

Fokker–Planck equations and maximal dissipativity for Kolmogorov operators with time dependent singular drifts in Hilbert spaces

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Abstract

We consider a Kolmogorov operator L_0 in a Hilbert space H , related to a stochastic PDE with a time-dependent singular quasi-dissipative drift $F = F(t, \cdot): H \rightarrow H$, defined on a suitable space of regular functions. We show that L_0 is essentially m -dissipative in the space $L^p([0, T] \times H; \nu)$, $p \geq 1$, where $\nu(dt, dx) = \nu_t(dx)dt$ and the family $(\nu_t)_{t \in [0, T]}$ is a solution of the Fokker–Planck equation given by L_0 . As a consequence, the closure of L_0 generates a Markov C_0 -semigroup. We also prove uniqueness of solutions to the Fokker–Planck equation for singular drifts F . Applications to reaction-diffusion equations with time-dependent reaction term are presented. This result is a generalization of the finite-dimensional case considered in [1], [2], and [5] to infinite dimensions.

2000 Mathematics Subject Classification AMS: 60H15, 60J35, 60J60, 47D07

Key words : Kolmogorov operators, stochastic PDEs, singular coefficients, parabolic equations for measures, Fokker–Planck equations, maximal dissipativity.

1 Introduction

Given a separable Hilbert space H (with norm $|\cdot|$ and inner product $\langle \cdot, \cdot \rangle$), we denote the space of all linear bounded operators in H by $L(H)$ and the set of all Borel probability measures on H by $\mathcal{P}(H)$.

We study non autonomous stochastic equations on H of the type

$$\begin{cases} dX(t) = (AX(t) + F(t, X(t)))dt + \sqrt{C} dW(t), \\ X(s) = x \in H, t \geq s, \end{cases} \quad (1.1)$$

where $A: D(A) \subset H \rightarrow H$ is the infinitesimal generator of a C_0 -semigroup e^{tA} in H , C is a linear positive definite operator in H and $F: D(F) \subset [0, T] \times H \rightarrow H$ is

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such that $F(t, \cdot)$ is quasi-dissipative for all $t \in [0, T]$ (see Sections 2 and 3 for the precise assumptions).

The case where no further regularity assumptions are made on F turns out to be very difficult because of the lack of parabolic regularity results in infinite dimensions. No existence (and uniqueness) results for solutions of (1.1) are known in this very general situation, in particular, when C is not of trace class.

Therefore, in order to get a first grip on the dynamics described by (1.1), we study the corresponding Kolmogorov operator L on $[0, T] \times H$ with the aim to prove that it generates a C_0 -semigroup on a Banach space \mathbb{B} of functions on $[0, T] \times H$. This semigroup is just the space-time homogenization of the family $P_{s,t}$, $0 \leq s \leq t \leq T$, of transition probabilities of the solution to (1.1) (if it exists), i.e., $P_{s,t}$ solve the Chapman–Kolmogorov equation corresponding to L .

The restriction L_0 of the Kolmogorov operator L to an initial domain of nice functions, specified below, is given on $[0, T] \times H$, with $T > 0$ fixed, as follows:

$$(L_0 u)(t, x) = \frac{\partial}{\partial t} u(t, x) + N(t)u(t, x), \quad (1.2)$$

where

$$N(t)u(t, x) = \frac{1}{2} \operatorname{Tr} [CD_x^2 u(t, x)] + \langle x, A^* D_x u(t, x) \rangle + \langle F(t, x), D_x u(t, x) \rangle \quad (1.3)$$

and A^* is the adjoint of A .

In order to define the initial domain of L_0 we introduce some functional spaces. We denote the linear span of all real and imaginary parts of functions $e^{i\langle x, h \rangle}$ where $h \in D(A^*)$ by $\mathcal{E}_A(H)$. Moreover, for any $\phi \in C^1([0, T])$ such that $\phi(T) = 0$ and any $h \in C^1([0, T]; D(A^*))$ we consider the function

$$u_{\phi, h}(t, x) = \phi(t)e^{i\langle x, h(t) \rangle}, \quad t \in \mathbb{R}, \quad x \in H,$$

and denote by $\mathcal{E}_A([0, T] \times H)$ the linear span of all real and imaginary parts of such functions $u_{\phi, h}$. We shall define the operator L_0 on the space $D(L_0) := \mathcal{E}_A([0, T] \times H)$.

Our strategy to achieve the above described goal is the following:

Step 1. Choose the Banach space \mathbb{B} as

$$\mathbb{B} := L^p([0, T] \times H; \nu),$$

for $p \geq 1$, with ν an *appropriate* measure on $[0, T] \times H$ of the form $\nu(dt dx) = \nu_t(dx)dt$, where ν_t are probability measures in H . It turns out that *appropriate* are all measures ν of the above type such that for some $\alpha > 0$

$$\int_{[0, T] \times H} L_0 u d\nu \leq \alpha \int_{[0, T] \times H} u d\nu, \quad \forall u \in D(L_0), \quad u \geq 0. \quad (1.4)$$

Then it follows that L_0 is quasi-dissipative on $L^p([0, T] \times H; \nu)$, hence closable. Let L_p denote its closure. So, the first task is to find such measures. One way to do this is to solve the Fokker–Planck equation corresponding to L_0 (i.e. the dual of the Kolmogorov equation). The resulting measure satisfies (1.4) with $\alpha = 0$.

Step 2. Prove that L_p is maximal-dissipative on $L^p([0, T] \times H; \nu)$. Hence it generates a C_0 -semigroup $e^{\tau L_p}$, $\tau \geq 0$, on $L^p([0, T] \times H; \nu)$ which turns out to be

Markov. Then $e^{\tau L_p}$, $\tau \geq 0$, is the desired space-time homogenization of the transition probabilities $P_{s,t}$, $0 \leq s \leq t \leq T$, of the process that (if it exists) should solve (1.1).

In this paper we realize both steps above, but emphasize that though this is already quite hard work, it constitutes only a partial result. It would be desirable to prove that $e^{\tau L_p}$ is given by a probability kernel on $[0, T] \times H$ and thus also get $P_{s,t}$ as probability kernels on H . And furthermore one should prove the existence of a weak solution to (1.1) having $P_{s,t}$ as transition probabilities.

This second part of the programme is under study and will be the subject of forthcoming work. This paper consists of two parts, namely the case of regular F and non regular F .

In the first part of the paper (Section 2) we assume that $F(t, x)$ is regular, see Hypothesis 2.1, and (extending [1], [2] and [5] to infinite dimensions) prove that, for any $\nu_0 \in \mathcal{P}(H)$, there exists a unique family of probability measures $(\nu_t)_{t \in [0, T]} \subset \mathcal{P}(H)$ with the same initial value ν_0 such that they solve the Fokker–Planck equation for L_0 , i.e., for each $u \in D(L_0)$ for almost all $t \in [0, T]$ one has

$$\frac{d}{dt} \int_H u(t, x) \nu_t(dx) = \int_H L_0 u(t, x) \nu_t(dx),$$

or, equivalently, for each $u \in D(L_0)$ for almost all $t \in [0, T]$ one has

$$\int_H u(t, x) \nu_t(dx) = \int_H u(0, x) \nu_0(dx) + \int_0^t \int_H L_0 u(s, x) \nu_s(dx). \quad (1.5)$$

Here we implicitly assume that the second integral on the right hand side exists for all $u \in D(L_0)$, which is e.g. the case if

$$\int_H |x| \nu_t(dx) < +\infty$$

and F is Lipschitz, or if

$$\langle x, A^* h \rangle, \quad \langle F, h \rangle \in L^1([0, T] \times H; \nu), \quad \forall h \in D(A^*),$$

where $\nu(dt, dx) = \nu_t(dx) dt$. The following remark is crucial in this paper.

Remark 1.1 (i) We note that even without F being regular, the relations $\nu_t(H) = 1$ for all $t \in [0, T]$ and $\lim_{t \rightarrow T} u(t, x) = 0$ for all $x \in H$ along with (1.5) imply

$$\int_0^T \int_H L_0 u(t, x) \nu_t(dx) dt = - \int_H u(0, x) \nu_0(dx), \quad \forall u \in D(L_0). \quad (1.6)$$

In particular,

$$\int_0^T \int_H L_0 u(t, x) \nu_t(dx) dt \leq 0, \quad \forall u \in D(L_0), \quad u \geq 0. \quad (1.7)$$

(ii) If $u \in D(L_0)$, then $u^2 \in D(L_0)$ and $L_0 u^2 = 2u L_0 u + |C^{1/2} D_x u|^2$. Hence by (1.6) we have

$$\begin{aligned} & \int_0^T \int_H L_0 u(t, x) u(t, x) \nu_t(dx) dt \\ &= -\frac{1}{2} \int_0^T \int_H |C^{1/2} D_x u(t, x)|^2 \nu_t(dx) dt - \int_H u^2(0, x) \nu_0(dx). \end{aligned} \quad (1.8)$$

If ν_t only satisfies (1.7) we still have

$$\int_0^T \int_H L_0 u(t, x) u(t, x) \nu_t(dx) dt \leq -\frac{1}{2} \int_0^T \int_H |C^{1/2} D_x u(t, x)|^2 \nu_t(dx) dt. \quad (1.9)$$

After having established existence and uniqueness of $(\nu_t)_{t \in [0, T]}$ satisfying (1.5) in the regular case, we show that L_0 is essentially m -dissipative in the space $L^p([0, T] \times H; \nu)$, i.e. $(L_0, D(L_0))$ is dissipative on $L^p([0, T] \times H; \nu)$ and $(\lambda - L_0, D(L_0))$ has dense range for all $\lambda > 0$. By the well known Lumer–Phillips Theorem this means that the closure L_p of L_0 generates a C_0 -semigroup e^{tL_p} , $t \geq 0$, on $L^p([0, T] \times H; \nu)$, which in our case is even Markov.

In the second part (Section 3), devoted to the case of irregular drifts, we prove (see Theorem 3.3 below) that L_0 is essentially m -dissipative in $L^p([0, T] \times H; \nu)$ where $\nu(dt, dx) = \nu_t(dx)dt$ and $(\nu_t)_{t \in [0, T]}$ is a suitable family of probability measures (see Hypothesis 3.1) as e.g. the solutions to the Fokker–Planck equation corresponding to L_0 ; sufficient conditions for the existence of the latter have been obtained in [4], to which we refer for the proofs. However, in this paper, we prove uniqueness (see Theorem 3.6 below). Then, in Section 4, we apply the obtained results to reaction-diffusion equations with time-dependent coefficients. In this case existence and uniqueness for equation (1.1) is known. However, the m -dissipativity of its Kolmogorov operator and the uniqueness result for the Fokker–Planck equation are new.

