Regular dependence on initial data for stochastic evolution equations with multiplicative Poisson noise

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Abstract

We prove existence, uniqueness and Lipschitz dependence on the initial datum for mild solutions of stochastic partial differential equations with Lipschitz coefficients driven by Wiener and Poisson noise. Under additional assumptions, we prove Gâteaux and Fréchet differentiability of solutions with respect to the initial datum. As an application, we obtain gradient estimates for the resolvent associated to the mild solution. Finally, we prove the strong Feller property of the associated semigroup.

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1 Introduction

We shall consider the mild formulation of a stochastic PDE of the form

$$du(t) = [Au(t)+f(t,u(t))]\,dt+B(t,u(t))\,dW(t)+\int_Z G(t,u(t),z)\,\bar{\mu}(dt,dz), \qquad u(0)=x, \quad (1.1)$$

where W and $\bar{\mu}$ are a Wiener process and a compensated Poisson measure, respectively, on a Hilbert space, thus including a large class of equations driven by Hilbert space-valued Lévy noise, thanks to the Lévy-Itô decomposition theorem. Precise assumptions on the data of (1.1) are given in the next section.

The main contribution of this paper is global well-posedness (i.e. existence, uniqueness, and regular dependence on the initial datum of a mild solution on any time interval $[0,T]$, $T<\infty$) in spaces of càdlàg predictable processes whose supremum (in time) has finite p-th moments. While the L_2 result was fully settled by Kotelenez [18] about twenty-five years ago, the lack of an L_p theory has been pointed out more recently in [2]. The new tool allowing the development of such a theory is an infinite dimensional Bichteler-Jacod inequality, which also holds for stochastic convolutions. The L_p existence result allows us to prove first and second order Fréchet differentiability of the solution with respect to the initial datum (for first order Gâteaux differentiability the L_2 theory is enough, see also [1]). Moreover, these differentibility results are a key tool to prove that, as long as the noise has a Brownian component, the semigroup associated to the SPDE is regularizing, in particular, that it has the strong Feller property. An essential ingredient to obtain this result is a formula of Bismut-Elworthy type, which only holds under non-degeneracy assumptions on B. Finally, we also obtain gradient estimates on the resolvent associated to the SPDE.

The issues considered in this paper are by now classical for stochastic PDE with Wiener noise (see e.g. [5, 6, 7, 11] and references therein), but comparable results do not seem to be available in the more general jump case considered here. In fact, it is fair to say that the theory of stochastic PDEs driven by jump noise is not yet fully developed, even though recent years have witnessed a growing interest in the area: let us just mention, without any claim of completeness, the recent monograph [24], where the semigroup approach is discussed, [20] for an analytic approach based on generalized Mehler semigroups, as well as the earlier important contributions [12, 22] for the variational approach.

Let us also mention that differentiability properties of the solution of stochastic PDE play an essential role in the study of the associated Kolmogorov equations. This direction of research, while thoroughly pursued in the case of Wiener noise (see e.g. [5, 6, 29]), is still in its infancy for equations with jumps, and we hope that our results will be useful in this respect.

The paper is organized as follows: in Section 2 precise assumptions on the SPDE (1.1) are given, and the main results on well-posedness and regular dependence on the initial datum are stated. In Section 3 we prove a Bichteler-Jacod inequality for infinite dimensional stochastic integrals with respect to Poisson random measures, and we extend it to corresponding stochastic convolutions. This result is essential in order to obtain L_p well-posedness, and it could be interesting in its own right. We also recall some results on the differentiability of implicit functions in Banach spaces, on which the proofs of regular dependence heavily rely. Section 4 contains the proofs of the well-posedness and differentiability results. In Section 5 we obtain an analytic consequence of these results, that is gradient estimates for the resolvent associated to the (solution of the) SPDE. Finally, in Section 6 we show that the semigroup associated to the SPDE is strong Feller, if B is not degenerate.

Finally, we would like to mention that a part of the results of this paper has been announced in [25] (based on [26]). There was, however, an error both in the formulation and proof in what was called there Burkholder-Davis-Gundy inequality for Poisson integrals, on which all subsequent results depended. One point of this paper is to correct this error. The corresponding inequality is contained in Proposition 3.3 below. Then, as described above, among other things we prove that all results announced in [25] hold.

Let us conclude this introduction with some words about notation. By $a \lesssim b$ we mean that there exists a constant N such that $a \leq Nb$. To emphasize that the constant N depends on a parameter p we shall write $N(p)$ and $a \lesssim_p b$. Generic constants, which may change from line to line, are denoted by N. Given two separable Banach spaces E, F we shall denote the space of linear bounded operators from E to F by $\mathcal{L}(E,F)$. Similarly, if H and K are Hilbert spaces, we shall denote the space of trace-class and Hilbert-Schmidt operators from K to H by $\mathcal{L}_1(K,H)$ and $\mathcal{L}_2(K,H)$, respectively. $\mathcal{L}_1^+$ stands for the subset of $\mathcal{L}_1$ consisting of all positive operators. We shall write $\mathcal{L}_1(H)$ in place of $\mathcal{L}_1(H,H)$, and similarly for the other spaces. Given a self-adjoint operator $Q \in \mathcal{L}_1^+(K)$, we denote by $\mathcal{L}_2^Q(K,H)$ the set of all (possibly unbounded) operators $B : Q^{1/2}K \to H$ such that $BQ^{1/2} \in \mathcal{L}_2(K,H)$. The norms in $\mathcal{L}_2(K,H)$ and $\mathcal{L}_2^Q(K,H)$ will be denoted by $|\cdot|_2$ and $|\cdot|_Q$, without explicitly indicating the dependence on the spaces K and H. Lebesgue measure is denoted by Leb, without mentioning the underlying space if no misunderstanding can arise. Given a function $\phi : E \to F$, we set

$$[\phi]_1 := \sup_{\substack{x,y\in E\\ x\neq y}} \frac{|\phi(x)-\phi(y)|}{|x-y|},$$

and we denote by $\partial\phi : E \times E \to F$ the map $(x,y) \mapsto \partial_y\phi(x)$, where the directional derivative $\partial_y\phi(x)$ is defined by

$$\partial_y\phi(x) := \lim_{h\to 0} \mathcal{Q}_y^h\phi(x) := \lim_{h\to 0} \frac{\phi(x+hy)-\phi(x)}{h}.$$

We shall also use the symbol $\partial\phi(x)$ to denote the Gâteaux derivative, so $\partial\phi(x) \in \mathcal{L}(E,F)$, defined by $y \mapsto \partial_y\phi(x)$. Analogously, given a function $\phi : E_1 \times E_2 \to F$, where E_1, E_2 are further Banach spaces, we define the following directional derivatives

$$\partial_{1,y}\phi(x_1,x_2) = \lim_{h\to 0} \mathcal{Q}^h_{1,y}\phi(x_1,x_2) := \lim_{h\to 0} \frac{\phi(x_1+hy,x_2)-\phi(x_1,x_2)}{h}$$
$$\partial_{2,z}\phi(x_1,x_2) = \lim_{h\to 0} \mathcal{Q}^h_{2,z}\phi(x_1,x_2) := \lim_{h\to 0} \frac{\phi(x_1,x_2+hz)-\phi(x_1,x_2)}{h},$$

and the corresponding maps

$$\partial_1\phi : E_1 \times E_2 \times E_1 \ni (x_1,x_2,y) \mapsto \partial_{1,y}\phi(x_1,x_2) \in F,$$
$$\partial_2\phi : E_1 \times E_2 \times E_2 \ni (x_1,x_2,z) \mapsto \partial_{2,z}\phi(x_1,x_2) \in F.$$

Partial Gâteaux derivatives are denoted by the same symbols. Fréchet differentials are denoted by D, with subscripts if necessary. Moreover, in view of the canonical isomorphism between $\mathcal{L}(E,\mathcal{L}(E,F))$ and $\mathcal{L}^{\otimes 2}(E,F)$, the space of bilinear maps from E to F, we can and will consider $D^2\phi$ as a map from E to $\mathcal{L}^{\otimes 2}(E,F)$. The space of k times continuously differentiable maps from E to F will be denoted by $C^k(E,F)$, and simply by $C^k(E)$ if $F=\mathbb{R}$.

We shall occasionally use the following standard notation for stochastic integrals with respect to semimartingales and random measures:

$$\phi \cdot M(t) := \int_0^t \phi(s)\,dM(s), \qquad \phi \star \mu(t) := \int_0^t \int \phi(s,z)\,\mu(ds,dz).$$

2 Main results

Let us begin with stating our precise assumptions on equation (1.1). We are given two real separable Hilbert spaces H, K and a filtered probability space $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$, $\mathbb{F}=(\mathcal{F}_t)_{t\in[0,T]}$, on which a Wiener process with covariance operator $Q \in \mathcal{L}_1^+(K)$ is defined. Moreover, we are given a measure space $(Z,\mathcal{Z},m)$ and a Poisson measure μ on $[0,T]\times Z$, independent of W, defined on the same stochastic basis. The compensator (dual predictable projection) of μ is $\mathrm{Leb}\otimes m$, and the compensated measure $\bar\mu$ is $\bar\mu := \mu - \mathrm{Leb}\otimes m$.

Denoting the predictable σ-field by $\mathcal{P}$, we shall assume throughout the paper that the following assumptions are satisfied:

- (i) A is the generator of a strongly continuous semigroup on H;
- (ii) $f : \Omega\times[0,T]\times H \to H$ and $B : \Omega\times[0,T]\times H \to \mathcal{L}_2^Q(K,H)$ are $\mathcal{P}\times\mathcal{B}(H)$-measurable functions;
- (iii) $G : \Omega\times[0,T]\times H\times Z \to Z$ is a $\mathcal{P}\times\mathcal{B}(H)\times\mathcal{Z}$-measurable function;
- (iv) x is an H-valued $\mathcal{F}_0$-measurable random variable.

Further assumptions on the data of the problem will be specified when needed. For simplicity, we shall suppress explicit dependence on $\omega\in\Omega$ of all random elements, if no confusion can arise. Let us also recall that, by (i), there exist M, $\sigma \geq 0$, such that $|e^{tA}| \leq Me^{\sigma t}$. We set, for future reference, $M_T := Me^{\sigma T}$.

The concept of solution we shall work with and the spaces where solutions are sought are defined next.

Definition 2.1. *A predictable process $u : [0,T]\to H$ is a mild solution of (1.1) if it satisfies*

$$\begin{aligned} u(t) =& e^{tA}x + \int_0^t e^{(t-s)A} f(s,u(s))\,ds + \int_0^t e^{(t-s)A} B(s,u(s))\,dW(s) \\ &+ \int_{(0,t]}\int_Z e^{(t-s)A} G(s,u(s),z)\,\bar\mu(ds,dz) \end{aligned}$$

$\mathbb{P}$-a.s. for all $t \in [0,T]$, where at the same time we assume that all integrals on the right-hand side exist.

We shall also write $u(x)$ to emphasize the dependence on the initial datum, and $u(t,x)$ will stand for the value of $u(x)$ at time $t \in [0,T]$.

In the following, for simplicity of notation, we shall often write $\int_0^t$ in place of $\int_{(0,t]}$.

Definition 2.2. *Let $p \geq 2$. We shall denote by $\mathcal{H}_p(T)$ and $\mathbb{H}_p(T)$ the spaces of all predictable processes $u : [0,T] \to H$ such that*

$$|[u]|_p := \Big(\sup_{t\in[0,T]} \mathbb{E}|u(t)|^p \Big)^{1/p} < \infty,$$

and

$$\|u\|_p := \big(\mathbb{E} \sup_{t\in[0,T]} |u(t)|^p \big)^{1/p} < \infty,$$

respectively.

For reasons that will become apparent later, we shall also need to consider the same spaces endowed with the equivalent norms

$$|[u]|_{p,\lambda} := \Big(\sup_{t\in[0,T]} \mathbb{E}|e^{-\lambda t}u(t)|^p \Big)^{1/p}, \qquad \|u\|_{p,\lambda} := \Big(\mathbb{E} \sup_{t\in[0,T]} |e^{-\lambda t}u(t)|^p \Big)^{1/p},$$

with $\lambda > 0$. We shall also use the notation $\mathbb{L}_p$ to denote the set $L_p(\Omega, \mathcal{F}, \mathbb{P}; H)$.

The following well-posedness result in $\mathcal{H}_2(T)$ is quite simple to prove and it essentially relies only on the isometric formula for stochastic integrals with respect to compensated Poisson measures (see also [1, 24] for similar results).

