THE FOCUSING NLS EQUATION WITH PERIODIC DIRICHLET DATA: ADMISSIBLE NEUMANN VALUES AND LONG-TIME ASYMPTOTICS

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Abstract. We consider the initial boundary value problem for the focusing nonlinear Schrödinger equation in the quarter plane $x > 0$, $t > 0$ in the case of decaying initial data (for $t = 0$, as $x \to +\infty$) and Dirichlet boundary data (for $x = 0$) approaching a periodic (single-frequency) background $a\mathrm{e}^{2\mathrm{i}\omega t}$ as $t \to +\infty$. We first provide admissibility conditions for the normal derivative of the solution on the boundary, under the assumption that it behaves asymptotically in a similar (single-frequency) manner. We then show that for the range $\omega \geq \frac{a^2}{2}$, the long-time asymptotics of the solution inside the quarter plane exhibits decaying oscillations of Zakharov–Manakov type.

1. Introduction

1.1. The RH method for IBV integrable problems.

One of the main advantages of the inverse scattering transform (IST) method to study initial value problems for certain nonlinear evolution equations — integrable nonlinear equations possessing a Lax pair representation — is that it allows a rigorous and complete analysis of the long-time behavior of solutions of such equations, in certain classes of initial data. It has been realized since the mid-1990s that the most appropriate form of the IST method for studying asymptotics is the Riemann–Hilbert (RH) problem method, based on the reformulation of the scattering problem for one of the Lax pair equations — the x-equation — in terms of an analytic (matrix) factorization problem of Riemann–Hilbert type. This reformulation allows to reduce the problem to the asymptotic study of time-dependent RH problem, whose jump matrices are time-oscillating. The appropriate tool for this study has been developed by several authors and has finally been formalized by Deift and Zhou [12] as the "nonlinear steepest descent" method.

A natural question arises: is it possible to adapt the RH method – and subsequently the nonlinear steepest descent method – to the initial-boundary value (IBV) problems. Such an adaptation was actually introduced by Fokas [13–15] and was further developed by several authors [2, 5, 9, 16]. The key feature of this method is that both of the linear equations of the associated Lax pair are treated in a similar way, in terms of the direct–inverse scattering formalism. Comparing to the initial value problems, the effectiveness of the RH method for IBV problems is reduced by the fact that in its implementation, more boundary values are required to construct the underlying RH problem than those given in the framework of a well-posed IBV problem. In particular, for the (focusing) NLS equation

$$\mathrm{i}q_t + q_{xx} + 2|q|^2 q = 0, \qquad x > 0,\ t > 0, \tag{1.1}$$

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a well-posed IBV problem requires only $q(x,0)$ and $q(0,t)$ to be specified as respective initial and boundary conditions (see, e.g., [17]), whereas the construction of the RH problem requires spectral functions associated with the scattering problem for the t-equation of the Lax pair

$$\Phi_t + 2\mathrm{i}k^2\sigma_3\Phi = \tilde{Q}\Phi, \tag{1.2}$$

where $\tilde{Q}$ involves q as well as q_x, and thus these spectral functions are uniquely determined by $q(0,t)$ and $q_x(0,t)$. This drawback is compensated, to some extent, by the fact that the consistency of the set $\{q(x,0), q(0,t), q_x(0,t)\}$ is completely characterized by a relation amongst the associated spectral functions — the "global relation" [2,16].

Using the RH problem approach, IBV problems have been studied for a number of nonlinear equations, for the case $0 < t < T$ with $T < \infty$ as well as for the case $0 < t < \infty$ under the assumption that the boundary values decay to zero when $t \to +\infty$. The long-time results in the latter case show that they are qualitatively similar to those in the case of the initial value problems: the asymptotics is dominated by solitons whereas in the solitonless cases (as well as in domains "between" solitons), the long-time behavior is described by decaying, modulated oscillations of Zakharov–Manakov type [11]. Quantitatively, the parameters of the solitons and of decaying oscillations are affected by the boundary values via the corresponding spectral functions.

Physically, more interesting problems arise when the boundary conditions do not decay with time but oscillate, in a certain prescribed way, reflecting the action of a wavemaker placed at the boundary of the space domain (half-line) and producing non-decaying oscillations. Various aspects of such a problem for the focusing nonlinear Schrödinger (NLS_+) equation are discussed in [5–7].