Finally, it would be interesting to prove existence and uniqueness for equation (1.5) when t varies on all \mathbb{R} , generalizing results in [11], [12]. This problem will be studied in a forthcoming paper. Some results of this work have been announced in our note [3].

We end this section by listing the assumptions on the linear operator A which we will assume throughout.

Hypothesis 1.2

(i) *There is $\omega \in \mathbb{R}$ such that $\langle Ax, x \rangle \leq \omega|x|^2$, $\forall x \in D(A)$.*

(ii) *$C \in L(H)$ is symmetric, nonnegative and such that the linear operator*

$$Q_t^{(\alpha)} := \int_0^t s^{-2\alpha} e^{sA} C e^{sA^*} ds$$

is of trace class for all $t > 0$ and some $\alpha \in (0, 1/2)$.

(iii) *Setting $Q_t := \int_0^t e^{sA} C e^{sA^*} ds$, one has $e^{tA}(H) \subset Q_t^{1/2}(H)$ for all $t > 0$ and there is $\Lambda_t \in L(H)$ such that $Q_t^{1/2} \Lambda_t = e^{tA}$ and*

$$\gamma_\lambda := \int_0^{+\infty} e^{-\lambda t} \|\Lambda_t\| dt < +\infty,$$

where $\|\cdot\|$ denotes the operator norm in $L(H)$.

We note that by our assumptions on F (see Hypothesis 2.1(ii) in the regular case and Hypothesis 3.1(ii) in the irregular case), by adding a constant times identity to

F , we may assume without loss of generality that ω in Hypothesis 1.2(i) is strictly negative.

We also note that Hypothesis 2.1(iii) implies that the Ornstein–Uhlenbeck operator associated to L_0 (that is when $F = 0$) is strong Feller. This assumption is not essential but it allows to simplify several proofs below. In the appendix we collect some results on the Ornstein–Uhlenbeck operator

$$U\varphi(x) = \frac{1}{2} \operatorname{Tr} [CD_x^2\varphi(x)] + \langle x, A^*D_x\varphi(x) \rangle, \quad \varphi \in \mathcal{E}_A(H), \quad x \in D(A^*),$$

needed throughout. In addition, we introduce the operator

$$V_0u(t, x) = D_tu(t, x) + Uu(t, x), \quad u \in \mathcal{E}_A([0, T] \times H),$$

and its maximal monotone extension V (see (A.1)). Then we prove that the space $\mathcal{E}_A([0, T] \times H)$ is a core (in a suitable sense) of V_0 , generalizing a similar result for the operator U in [13].

2 The case when F is regular

In this section we assume that

Hypothesis 2.1

- (i) *Hypothesis 1.2 is fulfilled.*
- (ii) $F: [0, T] \times H \rightarrow H$ is continuous together with $D_xF(t, \cdot): H \rightarrow L(H)$ for all $t \in [0, T]$. Moreover, there exists $K > 0$ such that

$$|F(t, x) - F(t, y)| \leq K|x - y|, \quad x, y \in H, \quad t \in [0, T].$$

This clearly implies that $x \rightarrow F(t, x) - Kx$ is m -dissipative for any $t \in [0, T]$.

It is known (see, e.g., [14]) that, under Hypothesis 2.1, for any $s \geq 0$, there exists a unique *mild* solution $X(\cdot, s, x)$ with \mathbb{P} -a.s. H -continuous sample paths of the stochastic differential equation

$$\begin{cases} dX(t) = (AX(t) + F(t, X(t)))dt + \sqrt{C} dW(t), \\ X(s) = x \in H, \quad t \geq s, \end{cases} \quad (2.1)$$

where $W(t)$, $t \in \mathbb{R}$, is a cylindrical Wiener process in H defined on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$. A mild solution $X(t, s, x)$ of (2.1) is an adapted stochastic process $X \in C([s, T]; L^2(\Omega, \mathcal{F}, \mathbb{P}))$ such that

$$X(t, s, x) = e^{(t-s)A}x + \int_s^t e^{(t-r)A}F(r, X(r, s, x))dr + W_A(t, s), \quad t \geq s,$$

where $W_A(t, s)$ is the *stochastic convolution*:

$$W_A(t, s) = \int_s^t e^{(t-r)A}\sqrt{C} dW(r), \quad t \geq s,$$

which is also \mathbb{P} -a.s. H -continuous under our assumptions on A . In view of Hypothesis 1.2(ii), $W_A(t, s)$ is a Gaussian random variable in H with mean 0 and covariance operator $Q_{s,t}$ given by

$$Q_{s,t}x = \int_s^t e^{sA} C e^{sA^*} x ds, \quad t \geq s, \quad x \in H.$$

The next result will be useful below.

Lemma 2.2 *For any $m > 1/2$ there is $C_m > 0$ such that for $\omega_1 := \omega - K$*

$$\mathbb{E}(|X(t, s, x)|^{2m}) \leq C_m(1 + e^{-m\omega_1(t-s)}|x|^{2m}), \quad t \geq s. \quad (2.2)$$

Proof. It is convenient to write equation (2.1) as a family of deterministic equations. Setting $Y(t) = X(t, s, x) - W_A(t, s)$, we see that $Y(t)$ satisfies the equation

$$\begin{cases} Y'(t) = AY(t) + F(t, Y(t) + W_A(t, s)), \\ Y(s) = x, \quad t \geq s, \end{cases} \quad (2.3)$$

again in the mild sense. Also we set

$$M = \sup_{t \in [0, T]} |F(t, 0)|.$$

Multiplying (2.3) by $|Y(t)|^{2m-2}Y(t)$ and taking into account Hypothesis 2.1, yields for a suitable constant C_m^1 that

$$\begin{aligned} \frac{1}{2m} \frac{d}{dt} |Y(t)|^{2m} &\leq -\omega |Y(t)|^{2m} + \langle F(t, W_A(t, s)), Y(t) \rangle |Y(t)|^{2m-2} \\ &\quad + \langle F(t, Y(t) + W_A(t, s)) - F(t, W_A(t, s)), Y(t) \rangle |Y(t)|^{2m-2} \\ &\leq -\omega |Y(t)|^{2m} + \langle F(t, W_A(t, s)), Y(t) \rangle |Y(t)|^{2m-2} + K |Y(t)|^{2m} \\ &\leq -\frac{K - \omega}{2} |Y(t)|^{2m} + C_m^1 |F(t, W_A(t, s))|^{2m}. \end{aligned}$$

This computation is formal, but can be made rigorous by approximation, cf. [14]. By a standard comparison result it follows that

$$|Y(t)|^{2m} \leq e^{-m\omega_1(t-s)}|x|^{2m} + 2mC_m^1 \int_s^t e^{-m\omega_1(s-\sigma)} |F(\sigma, W_A(\sigma, s))|^{2m} d\sigma,$$

and finally we find that, for some constant C_m^2 one has

$$\begin{aligned} |X(t, s, x)|^{2m} &\leq C_m^2 e^{-m\omega(t-s)}|x|^{2m} \\ &\quad + C_m^2 \left(\int_s^t e^{-m\omega(\sigma-s)} |F(\sigma, W_A(\sigma, s))|^{2m} d\sigma + |W_A(t, s)|^{2m} \right). \end{aligned} \quad (2.4)$$

Now the conclusion follows by taking the expectation and recalling that in view of Hypothesis 2.1 one has

$$|F(t, x)| \leq |F(t, 0)| + |F(t, x) - F(t, 0)| \leq M + K|x|, \quad t \in [0, T], \quad x \in H,$$

and using the fact that (see [15])

$$\sup_{t \in [0, T], t \geq s} \mathbb{E}|W_A(t, s)|^{2m} < +\infty.$$

The proof is complete. \square

We define the transition evolution operator

$$P_{s,t}\varphi(x) = \mathbb{E}[\varphi(X(t, s, x))], \quad t \geq s, \varphi \in C_u(H),$$

where $C_u(H)$ is the Banach space of all uniformly continuous and bounded functions $\varphi: H \rightarrow \mathbb{R}$ endowed with the usual supremum norm

$$\|\varphi\|_0 = \sup_{x \in H} |\varphi(x)|.$$

For $k \in \mathbb{N}$, $C_u^k(H)$ is the subspace of $C_u(H)$ consisting of all functions with uniformly continuous and bounded derivatives of order l for all $l \leq k$, equipped with its natural norm.

By $C_{u,2}(H)$ we denote the set of all functions $\varphi: H \rightarrow \mathbb{R}$ such that the function $x \rightarrow \frac{\varphi(x)}{1+|x|^2}$ is uniformly continuous and bounded. Endowed with the norm

$$\|\varphi\|_{u,2} := \sup_{x \in H} \frac{|\varphi(x)|}{1+|x|^2},$$

$C_{u,2}(H)$ is a Banach space. We notice that, in view of Lemma 2.2, the transition evolution operator $P_{s,t}$ acts in $C_{u,2}(H)$.

We recall that, since F is Lipschitz, we have for some constant $C > 0$,

$$P_{s,t}|\cdot|(x) \leq C(1+|x|), \quad \forall x \in H, 0 \leq s \leq t \leq T, \quad (2.5)$$

and, by Lemma 2.2, for all $m > 1/2$ and some $C_m > 0$ one has

$$P_{s,t}|\cdot|^{2m}(x) \leq C_m(1+|x|^{2m}), \quad \forall x \in H, 0 \leq s \leq t \leq T. \quad (2.6)$$

The following result is well-known (it follows from Itô's formula, see [5]).