Theorem 2.3. *Assume that $x \in \mathbb{L}_2$, $e^{sA}B(t,x) \in \mathcal{L}_2^Q(K,H)$, $e^{sA}G(t,x,\cdot) \in L_2(Z,m)$ for all $(s,t,x) \in [0,T]^2 \times H$, and there exist $h \in L_1([0,T])$ and $a \in H$ such that*

$$\begin{aligned} |e^{sA}(f(t,x) - f(t,y))|^2 + |e^{sA}(B(t,x) - B(t,y))|_Q^2 \\ + \int_Z |e^{sA}(G(t,x,z) - G(t,y,z))|^2 \, m(dz) \leq h(s)|x-y|^2, \end{aligned} \tag{2.1}$$

$$|e^{sA}f(t,a)|^2 + |e^{sA}B(t,a)|_Q^2 + \int_Z |e^{sA}G(t,a,z)|^2 \, m(dz) \leq h(s). \tag{2.2}$$

$\mathbb{P}$-a.s. for all x, $y \in H$ and s, $t \in [0,T]$. Then equation (1.1) admits a unique mild solution in $\mathcal{H}_2(T)$. Moreover, the solution map $x \mapsto u(x)$ is Lipschitz from $\mathbb{L}_2$ to $\mathcal{H}_2(T)$.

As briefly mentioned above, this theorem is stated and proved for its simplicity, even though a more refined result holds true. In fact, one can look for mild solutions of (1.1) in the smaller (and more regular) spaces $\mathbb{H}_p(T)$, obtaining also that solutions have càdlàg paths. The price to pay is that one has to find suitable estimates to replace the isometry of the stochastic integral. In order to obtain such estimates, we need to assume that A is η-m-dissipative, i.e. that $A - \eta I$ is m-dissipative for some $\eta \geq 0$ (this is equivalent to assuming that $|e^{tA}| \leq e^{\eta t}$ for all $t \geq 0$, i.e. that the semigroup generated by A is of quasi-contraction type). On the other hand, Theorem 2.3 above holds without the quasi-dissipativity condition on A.

Our first main result is the following theorem, where the solution map is defined from $\mathbb{L}_p$ to $\mathbb{H}_p(T)$.

Theorem 2.4. *Let $p \geq 2$. Assume that A is η-m-dissipative, $x \in \mathbb{L}_p$, and there exist $h \in L_p([0,T])$ and $a \in H$ such that*

$$\begin{aligned}|e^{sA}(f(t,x) - f(t,y))| + |B(s,x) - B(s,y)|_Q \\ + \max\big(|G(s,x,\cdot) - G(s,y,\cdot)|_{L_2(Z,m)}, |G(s,x,\cdot) - G(s,y,\cdot)|_{L_p(Z,m)}\big) \leq h(s)|x-y|,\end{aligned} \tag{2.3}$$

$$|e^{sA}f(t,a)|^2 + |B(s,a)|_Q^2 + \int_Z \big(|G(s,a,z)|^2 + |G(s,a,z)|^p\big)\, m(dz) \leq h(s) \tag{2.4}$$

$\mathbb{P}$-a.s. for all x, $y \in H$ and s, $t \in [0,T]$. Then equation (1.1) admits a unique càdlàg mild solution in $\mathbb{H}_p(T)$. Moreover, the solution map $x \mapsto u(x)$ is Lipschitz from $\mathbb{L}_p$ to $\mathbb{H}_p(T)$.

Remark 2.5. Before we proceed to state other results, we would like to make the following remarks:

(i) Much more general existence and uniqueness results in $\mathbb{H}_2(T)$ were proved by Kotelenez [18], where noise terms driven by general locally square integrable martingales are allowed, as well as locally Lipschitz coefficients with linear growth.

(ii) One could also consider equations driven by martingales with independent increments, "embedding" equations driven by compensated Poisson random measures, using the equivalence result of Gyöngy and Krylov [13]. This approach, however, even though very powerful, would be less transparent, and for this reason we prefer to work directly with equations driven by a Wiener process and a compensated Poisson measure. Let us also recall that, if one only wants to obtain results in $\mathbb{H}_2(T)$, then general stochastic martingale measures are also allowed, appealing to the results in [18] and to the above mentioned procedure developed in [13].

(iii) If the coefficients of (1.1) are independent of $\omega \in \Omega$, then one can obtain the Markov property of solutions in a standard way, e.g. following the method of [19, Sect. 2.9] – see also [12], [24].

(iv) It is not difficult to prove that mild solutions are weak solutions (in the sense of PDEs), as it follows, roughly speaking, by a suitable stochastic version of Fubini's theorem. More details can be found e.g. in [24, Sect. 9.3].

Under the additional assumption that the coefficients f, B and G are Gâteaux differentiable, we obtain that the solution map enjoys the same property. For this to hold, the simpler $\mathcal{H}_2(T)$ well-posedness suffices. In particular, no quasi-m-dissipativity assumption on A is needed.

From here until the end of this section we assume, for simplicity only, that f, B and G are deterministic maps that do not depend on time. Given a Banach space E, we shall denote the space of functions $\phi : Z \to E$ such that $\int_Z |\phi|_E^p \, dm < \infty$ by $L_p(Z,m;E)$.

Theorem 2.6. *Under the hypotheses of Theorem 2.3, assume that*

(i) f is Gâteaux differentiable with $\partial f \in C(H \times H, H)$;

(ii) B is Gâteaux differentiable and $\partial B \in C(H \times H, \mathcal{L}_2^Q(K,H))$.

(iii) the map $x \mapsto G(x,z)$ is Gâteaux differentiable for all $s \in]0,T]$ and $z \in Z$, and

$$x \mapsto e^{sA}\partial_{1,y}G(x,z) \in C(H,H)$$

for all $s \in]0,T]$, $y \in H$, and $z \in Z$;

(iv) *one has*

$$x \mapsto e^{sA}\partial_{1,y}G(x,\cdot) \in C(H, L_2(Z,m;H))$$

for all $s \in]0,T]$ *and* $y \in H$.

Then the solution map $x \mapsto u(x) : \mathbb{L}_2 \to \mathcal{H}_2(T)$ *is Gâteaux differentiable and* $\partial u : (x,y) \mapsto \partial_y u(x) \in C(\mathbb{L}_2 \times \mathbb{L}_2, \mathcal{H}_2(T))$ *is the mild solution of*

$$\begin{aligned} dv(t) = Av(t)\,dt + \partial f(u(t,x))v(t)\,dt + \partial B(u(t,x))v(t)\,dW(t) \\ + \int_Z \partial_1 G(u(t,x),z)v(t)\,\bar{\mu}(dt,dz), \qquad v(0) = y. \end{aligned} \tag{2.5}$$

Moreover, one has

$$|[\partial_y u(x)]|_2 \le N|y|_{\mathbb{L}_2}$$

for all x, $y \in \mathbb{L}_2$, *where the constant* N, *which does not depend on* x *and* y, *is the Lipschitz constant of the solution map* $\mathbb{L}_2 \ni x \mapsto u(x) \in \mathcal{H}_2(T)$.

On the other hand, in order to obtain Fréchet differentiability of the solution map, the full $\mathbb{H}_p(T)$ well-posedness result is needed. At this point we would like to stress that the following two theorems cannot be proved, to the best of our knowledge, on the basis of the already known $\mathbb{H}_2(T)$ well-posedness, even if one is interested only in the Fréchet differentiability of the solution map from H to $\mathcal{H}_2(T)$.

Theorem 2.7. *Let* $q > p \ge 2$, *and assume that the hypotheses of Theorem 2.4 are satisfied with p replaced by q. Moreover, assume that*

(i) $f \in C^1(H,H)$ *and* $B \in C^1(H, \mathcal{L}(K,H))$;

(ii) $x \mapsto e^{tA}DB(x) \in C\big(H, \mathcal{L}(H, \mathcal{L}_2^Q(K,H))\big)$ *for all* $t \in]0,T]$;

(iii) $x \mapsto G(x,z) \in C^1(H,H)$ *for all* $z \in Z$.

Then the solution map $\mathbb{L}_p \ni x \mapsto u(x) \in \mathbb{H}_p(T)$ *is Gâteaux differentiable and* $(x,y) \mapsto \partial_y u(x) \in C(\mathbb{L}_p \times \mathbb{L}_p, \mathbb{H}_p(T))$ *is the mild solution of (2.5) in* $\mathbb{H}_p(T)$. *Moreover, one has*

$$\|\partial_y u(x)\|_p \le N|y|_{\mathbb{L}_p}$$

for all x, $y \in \mathbb{L}_p$, *where* N *denotes the Lipschitz constant of* $\mathbb{L}_p \ni x \mapsto u(x) \in \mathbb{H}_p(T)$. *Finally, if* $x \in \mathbb{L}_q$, *then* $x \mapsto u(x) \in C^1(\mathbb{L}_q, \mathbb{H}_p(T))$. *In particular,* $x \mapsto u(x) \in C^1(H, \mathbb{H}_p(T))$.

Theorem 2.8. *Let* $q > 2p \ge 4$. *Under the hypotheses of Theorem 2.4 with p replaced by q, assume that*

(i) $f \in C^2(H,H)$, $B \in C^2(H, \mathcal{L}(K,H))$, *and there exists* $C_1 > 0$ *such that*

$$|D^2 f(x)| + |D^2 B(x)| \le C_1 \qquad \forall x \in H;$$

(ii) *the map* $x \mapsto G(x,z) : H \to H$ *is twice Fréchet differentiable for all* $z \in Z$ *and*

$$x \mapsto D_1^2 G(x,z) \in C\big(H, \mathcal{L}(H, \mathcal{L}(H))\big)$$

for all $z \in Z$;

(iii) there exists $h_1 \in L_p(Z,m) \cap L_2(Z,m)$ such that

$$|D_1^2 G(x,z)(y_1,y_2)| \le h_1(z)|y_1||y_2|$$

for all x, y_1, $y_2 \in H$ and $z \in Z$;

(iv) there exists $k \in L_2([0,T])$ such that

$$\left|e^{tA} D^2 B(x)(y,z)\right|_Q \le k(t)|y|\,|z|.$$

Then the Fréchet derivative $Du : \mathbb{L}_q \to \mathcal{L}(\mathbb{L}_q, \mathbb{H}_p(T))$ is Gâteaux differentiable. Let x, y_1, $y_2 \in \mathbb{L}_q$, and $w := [\partial Du(x)](y_1,y_2) \equiv [\partial^2 u(x)](y_1,y_2)$, $v_1 = Du(x)y_1$, $v_2 = Du(x)y_2$. Then w is the mild solution of

$$\begin{aligned} dw(t) &= \big[Aw(t) + Df(u(t))w(t) + D^2 f(u(t))(v_1(t), v_2(t))\big]\,dt \\ &\quad + \int_Z \big[D_1 G(u(t),z)w(t) + D_1^2 G(u(t),z)(v_1(t),v_2(t))\big]\,\bar\mu(dt,dz), \qquad w(0)=0. \end{aligned} \tag{2.6}$$

Moreover, there exists a constant $N = N(T,p,q) > 0$ such that

$$\|\partial Du(x)(y_1,y_2)\|_p \le N|y_1|_{\mathbb{L}_q}|y_2|_{\mathbb{L}_q}$$

for all y_1, $y_2 \in \mathbb{L}_q$. Finally, if $q > 4p \ge 8$, then

$$x \mapsto u(x) \in C^2(\mathbb{L}_q, \mathbb{H}_p(T)).$$

In particular, the solution map belongs to $C^2(H, \mathbb{H}_p(T))$.

3 Auxiliary results

3.1 L_p estimates for stochastic convolutions

In order to prove Theorem 2.4 we need to establish a maximal inequality for stochastic convolutions with respect to a compensated Poisson measure, which may be of independent interest. For related estimates (which hold only for stochastic integrals) in the finite dimensional case see [4] and references therein, and for the special case of stochastic integrals with respect to Lévy processes [16, 28]. Maximal inequalities for stochastic convolutions can be found e.g. in [15, 17, 18]. None of the latter results, however, seems to be useful to obtain the estimates we need to establish well-posedness in $\mathbb{H}_p(T)$.

Let us begin with a Bichteler-Jacod inequality for Poisson integrals.

Lemma 3.1. *Let $p \ge 2$. Assume that $g : [0,T] \times Z \to H$ is a predictable process such that the expectation on the right-hand side of (3.1) below is finite. Then one has*

$$\mathbb{E}\Big|\sup_{t\le T}\int_{(0,t]}\int_Z g(s,z)\,\bar\mu(ds,dz)\Big|^p \le N\mathbb{E}\int_0^T\Big[\int_Z |g(s,z)|^p\,m(dz) + \Big(\int_Z |g(s,z)|^2\,m(dz)\Big)^{p/2}\Big]\,ds, \tag{3.1}$$

where $N = N(p,T)$, and $(p,T) \mapsto N$ is continuous.