1.2. **NLS_+ with decaying initial data and Dirichlet data $a\mathrm{e}^{2\mathrm{i}\omega t}$ for $\omega < -3a^2$.** In [3,4], the long-time asymptotics has been studied for the IBV problem

$$\mathrm{i}q_t + q_{xx} + 2|q|^2 q = 0, \qquad x > 0,\ t > 0, \tag{1.3a}$$

$$q(x,0) = q_0(x), \qquad x \geq 0, \tag{1.3b}$$

$$q(0,t) = g_0(t), \qquad t \geq 0, \tag{1.3c}$$

where the initial and Dirichlet boundary data satisfy the conditions:

(i) $q_0(x)$ decays to 0 when $x \to +\infty$;

(ii) $g_0(t) = a\mathrm{e}^{2\mathrm{i}\omega t}$, with constant amplitude $a > 0$ and constant single-frequency $\omega \in \mathbb{R}$,

in the range

(iii) $\omega < -3a^2$,

and under the *additional condition* about the long-time behavior of the Neumann boundary values $q_x(0,t)$:

(iv) we assume

$$q_x(0,t) - 2\mathrm{i}ab\mathrm{e}^{2\mathrm{i}\omega t} \to 0, \quad t \to +\infty, \tag{1.4a}$$

where

$$b = \sqrt{\frac{a^2 - \omega}{2}} > 0. \tag{1.4b}$$

Remark. Assumption (iv) is motivated by the existence of the explicit solution of (1.1) satisfying (1.4):

$$q(x,t) = a\mathrm{e}^{2\mathrm{i}\omega t + 2\mathrm{i}bx}. \tag{1.5}$$

Notice, however, that for this solution (1.5), the initial data $q(x,0)$ does not decay to 0 when $x \to +\infty$.

The asymptotic result is as follows:

Theorem 1 ($\omega < -3a^2$, [3, 4]). *The quarter plane $x > 0, t > 0$ can be divided into three regions, $0 < \frac{x}{t} < C_1$, $C_1 < \frac{x}{t} < C_2$, $\frac{x}{t} > C_2$, with C_j expressed in terms of $\{a, \omega\}$, such that the long-time behavior of $q(x,t)$ is qualitatively different in these regions. In particular, in the first two regions, $q(x,t)$ exhibits nondecaying oscillations.*

Numerical simulations for the solution of (1.3) with assumptions (i)-(iii) are shown [8] to be in good agreement with the theoretical considerations, in particular with the additional assumption (iv).

Concerning this additional assumption (1.4), it is worth noting the following. The IBV problem (1.3) is well posed in the sense that given q_0 and g_0 from certain classes (see, e.g., [17]), a unique global solution to (1.3) exists for all $x > 0$ and $0 < t < T$ for any finite T. In particular, $q_x(0,t)$ is uniquely determined by $q_0(x)$ and $g_0(t)$. In the framework of the RH approach, the mapping $\{q_0, g_0\} \mapsto \{q_x(0,t)\}$ $(0 < t < T)$, is described in [1] for any $T > 0$. It is based on the analysis of the associated "global relation" and allows, in principle, to calculate $q_x(0,t)$ without solving the IBV for all (x,t). In particular, it could be used to study the long-time behavior of $q_x(0,t)$, given $g_0(t)$ for all $t > 0$. But this problem remains open. On the other hand, knowledge of the long-time behavior of $q_x(0,t)$ is crucial for the scattering formalism governed by (1.2).

The meaningfulness of assumption (1.4) is supported by the observation that if we do not fix $q(0,t)$ to be $a\mathrm{e}^{2\mathrm{i}\omega t}$ but restrict only its long-time behavior: $q(0,t) - a\mathrm{e}^{2\mathrm{i}\omega t} \to 0$ when $t \to +\infty$, then we can secure the assumption by taking, as a starting point of the construction of the RH problem, the set of spectral functions satisfying the global relation. Following this way one cannot guarantee that the constructed $q(x,t)$ will satisfy the Dirichlet boundary condition $q(0,t) = g_0(t)$ with a *prescribed* g_0, but one can guarantee that $q(0,t) - a\mathrm{e}^{2\mathrm{i}\omega t} \to 0$ and $q_x(0,t) - 2\mathrm{i}ab\mathrm{e}^{2\mathrm{i}\omega t} \to 0$ when $t \to +\infty$. This follows from the fact that the analysis in [3, 4] is valid for all $g_0(t)$ approaching $a\mathrm{e}^{2\mathrm{i}\omega t}$ sufficiently fast.

Therefore, at least a part of IBV problems (1.3) with $q_0(x) \to 0$ and $g_0(t) - a\mathrm{e}^{2\mathrm{i}\omega t} \to 0$ has solutions satisfying (1.4). An interesting open question is: how big is this part, and how does it lie in the set of all such problems?