Lemma 2.3 *For each $0 \leq s \leq t \leq T$, $P_{s,t}$ is Feller, and maps $C_{u,2}(H)$ into itself. Moreover, for any $u \in \mathcal{E}_A([0, T] \times H)$ we have*

$$\frac{\partial}{\partial t} P_{s,t}u(t, \cdot) = P_{s,t}L_0u(t, \cdot), \quad \forall 0 \leq s \leq t \leq T.$$

It is useful to introduce an extension of the operator L_0 in $C([0, T]; C_{u,2}(H))$. For any $\lambda \in \mathbb{R}$, $(s, x) \in [0, T] \times H$ set

$$F_\lambda f(s, x) := \int_s^T e^{-\lambda(r-s)} P_{s,r}f(r, \cdot)(x) dr, \quad f \in C([0, T]; C_{u,2}(H)).$$

Let us show that F_λ satisfies the resolvent identity

$$F_\lambda - F_{\lambda'} = (\lambda' - \lambda)F_{\lambda'}F_\lambda \quad (2.7)$$

for all real λ and λ' , whence it follows that the range $F_\lambda(C([0, T]; C_{u,2}(H)))$ is independent of λ . Identity (2.7) is verified as follows:

$$\begin{aligned} F_\lambda(F_{\lambda'}f)(s, x) &= \int_s^T e^{-\lambda(r-s)} P_{s,r}(F_{\lambda'}(r, \cdot))(x) dr \\ &= \int_s^T e^{-\lambda(r-s)} P_{s,r} \left(\int_r^T e^{-\lambda'(u-r)} P_{r,u} f(u, \cdot)(x) du \right) dr \\ &= \int_s^T e^{-(\lambda-\lambda')r} e^{\lambda s} \int_r^T e^{-\lambda' u} P_{s,u} f(u, \cdot)(x) du dr. \end{aligned}$$

Integrating by parts we obtain on the right

$$\begin{aligned} -\frac{e^{\lambda s}}{\lambda - \lambda'} \int_s^T e^{-\lambda r} P_{s,r} f(r, \cdot)(x) dr + \frac{e^{\lambda' s}}{\lambda - \lambda'} \int_s^T e^{-\lambda' u} P_{s,u} f(u, \cdot)(x) du \\ = \frac{1}{\lambda - \lambda'} \left(F_{\lambda'} f(s, x) - F_\lambda f(s, x) \right). \end{aligned}$$

Furthermore, as $\lambda \rightarrow \infty$, we have

$$\begin{aligned} \lambda F_\lambda f(s, x) &= \lambda \int_0^{T-s} e^{-\lambda r} P_{s, r+s} f(r+s, \cdot)(x) dr \\ &= \int_0^{\lambda(T-s)} e^{-r} P_{s, r\lambda^{-1}+s} f(r\lambda^{-1} + s, \cdot)(x) dr \rightarrow f(s, x). \end{aligned}$$

Hence F_λ is one-to-one, continuous with $D(F_\lambda) := C([0, T], C_{u,2}(H))$, so F_λ^{-1} exists and is closed on $F_\lambda(D(F_\lambda))$. Therefore, the operator $L := \lambda I - F_\lambda^{-1}$ is closed (as a densely defined operator on $C([0, T]; C_{u,2}(H))$) and does not depend on λ (which follows by (2.7)). In addition, we have

$$F_\lambda = (\lambda - L)^{-1}, \quad D(L) = F_\lambda(C([0, T]; C_{u,2}(H))), \quad \lambda \in \mathbb{R}. \quad (2.8)$$

By Lemma 2.3 it follows that L is indeed an extension of L_0 .

Finally, it is easy to check that the semigroup \mathcal{P}_τ , $\tau \geq 0$, in the space

$$C_T([0, T]; C_{u,2}(H)) = \{u \in C([0, T]; C_{u,2}(H)) : u(T, x) = 0, \forall x \in H\},$$

defined by

$$\mathcal{P}_\tau f(t, x) = \begin{cases} P_{t, t+\tau} f(t+\tau, \cdot)(x) & \text{if } t+\tau \leq T \\ 0 & \text{otherwise,} \end{cases} \quad (2.9)$$

is generated by L in the sense of π -semigroups (cf. [19]).

Arguing as in [19] one can show that $u \in D(L)$ and $Lu = f$ if and only if

$$\left\{ \begin{array}{l} \text{(i)} \quad \lim_{h \rightarrow 0} \frac{1}{h} (\mathcal{P}_h u(t, x) - u(t, x)) = f(t, x), \quad \forall (t, x) \in [0, T] \times H, \\ \text{(ii)} \quad \sup_{h \in (0, 1], (t, x) \in [0, T] \times H} \frac{(1 + |x|^2)^{-1}}{h} |(\mathcal{P}_h u(t, x) - u(t, x))| < +\infty. \end{array} \right. \quad (2.10)$$

2.1 Existence for problem (1.5)

We denote the topological dual of $C_{u,2}(H)$ by $C_{u,2}(H)^*$. If $0 \leq s < t \leq T$, let $P_{s,t}^*$ be the adjoint operator of $P_{s,t}$. It is easy to see that if $\nu_0 \in \mathcal{P}(H)$ we have $P_{s,t}^*\nu_0 \in \mathcal{P}(H)$ and

$$\int_H \varphi(x)(P_{s,t}^*\nu_0)(dx) = \int_H P_{s,t}\varphi(x)\nu_0(dx), \quad \forall \varphi \in C_{u,2}(H).$$

Proposition 2.4 *Let $\nu_0 \in \mathcal{P}(H)$ be such that $\int_H |x|\nu_0(dx) < +\infty$. Then, setting $\nu_t = P_{0,t}^*\nu_0$, (ν_t) is a solution of problem (1.5) for all (not just a.e.) $t \in [0, T]$ such that for any $m \geq 1/2$ there exists $C_m > 0$ such that*

$$\int_H |x|^{2m}\nu_t(dx) \leq C_m \left(1 + \int_H |x|^{2m}\nu_0(dx) \right). \quad (2.11)$$

In particular,

$$\int_H |x|\nu_t(dx) < +\infty, \quad \forall t \in [0, T]. \quad (2.12)$$

Proof. Let $u \in D(L_0)$, i.e., $u(t, x) = \phi(t)e^{i\langle x, h(t) \rangle}$, where $\phi \in C^1([0, T])$, $\phi(T) = 0$ and $h \in C^1([0, T]; D(A^*))$. Then by definition

$$\int_H u(t, x)\nu_t(dx) = \int_H P_{0,t}u(t, \cdot)(x)\nu_0(dx).$$

Hence by (2.5) and (2.6) we obtain (2.11) and (2.12). On the other hand, by Lemma 2.3 we have

$$\frac{\partial}{\partial t} P_{0,t}u(t, \cdot) = P_{0,t}L_0u(t, \cdot).$$

So, using (2.5) we obtain

$$\begin{aligned} \frac{\partial}{\partial t} \int_H u(t, x)\nu_t(dx) &= \int_H \frac{\partial}{\partial t} P_{0,t}u(t, \cdot)(x)\nu_0(dx) \\ &= \int_H P_{0,t}L_0u(t, \cdot)(x)\nu_0(dx) = \int_H L_0u(t, \cdot)(x)\nu_t(dx). \end{aligned}$$

The proof is complete. \square

2.2 Uniqueness for problem (1.5)

Lemma 2.5 *Let $f \in C([0, T]; C_u^1(H))$, $\lambda \in \mathbb{R}$, and let $u = (\lambda - L)^{-1}f$. Then*

(i) $D_x u \in C([0, T]; C_u(H; H))$,

$u \in D(V)$, where V is the operator defined in the Appendix by (A.3), and we have

$$\lambda u - Vu - \langle F, D_x u \rangle = f. \quad (2.13)$$

Proof. By the definition of u we have that $u \in D(L)$ and

$$u(t, x) = \int_t^T e^{-\lambda(r-t)} P_{t,r} f(r, \cdot)(x) dr, \quad t \in [0, T], \quad x \in H.$$

Let us prove (i). Since $P_{t,r}f(r, \cdot)(x) = \mathbb{E}[f(r, X(r, t, x))]$, and F is C^1 we have

$$D_x P_{t,r}f(r, \cdot)(x) = \mathbb{E}[D_x X(r, t, x)^* D_x f(r, X(r, t, x))],$$

which is also bounded in x since F is Lipschitz uniformly in t . Consequently, $D_x u \in C([0, T]; C_u(H; H))$ and

$$D_x u(t, x) = \int_t^T e^{-\lambda(r-t)} D_x P_{t,r}f(r, \cdot)(x) dr.$$

Let us now prove (ii). Fix $t \in [0, T]$ and $h > 0$ such that $t + h \leq T$. Then

$$X(t + h, t, x) = Z(t + h, t, x) + \int_t^{t+h} e^{(t+h-s)A} F(s, X(s, t, x)) ds, \quad (2.14)$$

where

$$Z(t + h, t, x) = e^{hA} x + \int_t^{t+h} e^{(t+h-s)A} \sqrt{C} dW(s).$$

Therefore, we have

$$\begin{aligned} \mathcal{R}_h u(t, x) &= R_h u(t + h, \cdot)(x) = \mathbb{E}[u(t + h, Z(t + h, t, x))] \\ &= \mathbb{E}[u(t + h, X(t + h, t, x))], \end{aligned}$$

where \mathcal{R}_h is defined by (A.4). Set

$$g(t + h, t, x) = \int_t^{t+h} e^{(t+h-s)A} F(s, X(s, t, x)) ds.$$

Then, taking into account (2.14), we have for any $h > 0$

$$\begin{aligned} &\mathcal{R}_h u(t, x) \\ &= \mathbb{E} \left[u \left(t + h, X(t + h, t, x) - \int_t^{t+h} e^{(t+h-s)A} F(s, X(s, t, x)) ds \right) \right] \\ &= \mathbb{E}[u(t + h, X(t + h, t, x))] \\ &\quad - \int_0^1 \mathbb{E} [\langle Du(t + h, X(t + h, t, x)) - \xi g(t + h, t, x), g(t + h, t, x) \rangle] d\xi \\ &= \mathcal{P}_h u(t, x) \\ &\quad - \int_0^1 \mathbb{E} [\langle Du(t + h, X(t + h, t, x)) - \xi g(t + h, t, x), g(t + h, t, x) \rangle] d\xi. \end{aligned}$$

It follows that

$$\begin{aligned} \frac{1}{h} (\mathcal{R}_h u(t, x) - u(t, x)) &= \frac{1}{h} (\mathcal{P}_h u(t, x) - u(t, x)) \\ &\quad - \frac{1}{h} \int_0^1 \mathbb{E} [\langle Du(t + h, X(t + h, t, x)) - \xi g(t + h, t, x), g(t + h, t, x) \rangle] d\xi. \end{aligned} \quad (2.15)$$