Proof. Setting $\phi : H \to \mathbb{R}$, $\phi(x) = |x|^p$, we have that ϕ is twice Fréchet differentiable with derivatives

$$\phi'(x) : \eta \to p|x|^{p-2}\langle x, \eta\rangle$$

and

$$\phi''(x) : (\eta,\zeta) \mapsto p(p-2)|x|^{p-4}\langle x,\eta\rangle\langle x,\zeta\rangle + p|x|^{p-2}\langle \eta,\zeta\rangle, \qquad x \neq 0,$$

$\phi''(0) = 0$. Let us set $X = g \star \bar{\mu}$. Then Itô's formula (see e.g. [21]) yields

$$\begin{aligned} |X(t)|^p = p &\int_0^t \langle |X(s-)|^{p-2}X(s-), dX(s)\rangle \\ &+ \sum_{s\le t} \big(|X(s)|^p - |X(s-)|^p - p|X(s-)|^{p-2}\langle X(s-), \Delta X(s)\rangle\big) \end{aligned} \tag{3.2}$$

$\mathbb{P}$-a.s. for all $t \le T$, where, as usual, $\Delta X(s) := X(s) - X(s-)$. Applying Taylor's formula to the function ϕ we obtain

$$\begin{aligned} &|X(s)|^p - |X(s-)|^p - p|X(s-)|^{p-2}\langle X(s-), \Delta X(s)\rangle \\ &\quad= \frac{1}{2}p(p-2)\big|X(s-) + \xi\Delta X(s)\big|^{p-4}\big\langle X(s-) + \xi\Delta X(s), \Delta X(s)\big\rangle^2 \\ &\quad\quad+ \frac{1}{2}p\big|X(s-) + \xi\Delta X(s)\big|^{p-2}|\Delta X(s)|^2 \\ &\quad\le \frac{1}{2}p(p-1)\big|X(s-) + \xi\Delta X(s)\big|^{p-2}|\Delta X(s)|^2, \end{aligned}$$

where $\xi \equiv \xi(s) \in]0,1[$ (see e.g. [10, Thm. 4.18.1]). Since $|X(s-) + \xi\Delta X(s)| \le |X(s-)| + |\Delta X(s)|$, we also have

$$\big|X(s-) + \xi\Delta X(s)\big|^{p-2} \lesssim_p |X(s-)|^{p-2} + |\Delta X(s)|^{p-2} \le X^*(s-)^{p-2} + |\Delta X(s)|^{p-2},$$

where $X^*(s) := \sup_{r\le s}|X(r)|$.

Let us now assume, for the time being, that X is bounded $\mathbb{P}$-a.s.. Then the first term on the right-hand side of (3.2) is a martingale with expectation zero, and we obtain

$$\mathbb{E}|X(t)|^p \le N(p)\mathbb{E}\sum_{s\le t}\big(X^*(s-)^{p-2}|\Delta X(s)|^2 + |\Delta X(s)|^p\big).$$

Therefore, recalling that the compensator of μ is $m \otimes \mathrm{Leb}$ and using Young's inequality

$$ab \le \frac{a^{\frac{p}{p-2}}}{\frac{p}{p-2}} + \frac{b^{p/2}}{p/2},$$

we get

$$\begin{aligned} \mathbb{E}|X(t)|^p &\lesssim_p \mathbb{E}\int_0^t \big[X^*(s-)^{p-2}|g(s,\cdot)|^2_{L_2(Z,m)} + |g(s,\cdot)|^p_{L_p(Z,m)}\big]\,ds \\ &\lesssim_p \mathbb{E}\int_0^t \big[X^*(s)^p + |g(s,\cdot)|^p_{L_2(Z,m)} + |g(s,\cdot)|^p_{L_p(Z,m)}\big]\,ds. \end{aligned}$$

Doob's inequality then yields

$$\mathbb{E}X^*(t)^p \lesssim_p \mathbb{E}\int_0^t \big[X^*(s)^p + |g(s,\cdot)|^p_{L_2(Z,m)} + |g(s,\cdot)|^p_{L_p(Z,m)}\big]\,ds,$$

hence, thanks to Gronwall's inequality, we obtain (3.1).

In order to remove the assumption that X is bounded almost surely, we shall proceed in two steps. Assume first that $|g(s,z)| \le N$ a.s. for all $(s,z) \in [0,T] \times Z$, and define the stopping times

$$\tau_n = \inf\{t \ge 0 : \ |X(t)| > n\} \wedge T.$$

We clearly have $\tau_n \to T$ a.s. as $n \to \infty$ because X is bounded on compact intervals. We also have

$$|\Delta X(t)| \leq \sup_{(s,z)\in[0,T]\times Z} |g(s,z)| \leq N \qquad \text{a.s.}$$

hence, setting $Y_n = (1_{]0,\tau_n]} g) \star \bar{\mu} \equiv X(t \wedge \tau_n)$, one easily sees that

$$|Y_n(t)| \leq |Y_n(t-)| + \sup_{t\leq T} |\Delta X(t)| \leq n + N,$$

and, by Fatou's lemma and passing to the limit,

$$\begin{aligned}
\mathbb{E}X^*(T)^p &\leq \liminf_{n\to\infty} \mathbb{E}Y_n^*(T)^p \\
&\lesssim_p \liminf_{n\to\infty} \mathbb{E}\int_0^T \Big[\int_Z |1_{]0,\tau_n]} g(s,z)|^p\, m(dz) + \Big(\int_Z |1_{]0,\tau_n]} g(s,z)|^2\, m(dz)\Big)^{p/2}\Big]\, ds \\
&\leq \mathbb{E}\int_0^T \Big[\int_Z |g(s,z)|^p\, m(dz) + \Big(\int_Z |g(s,z)|^2\, m(dz)\Big)^{p/2}\Big]\, ds,
\end{aligned}$$

which proves the claim if g is a.s. bounded. The general case can be proved by setting

$$g_n(s,z) := \begin{cases} g(s,z), & \text{if } |g(s,z)| \leq n \\ n\dfrac{g(s,z)}{|g(s,z)|}, & \text{if } |g(s,z)| > n, \end{cases}$$

and $X_n := g_n \star \bar{\mu}$, from which it is easy to prove that, by (3.1), $\{X_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{H}_p(T)$ and $X_n \to X$ in $\mathbb{H}_p(T)$, with $X = g \star \bar{\mu}$. Using again Fatou's lemma and recalling that (3.1) holds for bounded integrands, we have

$$\begin{aligned}
\mathbb{E}X^*(T)^p &\leq \liminf_{n\to\infty} \mathbb{E}X_n^*(T)^p \\
&\lesssim_p \liminf_{n\to\infty} \mathbb{E}\int_0^T \Big[\int_Z |g_n(s,z)|^p\, m(dz) + \Big(\int_Z |g_n(s,z)|^2\, m(dz)\Big)^{p/2}\Big]\, ds \\
&\leq \mathbb{E}\int_0^T \Big[\int_Z |g(s,z)|^p\, m(dz) + \Big(\int_Z |g(s,z)|^2\, m(dz)\Big)^{p/2}\Big]\, ds,
\end{aligned}$$

which concludes the proof. □

Remark 3.2. The "usual" Bichteler-Jacod inequality for Lévy integrals (see e.g. [28]) follows immediately by (3.1) and the Lévy-Itô decomposition. Moreover, the proof we gave is different from the ones in the literature and, apart from holding also in infinite dimensions, has the peculiarity of avoiding completely the use of the Burkholder-Davis-Gundy's inequality.

Inequality (3.1) can be extended also to stochastic convolutions, even though in general they are not martingales.

Proposition 3.3. *Let A be m-dissipative on H and g satisfies the hypotheses of Lemma 3.1. Then for all $p \in [2,\infty)$ there exists a constant N such that*

$$\begin{aligned}
&\mathbb{E}\sup_{t\leq T}\Big|\int_0^t\int_Z e^{(t-s)A} g(s,z)\,\bar{\mu}(ds,dz)\Big|^p \\
&\qquad\leq N\mathbb{E}\int_0^T \Big[\int_Z |g(s,z)|^p\, m(dz) + \Big(\int_Z |g(s,z)|^2\, m(dz)\Big)^{p/2}\Big]\, ds, \quad (3.3)
\end{aligned}$$

where N depends continuously on p and T only.

Proof. We shall follow the approach of [14]. In particular, by Sz.-Nagy's theorem on unitary dilations, there exists a Hilbert space $\bar{H}$, with H isometrically embedded into $\bar{H}$, and a unitary strongly continuous group $T(t)$ on $\bar{H}$ such that $\pi T(t)x = e^{tA}x$ for all $x \in H$, $t \in \mathbb{R}$, where π denotes the orthogonal projection from $\bar{H}$ to H. Then we have, recalling that the operator norms of π and $T(t)$ are less than or equal to one,

$$\begin{aligned}
\mathbb{E}\sup_{t\le T}\Big|\int_0^t\int_Z e^{(t-s)A}g(s,z)\,\bar{\mu}(ds,dz)\Big|_H^p &= \mathbb{E}\sup_{t\le T}\Big|\pi T(t)\int_0^t\int_Z T(-s)g(s,z)\,\bar{\mu}(ds,dz)\Big|_{\bar{H}}^p\\
&\le |\pi|^p\sup_{t\le T}|T(t)|^p\,\mathbb{E}\sup_{t\le T}\Big|\int_0^t\int_Z T(-s)g(s,z)\,\bar{\mu}(ds,dz)\Big|_{\bar{H}}^p\\
&\le \mathbb{E}\sup_{t\le T}\Big|\int_0^t\int_Z T(-s)g(s,z)\,\bar{\mu}(ds,dz)\Big|_{\bar{H}}^p
\end{aligned}$$

Since the integral in the last expression is a martingale, inequality (3.1) implies that there exists a constant $N = N(p,T)$ such that

$$\begin{aligned}
&\mathbb{E}\sup_{t\le T}\Big|\int_0^t\int_Z e^{(t-s)A}g(s,z)\,\bar{\mu}(ds,dz)\Big|^p\\
&\le N\mathbb{E}\int_0^T\Big[\int_Z|T(-s)g(s,z)|^p\,m(dz)+\Big(\int_Z|T(-s)g(s,z)|^2\,m(dz)\Big)^{p/2}\Big]\,ds\\
&\le N\mathbb{E}\int_0^T\Big[\int_Z|g(s,z)|^p\,m(dz)+\Big(\int_Z|g(s,z)|^2\,m(dz)\Big)^{p/2}\Big]\,ds
\end{aligned}$$

where we have used again that $T(t)$ is a unitary group and that the norms of in $\bar{H}$ and H are equal. □

Corollary 3.4. *Let A be η-m-dissipative. Then inequality (3.3) holds, with N a continuous function of p, T, and η.*

Proof. Follows by exactly the same arguments used above applied to the m-dissipative operator $A - \eta I$. □

3.2 Differentiability of implicit functions

In order to prove regular dependence of solutions with respect to the initial datum, we shall need the following versions of the implicit function theorem. Similar results can be found in the literature (see e.g. [5, 8, 9]), but we have included the complete statements here for the reader's convenience. A proof of these specific versions can be found in [11].

Let E, Λ be two Banach spaces, and $\Phi : \Lambda \times E \to E$ a function such that

$$|\Phi(\lambda,x)-\Phi(\lambda,y)| \le \alpha|x-y|$$

for all $\lambda \in \Lambda$ and all x, $y \in E$, with $\alpha \in [0,1[$. Banach's fixed point theorem implies the existence and uniqueness of a function $\phi : \Lambda \to E$ such that $\Phi(\lambda,\phi(\lambda)) = \phi(\lambda)$ for all $\lambda \in \Lambda$.

Theorem 3.5. *Assume that $\lambda \mapsto \Phi(\lambda,x)$ is continuous for all $x \in E$. Then $\phi \in C(\Lambda,E)$. Moreover, if Φ is Lipschitz with respect to λ uniformly over $x \in E$, then ϕ is Lipschitz.*

Theorem 3.6. *Assume that $\Phi(\cdot,x) : \Lambda \to E$ is continuous for all $x \in E$, and that the maps $\partial_1\Phi : \Lambda \times E \times \Lambda \to E$, $\partial_2\Phi : \Lambda \times E \times E \to E$ are continuous. Then ϕ is Gâteaux differentiable and $(\lambda,\mu) \mapsto \partial_\mu\phi(\lambda)$ is continuous from $\Lambda \times \Lambda$ to E. Moreover, one has*

$$\partial_\mu\phi(\lambda) = \big(I - \partial_2\Phi(\lambda,\phi(\lambda))\big)^{-1}\partial_{1,\mu}\Phi(\lambda,\phi(\lambda)).$$

In the formulation of the following theorems we shall denote by Λ_0 and E_0 two Banach spaces continuously embedded in Λ and E, respectively. Moreover, Λ_1 will denote a further Banach space continuously embedded in Λ_0.