For all such problems with $\omega < -3a^2$, the asymptotics of $q(x,t)$ is as described in [3, 4], showing in particular large regions with nondecaying oscillations. On the other hand, numerical simulations for positive values of ω, in particular for ω with the same absolute value as that considered in [3, 4], show completely different pictures: $q(x,t)$ decays fast for all (x,t) outside a thin layer near the t-axis.

1.3. **Admissible parameter sets and asymptotics for $\omega > a^2/2$.** In this paper, which is the continuation of the studies in [3, 4], we are interested

i) in describing possible *joint* asymptotic behavior of the Dirichlet and Neumann boundary values of a solution to the NLS equation (1.1), and, subsequently,

ii) in the large time behavior of this solution in the quarter plane $x > 0$, $t > 0$ – in a parameter range different from that considered in [3, 4].

Definition (admissible parameter set)**.** We say that a parameter set $\{a, \omega, c\}$, where $a > 0$, $\omega \in \mathbb{R}$, and $c \in \mathbb{C}$, is *admissible* if there exists a smooth function $q(x,t)$ defined for $x \geq 0$, $t \geq 0$ such that:

1) $q(x,t)$ is a solution of the NLS equation (1.1) for $x > 0$, $t > 0$;
2) $q(x,0) \to 0$ sufficiently fast as $x \to +\infty$;
3) $q(0,t) - a\mathrm{e}^{2\mathrm{i}\omega t} \to 0$ and $q_x(0,t) - c\mathrm{e}^{2\mathrm{i}\omega t} \to 0$ sufficiently fast as $t \to +\infty$.

In the present paper, we study

(i) the admissibility of parameter sets $\{a, \omega, c\}$;
(ii) the asymptotics of $q(x,t)$ in the admissible cases for $\omega \geq \frac{a^2}{2}$.

Concerning (i), the main result is Theorem 2, which states, in particular, that for a given a, the admissible values for ω are in $(-\infty, -3a^2]$ and $[\frac{a^2}{2}, \infty)$. More precisely, the admissible parameter values are

- $\omega > a^2/2$ with c given by the (two-valued) function of (a, ω) defined by $c^2 = a^2(2\omega - a^2)$,
- $\omega < -3a^2$ with c given by the function of (a, ω) defined by $c^2 = 2a^2(\omega - a^2)$ and $\operatorname{Im} c > 0$.

As for (ii), the result is that in the whole quarter plane (except, possibly, for a thin layer near the t axis), $q(x,t)$ decays to 0 exhibiting oscillations of Zakharov–Manakov type, see Theorem 3.

Remark. Notice that for $\omega \leq -3a^2$, Theorem 2 *justifies* the choice of the value of c, see (1.4), adopted in [3, 4].

The paper is organized as follows.

In Section 2, we present the formalism of the RH approach under assumption that the boundary values satisfy $q(0,t) - a\mathrm{e}^{2\mathrm{i}\omega t} \to 0$ and $q_x(0,t) - c\,\mathrm{e}^{2\mathrm{i}\omega t} \to 0$ when $t \to +\infty$, with any $c \in \mathbb{C}$.

In Section 3, we analyze the consistency of these assumptions with the global relation, which leads us to the determination of admissible parameter sets $\{a, \omega, c\}$.

Section 4 introduces to the second part of the paper, devoted to the asymptotic analysis in the range $\omega \geq \frac{a^2}{2}$.

In Sections 5 and 6, we discuss specific features of the RH formalism corresponding to admissible sets of parameters with $\omega \geq \frac{a^2}{2}$.

In Section 7, we show that for all admissible values of the parameters with $\omega \geq \frac{a^2}{2}$, the asymptotics of $q(x,t)$ in the solitonless case has a Zakharov–Manakov type decaying behavior.

Section 8 presents concluding remarks.

2. The Riemann–Hilbert formalism for IBV problems

2.1. **The Lax pair.** As we pointed out in Section 1, the implementation of the Riemann–Hilbert approach for the IBV problems depends strongly on the character of the long-time behavior of the boundary values.

In this section, we assume that the long-time asymptotics of $q_x(0,t)$ is single-frequency periodic, similarly to the Dirichlet boundary data $q(0,t)$, but with a priori arbitrary complex amplitude. Thus we consider the IBV problem (1.3), where $q_0(x)$ and $g_0(t)$ are sufficiently smooth functions such that

- $q_0(x) \to 0$ fast enough when $x \to +\infty$,
- $g_0(t) - a\mathrm{e}^{2\mathrm{i}\omega t} \to 0$ fast enough when $t \to +\infty$,
- and we assume that
$$q_x(0,t) - c\mathrm{e}^{2\mathrm{i}\omega t} \to 0 \quad \text{as } t \to +\infty \tag{2.1}$$
with some constant $c = c_1 + \mathrm{i}c_2 \in \mathbb{C}$.