Since $u \in D(L)$ due to the equality $u = (\lambda - L)^{-1} f$, Lemma 2.2 yields

$$\lim_{h \rightarrow 0} \frac{1}{h} (\mathcal{R}_h u(t, x) - u(t, x)) = Lu(t, x) - \langle D_x u(t, x), F(t, x) \rangle, \quad (t, x) \in [0, T] \times H.$$

To show that $u \in D(V)$ and $Vu = Lu - \langle F, Du \rangle$, it remains to prove (see (A.5)) that

$$\sup_{h \in (0,1], (t,x) \in [0,T] \times H} \frac{(1 + |x|^2)^{-1}}{h} |\mathcal{R}_h u(t, x) - u(t, x)| < +\infty.$$

By (2.5) and (2.15) we have

$$\begin{aligned} \frac{1}{h} \left| \int_0^1 \mathbb{E} [\langle Du(t+h, X(t+h, t, x)) - \xi g(t+h, t, x), g(t+h, t, x) \rangle] d\xi \right| \\ \leq c \|Du\|_0 \left(1 + \sup_{t \geq s} \mathbb{E}[|X(t, s, x)|] \right) \leq c'(1 + |x|). \end{aligned}$$

The proof is complete. \square

(ii) **Corollary 2.6** *Let $f \in C([0, T]; C_u^1(H))$, $\lambda \in \mathbb{R}$ and $u = (\lambda - L)^{-1}f$. Then, for every bounded Borel measure μ on $[0, T] \times H$, there exists a sequence $(u_n) \subset D(L_0)$ such that $u_n \rightarrow u$, $D_x u_n \rightarrow D_x u$, $V_0 u_n \rightarrow V_0 u$, hence one has $L_0 u_n \rightarrow Lu$ in measure μ and*

$$|u_n(t, x)| + |V_0 u_n(t, x)| + |D_x u_n(t, x)| \leq c_1(1 + |x|^2), \quad \forall (t, x) \in [0, T] \times H,$$

for some constant c_1 .

Proof. By Lemma 2.5 we know that $u = (\lambda - L)^{-1}f$ belongs to $D(V)$ and

$$Lu = Vu + \langle D_x u, F \rangle.$$

Note that $\langle D_x u, F \rangle \in C_{u,2}(H)$ (this space is defined before Lemma 2.3) since F is Lipschitz continuous and consequently sub-linear. On the other hand, by Corollary A.3 there exists a sequence of elements $u_n \in D(L_0)$ and a constant $c_2 > 0$ such that

$$|u_n(t, x)| + |V_0 u_n(t, x)| + |D_x u_n(t, x)| \leq c_2(1 + |x|^2), \quad \forall (t, x) \in [0, T] \times H,$$

and $u_n \rightarrow u$, $V_0 u_n \rightarrow V_0 u$, $D_x u_n \rightarrow D_x u$ in measure μ . It follows that $L_0 u_n \rightarrow Lu$ in μ -measure. \square

Proposition 2.7 *Let $(\zeta_t)_{t \in [0, T]}$ be a solution of (1.5) such that*

$$\sup_{t \in [0, T]} \int_H |x|^2 \zeta_t(dx) < +\infty.$$

Then $\zeta_t = P_{0,t}^* \nu_0$ for all $t \in [0, T]$.

Proof. Set $\gamma_t = \nu_t - \zeta_t$, where $\nu_t = P_{0,t}^* \nu_0$ for all $t \in [0, T]$ and $\gamma(dt, dx) = \gamma_t(dx)dt$. Then for any $u \in D(L_0)$ by Remark 1.1(i) we have

$$\int_0^T \int_H L_0 u \gamma_t(dx) dt = 0. \quad (2.16)$$

Let now $f \in C([0, T]; C_u^1(H))$ and set $u = L^{-1}f$. Then by Corollary 2.6 there exists a sequence $(u_n) \subset D(L_0)$ such that

$$u_n \rightarrow u, \quad L_0 u_n \rightarrow L_0 u \quad \text{in } \gamma \text{ measure,}$$

and

$$|u_n(t, x)| + |L_0 u_n(t, x)| \leq c(1 + |x|^2), \quad \forall (t, x) \in [0, T] \times H, \quad \rho \in \Gamma,$$

for a suitable constant $c > 0$. Then by (2.16) we find by the dominated convergence theorem

$$\int_0^T \int_H f(t, x) \gamma_t(dx) dt = \int_0^T \int_H L u(t, x) \gamma_t(dx) dt = 0.$$

This implies that $\gamma_t dt = 0$ since the set $C([0, T]; C_u^1(H))$ is dense in the space $L^1([0, T] \times H; \gamma)$. \square

2.3 m -Dissipativity of L_0

Theorem 2.8 *Let $p \in [1, \infty)$ and let ν be a positive bounded Borel measure on $[0, T] \times H$ such that there exists a constant $\alpha > 0$ such that*

$$\int_0^T \int_H L_0 u(t, x) \nu(dt, dx) \leq \alpha \int_0^T \int_H u(t, x) \nu(dt, dx), \quad \forall u \in D(L_0), \quad u \geq 0,$$

and

$$\int_0^T \int_H |x|^{2p} (1 + |F(t, x)|^p) \nu(dt, dx) < \infty.$$

Then under Hypotheses 1.2 and 2.1, $L_0 - \alpha/p$ is dissipative in the space $L^p([0, T] \times H; \nu)$. Consequently, $L_0 - \alpha/p$ is closable. Its closure $L_p - \alpha/p$ is m -dissipative in the space $L^p([0, T] \times H; \nu)$. Hence L_p generates a C_0 -semigroup $e^{\tau L_p}$, $\tau \geq 0$, on $L^p([0, T] \times H; \nu)$. Furthermore, this semigroup is Markov. In particular this holds for $\nu(dt, dx) = \nu(dx) dt$ from Proposition 2.4 with $\alpha = 0$, provided $\int_H |x|^{3p} \nu_0(dx) < \infty$.

Proof. By [17, Lemma 1.8 in Appendix B], the operator $(L_0 - \alpha/p, D(L_0))$ is dissipative in $L^p([0, T] \times H; \nu)$ for all $p \in [1, \infty)$. Let $f \in C([0, T]; C_u^1(H))$, $\lambda \in \mathbb{R}$, and let $u = (\lambda - L)^{-1} f$. By Lemma 2.5 we know that $u \in D(V) \cap C([0, T]; C_u^1(H))$ and

$$\lambda u - V u - \langle D_x u, F \rangle = f.$$

By Corollary A.3 there exists a sequence $(u_n) \subset \mathcal{E}_A([0, T] \times H)$ such that for some $c_1 > 0$ one has

$$|u_n(t, x)| + |D_x u_n(t, x)| + |V u_n(t, x)| \leq c_1(1 + |x|^2), \quad \forall n \in \mathbb{N},$$

and $u_n \rightarrow u$, $V u_n \rightarrow V u$, $D_x u_n \rightarrow D_x u$ in measure ν . Set

$$f_n = \lambda u_n - V u_n - \langle D_x u_n, F \rangle = \lambda u_n - L_0 u_n.$$

Then we have $f_n \rightarrow f$ in measure ν and there exists $c_2 > 0$ such that

$$|f_n(t, x)| \leq c_1(1 + |x|^2 + |x|^2 |F(t, x)|), \quad (t, x) \in [0, T] \times H.$$

By assumption and the dominated convergence theorem it follows that $f_n \rightarrow f$ in $L^p([0, T] \times H; \nu)$. So we have proved that the closure of the range of $\lambda - L_0$ includes $C([0, T]; C_u^1(H))$ which is dense in $L^p([0, T] \times H; \nu)$. The remaining part of the assertion is proved as Theorem 3.3 below. \square

3 General coefficients

Suppose we are given a family $\{F(t, \cdot)\}_{t \in [0, T]}$ of m -quasi-dissipative mappings

$$F(t, \cdot): D(F(t, \cdot)) \subset H \rightarrow 2^H.$$

This means that $D(F(t, \cdot))$ is a Borel set in H and for some $K > 0$

$$\langle u - v, x - y \rangle \leq K|x - y|^2, \quad \forall x, y \in D(F(t, \cdot)), \quad u \in F(t, x), \quad v \in F(t, y), \quad (3.1)$$

and $\text{Range}(\lambda - F(t, \cdot)) := \bigcup_{x \in D(F(t, \cdot))} (x - F(t, x)) = H$ for any $\lambda > K$. We assume additionally that K is independent of t .

For any $x \in D(F(t, \cdot))$ the set $F(t, x)$ is closed, non empty, and convex; we set $F_0(t, x) := y_0(t)$, where $y_0(t) \in F(t, x)$ is such that $|y_0(t)| = \min_{y \in F(t, x)} |y|$, $x \in D(F(t, \cdot))$.

We are concerned with the Kolmogorov operator

$$L_0 u(t, x) := D_t u(t, x) + U u(t, x) + \langle F_0(t, x), D_x u(t, x) \rangle, \quad u \in D(L_0),$$

where $D(L_0) = \mathcal{E}_A([0, T] \times H)$ and U is the Ornstein–Uhlenbeck operator defined by (A.1) in the Appendix.

Our goal is to prove that the closure of $L_0 - \alpha/p$ is m -dissipative in the space $L^p([0, T] \times H, \nu)$, $p \in [1, \infty)$, where $\nu(dt, dx) = \nu_t(dx)dt$ and $(\nu_t)_{t \in [0, T]}$ is a given family of finite positive Borel measures on H such that for some $\alpha > 0$ one has

$$\begin{aligned} \int_0^T \int_H L_0 u(t, x) \nu_t(dx) dt \\ \leq \alpha \int_0^T \int_H u(t, x) \nu_t(dx) dt, \quad \forall u \in D(L_0), \quad u \geq 0. \end{aligned} \quad (3.2)$$

We shall assume, in addition to Hypothesis 1.2, that

Hypothesis 3.1 (i) *There is a family $\{F(t, \cdot)\}_{t \in [0, T]}$ of m -quasi-dissipative mappings in H such that $0 \in D(F(t, \cdot))$ and $F_0(t, 0) = 0$ for all $t \in \mathbb{R}$.*

(ii) *There is a family $(\nu_t)_{t \in [0, T]}$ of Borel probability measures on H such that for some $p \in [1, \infty)$,*

$$\int_0^T dt \int_H (|x|^{2p} + |F_0(t, x)|^p + |x|^{2p} |F_0(t, x)|^p) \nu_t(dx) < +\infty.$$

(iii) *For all $u \in D(L_0)$ we have $L_0 u \in L^p([0, T] \times H, \nu)$ and (3.2) is fulfilled.*

(iv) $\nu_t(D(F(t, \cdot))) = 1$, $\forall t \in [0, T]$.