Theorem 3.7. *Assume that Φ satisfies the hypotheses of Theorem 3.6, also with Λ_0 and E_0 replacing Λ and E, respectively. Moreover, assume that $\partial_1\Phi \in C(\Lambda_0 \times E_0, \mathcal{L}(\Lambda_0, E))$ and $\partial_2\Phi \in C(\Lambda_0 \times E_0, \mathcal{L}(E_0, E))$. Then $\partial\phi \in C(\Lambda_0, \mathcal{L}(\Lambda_0, E))$, hence $\phi \in C^1(\Lambda_0, E)$.*

Theorem 3.8. *Assume that both $\Phi : \Lambda \times E \to E$ and $\Phi : \Lambda_0 \times E_0 \to E_0$ satisfy the hypotheses of Theorem 3.6. If $\Phi : \Lambda_0 \times E_0 \to E$ admits second-order directional derivatives, then $\phi : \Lambda_0 \to E$ is twice Gâteaux differentiable with $\partial^2\phi \in C(\Lambda_0^3, E)$ and*

$$\begin{aligned}\partial^2\phi(\lambda_0) : (\mu_0, \nu_0) \mapsto \big(I - \partial_2\Phi(\lambda_0, \phi(\lambda_0))\big)^{-1} \big[&\partial_1^2\Phi(\lambda_0, \phi(\lambda_0))(\mu_0, \nu_0)\\ &+ \partial_1\partial_2\Phi(\lambda_0, \phi(\lambda_0))(\partial_{\mu_0}\phi(\lambda_0), \nu_0)\\ &+ \partial_2\partial_1\Phi(\lambda_0, \phi(\lambda_0))(\mu_0, \partial_{\nu_0}\phi(\lambda_0))\\ &+ \partial_2^2\Phi(\lambda_0, \phi(\lambda_0))(\partial_{\mu_0}\phi(\lambda_0), \partial_{\nu_0}\phi(\lambda_0))\big].\end{aligned}$$

Theorem 3.9. *Assume that both $\Phi : \Lambda \times E \to E$ and $\Phi : \Lambda_0 \times E_0 \to E_0$ satisfy the hypotheses of Theorem 3.6. Moreover, assume that $\Phi \in C^2(\Lambda_0 \times E_0, E)$ and that $\phi \in C^1(\Lambda_1, E_0)$. Then the Fréchet derivative $D\phi : \Lambda_1 \to \mathcal{L}(\Lambda_1, E)$ is Gâteaux differentiable. Furthermore, if $\partial D\phi$ can be realized as a map $\Lambda_1 \to \mathcal{L}\big(\Lambda_1, \mathcal{L}(\Lambda_1, E_0)\big)$, then $\phi \in C^2(\Lambda_1, E)$.*

Corollary 3.10. *Let Φ be as in the previous theorem and $\phi \in C^1(\Lambda_1, E_0)$. Moreover, assume that $D\phi$ and $D_iD_j\Phi$, $i, j \in \{1, 2\}$, are bounded. Then $\partial D\phi : \Lambda_1 \to \mathcal{L}(\Lambda_1, \mathcal{L}(\Lambda_1, E))$.*

3.3 Some regularization results

We record for future reference some simple regularization and approximation results which are used in the proofs of the main results.

Proposition 3.11. *Let u be the mild solution of (1.1) in $\mathcal{H}_2(T)$, and u_λ the strong solution of the equation*

$$du(t) = (A_\lambda u(t) + f(u(t))\,dt + B(u(t))\,dW(t) + \int_Z G(u(t-), z)\,\bar{\mu}(dt, dz), \qquad u(0) = x, \quad (3.4)$$

where A_λ stands for the Yosida approximation of A. Then $u_\lambda \to u$ in $\mathcal{H}_2(T)$ as $\lambda \to 0$.

Proof. We sketch the proof only, as it resembles the corresponding proof for equations driven by Wiener noise only. In fact, the strong solution u_λ of (3.4) is an adapted càdlàg process, and the predictable process $t \mapsto u_\lambda(t-)$ is a mild solution of (3.4). Recalling that, for a fixed $t \in [0, T]$, one has $u_\lambda(t) - u_\lambda(t-) = 0$ almost surely (no jumps at a fixed time can occurr), we can proceed along the lines of e.g. [6, Thm. 3.5]. □

In the following proposition we take f, f_ε, B, B_ε, G, G_ε independent of $t \in [0, T]$ and $\omega \in \Omega$.

Proposition 3.12. *Let u and u_ε be, respectively, the mild solutions in $\mathcal{H}_2(T)$ of (1.1) and of the equation obtained replacing f, B, and G with f_ε, B_ε, and G_ε in (1.1), where $f_\varepsilon(x) \to f(x)$ in H, $e^{tA}B_\varepsilon(x) \to e^{tA}B(x)$ in $\mathcal{L}_2^Q(H)$, and*

$$\int_Z \big|e^{tA}(G_\varepsilon(x, z) - G(x, z))\big|^2\, m(dz) \to 0$$

for all $x \in H$ as $\varepsilon \to 0$. Moreover, assume that there exists $K > 0$ such that

$$[f_\varepsilon]_1^2 + \left|e^{tA}(B_\varepsilon(x) - B_\varepsilon(y))\right|_Q^2 + \int_Z \left|e^{tA}(G_\varepsilon(x,z) - G_\varepsilon(y,z))\right|^2 m(dz) \leq K|x-y|^2$$

for all $t \in [0,T]$. Then $u_\varepsilon \to u$ in $\mathcal{H}_2(T)$.

Proof. We have

$$\begin{aligned}
\mathbb{E}|u_\varepsilon(t) - u(t)|^2 \;\lesssim\; & \mathbb{E}\int_0^t \left|e^{(t-s)A}[f_\varepsilon(u_\varepsilon(s)) - f(u(s))]\right|^2 ds \\
& + \mathbb{E}\int_0^t \left|e^{(t-s)A}[B_\varepsilon(u_\varepsilon(s)) - B(u(s))]\right|_Q^2 ds \\
& + \mathbb{E}\int_0^t\int_Z \left|e^{(t-s)A}[G_\varepsilon(u_\varepsilon(s),z) - G(u(s),z)]\right|^2 m(dz)\, ds \\
=:\; & I_1(\varepsilon) + I_2(\varepsilon) + I_3(\varepsilon),
\end{aligned}$$

and

$$\begin{aligned}
I_2(\varepsilon) &\lesssim \mathbb{E}\int_0^t \left|e^{(t-s)A}[B_\varepsilon(u_\varepsilon(s)) - B_\varepsilon(u(s))]\right|_Q^2 ds + \mathbb{E}\int_0^t \left|e^{(t-s)A}[B_\varepsilon(u(s)) - B(u(s))]\right|_Q^2 ds \\
&\leq K\mathbb{E}\int_0^t |u_\varepsilon(s) - u(s)|^2\, ds + \delta_2(\varepsilon),
\end{aligned}$$

where $\delta_2(\varepsilon) \to 0$ as $\varepsilon \to 0$, in view of the assumptions on B_ε and by dominated convergence. Completely similar estimates can be obtained for $I_1(\varepsilon)$ and $I_3(\varepsilon)$. We thus get

$$\mathbb{E}|u_\varepsilon(t) - u(t)|^2 \leq N\int_0^t \mathbb{E}|u_\varepsilon(t) - u(t)|^2 + \delta(\varepsilon),$$

with $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$, and the conclusion follows by Gronwall's lemma. □

4 Proofs

Proof of Theorem 2.3. We sketch the proof only, as we follow the well-known approach based on Banach's fixed point theorem. We have to prove that the mapping $\mathfrak{F} : \mathcal{H}_2(T) \to \mathcal{H}_2(T)$ defined by

$$\begin{aligned}
\mathfrak{F}u(t) \;=\; & e^{tA}x + \int_0^t e^{(t-s)A} f(s,u(s))\, ds + \int_0^t e^{(t-s)A} B(s,u(s))\, dW(s) \\
& + \int_0^t\int_Z e^{(t-s)A} G(s,u(s),z)\, \bar{\mu}(ds,dz)
\end{aligned} \tag{4.1}$$

is well defined and is a contraction, after which the result follows easily. Let us show that, for any $u \in \mathcal{H}_2(T)$, $\mathfrak{F}u$ admits a predictable modification such that $\|[\mathfrak{F}u]\|_2 < \infty$. Predictability of $\mathfrak{F}u$ follows by the mean-square continuity of the stochastic convolution term with respect to $\bar{\mu}$ in (4.1): in fact, setting $M_A(t) = \int_0^t\int_Z e^{(t-s)A}G(s,u(s),z)\,\bar{\mu}(ds,dz)$, a simple calculation shows that, for $0 \leq s \leq t \leq T$,

$$\begin{aligned}
\mathbb{E}|M_A(t) - M_A(s)|^2 \;\lesssim\; & \mathbb{E}\int_0^s\int_Z |e^{(t-r)A} - e^{(s-r)A}|^2 |G(r,u(r),z)|^2\, m(dz)\, dr \\
& + \mathbb{E}\int_s^t\int_Z |e^{(t-r)A}|^2 |G(r,u(r),z)|^2\, m(dz)\, dr,
\end{aligned}$$

which converges to zero as $s \to t$.
Moreover, we have

$$
\begin{aligned}
|[\mathfrak{F}u]|_2^2 \lesssim\ & \sup_{t\le T} \mathbb{E}|e^{tA}x|^2 + \sup_{t\le T}\mathbb{E}\Big|\int_0^t e^{(t-s)A} f(s,u(s))\,ds\Big|^2 \\
& + \sup_{t\le T}\mathbb{E}\Big|\int_0^t e^{(t-s)A}B(s,u(s))\,dW(s)\Big|^2 + \sup_{t\le T}\mathbb{E}\Big|\int_0^t\int_Z e^{(t-s)A}G(s,u(s),z)\,\bar\mu(ds,dz)\Big|^2.
\end{aligned}
$$

Using the isometry for stochastic integrals, and noting that hypotheses (2.2) and (2.1) imply the estimate

$$\int_Z |e^{sA}G(t,x,z)|^2\,m(dz) \le Nh(s)(1+|x|)^2,$$

we have

$$
\begin{aligned}
&\sup_{t\le T}\mathbb{E}\Big|\int_0^t\int_Z e^{(t-s)A}G(s,u(s),z)\,\bar\mu(ds,dz)\Big|^2 \\
&= \sup_{t\le T}\mathbb{E}\int_0^t\int_Z |e^{(t-s)A}G(s,u(s),z)|^2\,m(dz)\,ds \lesssim \sup_{t\le T}\mathbb{E}\int_0^t h(t-s)(1+|u(s)|)^2\,ds \\
&\le 2|h|_{L_1}(1+\sup_{t\le T}\mathbb{E}|u(t)|^2) < \infty.
\end{aligned}
$$

Analogous estimates for the remaining terms in (4.1) are classical (see e.g. [8]), hence $|[\mathfrak{F}u]|_2^2 < \infty$.