The RH problem formalism follows basically the same lines as in [7]. The focusing NLS equation is the compatibility condition for a pair of linear equations (the Lax pair)

$$\Phi_x + \mathrm{i}k\sigma_3\Phi = Q\Phi, \tag{2.2a}$$
$$\Phi_t + 2\mathrm{i}k^2\sigma_3\Phi = \tilde{Q}\Phi, \tag{2.2b}$$

where $\Phi = \Phi(x,t,k)$ is a 2×2 matrix valued function, $\sigma_3 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, and

$$Q = \begin{pmatrix} 0 & q \\ -\bar{q} & 0 \end{pmatrix}, \qquad \tilde{Q} = \begin{pmatrix} \mathrm{i}|q|^2 & 2kq + \mathrm{i}q_x \\ -2k\bar{q} + \mathrm{i}\bar{q}_x & -\mathrm{i}|q|^2 \end{pmatrix}, \tag{2.3}$$

with $q = q(x,t)$.

2.2. **The background eigenfunction.** Let us introduce the background eigenfunction $\Psi(t,k)$, which is a solution of the t-equation (2.2b) with $q \equiv a\mathrm{e}^{2\mathrm{i}\omega t}$ and $q_x \equiv c\,\mathrm{e}^{2\mathrm{i}\omega t}$:

$$\Psi(t,k) := \mathrm{e}^{\mathrm{i}\omega t\sigma_3}E(k)\mathrm{e}^{-\mathrm{i}\Omega(k)t\sigma_3}, \tag{2.4}$$

where $\Omega(k)$ and $E(k)$ are determined by substituting (2.4) into (2.2b), which gives

$$\Omega^2(k) = 4k^4 + 4\omega k^2 + \beta k + \gamma, \tag{2.5a}$$

with

$$\beta = -4ac_2, \tag{2.5b}$$
$$\gamma = (a^2-\omega)^2 + c_1^2 + c_2^2. \tag{2.5c}$$

We specify E to be unimodular, i.e., $\det E(k) \equiv 1$:

$$E(k) = \sqrt{\frac{2\Omega - H}{2\Omega}} \begin{pmatrix} 1 & -\frac{\mathrm{i}H}{2ak - \mathrm{i}\bar{c}} \\ -\frac{\mathrm{i}H}{2ak+\mathrm{i}c} & 1 \end{pmatrix}, \tag{2.6a}$$

where

$$H(k) = \Omega(k) - 2k^2 + a^2 - \omega. \tag{2.6b}$$

The branches of $\Omega(k)$ and $\sqrt{\frac{2\Omega-H}{2\Omega}}$ are fixed by their asymptotics, for $k \to \infty$:

$$\Omega(k) \sim 2k^2 + \omega, \tag{2.7a}$$
$$\sqrt{\frac{2\Omega - H}{2\Omega}} \sim 1, \tag{2.7b}$$

on the cut plane, with finite branch cuts connecting the roots of $\Omega^2(k)$ and the points $\{\frac{\mathrm{i}\bar{c}}{2a}, -\frac{\mathrm{i}c}{2a}\}$. The equality

$$(2\Omega - H)H = (2ak - \mathrm{i}\bar{c})(2ak + \mathrm{i}c), \tag{2.8}$$

shows that the zeros of $2\Omega - H$ and H are amongst $\{\frac{\mathrm{i}\bar{c}}{2a}, -\frac{\mathrm{i}c}{2a}\}$.

Remark. Although global analytic properties of the functions introduced above depend on the branch cuts position, certain properties of functions involved in the spectral analysis of the Lax pair equation (2.2a) must hold for any choice of the branch cuts. In what follows, this fact will allow us, by choosing appropriately a particular position of a branch cut, "filter out" the values of the parameters of the long time asymptotics of $q(0,t)$ and $q_x(0,t)$ which cannot take place for any solution $q(x,t)$ of the NLS equation. In particular, we will see that it is helpful to have a part of a branch cut going along the curve $\operatorname{Im}\Omega(k) = 0$.