Remark 3.2 (i) For simplicity below we shall assume that K in (3.1) is zero. This is, however, no restriction since all our arguments below immediately extend to the case when we add a C^∞ -Lipschitz map to F and clearly $F = \tilde{F} + KId$ with \tilde{F} satisfying (3.1) with $K = 0$.

(ii) Obviously, in Hypothesis 3.1(iii) we have $L_0 u \in L^p([0, T] \times H, \nu)$ if and only if the maps $x \mapsto \langle x, A^* h \rangle$, $(t, x) \mapsto \langle F(t, x), h \rangle$ are in $L^p([0, T] \times H, \nu)$ for all $h \in D(A^*)$.

(iii) In [4] a number of results have been proved that ensure the existence of measures $\nu(dt dx) = \nu_t(dx)dt$ satisfying the required properties in Hypothesis 3.1. More precisely, it was proved that they even satisfy (1.5) which by Remark 1.1(i) is stronger than (3.2).

Let us introduce the Yosida approximations of $F(t, \cdot)$, $t \in \mathbb{R}$. For any $\alpha > 0$ we set

$$F_\alpha(t, x) := \frac{1}{\alpha} (J_\alpha(t, x) - x), \quad x \in H,$$

where

$$J_\alpha(t, x) := (I - \alpha F(t, \cdot))^{-1}(x), \quad x \in H, \quad t \in \mathbb{R}, \quad \alpha > 0.$$

It is well-known that

$$\lim_{\alpha \rightarrow 0} F_\alpha(t, x) = F_0(t, x), \quad \forall x \in D(F(t, \cdot)),$$

and

$$|F_\alpha(t, x)| \leq |F_0(t, x)|, \quad \forall x \in D(F(t, \cdot)).$$

Moreover, $F_\alpha(t, \cdot)$ is Lipschitzian with constant $2/\alpha$ and $F_\alpha(t, 0) = 0$.

Since $F_\alpha(t, \cdot)$ is not differentiable in general, we introduce a further regularization, as in [10], by setting

$$F_{\alpha, \beta}(t, x) = \int_H e^{\beta B} F_\alpha(t, e^{\beta B} x + y) N_{\frac{1}{2} B^{-1}(e^{2\beta B} - 1)}(dy), \quad \alpha, \beta > 0,$$

where $B: D(B) \subset H \rightarrow H$ is a self-adjoint negative definite operator such that B^{-1} is of trace class.

The mapping $F_{\alpha, \beta}(t, \cdot)$ is dissipative, of class C^∞ , possesses bounded derivatives of all orders, and $F_{\alpha, \beta}(t, \cdot) \rightarrow F_\alpha(t, \cdot)$ pointwise as $\beta \rightarrow 0$, see [14, Theorem 9.19]. Moreover, $F_{\alpha, \beta}(t, \cdot)$ satisfies Hypothesis 2.1(ii) since it is Lipschitz continuous with Lipschitz constant $2/\alpha$ and

$$|F_{\alpha, \beta}(t, 0)| \leq \int_H |F_\alpha(t, y)| N_{\frac{1}{2} B^{-1}(e^{2\beta B} - 1)}(dy) \leq \frac{2}{\alpha} \int_H |y| N_{\frac{1}{2} B^{-1}(e^{2\beta B} - 1)}(dy).$$

3.1 m -dissipativity of $L_p - \alpha/p$

We assume here that Hypotheses 1.2 and 3.1 hold for some $p \in [1, \infty)$. As in the regular case, (3.2) implies that $(L_p - \alpha/p, D(L_0))$ is dissipative, hence closable in $L^p([0, T] \times H, \nu)$ for all $p \in [1, \infty)$. We shall denote its closure with domain $D(L_p)$ by $L_p - \alpha/p$. We are going to show that L_p is m -dissipative.

Let us consider the approximating equation

$$\lambda u_{\alpha, \beta} - V u_{\alpha, \beta} - \langle F_{\alpha, \beta}, D_x u_{\alpha, \beta} \rangle = f, \quad \alpha, \beta > 0, \quad (3.3)$$

where $\lambda > 0$ and $f \in C([0, T]; C_u^1(H))$. In view of Lemma 2.5, equation (3.3) has a unique solution $u_{\alpha, \beta} \in D(V) \cap C([0, T]; C_u^1(H))$ given by

$$u_{\alpha, \beta}(t, x) = \int_t^T e^{-\lambda(s-t)} \mathbb{E}[f(s, X_{\alpha, \beta}(s, t, x))] ds, \quad t \in \mathbb{R}, \quad x \in H,$$

where $X_{\alpha, \beta}$ is the mild solution of the problem

$$\begin{cases} dX_{\alpha, \beta}(s, t, x) = (AX_{\alpha, \beta}(s, t, x) + F_{\alpha, \beta}(t, X_{\alpha, \beta}(s, t, x))) ds + \sqrt{C} dW(s), \\ X_{\alpha, \beta}(t, t, x) = x \in H. \end{cases} \quad (3.4)$$

For all $h \in H$ we have

$$\langle D_x u_{\alpha,\beta}(t, x), h \rangle = \int_t^T e^{-\lambda(s-t)} \mathbb{E}[\langle D_x f(s, X_{\alpha,\beta}(s, t, x)), \eta_{\alpha,\beta}^h(s, t, x) \rangle] ds, \quad (3.5)$$

where $\eta_{\alpha,\beta}^h(s, t, x) := \langle D_x X_{\alpha,\beta}(s, t, x), h \rangle$ is the mild solution of the problem

$$\begin{cases} \frac{d}{ds} \eta_{\alpha,\beta}^h(s, t, x) = A \eta_{\alpha,\beta}^h(s, t, x) + D_x F_{\alpha,\beta}(s, X_{\alpha,\beta}(s, t, x)) \eta_{\alpha,\beta}^h(s, t, x), & s \geq t, \\ \eta_{\alpha,\beta}^h(t, t, x) = h. \end{cases}$$

By a standard argument, based on approximations (see e.g. [9, Section 3.2]) and on the Gronwall lemma, we see that for some constant $c > 0$ one has

$$|\eta_{\alpha,\beta}^h(s, t, x)| \leq e^{c(s-t)} |h|, \quad T \geq s \geq t \geq 0.$$

Consequently, by (3.5) it follows that for $\lambda > c$ we have

$$|D_x u_{\alpha,\beta}(t, x)| \leq \frac{1}{\lambda - c} \sup_{t \in [0, T], x \in H} |D_x f(t, x)|, \quad t \in [0, T], x \in H. \quad (3.6)$$

Now we can prove the main result of this section.

Theorem 3.3 *Under Hypotheses 1.2 and 3.1, $L_p - \alpha/p$ is m -dissipative in the space $L^p([0, T] \times H, \nu)$. Hence L_p generates a C_0 -semigroup $e^{\tau L_p}$, $\tau \geq 0$, on the space $L^p([0, T] \times H, \nu)$. Furthermore, this semigroup is Markov, i.e. positivity preserving and $e^{\tau L_p} \mathbb{1} = \mathbb{1}$ for all $\tau \geq 0$. Finally, the resolvent set $\rho(L_p)$ of L_p coincides with \mathbb{R} .*

Proof. Let $f \in C([0, T]; C_u^1(H))$ and let $u_{\alpha,\beta}$ be the solution to equation (3.3).

Claim 1 One has

$$\lim_{\alpha \rightarrow 0} \limsup_{\beta \rightarrow 0} \langle F_{\alpha,\beta}(t, \cdot) - F_0(t, \cdot), D_x u_{\alpha,\beta} \rangle = 0 \quad \text{in } L^p([0, T] \times H, \nu).$$

In fact, it follows by (3.6) that for $\lambda > c$

$$\begin{aligned} I_{\alpha,\beta} &:= \int_0^T dt \int_H |\langle F_{\alpha,\beta}(t, \cdot) - F_0, D_x u_{\alpha,\beta}(t, \cdot) \rangle|^p d\nu \\ &\leq \frac{1}{(\lambda - c)^p} \sup_{t \in [0, T], x \in H} |D_x f(t, x)|^p \int_0^T dt \int_H |F_{\alpha,\beta}(t, \cdot) - F_0(t, \cdot)|^p d\nu_t. \end{aligned}$$

Now, since for fixed $\alpha > 0$, $F_{\alpha,\beta}(t, \cdot)$ is Lipschitz continuous with Lipschitz constant $2/\alpha$, we see that for any $\alpha > 0$ there is $c_\alpha > 0$ such that

$$|F_{\alpha,\beta}(t, x)| \leq c_\alpha(1 + |x|), \quad x \in H,$$

and so

$$\limsup_{\beta \rightarrow 0} I_{\alpha,\beta} \leq \frac{1}{(\lambda - c)^p} \sup_{t \geq 0, x \in H} |D_x f(t, x)|^p \int_0^T dt \int_H |F_\alpha(t, \cdot) - F_0(t, \cdot)|^p d\nu_t.$$

Now the claim follows, in view of the dominated convergence theorem.

Claim 2 One has $u_{\alpha,\beta} \in D(L_p)$ and for $\lambda > c$

$$\lambda u_{\alpha,\beta} - L_p u_{\alpha,\beta} = f + \langle F_{\alpha,\beta} - F_0, D_x u_{\alpha,\beta} \rangle. \quad (3.7)$$

Applying Corollary 2.6 with L being the Kolmogorov operator corresponding to (3.4) we can find $u_n \in \mathcal{E}([0, T] \times H)$, $n \in \mathbb{N}$, such that

$$u_n \rightarrow u_{\alpha,\beta}, \quad D_x u_n \rightarrow D_x u_{\alpha,\beta}, \quad \text{and } V_0 u_n \rightarrow V u_{\alpha,\beta} \quad \text{in } \nu\text{-measure}$$

and for all $(t, x) \in [0, T] \times H$ one has

$$|u_n(t, x)| + |V_0 u_n(t, x)| + |D_x u_n(t, x)| \leq c_1(1 + |x|^2).$$

By Hypothesis 3.1(ii) it follows that the sequence $\{L_0 u_n\}$ is bounded in the space $L^p([0, T] \times H, \nu)$. Hence Claim 2 follows by the closability of L_0 on $L^p([0, T] \times H, \nu)$.