We shall now prove that there exists λ such that $|[\mathfrak{F}u-\mathfrak{F}v]|_{2,\lambda} \le N|[u-v]|_{2,\lambda}$, with $N<1$. In fact we have

$$
\begin{aligned}
|[\mathfrak{F}u-\mathfrak{F}v]|_{2,\lambda}^2 \lesssim\ & \sup_{t\le T} e^{-2\lambda t}\mathbb{E}\Big|\int_0^t e^{(t-s)A}\big(f(s,u(s))-f(s,v(s))\big)\,ds\Big|^2 \\
& + \sup_{t\le T} e^{-2\lambda t}\mathbb{E}\Big|\int_0^t e^{(t-s)A}\big(B(s,u(s))-B(s,v(s))\big)\,dW(s)\Big|^2 \\
& + \sup_{t\le T} e^{-2\lambda t}\mathbb{E}\Big|\int_0^t\int_Z e^{(t-s)A}\big(G(s,u(s),z)-G(s,v(s),z)\big)\,\bar\mu(ds,dz)\Big|^2,
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathbb{E}\Big|\int_0^t\int_Z e^{(t-s)A}\big(G(s,u(s),z)-G(s,v(s),z)\big)\,\bar\mu(ds,dz)\Big|^2 \\
&\quad= \mathbb{E}\int_0^t\int_Z \big|e^{(t-s)A}\big(G(s,u(s),z)-G(s,v(s),z)\big)\big|^2\,m(dz)\,ds \\
&\quad\le \mathbb{E}\int_0^t e^{2\lambda s}h(t-s)e^{-2\lambda s}|u(s)-v(s)|^2\,ds \le |u-v|_{2,\lambda}^2\int_0^t e^{2\lambda s}h(t-s)\,ds \\
&\quad\le e^{2\lambda t}|u-v|_{2,\lambda}^2\int_0^t e^{-2\lambda s}h(s)\,ds,
\end{aligned}
$$

which implies that the third summand on the right-hand side of the previous estimate of $|[\mathfrak{F}u-\mathfrak{F}v]|_{2,\lambda}^2$ is bounded by $|u-v|_{2,\lambda}^2\int_0^T e^{-2\lambda s}h(s)\,ds$, which converges to zero as $\lambda\to\infty$. Completely analogous calculations for the other summands show that there exists $N = N(T,h,\lambda)$ such that $|[\mathfrak{F}u-\mathfrak{F}v]|_{2,\lambda}^2 \le N|[u-v]|_{2,\lambda}^2$, and that one can find $\lambda_0>0$ so that $N(T,h,\lambda_0)<1$, thus obtaining, by Banach's fixed point theorem, existence and uniqueness of a mild solution to (1.1). Finally, Lipschitz continuity of the solution map follows by Theorem 3.5. □

Remark 4.1. One can also prove by a direct calculation that $x \mapsto u(x)$ is Lipschitz. This method has the advantage of yielding explicit estimates on the Lipschitz constant, and will be useful to establish the strong Feller property. In fact, one has

$$\begin{aligned} u(t,x) - u(t,y) &= e^{tA}(x-y) + \int_0^t e^{(t-s)A}[f(s,u(s,x)) - f(s,u(s,y))]\,ds \\ &\quad + \int_0^t e^{(t-s)A}[B(s,u(s,x)) - B(s,u(s,y))]\,dW(s) \\ &\quad + \int_0^t \int_Z e^{(t-s)A}[G(s,u(s,x),z) - G(s,u(s,y),z)]\,\bar{\mu}(ds,dz), \end{aligned}$$

hence, squaring both sides and taking expectations,

$$\mathbb{E}|u(t,x) - u(t,y)|^2 \leq 2M^2 e^{2\eta t}\mathbb{E}|x-y|^2 + (2t+1)\int_0^t h(t-s)\mathbb{E}|u(s,x) - u(s,y)|^2\,ds,$$

which yields, via Gronwall's inequality,

$$|[u(x) - u(y)]|_2 \leq \sqrt{2}Me^{(\eta+|h|_1)T+|h|_1/2}|x-y|_{\mathbb{L}_2}.$$

Proof of Theorem 2.4. We shall use a fixed point argument in the space $\mathbb{H}_p(T)$. In particular, we want to prove that the mapping $\mathfrak{F}$ defined as in (4.1) is a well-defined contraction on $\mathbb{H}_p(T)$. Here we limit ourselves to prove that there exists $N < 1$ such that $\|\mathfrak{F}u - \mathfrak{F}v\|_{p,\lambda} \leq N\|u-v\|_{p,\lambda}$ for all u, $v \in \mathbb{H}_p(T)$, with a suitably chosen $\lambda \geq 0$. In fact, this implies

$$\|\mathfrak{F}u\|_p \lesssim \|u-a\|_p + \|\mathfrak{F}a\|_p < \infty$$

for all $u \in \mathbb{H}_p(T)$, thanks to (2.4). Moreover, predictability of $\mathfrak{F}u$, $u \in \mathbb{H}_p(T)$, follows as in the proof of Theorem 2.3.

We have

$$\begin{aligned} \|\mathfrak{F}u - \mathfrak{F}v\|_{p,\lambda}^p &\lesssim_p \mathbb{E}\sup_{t\leq T}\Big|e^{-\lambda t}\int_0^t e^{(t-s)A}(f(s,u(s)) - f(s,v(s)))\,ds\Big|^p \\ &\quad + \mathbb{E}\sup_{t\leq T}\Big|e^{-\lambda t}\int_0^t e^{(t-s)A}(B(s,u(s)) - B(s,v(s)))\,dW(s)\Big|^p \\ &\quad + \mathbb{E}\sup_{t\leq T}\Big|e^{-\lambda t}\int_0^t\int_Z e^{(t-s)A}(G(s,u(s),z) - G(s,v(s),z))\,\bar{\mu}(ds,dz)\Big|^p \\ &=: A_1 + A_2 + A_3. \end{aligned}$$

The term A_1 on the right-hand side is bounded from above, thanks to (2.3) and Hölder's inequality, by

$$\begin{aligned} &T^{p-1}\mathbb{E}\sup_{t\leq T} e^{-p\lambda t}\int_0^t h(t-s)^p e^{p\lambda s}\big(e^{-\lambda s}|u(s) - v(s)|\big)^p\,ds \\ &\leq T^{p-1}\|u-v\|_{p,\lambda}^p \sup_{t\leq T}\int_0^t h(t-s)^p e^{-p\lambda(t-s)}\,ds \\ &\leq T^{p-1}|h_\lambda|^p_{L_p([0,T])}\|u-v\|_{p,\lambda}^p, \end{aligned}$$

where $h_\lambda(s) := e^{-\lambda s}h(s)$.

Moreover, since

$$A_3 = \mathbb{E}\sup_{t\leq T}\Big|\int_0^t\int_Z e^{(t-s)(A-\lambda I)}e^{-\lambda s}(G(s,u(s),z) - G(s,v(s),z))\,\bar{\mu}(ds,dz)\Big|^p$$

and, by a slight modification of the proof of (3.3),

$$\begin{aligned}\mathbb{E}\sup_{t\le T}\Big|\int_0^t\int_Z e^{(t-s)(A-\lambda I)}\phi(s,z)\,\bar\mu(ds,dz)\Big|^p\\ \lesssim_{p,T,\eta} e^{-\lambda pT}\mathbb{E}\int_0^T e^{\lambda ps}\big(|\phi(s,\cdot)|^p_{L_p(Z,m)}+|\phi(s,\cdot)|^p_{L_2(Z,m)}\big)\,ds,\end{aligned}$$

we obtain, thanks to (2.3),

$$\begin{aligned}A_3 &\lesssim_{p,T,\eta} e^{-\lambda pT}\mathbb{E}\Big[\int_0^T e^{\lambda ps}e^{-\lambda ps}|G(s,u(s),\cdot)-G(s,v(s),\cdot)|^p_{L_2(Z,m)}\,ds\\ &\qquad+\int_0^T e^{\lambda ps}e^{-\lambda ps}|G(s,u(s),\cdot)-G(s,v(s),\cdot)|^p_{L_p(Z,m)}\,ds\Big]\\ &\le \|u-v\|^p_{p,\lambda}e^{-\lambda pT}\int_0^T e^{\lambda ps}h(s)^p\,ds \le |\tilde h_\lambda|^p_{L_p([0,T])}\|u-v\|^p_{p,\lambda},\end{aligned}$$

where $\tilde h_\lambda(s) := e^{-\lambda s}h(T-s)$. Classical maximal inequalities for stochastic convolutions with respect to Wiener processes yield a completely analogous estimate for A_2. Observing that the norms of h_λ and $\tilde h_\lambda$ appearing in the above estimates tend to zero as $\lambda\to\infty$, we conclude that there exists a constant $N=N(\lambda,T,p,\eta)$ such that $\|\mathfrak{F}u-\mathfrak{F}v\|_{p,\lambda}\le N\|u-v\|_{p,\lambda}$, with $N<1$ for some $\lambda>0$ sufficiently large. The existence and uniqueness of a solution, as well as its Lipschitz continuity with respect to the initial datum, follows then by Banach's fixed point theorem, as for Theorem 2.3, and by the equivalence of the norms $\|\cdot\|_{p,\lambda}$ for $\lambda\ge 0$. □

Remark 4.2. The Lipschitz continuity of the solution map, in analogy to the previous remark, could also be obtained by a direct calculation. However, in this case the norm of $\mathbb{H}_p(T)$ is somewhat more difficult to work with. In Section 5 below we shall obtain some estimates for the Lipschitz constant of the solution map under additional assumptions on the coefficient of the SPDE.

Proof of Theorem 2.6. It is enough to prove the statements in the case $B\equiv 0$. We are going to apply Theorem 3.6, with $\Lambda=\mathbb{L}_2$, $E=\mathcal{H}_2(T)$. The latter space needs to be endowed with a norm $|[\cdot]|_{2,\lambda}$, where $\lambda>0$ is chosen in such a way that $\mathfrak{F}:\mathbb{L}_2\times\mathcal{H}_2(T)\to\mathcal{H}_2(T)$ is a contraction in its second argument. However, in view of the equivalence of the norms $|[\cdot]|_{2,\lambda}$, we shall perform the calculations assuming $\lambda=0$, without loss of generality.

It is immediate that the directional derivative $\partial_{1,y}\mathfrak{F}(x,u)$ coincides with the map $t\mapsto e^{tA}y$, which clearly belongs to $\mathcal{H}_2(T)$. Moreover, we have

$$\begin{aligned}\Big|\Big[\mathcal{Q}^h_{2,v}\mathfrak{F}(x,u)-\int_0^\cdot e^{(\cdot-s)A}\partial_{v(s)}f(u(s))\,ds-\int_0^\cdot\int_Z e^{(\cdot-s)A}\partial_{1,v(s)}G(u(s),z)\,\bar\mu(ds,dz)\Big]\Big|_2\\ \le\Big|\Big[\int_0^\cdot e^{(\cdot-s)A}\big[\mathcal{Q}^h_{v(s)}f(u(s))-\partial_{v(s)}f(u(s))\big]\,ds\Big]\Big|_2\\ +\Big|\Big[\int_0^\cdot\int_Z e^{(\cdot-s)A}\big[\mathcal{Q}^h_{1,v(s)}G(u(s),z)-\partial_{1,v(s)}G(u(s),z)\big]\,\bar\mu(ds,dz)\Big]\Big|_2\end{aligned}$$

The first term on the right hand side of this inequality tends to zero as $h\to 0$ by obvious estimates and the dominated convergence theorem. Using the isometric property of the stochastic integral, the square of the second term is equal to

$$\mathbb{E}\int_0^T\int_Z\big|e^{(T-s)A}\mathcal{Q}^h_{1,v(s)}G(u(s),z)-e^{(T-s)A}\partial_{1,v(s)}G(u(s),z)\big|^2\,m(dz)\,ds.$$

In view of assumptions (ii)–(iv), a simple computation shows that

$$\frac{e^{tA}G(x+hy,\cdot)-e^{tA}G(x,\cdot)}{h}\to e^{tA}\partial_{1,y}G(x,\cdot)\qquad\forall x\in H$$

in $L_2(Z,m;H)$ as $h\to 0$, whence we obtain convergence to zero of the second term in the above estimate again by dominated convergence. Therefore we have

$$[\partial_{2,v}\mathfrak{F}(x,u)](t)=\int_0^t e^{(t-s)A}\partial_{v(s)}f(u(s))\,ds+\int_0^t\int_Z e^{(t-s)A}\partial_{1,v(s)}G(u(s),z)\,\bar{\mu}(ds,dz).$$

The continuity of $\partial_1\mathfrak{F}$ and $\partial_2\mathfrak{F}$, considered as maps $\mathbb{L}_2\times\mathcal{H}_2(T)\times\mathbb{L}_2\to\mathcal{H}_2(T)$ and $\mathbb{L}_2\times\mathcal{H}_2(T)\times\mathcal{H}_2(T)\to\mathcal{H}_2(T)$, respectively, can be proved in a completely similar way, and we omit the details.

Let us now prove the second assertion of the theorem: let x, y in $\mathbb{L}_2$. Then Theorem 3.6 yields

$$\partial_y u(x)=\bigl(I-\partial_2\mathfrak{F}(x,u(x))\bigr)^{-1}\partial_{1,y}\mathfrak{F}(x,u(x)),$$

thus also

$$\partial_y u(x)=\partial_{1,y}\mathfrak{F}(x,u(x))+\partial_2\mathfrak{F}(x,u(x))\partial_y u(x),$$

and the result follows substituting in the previous formula the expressions for the directional derivatives of $\mathfrak{F}$ found above.

The last assertion of the theorem is a direct consequence of the definition of directional derivative and the fact that the solution map $x\mapsto u(x)$ is Lipschitz. □

Proof of Theorem 2.7. One can prove that the solution map is Gâteaux differentiable as in the proof of Theorem 2.6, except for the fact that one cannot use the isometric property of the stochastic integral, but has to rely on the estimate (3.3). In particular, one has

$$\begin{aligned}\Bigl\|\int_0^\cdot\int_Z e^{(t-s)A}\bigl[\mathcal{Q}^h_{1,v(s)}G(u(s),z)-\partial_{1,v(s)}G(u(s),z)\bigr]\,\bar{\mu}(ds,dz)\Bigr\|_p^p\\
\lesssim\mathbb{E}\int_0^T\bigl|\mathcal{Q}^h_{1,v(s)}G(u(s),\cdot)-\partial_{1,v(s)}G(u(s),\cdot)\bigr|^p_{L_p(Z,m)}\,ds\\
+\mathbb{E}\int_0^T\bigl|\mathcal{Q}^h_{1,v(s)}G(u(s),\cdot)-\partial_{1,v(s)}G(u(s),\cdot)\bigr|^p_{L_2(Z,m)}\,ds,\end{aligned}$$

which converges to zero by dominated convergence, thanks to the assumptions on G.

