}(\mathscr{O})$. Then, denoting by B_R^c the complement of B_R in H, we write

$$\mu_t(B_R^c) = \frac{1}{t}\, \int_0^t \pi_s(x, B_R^c) ds = \frac{1}{t}\, \int_0^t \int_{B_R^c} \pi_s(x, dy) ds$$

$$\leq \frac{1}{t}\, \frac{1}{R^{m+1}}\, \int_0^t \int_H |y|_{m+1}^{m+1} \pi_s(x, dy) ds.$$

Recalling (6.1) we deduce that

$$\mu_t(B_R^c) \leq \frac{1}{t}\, \frac{1}{R^{m+1}}\, \int_0^t P_s(|x|_{m+1}^{m+1}) ds = \frac{1}{t}\, \frac{1}{R^{m+1}}\, \int_0^t \mathbb{E}(|X(s,x)|_{m+1}^{m+1}) ds.$$

Finally, we deduce from (6.7) that

$$\mu_t(B_R^c) \leq \frac{1}{R^{m+1}}\, (\mathbb{E}|x|_{-1}^2 + 2\kappa_1\kappa_1^2).$$

Since R is arbitrary, this implies the claim. The proof is complete. □

Remark 6.2 We do not know whether the invariant measure is unique or not. In the case of additive noise this was proved in [6].

Remark 6.3 A different prove of existence of invariant measure, based on dissipativity of the equation, was given in [9].

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