Claim 3 One has $f \in R(\lambda - L_p)$ for $\lambda > c$.

Claim 3 immediately follows from Claim 1 and (3.7).

Since $C([0, T]; C_u^1(H))$ is dense in $L^p([0, T] \times H, \nu)$ the first assertion of the theorem follows. The second one follows from the well-known Lumer–Phillips Theorem. The final statement is now a consequence of [17, Lemma 1.9] and the fact that $L_p \mathbb{1} = 0$.

Claim 4 $\rho(L_p) = \mathbb{R}$.

Let $\beta \in \mathbb{R}$ and $\lambda > 0$ such that $\lambda + \beta > \frac{c}{p}$. Let $f \in L^p([0, T] \times H, \nu)$. Then by what we proved above there exists $v \in D(L_p)$ such that

$$(\beta + \lambda - L_p)v = e_\lambda f,$$

where $e_\lambda(t, x) := e^{\lambda t}$, $(t, x) \in [0, T] \times H$. Define $u := e_{-\lambda}v$. Then an easy approximation argument proves that $u \in D(L_p)$ and

$$(\beta - L_p)u = e_{-\lambda}(\beta + \lambda - L_p)v = f.$$

So, $(\beta - L_p, D(L_p))$ is surjective. It is also injective because so is $(\beta + \lambda - L_p, D(L_p))$. Hence $\beta \in \rho(L_p)$, since $(\beta - L_p, D(L_p))$ is closed. \square

Remark 3.4 It immediately follows from Claim 1 above that

$$\lim_{\alpha, \beta \rightarrow 0} u_{\alpha, \beta} = (\lambda - L_p)^{-1} f \quad \text{in } L^p([0, T] \times H, \nu),$$

that is, the space-time resolvent corresponding to (3.4) converges to the one of (1.1) in $L^p([0, T] \times H, \nu)$ on functions in $C([0, T]; C_u^1(H))$.

3.2 Uniqueness for problem (1.5) in the irregular case

Let us fix a Borel probability measure ν_0 on H . We introduce the set \mathcal{M}_{ν_0} of all Borel measures ν on $[0, T] \times H$ having the following properties:

- (i) $\nu(dt dx) = \nu_t(dx)dt$, where for $t \in [0, T]$, ν_t is a Borel probability on H such that $\nu_t(D(F(t, \cdot))) = 1$ for all $t \in [0, T]$.
- (ii) $L_0 u \in L^1([0, T] \times H, \nu)$ for all $u \in \mathcal{E}_A([0, T] \times H)$ and $(\nu_t)_{t \in [0, T]}$ satisfies (1.5).

$$(iii) \int_0^T \int_H (|x|^2 + |F_0(t, x)| + |x|^2 |F_0(t, x)|) \nu_t(dx) dt < \infty.$$

The aim of this subsection is to prove that under Hypotheses 1.2 and 3.1 \mathcal{M}_{ν_0} contains at most one element, i.e. $\#\mathcal{M}_{\nu_0} \leq 1$.

Remark 3.5 As mentioned above, the existence of solutions of (1.5) under suitable conditions has been proved in [4]. There, however, (1.5) has been written equivalently as follows:

$$\int_0^T \int_H L_0 u(t, x) \nu_t(dx) dt = 0$$

for all $u \in \mathcal{E}_A([0, T] \times H)$ such that $u(t, x) = 0$ if $t \leq \epsilon$ or $t \geq T - \epsilon$ for some $\epsilon > 0$ and

$$\lim_{t \rightarrow 0} \int_H \zeta(x) \nu_t(dx) = \int_H \zeta(x) \nu_0(dx), \quad \forall \zeta \in \mathcal{E}_A(H).$$

The same proof as that of [5, Lemma 2.7] shows that this formulation is indeed equivalent to (1.5). Clearly, the above formulation is nothing but a generalization of the classical Fokker–Planck equation corresponding to the Kolmogorov operator L_0 . So, as already mentioned in the introduction, our results can be summarized as follows: first solve the Fokker–Planck equation (for measures) corresponding to L_0 and using its solution solve the Kolmogorov equation for L_0 on $L^p([0, T] \times H, \nu)$ (for functions) which is possible according to Theorem 3.3 above.

Theorem 3.6 *Let ν_0 be a Borel probability measure on H . Under Hypotheses 1.2 and 3.1(i) we have $\#\mathcal{M}_{\nu_0} \leq 1$.*

Proof. Let $\nu^{(1)}, \nu^{(2)} \in \mathcal{M}_{\nu_0}$ and set

$$\mu := \frac{1}{2} \nu^{(1)} + \frac{1}{2} \nu^{(2)}.$$

Then $\mu \in \mathcal{M}_{\nu_0}$ and $\nu^{(i)} = \sigma_i \mu$ for some measurable functions $\sigma_i: [0, T] \times H \rightarrow [0, 2]$. By (1.6) we have

$$\int_{[0, T] \times H} L_0 u d\nu^{(1)} = \int_{[0, T] \times H} L_0 u d\nu^{(2)}, \quad \forall u \in D(L_0),$$

that is

$$\int_{[0, T] \times H} L_0 u (\sigma_1 - \sigma_2) d\mu = 0, \quad \forall u \in D(L_0).$$

Since by the last statement of Theorem 3.3, the range of $(L_0, D(L_0))$ is dense in $L^1([0, T] \times H, \mu)$ and $(\sigma_1 - \sigma_2)$ is bounded, we conclude that $\sigma_1 = \sigma_2$. \square

4 Application to reaction-diffusion equations

We shall consider here a stochastic heat equation perturbed by a polynomial drift, with time dependent coefficients, of odd degree $d > 1$ of the form

$$\lambda\xi - p(t, \xi), \quad \xi \in \mathbb{R}, \quad t \in [0, T],$$

where $\lambda \in \mathbb{R}$ is given, $p(t, 0) = 0$ and $D_\xi p(t, \xi) \geq 0$ for all $\xi \in \mathbb{R}$ and $t \in [0, T]$.

We set $H = L^2(\mathcal{O})$ where $\mathcal{O} = (0, 1)^n$, $n \in \mathbb{N}$, and denote by $\partial\mathcal{O}$ the boundary of \mathcal{O} .

We are concerned with the following stochastic PDE on \mathcal{O} :

$$\begin{cases} dX(t, s, \xi) = [\Delta_\xi X(t, s, \xi) + \lambda X(t, s, \xi) - p(t, X(t, s, \xi))]dt + \sqrt{C} dW(t, \xi), \\ X(t, s, \xi) = 0, \quad t \geq s, \quad \xi \in \partial\mathcal{O}, \\ X(s, s, \xi) = x(\xi), \quad \xi \in \mathcal{O}, \quad x \in H, \end{cases} \quad (4.1)$$

where Δ_ξ is the Laplace operator, $C \in L(H)$ is positive, and W is a cylindrical Wiener process with respect to $(\mathcal{F}_t)_{t \in \mathbb{R}}$ in H defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}}, \mathbb{P})$. We choose W of the form

$$W(t, \xi) = \sum_{k=1}^{\infty} e_k(\xi) \beta_k(t), \quad \xi \in \mathcal{O}, \quad t \geq 0,$$

where (e_k) is a complete orthonormal system in H and (β_k) is a sequence of independent standard Brownian motions on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then we extend $W(t)$ to $(-\infty, 0)$ by symmetry.

Let us write problem (4.1) as a stochastic differential equation in the Hilbert space H . For this we denote by A the realization of the Laplace operator with Dirichlet boundary conditions, i.e.,

$$\begin{cases} Ax = \Delta_\xi x, \quad x \in D(A), \\ D(A) = H^2(\mathcal{O}) \cap H_0^1(\mathcal{O}). \end{cases}$$

The operator A is self-adjoint and possesses a complete orthonormal system of eigenfunctions, namely

$$e_k(\xi) = (2/\pi)^{n/2} \sin(\pi k_1 \xi_1) \cdots (\sin \pi k_n \xi_n), \quad \xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n,$$

where $k = (k_1, \dots, k_n)$, $k_i \in \mathbb{N}$.

For any $x \in H$ we set $x_k = \langle x, e_k \rangle$, $k \in \mathbb{N}^n$. Notice that

$$Ae_k = -\pi^2 |k|^2 e_k, \quad k \in \mathbb{N}^n, \quad |k|^2 = k_1^2 + \cdots + k_n^2.$$

Therefore, we have

$$\|e^{tA}\| \leq e^{-\pi^2 t}, \quad t \geq 0.$$

Concerning the operator C , we shall assume for simplicity that $C = (-A)^{-\gamma}$ with $n/2 - 1 < \gamma < 1$ (which implies $n < 4$). Now it is easy to check that Hypothesis 1.2 is fulfilled. In fact we have

$$\begin{aligned} Q_t x &= \int_0^t e^{sA} C e^{sA^*} x ds = \int_0^t (-A)^{-\gamma} e^{2sA} x ds \\ &= \frac{1}{2} (-A)^{-(1+\gamma)} (1 - e^{2tA}) x, \quad t \geq 0, x \in H. \end{aligned}$$

Then

$$\text{Tr} [(-A)^{-(1+\gamma)}] = \sum_{k \in \mathbb{N}^n} |k|^{-2(1+\gamma)} < +\infty,$$

since $\gamma > \frac{n}{2} - 1$. Similarly, one obtains that for any $\alpha \in (0, 1/2)$

$$\text{Tr} \int_0^1 s^{-2\alpha} e^{sA} C e^{sA^*} ds < +\infty,$$

Hence part (i) and (ii) of Hypothesis 1.2 hold. Part (iii) can also be derived easily. We refer to [7] for details.