In order to prove that $x\mapsto u(x)$ is also Fréchet differentiable, let us set $\Lambda_0=\mathbb{L}_q$, $\Lambda=\mathbb{L}_p$, $E_0=\mathbb{H}_q(T)$, and $E=\mathbb{H}_p(T)$, and apply Theorem 3.7. We are going to prove that the partial Gâteaux derivatives

$$\begin{aligned}\partial_1\mathfrak{F}&:\mathbb{L}_q\times\mathbb{H}_q(T)\to\mathcal{L}\bigl(\mathbb{L}_q,\mathbb{H}_p(T)\bigr),\\
\partial_2\mathfrak{F}&:\mathbb{L}_q\times\mathbb{H}_q(T)\to\mathcal{L}\bigl(\mathbb{H}_q(T),\mathbb{H}_p(T)\bigr)\end{aligned}$$

are continuous, which implies that $\mathfrak{F}$ is Frechét differentiable (see e.g. [3]). Since $\partial_1\mathfrak{F}:(x,u)\mapsto e^{tA}x$ is clearly continuous, it suffices to show that $\partial_2\mathfrak{F}$ is continuous. To this purpose, let $\{x_n\}\subset\mathbb{L}_q$, $x\in\mathbb{L}_q$, $\{u_n\}\subset\mathbb{H}_q(T)$, u, $w\in\mathbb{H}_q(T)$ such that $x_n\to x$ in $\mathbb{L}_q$, $u_n\to u$ in $\mathbb{H}_p(T)$, and $\|w\|_q\leq 1$. We shall prove that

$$\bigl\|\partial_2\mathfrak{F}(x_n,u_n)w-\partial_2\mathfrak{F}(x,u)w\bigr\|_p\to 0$$

as $n \to \infty$. In fact, the above expression is no greater than

$$\Big\| \int_0^\cdot e^{(\cdot-s)A} \big[Df(u_n(s))w(s) - Df(u(s))w(s)\big]\,ds\Big\|_p$$
$$+ \Big\| \int_0^\cdot \int_Z e^{(\cdot-s)A} \big[D_1G(u_n(s),z)w(s) - D_1G(u(s),z)w(s)\big]\,\bar\mu(ds,dz)\Big\|_p =: I_1(n) + I_2(n),$$

and, using Hölder's inequality with conjugate exponents $r = q/p$ and r',

$$\begin{aligned} I_1(n)^p &\lesssim \mathbb{E}\int_0^T \big|[Df(u_n(s)) - Df(u(s))]w(s)\big|^p\,ds \\ &\leq \biggl(\mathbb{E}\int_0^T \big|Df(u_n(s)) - Df(u(s))\big|^{pr'}\,ds\biggr)^{1/r'} \biggl(\mathbb{E}\int_0^T |w(s)|^{pr}\,ds\biggr)^{1/r} \\ &\lesssim \|w\|_q^p \biggl(\mathbb{E}\int_0^T \big|Df(u_n(s)) - Df(u(s))\big|^{\frac{pq}{q-p}}\,ds\biggr)^{\frac{q-p}{q}} \end{aligned}$$

Since $|Df(u_n(s)) - Df(u(s))| \leq 2[f]_1$, the dominated convergence theorem and the continuity of Df imply that $I_1(n) \to 0$ as $n \to \infty$.
Similarly, using the maximal inequality (3.3), we obtain

$$\begin{aligned} I_2(n)^p &\leq \mathbb{E}\biggl|\sup_{t\leq T}\int_0^t \int_Z e^{(t-s)A}\big[D_1G(u_n(s),z)w(s) - D_1G(u(s),z)w(s)\big]\,\bar\mu(ds,dz)\biggr|^p \\ &\lesssim \mathbb{E}\int_0^T \Big(\big|D_1G(u_n(s),\cdot) - D_1G(u(s),\cdot)\big|^p_{L_p(Z,m)} \\ &\qquad\qquad + \big|D_1G(u_n(s),\cdot) - D_1G(u(s),\cdot)\big|^p_{L_2(Z,m)}\Big)|w(s)|^p\,ds, \end{aligned}$$

which converges to zero as $n \to \infty$ by arguments completely analogous to the above ones. □

Proof of Theorem 2.8. We are going to apply Theorem 3.9, with $\Lambda = \mathbb{L}_p$, $\Lambda_0 = \mathbb{L}_{q'}$, $\Lambda_1 = \mathbb{L}_q$, and $E = \mathbb{H}_p(T)$, $E_0 = \mathbb{H}_{q'}(T)$. Here $q' \in]2p, q[$. In analogy to what we have done before, we shall endow E and E_0 with norms $\|\cdot\|_{q',\lambda(q')}$ and $\|\cdot\|_{p,\lambda(p)}$, respectively, where the λ are chosen in a such a way that $\mathfrak{F} : \mathbb{L}_r \times \mathbb{H}_r(T) \to \mathbb{H}_r(T)$ are contractions in the second argument, but we shall perform the calculations assuming $\lambda \equiv 0$, without loss of generality.

It is clear that, in view of Theorem 2.7, it is enough to prove that $\mathfrak{F} \in C^2(\mathbb{L}_{q'} \times \mathbb{H}_{q'}(T), H_p(T))$. Since $\partial_1 D_1\mathfrak{F} \equiv \partial_1 D_2\mathfrak{F} \equiv \partial_2 D_1\mathfrak{F} \equiv 0$, we only have to consider $\partial_2 D_2\mathfrak{F}$. Let us first prove that

$$\begin{aligned} [\partial_{2,v} D_2\mathfrak{F}(x,u)w](t) &= \int_0^t e^{(t-s)A} D^2 f(u(s))(v(s),w(s))\,ds \\ &\quad + \int_0^t \int_Z e^{(t-s)A} D_1^2 G(u(s),z)(v(s),w(s))\,\bar\mu(ds,dz), \qquad t \in [0,T], \end{aligned} \tag{4.2}$$

for all v, $w \in \mathbb{H}_{q'}(T)$. In fact, the p-th power of the $\mathbb{H}_p(T)$ norm of the difference between $\mathcal{Q}^h_{2,v} D_2\mathfrak{F}(x,u)w$ and the right-hand side of (4.2) is not greater than a constant times

$$\begin{aligned} &\mathbb{E}\sup_{t\leq T}\biggl|\int_0^t e^{(t-s)A}\big[\mathcal{Q}^h_{v(s)} Df(u(s))w(s) - D^2 f(u(s))(v(s),w(s))\big]\,ds\biggr|^p \\ &+ \mathbb{E}\sup_{t\leq T}\biggl|\int_0^t \int_Z e^{(t-s)A}\big[\mathcal{Q}^h_{1,v(s)} D_1G(u(s),z)w(s) - D_1^2 G(u(s),z)(v(s),w(s))\big]\,\bar\mu(ds,dz)\biggr|^p \\ &=: I_1(h) + I_2(h). \end{aligned}$$

We have, by Hölder's inequality,

$$\begin{aligned}I_1(h) &\lesssim \mathbb{E}\int_0^T \big|\big(\mathcal{Q}^h_{v(s)}Df(u(s)) - D^2f(u(s))(v(s),\cdot)\big)w(s)\big|^p\,ds\\&\lesssim \|w\|^p_{q'}\Big(\mathbb{E}\int_0^T \big|\mathcal{Q}^h_{v(s)}Df(u(s)) - D^2f(u(s))(v(s),\cdot)\big|^{2p}_{\mathcal{L}(H)}\,ds\Big)^{1/2},\end{aligned}$$

which converges to zero as $h \to 0$ by dominated convergence, thanks to the boundedness of D^2f. Similarly, using inequality (3.3), we get

$$\begin{aligned}I_2(h) &\lesssim \mathbb{E}\int_0^T \big|\mathcal{Q}^h_{1,v(s)}D_1G(u(s),\cdot)w(s) - D_1^2G(u(s),\cdot)(v(s),w(s))\big|^p_{L_p(Z,m)}\,ds\\&\quad + \mathbb{E}\int_0^T \big|\mathcal{Q}^h_{1,v(s)}D_1G(u(s),\cdot)w(s) - D_1^2G(u(s),\cdot)(v(s),w(s))\big|^p_{L_2(Z,m)}\,ds\\&= I_{21}(h) + I_{22}(h),\end{aligned}$$

and, by Hölder's inequality,

$$\begin{aligned}I_{21}(h) &\leq \mathbb{E}\int_0^T \big|\mathcal{Q}^h_{1,v(s)}D_1G(u(s),\cdot) - D_1^2G(u(s),\cdot)(v(s),\cdot)\big|^p_{L_p(Z,m;\mathcal{L}(H))}ds|w(s)|^p\,ds\\&\lesssim \|w\|^p_{q'}\,\mathbb{E}\int_0^T \big|\mathcal{Q}^h_{1,v(s)}D_1G(u(s),\cdot) - D_1^2G(u(s),\cdot)(v(s),\cdot)\big|^{2p}_{L_p(Z,m;\mathcal{L}(H))}ds.\end{aligned}$$

By hypothesis (ii) we have that

$$\big|\mathcal{Q}^h_{1,v(s)}D_1G(u(s),z) - D_1^2G(u(s),z)(v(s),\cdot)\big|^{2p}_{\mathcal{L}(H)} \to 0$$

as $h \to 0$ for all $(s,z) \in [0,T]\times Z$, and, by (iii),

$$\begin{aligned}&\mathbb{E}\int_0^T \big|\mathcal{Q}^h_{1,v(s)}D_1G(u(s),\cdot) - D_1^2G(u(s),\cdot)(v(s),\cdot)\big|^{2p}_{L_p(Z,m;\mathcal{L}(H))}ds\\&\qquad\lesssim |h_1|^{2p}_{L_p(Z,m)}\mathbb{E}\int_0^T |v(s)|^{2p}\,ds \lesssim |h_1|^{2p}_{L_p(Z,m)}\|v\|^p_{q'} < \infty,\end{aligned}$$

hence, by dominated convergence, $I_{21}(h) \to 0$ as $h \to 0$. In a completely similar way one can prove that $I_{22}(h) \to 0$ as $h \to 0$ as well. We have thus proved that $I_2(h) \to 0$, hence that (4.2) holds.

Let us now show that

$$v \mapsto \partial_{2,v}D_2\mathfrak{F}(x,u) \in \mathcal{L}\big(\mathbb{H}_{q'}(T), \mathcal{L}(\mathbb{H}_{q'}(T), \mathbb{H}_p(T))\big) \tag{4.3}$$

for all $x \in \mathbb{L}_{q'}$ and $u \in \mathbb{H}_{q'}(T)$. In fact, for $w \in \mathbb{H}_{q'}(T)$, we have

$$\begin{aligned}\big\|\partial_{2,v}D_2\mathfrak{F}(x,u)w\big\|^p_p &\lesssim \mathbb{E}\int_0^T \big|D^2f(u(s))(v(s),w(s))\big|^p\,ds\\&\quad + \mathbb{E}\int_0^T\int_Z \big|D_1^2G(u(s),z)(v(s),w(s))\big|^p\,m(dz)\,ds\\&\quad + \mathbb{E}\int_0^T \Big(\int_Z \big|D_1^2G(u(s),z)(v(s),w(s))\big|^2\,m(dz)\Big)^{p/2}\,ds\\&\lesssim \mathbb{E}\int_0^T |v(s)|^p|w(s)|^p\,ds\\&\quad + \big(|h_1|^p_{L_p(Z,m)} + |h_1|^p_{L_2(Z,m)}\big)\mathbb{E}\int_0^T |v(s)|^p|w(s)|^p\,ds\\&\lesssim \big(1 + |h_1|^p_{L_p(Z,m)} + |h_1|^p_{L_2(Z,m)}\big)\|v\|^p_{q'}\|w\|^p_{q'},\end{aligned}$$

which establishes the continuity of $(v,w) \mapsto \partial_{2,v} D_2\mathfrak{F}(x,u)w$, and hence ensures that (4.3) holds true.