Now, setting $X(t, s) = X(t, s, \cdot)$ and $W(t) = W(t, \cdot)$, we shall write problem (4.1) as

$$\begin{cases} dX(t, s) = [AX(t, s) + F(t, X(t, s))]dt + (-A)^{-\gamma/2} dW(t), \\ X(s, s) = x. \end{cases} \quad (4.2)$$

where F is the mapping

$$F: D(F) = [0, T] \times L^{2d}(\mathcal{O}) \subset [0, T] \times H \rightarrow H, \quad x(\xi) \mapsto \lambda\xi - p(t, x(\xi)).$$

It is convenient, following [14], to introduce two different notions of solution of (4.2). For this purpose, for any $s \in [0, T]$, we consider the space

$$C_W([s, T]; H) := C_W([s, T]; L^2(\Omega, \mathcal{F}, \mathbb{P}; H))$$

consisting of all continuous mappings $F: [s, T] \rightarrow L^2(\Omega, \mathcal{F}, \mathbb{P}; H)$ adapted to the filtration $(\mathcal{F}_t)_{t \in \mathbb{R}}$, endowed with the norm

$$\|F\|_{C_W([s, T]; H)} = \left(\sup_{t \in [s, T]} \mathbb{E}(|F(t)|^2) \right)^{1/2};$$

the space $C_W([s, T]; H)$ is complete.

Definition 4.1 (i) *Let $x \in L^{2d}(\mathcal{O})$. We say that $X(\cdot, s, x) \in C_W([s, T]; H)$ is a mild solution of problem (4.1) if $X(t, s, x) \in L^{2d}(\mathcal{O})$ for all $t \in [s, T]$ and the following integral equation holds:*

$$X(t, s, x) = e^{(t-s)A} x + \int_s^t e^{(t-r)A} F(r, X(r, s, x)) dr + W_A(s, t), \quad t \geq s,$$

where $W_A(s, t)$ is the stochastic convolution

$$W_A(s, t) = \int_s^t e^{(t-r)A} (-A)^{-\gamma/2} dW(s), \quad t \geq s.$$

(ii) Let $x \in H$ and $s \in [0, T]$. We say that $X(\cdot, s, x) \in C_W([s, T]; H)$ is a generalized solution of problem (4.2) if there exists a sequence $(x_n) \subset L^{2d}(\mathcal{O})$ such that

$$\lim_{n \rightarrow \infty} x_n = x \quad \text{in } H$$

and the mappings $X(\cdot, s, x_n)$ from (i) satisfy

$$\lim_{n \rightarrow \infty} X(\cdot, s, x_n) = X(\cdot, s, x) \quad \text{in } C_W([s, T]; H).$$

One can show that Definition 4.1(ii) does not depend on (x_n) , see [8, §4.2].

We shall denote both mild and generalized solutions of (4.1) by $X(t, s, x)$.

The following result can be proved arguing as in [14], see also [8, Theorem 4.8].

Theorem 4.2 *The following statements are true.*

(i) If $x \in L^{2d}(\mathcal{O})$, problem (4.2) has a unique mild solution $X(\cdot, s, x)$. Moreover, for any $m \in \mathbb{N}$ there is $c_{m,d,T} > 0$ such that

$$\mathbb{E}\left(|X(t, s, x)|_{L^{2d}(\mathcal{O})}^{2m}\right) \leq c_{m,d,T} \left(1 + |x|_{L^{2d}(\mathcal{O})}^{2m}\right), \quad 0 \leq s \leq t \leq T.$$

(ii) If $x \in H$, problem (4.1) has a unique generalized solution $X(\cdot, s, x)$.

For any $0 \leq s \leq t \leq T$, let us consider the transition evolution operator

$$P_{s,t}\varphi(x) = \mathbb{E}[\varphi(X(t, s, x))], \quad \varphi \in C_u(H),$$

where $X(t, s, x)$ is a generalized solution of (4.2).

Then, given $\nu_0 \in \mathcal{P}(H)$, as in Section 2 we set

$$\nu_t := P_{0,t}^* \nu_0, \quad t \in [0, T]. \quad (4.3)$$

By Theorem 4.2 we find immediately the following result.

Proposition 4.3 *Let $m \in \mathbb{N}$ and assume that $\nu_0 \in \mathcal{P}(H)$ satisfies*

$$\int_H |x|_{L^{2d}(\mathcal{O})}^{2m} \nu_0(dx) < +\infty. \quad (4.4)$$

Then we have

$$\int_H |x|_{L^{2d}(\mathcal{O})}^{2m} \nu_t(dx) \leq c_{m,d,T} \int_H |x|_{L^{2d}(\mathcal{O})}^{2m} \nu_0(dx). \quad (4.5)$$

Now let us consider the operator L_0 defined by (1.2), (1.3) and associated with (4.2). Notice that if $u \in \mathcal{E}_A([0, T] \times H)$ then $L_0 u$ does not belong to $C([0, T]; C_u(H))$ in general. However, if $\nu_0 \in \mathcal{P}(H)$ satisfies (4.4), then by (4.5) one has $L_0 u \in L^2([0, T] \times H, \nu)$, where $\nu(dt, dx) = \nu_x(dt)dt$.

By Proposition 4.3 the family $(\nu_t)_{t \in [0, T]}$ obviously satisfies Hypothesis 3.1 for $p = 2$, provided ν_0 satisfies (4.4) with $m = 2d$. Then by Theorems 3.3 and 3.6 we deduce the following result.

Theorem 4.4 *Assume that $\nu_0 \in \mathcal{P}(H)$ satisfies (4.4) with $m = 2d$. Then the operator L_0 with domain $D(L_0) = \mathcal{E}_A([0, T] \times H)$ associated with (4.2) is closable on $L^2([0, T] \times H; \nu)$, where $\nu(dt, dx) = \nu_x(dt)dt$ and ν_t is defined by (4.3), and its closure L_2 is m -dissipative. Furthermore, L_2 generates a Markov C_0 -semigroup of contractions on $L^2([0, T] \times H; \nu)$ and ν is the unique measure satisfying the Fokker-Planck equation (1.5) and having properties (i)–(iii) in Subsection 3.2.*

A The Ornstein–Uhlenbeck semigroup

In this section Hypothesis 1.2 is still in force. We denote by R_t the Ornstein–Uhlenbeck semigroup

$$R_t\varphi(x) := \int_H \varphi(e^{tA}x + y)N_{Q_t}(dy), \quad \varphi \in C_{u,2}(H),$$

where

$$Q_t x := \int_0^t e^{sA} C e^{sA^*} x ds, \quad x \in H, \quad t \geq 0,$$

and N_{Q_t} is the Gaussian measure in H with mean 0 and covariance operator Q_t . Then (cf. [12]) we have

$$R_t\varphi(x) = \mathbb{E}[\varphi(Z(t, 0, x))].$$

We shall consider R_t acting in the Banach space $C_{u,2}(H)$ defined in Section 2. This will be needed in the proof of Proposition A.2 below.

Let us define the infinitesimal generator U of R_t through its resolvent by setting, following [6], $U := \lambda - \widetilde{G}_\lambda^{-1}$, $D(U) = \widetilde{G}_\lambda(C_{u,2}(H))$, where

$$\widetilde{G}_\lambda f(x) = \int_0^{+\infty} e^{-\lambda t} R_t f(x) dt, \quad x \in H, \quad \lambda > 0, \quad f \in C_{u,2}(H).$$

It is easy to see that for any $h \in D(A^*)$ the function $\varphi_h(x) = e^{i\langle x, h \rangle}$ belongs to the domain of U in $C_{u,2}(H)$ and we have

$$U\varphi_h = \frac{1}{2} \text{Tr} [CD^2\varphi_h] + \langle x, A^*D\varphi_h \rangle. \quad (\text{A.1})$$

A.1 The strong Feller property

The following identity for the derivative of $R_t\varphi$ is well known, see [16]:

$$\langle D_x R_t\varphi(x), h \rangle = \int_H \langle \Lambda_t h, Q_t^{-1/2} y \rangle \varphi(e^{tA}x + y) N_{Q_t}(dy), \quad \varphi \in C_{u,2}(H), \quad (\text{A.2})$$

where $\Lambda_t = Q_t^{-1/2} e^{tA}$. By Hölder's inequality it follows that

$$|\langle D_x R_t\varphi(x), h \rangle|^2 \leq |\Lambda_t h|^2 \int_H \varphi^2(e^{tA}x + y) N_{Q_t}(dy).$$

So, since h is arbitrary, one has

$$|D_x R_t\varphi(x)|^2 \leq \|\Lambda_t\|^2 \int_H \varphi^2(e^{tA}x + y) N_{Q_t}(dy).$$

It follows that

$$\begin{aligned} \frac{|D_x R_t\varphi(x)|^2}{(1 + |x|^2)^2} &\leq \|\Lambda(t)\|^2 \int_H \frac{\varphi^2(e^{tA}x + y)}{(1 + |x|^2)^2} N_{Q_t}(dy) \\ &\leq \|\Lambda(t)\|^2 \|\varphi\|_{C_{u,2}(H)}^2 \int_H \frac{(1 + |e^{tA}x + y|)^2}{(1 + |x|^2)^2} N_{Q_t}(dy) \\ &\leq 4\|\Lambda(t)\|^2 \|\varphi\|_{C_{u,2}(H)}^2 \int_H (1 + |y|^2)^2 N_{Q_t}(dy) \\ &\leq 4c_1^2 \|\Lambda(t)\|^2 \|\varphi\|_{C_{u,2}(H)}^2, \end{aligned}$$

where c_1 is a positive constant. Now, recalling Hypothesis 1.2(iii) and using the Laplace transform we obtain the following result.

Lemma A.1 *Let $\varphi \in D(U)$. Then there exists $c_2 > 0$ such that*

$$|D_x \varphi(x)| \leq c_2(\|\varphi\|_{C_{u,2}(H)} + \|U\varphi\|_{C_{u,2}(H)})(1 + |x|^2), \quad x \in H.$$

A.2 The Ornstein–Uhlenbeck semigroup in $C([0, T]; C_{u,2}(H))$

Let

$$V_0 u(t, x) = D_t u(t, x) + U u(t, x), \quad u \in \mathcal{E}_A([0, T] \times H).$$

It is clear that $V_0 u \in C([0, T]; C_{u,2}(H))$ (note that $U u(t, x)$ contains a term growing as $|x|$). Let us introduce an extension of the operator V_0 . For any $\lambda \in \mathbb{R}$ set

$$G_\lambda f(t, x) = \int_t^T e^{-\lambda(s-t)} R_{t-s} f(s, x) ds, \quad f \in C([0, T]; C_{u,1}(H)).$$

It is easy to see that G_λ satisfies the resolvent identity, so that there exists a unique linear closed operator V in $C([0, T]; C_{u,1}(H))$ such that

$$G_\lambda = (\lambda - V)^{-1}, \quad D(V) = G_\lambda(C([0, T]; C_{u,2}(H))), \quad \lambda \in \mathbb{R}. \quad (\text{A.3})$$

It is clear that V is an extension of V_0 .

Finally, it is easy to check that the semigroup \mathcal{R}_τ , $\tau \geq 0$, generated by V in $C_T([0, T]; C_{u,1}(H)) := \{u \in C_T([0, T]; C_{u,1}(H)) : u(T, x) = 0\}$ is given by

$$\mathcal{R}_\tau f(t, x) = \begin{cases} R_\tau f(t + \tau, \cdot)(x) & \text{if } t + \tau \leq T \\ 0 & \text{otherwise.} \end{cases} \quad (\text{A.4})$$

Arguing as in [19] one can show that $u \in D(V)$ and $Vu = f$ if and only if

$$\left\{ \begin{array}{l} \text{(i)} \quad \lim_{h \rightarrow 0} \frac{1}{h} (\mathcal{R}_h u(t, x) - u(t, x)) = f(t, x), \quad \forall (t, x) \in [0, T] \times H, \\ \text{(ii)} \quad \sup_{h \in (0, 1], (t, x) \in [0, T] \times H} \frac{(1 + |x|^2)^{-1}}{h} |\mathcal{R}_h u(t, x) - u(t, x)| < +\infty. \end{array} \right. \quad (\text{A.5})$$

A.3 A core for V

The following result is a generalization of [13].

Proposition A.2 *Let $u \in D(V)$ and let ν be a finite nonnegative Borel measure on $[0, t] \times H$. Then there exists a sequence $(u_n) \subset \mathcal{E}_A([0, T] \times H)$ such that for some $c_1 > 0$ one has*

$$|u_n(t, x)| + |V_0 u_n(t, x)| \leq c_1(1 + |x|^2), \quad \forall (t, x) \in [0, T] \times H$$

and $u_n \rightarrow u$, $V_0 u_n \rightarrow V_0 u$ in measure ν .

Proof. Let $f \in C([0, T; C_u(H)))$ and set $u = V^{-1}f$ so that

$$u(t, x) = - \int_t^T R_{s-t} f(s, x) ds = - \int_0^1 R_{(T-t)r} f((T-t)r + t, x) dr.$$

Arguing as in the proof of Proposition 1.2 in [8], it is easy to find a sequence $(f_{n_1, n_2}) \subset \mathcal{E}_A([0, T] \times H)$ such that

$$\lim_{n_1 \rightarrow \infty} \lim_{n_2 \rightarrow \infty} f_{n_1, n_2}(t, x) = f(t, x), \quad |f_{n_1, n_2}(t, x)| \leq c_1, \quad (\text{A.6})$$

where c_1 is independent of n_1, n_2 . Set

$$\begin{aligned} u_{n_1, n_2}(t, x) &= - \int_t^T R_{s-t} f_{n_1, n_2}(s, x) ds \\ &= - \int_0^1 R_{(T-t)r} f_{n_1, n_2}((T-t)r + t, x) dr, \end{aligned}$$

so that $Vu_{n_1, n_2} = f_{n_1, n_2}$. By (A.6) it follows that

$$\lim_{n_1 \rightarrow \infty} \lim_{n_2 \rightarrow \infty} u_{n_1, n_2}(t, x) = u(t, x), \quad |u_{n_1, n_2}(t, x)| \leq c_1 T. \quad (\text{A.7})$$

Moreover,

$$\begin{aligned} Vu_{n_1, n_2}(t, x) &= - \int_t^T R_{s-t} V f_{n_1, n_2}(s, x) ds \\ &= - \int_0^1 R_{(T-t)r} V f_{n_1, n_2}((T-t)r + t, x) dr, \end{aligned}$$

so that

$$\lim_{n_1 \rightarrow \infty} \lim_{n_2 \rightarrow \infty} Vu_{n_1, n_2}(t, x) = Vu(t, x), \quad |Vu_{n_1, n_2}(t, x)| \leq c_1 T. \quad (\text{A.8})$$

Now we want to approximate u_{n_1, n_2} by functions from $\mathcal{E}_A([0, T] \times H)$. For this we consider the set Σ of all partitions $\sigma = \{t_0, t_1, \dots, t_N\}$ of $[0, 1]$ with $0 = t_0 < t_1 < \dots < t_{N-1} < t_N = T$. We set

$$|\sigma| = \max_{1 \leq i \leq n} |t_i - t_{i-1}|$$

and endow Σ with the usual partial ordering

$$\sigma_1 < \sigma_2 \quad \text{if and only if } |\sigma_1| < |\sigma_2|.$$

Finally, for any $\sigma = \{t_0, t_1, \dots, t_N\} \in \Sigma$ we set

$$u_{n_1, n_2, \sigma}(t, x) = \sum_{k=1}^N R_{(T-t)r_k} f_{n_1, n_2}((T-t)r_k + t, x)(r_k - r_{k-1}), \quad (\text{A.9})$$

so that

$$Vu_{n_1, n_2, \sigma}(t, x) = \sum_{k=1}^N R_{(T-t)r_k} V f_{n_1, n_2}((T-t)r_k + t, x)(r_k - r_{k-1}). \quad (\text{A.10})$$

By (A.9) taking into account (A.7) it follows that

$$\lim_{n_1 \rightarrow \infty} \lim_{n_2 \rightarrow \infty} \lim_{|\sigma| \rightarrow 0} u_{n_1, n_2, \sigma}(t, x) = u(t, x), \quad |u_{n_1, n_2, \sigma}(t, x)| \leq c_1 T.$$

Similarly we see that

$$\lim_{n_1 \rightarrow \infty} \lim_{n_2 \rightarrow \infty} \lim_{|\sigma| \rightarrow 0} V u_{n_1, n_2, \sigma}(t, x) = V u(t, x).$$

However, (A.10) does not guarantee an estimate

$$|V u_{n_1, n_2, \sigma}(t, x)| \leq c_2 T,$$

with c_2 independent of n . Then we argue as follows. Note that if $z \in \mathcal{E}_A([0, T] \times H)$, then the function

$$F: [0, T] \times [0, T] \rightarrow H, \quad (t, s) \mapsto R_s z(t, x),$$

which is not continuous in the topology of $C_u(H)$, is continuous in that of $C_{u,2}(H)$, consequently the integral

$$\int_0^1 R_{(T-t)r} f((T-t)r + t, x) dr$$

is convergent in that topology. Therefore, for any $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that if $|\sigma| < \delta_\varepsilon$ we have

$$\left| V u_{n_1, n_2}(t, x) - \sum_{k=1}^N R_{(T-t)r_k} V f_{n_1, n_2}((T-t)r_k + t, x)(r_k - r_{k-1}) \right| \leq \varepsilon(1 + |x|^2).$$

Consequently,

$$|V u_{n_1, n_2, \sigma}(t, x)| \leq |V u_{n_1, n_2}(t, x)| + \varepsilon(1 + |x|^2)$$

and, taking into account (A.8), we find

$$|V u_{n_1, n_2, \sigma}(t, x)| \leq c_1 T + \varepsilon(1 + |x|^2).$$

Let σ_n denote the partition formed by the points $0, 2^{-n}T, 2^{1-n}T, \dots, T$. We can find functions u_{n_1, n_2, σ_n} indexed by the triples (n_1, n_2, σ_n) such that

$$u_{n_1, n_2, \sigma_n}(t, x) \rightarrow u(t, x)$$

in the following sense: keeping n_1, n_2 fixed, one has

$$\lim_{n \rightarrow \infty} u_{n_1, n_2, \sigma_n}(t, x) = u_{n_1, n_2, \sigma_n}(t, x),$$

next there is a limit u_{n_1} for any n_1 fixed as $n_2 \rightarrow \infty$, and finally, $u_{n_1} \rightarrow u$ as $n_1 \rightarrow \infty$. Convergence $V_0 u_{n_1, n_2, \sigma_n} \rightarrow V_0 u$ takes place in the same sense. Clearly, we may assume that $|x|^2$ is ν -integrable (just by multiplying ν by $(|x|^2 + 1)^{-1}$). By the dominated convergence theorem this yields $L^1(\nu)$ -convergence $u_{n_1, n_2, \sigma_n} \rightarrow u$ and $V_0 u_{n_1, n_2, \sigma_n} \rightarrow V_0 u$ in the same sense as above (first for any n_1, n_2 fixed etc.) and enables us to find a sequence of elements u_n in the net u_{n_1, n_2, σ_n} convergent in $L^1(\nu)$, hence in measure ν . \square

As in the proof of Lemma 2.5 one proves that if $u \in D(V)$ then u is differentiable in x . Hence the following result is a consequence of (A.2) and Lemma A.1.

Corollary A.3 *Let $u \in D(V)$ and let ν be a finite nonnegative Borel measure on $[0, T] \times H$. Then there exists a sequence $(u_n) \subset \mathcal{E}_A([0, T] \times H)$ such that for some $c_1 > 0$ one has*

$$|u_n(t, x)| + |D_x u_n(t, x)| + |V_0 u_n(t, x)| \leq c_1(1 + |x|^2), \quad \forall (t, x) \in [0, T] \times H,$$

and $u_n \rightarrow u$, $D_x u_n \rightarrow D_x u$, $V_0 u_n \rightarrow V_0 u$ in measure ν .

Acknowledgement. This work has been supported in part by the RFBR project 07-01-00536, the Russian–Japanese Grant 08-01-91205-JF, the Russian–Chinese Grant 06-01-39003, ARC Discovery Grant DP0663153, the DFG Grant 436 RUS 113/343/0(R), SFB 701 at the University of Bielefeld, the research programme “Analisi e controllo di equazioni di evoluzione deterministiche e stocastiche” of the Italian “Ministero della Ricerca Scientifica e Tecnologica”. Most of the work was done during visits of the first and third authors to the Scuola Normale Superiore di Pisa and visits of the first and second authors to the University of Bielefeld.

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