Our next goal is to prove that

$$u \mapsto \partial_2 D_2\mathfrak{F}(x,u) \in C\big(\mathbb{H}_{q'}(T), \mathcal{L}^{\otimes 2}(\mathbb{H}_{q'}(T), \mathbb{H}_p(T))\big), \tag{4.4}$$

for all $x \in \mathbb{L}_{q'}$, which implies the twice continuous differentiability of $\mathfrak{F}$ by a well-known criterion. Let $u_n \to u$ in $\mathbb{H}_{q'}(T)$. Then we have

$$\begin{aligned}
&\big\|\partial_2 D_2\mathfrak{F}(x,u_n)(v,w) - \partial_2 D_2\mathfrak{F}(x,u)(v,w)\big\|_p^p \\
&\quad\lesssim \mathbb{E}\sup_{t\le T}\Big|\int_0^t e^{(t-s)A}[D^2 f(u_n(s))(v(s),w(s)) - D^2 f(u(s))(v(s),w(s))]\,ds\Big|^p \\
&\qquad + \mathbb{E}\sup_{t\le T}\Big|\int_0^t\int_Z e^{(t-s)A}[D_1^2 G(u_n(s),z)(v(s),w(s)) - D_1^2 G(u(s),z)(v(s),w(s))]\,\bar{\mu}(ds,dz)\Big|^p \\
&\quad=: I_1(n) + I_2(n),
\end{aligned}$$

where, using Hölder's inequality with conjugate exponents $q'/(2p)$ and $q'/(q'-2p)$,

$$\begin{aligned}
I_1(n) &\lesssim \mathbb{E}\int_0^T \big|D^2 f(u_n(s))(v(s),w(s)) - D^2 f(u(s))(v(s),w(s))\big|^p\,ds \\
&\le \Big(\mathbb{E}\int_0^T \big|D^2 f(u_n(s)) - D^2 f(u(s))\big|_{\mathcal{L}^{\otimes 2}}^{\frac{pq'}{q'-2p}}\Big)^{\frac{q'-2p}{q'}} \Big(\mathbb{E}\int_0^T (|v(s)|\,|w(s)|)^{q'/2}\,ds\Big)^{\frac{2p}{q'}} \\
&\le \Big(\mathbb{E}\int_0^T \big|D^2 f(u_n(s)) - D^2 f(u(s))\big|_{\mathcal{L}^{\otimes 2}}^{\frac{pq'}{q'-2p}}\Big)^{\frac{q'-2p}{q'}} \|v\|_{q'}^p \|w\|_{q'}^p,
\end{aligned}$$

which converges to zero as $n \to \infty$ by dominated convergence, thanks to the assumption of boundedness of $D^2 f$. Applying again inequality (3.3) yields

$$\begin{aligned}
I_2(n) &\lesssim \mathbb{E}\int_0^T\int_Z \big|D_1^2 G(u_n(s),z)(v(s),w(s)) - D_1^2 G(u(s),z)(v(s),w(s))\big|^p\,m(dz)\,ds \\
&\quad + \mathbb{E}\int_0^T\Big(\int_Z \big|D_1^2 G(u_n(s),z)(v(s),w(s)) - D_1^2 G(u(s),z)(v(s),w(s))\big|^2\,m(dz)\Big)^{p/2}\,ds \\
&=: I_{21}(n) + I_{22}(n),
\end{aligned}$$

where, again by Hölder's inequality,

$$\begin{aligned}
I_{21}(n) &\lesssim \mathbb{E}\int_0^T |v(s)|^p |w(s)|^p \big|D_1^2 G(u_n(s),\cdot) - D_1^2 G(u(s),\cdot)\big|_{L_p(Z,m;\mathcal{L}^{\otimes 2})}^p\,ds \\
&\lesssim \|v\|_{q'}^p \|w\|_{q'}^p \Big(\mathbb{E}\int_0^T \big|D_1^2 G(u_n(s),\cdot) - D_1^2 G(u(s),\cdot)\big|_{L_p(Z,m;\mathcal{L}^{\otimes 2})}^{\frac{pq'}{q'-2p}}\,ds\Big)^{\frac{q'-2p}{q'}},
\end{aligned}$$

which converges to zero as $n \to \infty$ by continuity of $D_1^2 G$ in its first argument and dominated convergence, thanks to hypothesis (iii). An analogous argument shows that $I_{22}(n) \to 0$ as $n \to \infty$. We have thus proved (4.4). This concludes the proof that $\mathfrak{F} \in C^2$, hence that the Fréchet derivative $Du : \mathbb{L}_q \to \mathcal{L}(\mathbb{L}_q, \mathbb{H}_p(T))$ is Gâteaux differentiable.

By Theorem 3.8 we have

$$\partial Du(x)(y_1,y_2) = \big(I - D_2\mathfrak{F}(x,u(x))\big)^{-1} D_2^2\mathfrak{F}(x,u(x))(Du(x)y_1, Du(x)y_2),$$

hence

$$\partial Du(x)(y_1,y_2) = D_2\mathfrak{F}(x,u(x))\partial Du(x)(y_1,y_2) + D_2^2\mathfrak{F}(x,u(x))(Du(x)y_1, Du(x)y_2),$$

and (2.6) now follows substituting in the previous identity the expressions for $D_2\mathfrak{F}$ and $D_2^2\mathfrak{F}$ obtained above and in the proof of Theorem 2.7.

The bound for the bilinear form $\partial Du(x)$ can be established as an application of Corollary 3.10. Since $Du(x)$ is bounded by Theorem 2.7, it is enough to show that $D_2^2\mathfrak{F} : \mathbb{L}_{q'} \times \mathbb{H}_{q'}(T) \to \mathcal{L}^{\otimes 2}(\mathbb{H}_{q'}(T), \mathbb{H}_p(T))$ is bounded. In fact, by a computation completely analogous to the above ones, we have

$$\begin{aligned}\big\|D_2^2\mathfrak{F}(x, u(x)(v, w)\big\|_p^p &\lesssim_{p,T,\eta} \int_0^T \big|D^2 f(u(s))(v(s), w(s))\big|^p \, ds \\ &+ \int_0^T |v(s)|^p |w(s)|^p \Big(\big|D_1^2 B(u(s), \cdot)\big|^p_{L_p(Z,m;\mathcal{L}^{\otimes 2})} + \big|D_1^2 B(u(s), \cdot)\big|^p_{L_2(Z,m;\mathcal{L}^{\otimes 2})}\Big) ds \\ &\lesssim_T C_1^p \|v\|_{q'}^p \|w\|_{q'}^p + \big(|h_1|^p_{L_p(Z,m)} + |h_1|^p_{L_2(Z,m)}\big) \|v\|_{q'}^p \|w\|_{q'}^p.\end{aligned}$$

Let us now assume that $q > 4p \geq 8$, and take $q' \in]2p, q/2[$. We shall deduce the twice Fréchet differentiability of the solution map applying Theorem 3.9, with all spaces defined as before. Since $q > 2q'$, the first part of the proof guarantees that $Du : \mathbb{L}_q \to \mathcal{L}(\mathbb{L}_q, \mathbb{H}_{q'}(T))$ is Gâteaux differentiable with derivative $\partial Du : \mathbb{L}_q \to \mathcal{L}(\mathbb{L}_q, \mathcal{L}(\mathbb{L}_q, \mathbb{H}_{q'}(T))$. Therefore, by the last statement of Theorem 3.9, we infer that $x \mapsto u(x) \in C^2(\mathbb{L}_q, \mathbb{H}_p(T))$. □

5 Application: gradient estimates for the resolvent

In this section we assume that the coefficients f, B, and G do not depend on t and ω. This assumption allows us to define the semigroup and resolvent associated to the mild solution:

$$P_t\varphi(x) = \mathbb{E}\varphi(u(t, x)), \qquad R_\alpha\varphi(x) = \int_0^\infty e^{-\alpha t} P_t\varphi(x) \, dt,$$

where $\varphi \in C_b(H)$ and $\alpha > 0$.

In order to prove gradient estimate for the resolvent R_α we need the following lemma, which gives an explicit bound on the Lipschitz constant of the solution map.

Lemma 5.1. *Let A be η-m-dissipative, and set*

$$[B]_{1,Q} = \sup_{\substack{x,y \in H \\ x \neq y}} \frac{|B(x) - B(y)|_Q}{|x - y|}, \qquad [G]_{1,m} = \sup_{\substack{x,y \in H \\ x \neq y}} \frac{|G(x, z) - G(y, z)|_{L_2(Z,m)}}{|x - y|}.$$

Then we have

$$\mathbb{E}|u(t, x) - u(t, y)| \leq e^{\omega_1 t} |x - y|_{\mathbb{L}_2}, \tag{5.1}$$

where

$$\omega_1 := \eta + [f]_1 + \frac{1}{2}[B]_{1,Q}^2 + \frac{1}{2}[G]_{1,m}^2$$

Proof. Let A_λ be the Yosida approximation of A, $A_\lambda \to A$ as $\lambda \to 0$. Let u_λ be the solution of

$$du_\lambda(t) = (A_\lambda u_\lambda(t) + f(u_\lambda(t))) \, dt + B(u_\lambda(t)) \, dW(t) + \int_Z G(u_\lambda(t), z) \, \bar{\mu}(dt, dz).$$

Then Itô's formula yields

$$\begin{aligned}|u_\lambda(t)|^2 =& |u_\lambda(0)|^2 + 2\int_0^t \langle A_\lambda u_\lambda(s), u_\lambda(s)\rangle \, ds + 2\int_0^t \langle f(u_\lambda(s)), u_\lambda(s)\rangle \, ds \\ &+ 2\int_0^t \langle u_\lambda(s), B(u_\lambda(s)) dM(s)\rangle + \big[B(u_\lambda) \cdot W\big](t) + \big[G(u_\lambda) \star \bar{\mu}\big](t)\end{aligned}$$

hence

$$
\begin{aligned}
&\mathbb{E}|u_\lambda(t,x) - u_\lambda(t,y)|^2 \\
&\quad = \mathbb{E}|x-y|^2 + 2\mathbb{E}\int_0^t \langle A_\lambda u_\lambda(s,x) - A_\lambda u_\lambda(s,y), u_\lambda(s,x) - u_\lambda(s,y)\rangle\, ds \\
&\qquad + 2\mathbb{E}\int_0^t \langle f(u_\lambda(s,x)) - f(u_\lambda(s,y)), u_\lambda(s,x) - u_\lambda(s,y)\rangle\, ds \\
&\qquad + \mathbb{E}\int_0^t \big|B(u_\lambda(s,x)) - B(u_\lambda(s,y))\big|_Q^2\, ds \\
&\qquad + \mathbb{E}\int_0^t \big|G(u_\lambda(s,x),\cdot) - G(u_\lambda(s,y),\cdot)\big|_{L_2(Z,m)}^2 ds \\
&\quad \leq \mathbb{E}|x-y|^2 + \big(2\eta + 2[f]_1 + [B]_{1,Q}^2 + [G]_{1,m}^2\big)\int_0^t \mathbb{E}|u_\lambda(s,x) - u_\lambda(s,y)|^2\, ds
\end{aligned}
$$

and Gronwall's inequality implies that

$$
\mathbb{E}|u_\lambda(t,x) - u_\lambda(t,y)|^2 \leq e^{2\omega_1 t}\mathbb{E}|x-y|^2,
$$

therefore, in view of Proposition 3.11, we can pass to the limit as $\lambda \to 0$, and applying Cauchy-Schwarz' inequality, we obtain (5.1). □

Remark 5.2. Note that (5.1) also implies that $|u(x) - u(y)|_{\mathcal{H}_2(T)} \leq e^{\omega_1 T}|x-y|_{\mathcal{H}_2}$ and

$$
\mathbb{E}|u(t,x) - u(t,y)| \leq e^{\omega_1 t}|x-y|
$$

if x, $y \in H$ are nonrandom. Moreover, if the noise is additive, i.e. if B is constant, then one can prove, solving differential inequalities ω-by-ω, that $|u(t,x) - u(t,y)| \leq e^{\omega_1 t}|x-y|$ almost surely.

Obtaining an explicit estimate for the Lipschitz constant of the solution map in the $\mathcal{H}_p(T)$ setting seems considerably more difficult in the general case of multiplicative noise. It reduces instead to a simple computation in the case of additive noise, as we show next.

Lemma 5.3. *Let A be η-m-dissipative and assume that B and G do not depend on x. If (1.1) is well-posed in $\mathbb{H}_p(T)$, then for any x, $y \in \mathcal{L}_p$ we have*

$$
\mathbb{E}|u(t,x) - u(t,y)|^p \leq e^{p\omega_1 t}\mathbb{E}|x-y|^p,
$$

where $\omega_1 = \eta + [f]_1$.

Proof. One has

$$
\frac{d}{dt}\big(u(t,x) - u(t,y)\big) = Au(t,x) - Au(t,y) + f(u(t,x)) - f(u(t,y))
$$

$\mathbb{P}$-a.s., hence, multiplying both sides by $|u(t,x) - u(t,y)|^{p-2}(u(t,x) - u(t,y))$, we obtain

$$
\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\big|u(t,x) - u(t,y)\big|^p \\
&\quad = \big\langle Au(t,x) - Au(t,y), |u(t,x) - u(t,y)|^{p-2}(u(t,x) - u(t,y))\big\rangle \\
&\qquad + |u(t,x) - u(t,y)|^{p-2}\big\langle f(u(t,x)) - f(u(t,y)), (u(t,x) - u(t,y))\big\rangle \\
&\quad \leq \big\langle A|u(t,x) - u(t,y)|^{\frac{p-2}{2}}(u(t,x) - u(t,y)), |u(t,x) - u(t,y)|^{\frac{p-2}{2}}(u(t,x) - u(t,y))\big\rangle \\
&\qquad + [f]_1|u(t,x) - u(t,y)|^p \\
&\quad \leq (\eta + [f]_1)|u(t,x) - u(t,y)|^p.
\end{aligned}
$$

Writing in integral form and taking expectations, we obtain

$$\mathbb{E}|u(t,x)-u(t,y)|^p \leq \mathbb{E}|x-y|^p + p(\eta+[f]_1)\int_0^t \mathbb{E}|u(s,x)-u(s,y)|^p\,ds,$$

from which the result follows by Gronwall's inequality. □

The following gradient estimate for the resolvent associated to the mild solution of the stochastic PDE is a consequence of (5.1).

Theorem 5.4. *Assume that $\varphi \in C_b(H)$ is Gâteaux differentiable and Lipschitz, and that f, B, G satisfy the assumptions of Theorem 2.6. Let $\alpha > \omega_1$. Then $x \mapsto R_\alpha\varphi(x)$ is Gâteaux differentiable with*

$$\partial_y R_\alpha\varphi(x) = \int_0^\infty e^{-\alpha t}\mathbb{E}[\partial\varphi(u(t,x))\partial_{2,y}u(t,x)]\,dt, \tag{5.2}$$

and it satisfies the estimate

$$\big|\partial R_\alpha\varphi(x)\big|_H \leq \frac{1}{\alpha-\omega_1}[\varphi]_1.$$

Proof. We have, setting $v(t) = \partial_{2,y}u(t,x)$,

$$\begin{aligned}
\Big|&\frac{R_\alpha\varphi(x+hy)-R_\alpha(x)}{h} - \int_0^\infty e^{-\alpha t}\mathbb{E}[\partial\varphi(u(t,x))\partial_{2,y}u(t,x)]\,dt\Big|\\
&\leq \int_0^\infty e^{-\alpha t}h^{-1}\mathbb{E}[\varphi(u(t,x+hy))-\varphi(u(t,x))-h\partial\varphi(u(t,x))v(t)]\,dt\\
&\leq \int_0^\infty e^{-\alpha t}\mathbb{E}h^{-1}|\varphi(u(t,x+hy))-\varphi(u(t,x)+hv(t))|\,dt\\
&\quad+\int_0^\infty e^{-\alpha t}\mathbb{E}h^{-1}|\varphi(u(t,x)+hv(t))-\varphi(u(t,x))-h\partial\varphi(u(t,x))v(t)|\,dt\\
&=: I_1(h)+I_2(h),
\end{aligned}$$

and

$$I_1(h) \leq [\varphi]_1\int_0^\infty e^{-\alpha t}\mathbb{E}|h^{-1}(u(t,x+hy)-u(t,x))-\partial_{2,y}u(t,x)|\,dt.$$

Since, by Cauchy-Schwartz' inequality and the differentiability of the solution map from H to $\mathcal{H}_2(T)$, we have

$$\begin{aligned}
&\mathbb{E}|h^{-1}(u(t,x+hy)-u(t,x))-\partial_{2,y}u(t,x)|\\
&\qquad\leq \big(\mathbb{E}|h^{-1}(u(t,x+hy)-u(t,x))-\partial_{2,y}u(t,x)|^2\big)^{1/2} \to 0,
\end{aligned}$$

we conclude that $I_1(h) \to 0$ as $h \to 0$ by dominated convergence. On the other hand, $I_2(h)$ converges to zero as $h \to 0$ by definition of directional derivative and dominated convergence. This establishes (5.2).

Note that $x \mapsto P_t\varphi(x)$ is Lipschitz: in fact, the previous lemma yields

$$\begin{aligned}
\big|P_t\varphi(x)-P_t\varphi(y)\big| &\leq \mathbb{E}\big|\varphi(u(t,x))-\varphi(u(t,y))\big|\\
&\leq [\varphi]_1\mathbb{E}|u(t,x)-u(t,y)| \leq [\varphi]_1 e^{\omega_1 t}|x-y|.
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{R_\alpha\varphi(x+hy)-R_\alpha\varphi(x)}{h} &= \int_0^\infty e^{-\alpha t}h^{-1}\big(P_t\varphi(x+hy)-P_t\varphi(x)\big)\,dt\\
&\leq [\varphi]_1|y|\int_0^\infty e^{\omega_1 t}e^{-\alpha t}\,dt < \infty,
\end{aligned}$$

hence, in view of (5.2),

$$\left|\partial_y R_\alpha \varphi(x)\right| \leq \frac{1}{\alpha - \omega_1}[\varphi]_1 |y|,$$

thus also $|\partial R_\alpha \varphi(x)| \leq (\alpha - \omega_1)^{-1}[\varphi]_1$. □

6 Strong Feller property

The purpose of this section is to establish a Bismut-Elworthy formula for the semigroup associated to (1.1), and to deduce from it the strong Feller property, adapting an argument of [27] to the infinite dimensional case. We would like to emphasize that the proof depends essentially on the second order differentiability of the solution with respect to the initial datum established in Theorem 2.8 above. In the following we shall denote the set of bounded Borel functions from H to $\mathbb{R}$ by $B_b(H)$.

Theorem 6.1. *Assume that $Q \in \mathcal{L}_2(K)$, $B(x)$ is invertible with $|B(x)^{-1}| \leq C$ for all $x \in H$, for some $C > 0$, and the hypotheses of Theorem 2.6 are satisfied. Then the semigroup P_t is strong Feller, i.e. $\varphi \in B_b(H)$ implies $P_t\varphi \in C_b(H)$.*

Proof. We first assume that the coefficients f, B and G satisfy the hypotheses of Theorem 2.8, so that $x \mapsto u(x) \in C^2(H, \mathbb{H}_2(T))$. This assumption will be removed in the last part of the proof.

A formal application of Itô's formula shows that the generator L of the semigroup P_t associated to the mild solution of (1.1) takes the form, for $\varphi \in C_b^2(H)$,

$$\begin{aligned} L\varphi(x) =& \langle Ax + f(x), D\varphi(x)\rangle + \frac{1}{2}\operatorname{Tr}(QB(x)B^*(x)D^2\varphi(x)) \\ &+ \int_Z \left[\varphi(x + G(x,z)) - \varphi(x) - \langle D\varphi(x), G(x,z)\rangle\right] m(dz). \end{aligned}$$

Let u_λ be the solution of (1.1) with A replaced by its Yosida approximation A_λ, P_t^λ the associated semigroup, and L_λ the generator of P_t^λ. Then the action of L_λ on $\varphi \in C_b^2(H)$ is exactly as for L, with A replaced by A_λ. Let $\varphi \in C_b^2(H)$, $s \in [0,t]$, and set $v(s,x) = P_{t-s}^\lambda \varphi(x) \equiv \mathbb{E}\varphi(u_\lambda(t-s,x))$. Then $v \in C^{1,2}([0,T] \times H)$ and Itô's formula implies

$$\begin{aligned} v(s, u_\lambda(s)) = v(0,x) &+ \int_0^s (\partial_r + L_\lambda)v(r, u_\lambda(r))\,dr + \int_0^s \langle Dv(r, u_\lambda(r)), B(u_\lambda(r))\,dW(r)\rangle \\ &+ \int_0^s \int_Z \left[v(r-, u_\lambda(r-) + G(u_\lambda(r-), z)) - v(r-, u_\lambda(r-))\right] \bar{\mu}(dr, dz) \end{aligned}$$

Since $(\partial_t + L_\lambda)v = 0$, the previous identity evaluated at $s = t$ implies

$$\varphi(u_\lambda(t)) = P_t^\lambda \varphi(x) + M_1^\lambda(t) + M_2^\lambda(t),$$

where

$$\begin{aligned} M_1^\lambda(t) &= \int_0^t \langle DP_{t-r}^\lambda \varphi(u_\lambda(r)), B(u_\lambda(r))dW(r)\rangle, \\ M_2^\lambda(t) &= \int_0^t \int_Z \left[P_{t-s}^\lambda \varphi(u_\lambda(s-) + G(u_\lambda(s-))z) - P_{t-s}^\lambda \varphi(u_\lambda(s-))\right] \bar{\mu}(ds, dz). \end{aligned}$$

Letting $\lambda \to 0$ and recalling Proposition 3.11 we obtain

$$\varphi(u(t)) = P_t\varphi(x) + M_1(t) + M_2(t), \tag{6.1}$$

with M_1 and M_2 defined in the obvious way. Moreover, setting $w(t) = \partial_{2,y} u(t,x)$ and

$$M_3(t) = \int_0^t \langle B^{-1}(u(s))w(s), dW(s)\rangle,$$

multiplying both sides of (6.1) by $M_3(t)$ and taking expectations yields

$$\begin{aligned}\mathbb{E}\varphi(u(t))M_3(t) &= \mathbb{E}M_1(t)M_3(t) = \mathbb{E}\int_0^t \langle DP_{t-s}\varphi(u(s)), w(s)\rangle\, ds \\ &= \mathbb{E}\int_0^t D[P_{t-s}\varphi(u(s))]y\, ds = \int_0^t DP_t\varphi(x)y\, ds = tDP_t\varphi(x)y.\end{aligned}$$

Here $\mathbb{E}M_2(t)M_3(t) = 0$ because W and $\bar{\mu}$ are independent, and we have used the Markov property of solutions in the second to last step. In particular, we have proved the Bismut-Elworthy-type formula

$$DP_t\varphi(x)y = \frac{1}{t}\mathbb{E}\Big[\varphi(u(t,x))\int_0^t \langle B^{-1}(u(s,x))\partial_{2,y}u(s,x), dW(s)\rangle\Big].$$

We shall now remove the assumptions on f, B and G. Let us assume for a moment that we can find sequences f_ε, B_ε, G_ε satisfying the hypotheses of Theorem 2.8 and Proposition 3.12, and denote the mild solution of

$$du(t) = [Au(t) + f_\varepsilon(u(t))]\, dt + B_\varepsilon(u(t))\, dW(t) + \int_Z G_\varepsilon(u(t), z)\,\bar{\mu}(dt, dz), \qquad u(0) = x,$$

by u_ε, so that $x \mapsto u_\varepsilon(x) \in C^2(H, \mathcal{H}_2(T))$ and $u_\varepsilon \to u$ in $\mathcal{H}_2(T)$. In particular we also have $P_t^\varepsilon\varphi(x) \to P_t\varphi(x)$ for all $x \in H$ and $t \leq T$, where $P_t^\varepsilon\varphi(x) := \mathbb{E}\varphi(u_\varepsilon(t,x))$, $\varphi \in C_b(H)$. Then Cauchy-Schwartz' inequality yields

$$|DP_t^\varepsilon\varphi(x)y|^2 \leq \frac{1}{t^2}|\varphi|_\infty^2 C^2\mathbb{E}\int_0^t |\partial_{2,y}u_\varepsilon(s,x)|^2\, ds,$$

where $|\varphi|_\infty := \sup_{x\in H}|\varphi(x)|$. In view of Remark 4.1 it is not difficult to see that there exists a constant N, which does not depend on x, y, and ε, such that $|[u_\varepsilon(x_1) - u_\varepsilon(x_2)|_2 \leq N|x_1 - x_2|_H$, hence, by Theorem 2.6, $|\partial_{2,y}u_\varepsilon(s,x)| \leq N|y|$. We obtain

$$|DP_t^\varepsilon\varphi(x)| \leq \frac{NC}{t^{1/2}}|\varphi|_\infty,$$

thus also $|P_t^\varepsilon\varphi(x_1) - P_t^\varepsilon\varphi(x_2)| \leq t^{-1/2}NC|\varphi|_\infty|x_1 - x_2|$, and letting $\varepsilon \to 0$,

$$|P_t\varphi(x_1) - P_t\varphi(x_2)| \leq t^{-1/2}NC|\varphi|_\infty|x_1 - x_2|.$$

The same Lipschitz property continues to hold also for $\varphi \in B_b(H)$ by a simple regularization argument (see e.g. [23, Lemma 2.2]).

In order to complete the proof, we have to show that we can find sequences f_ε, B_ε, G_ε satisfying the hypotheses of Theorem 2.8 and Proposition 3.12. The existence of such f_ε and B_ε is well-known (see e.g. [6, Sect. 3.3.1], [23]), and the construction of G_ε can be carried out in a completely similar way, hence we omit it. □

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