2.3. **The RH approach.** The RH approach to IBV problems involves the following steps:

1. Analysis of the direct–inverse scattering mappings for both equations of the Lax pair: for (2.2a), for $t = 0$, and for (2.2b), for $x = 0$.
2. Introduction of appropriate eigenfunctions — simultaneous solutions to (2.2a), (2.2b) for all $x \geq 0$ and $t \geq 0$, which, being evaluated at $t = 0$ and at $x = 0$, can be related to those eigenfunctions involved in the scattering mappings for (2.2a) and (2.2b), respectively.
3. Construction of an associated RH problem —actually, a family of problems parametrized by (x,t), such that, on one hand, it is determined by the spectral functions from the scattering mappings at $t = 0$ and $x = 0$, and, on the other hand, its solution evaluated at $t = 0$ and at $x = 0$ allows well-controlled reduction to the eigenfunctions involved in the above mentioned scattering mappings on the boundary. The latter property is needed in order to prove that $q(x,t)$ obtained via the solution of the RH problem, being evaluated at $t = 0$ and at $x = 0$, indeed gives the prescribed initial and boundary data.

2.4. **Eigenfunctions.** The latter item suggest introducing eigenfunctions in terms of Volterra type integral equations, where the integration paths go along the x and t axes [16]. Explicitly, introduce $\Phi_j(x,t,k)$, $j = 1,2,3$ as follows (cf. [7]):

$$\begin{aligned}
\Phi_1(x,t,k) &= \mu_1(x,t,k)\mathrm{e}^{-\mathrm{i}kx\sigma_3}\mathrm{e}^{-\mathrm{i}(\Omega(k)-\omega)t\sigma_3}, \\
\Phi_j(x,t,k) &= \mu_j(x,t,k)\mathrm{e}^{-\mathrm{i}kx\sigma_3}\mathrm{e}^{-2\mathrm{i}k^2t\sigma_3}, \quad j = 2,3,
\end{aligned}$$

where the μ_j's are 2×2 matrix valued solutions of the integral equations

$$\begin{aligned}\mu_1(x,t,k) &= \mathrm{e}^{-\mathrm{i}kx\sigma_3}\mathcal{E}(t,k)\mathrm{e}^{\mathrm{i}kx\sigma_3} - \mathrm{e}^{(-\mathrm{i}kx+\mathrm{i}\omega t)\sigma_3}E(k)\int_t^\infty \mathrm{e}^{-\mathrm{i}\Omega(k)(t-\tau)\sigma_3}E^{-1}(k)\times\\ &\quad\times \mathrm{e}^{-\mathrm{i}\omega\tau\sigma_3}(\tilde{Q}-\tilde{Q}_0)(0,\tau,k)\mu_1(0,\tau,k)\mathrm{e}^{\mathrm{i}(\Omega(k)-\omega)(t-\tau)\sigma_3}\mathrm{d}\tau\mathrm{e}^{\mathrm{i}kx\sigma_3}+\\ &\quad+\int_0^x \mathrm{e}^{-\mathrm{i}k(x-y)\sigma_3}Q(y,t)\mu_1(y,t,k)\mathrm{e}^{\mathrm{i}k(x-y)\sigma_3}\mathrm{d}y, \qquad (2.9\mathrm{a})\\ \mu_2(x,t,k) &= I + \mathrm{e}^{-\mathrm{i}kx\sigma_3}\int_0^t \mathrm{e}^{-2\mathrm{i}k^2(t-\tau)\sigma_3}\tilde{Q}(0,\tau,k)\mu_2(0,\tau,k)\mathrm{e}^{2\mathrm{i}k^2(t-\tau)\sigma_3}\mathrm{e}^{\mathrm{i}kx\sigma_3}\\ &\quad+\int_0^x \mathrm{e}^{-\mathrm{i}k(x-y)\sigma_3}Q(y,t)\mu_2(y,t,k)\mathrm{e}^{\mathrm{i}k(x-y)\sigma_3}\mathrm{d}y, \qquad (2.9\mathrm{b})\\ \mu_3(x,t,k) &= I - \int_x^\infty \mathrm{e}^{-\mathrm{i}k(x-y)\sigma_3}Q(y,t)\mu_3(y,t,k)\mathrm{e}^{\mathrm{i}k(x-y)\sigma_3}\mathrm{d}y, \qquad (2.9\mathrm{c})\end{aligned}$$

with

$$\mathcal{E}(t,k) = \mathrm{e}^{\mathrm{i}\omega t\sigma_3}E(k)\mathrm{e}^{-\mathrm{i}\omega t\sigma_3} \qquad (2.10)$$

and $\tilde{Q}_0$ is $\tilde{Q}$ from (2.3) with $q = a\mathrm{e}^{2\mathrm{i}\omega t}$ and $q_x = c\mathrm{e}^{2\mathrm{i}\omega t}$. These eigenfunctions are normalized in such a way that $\mu_1(0,t,k)\sim\mathcal{E}(t,k)$ when $t\to+\infty$, $\mu_2(0,0,k)=\mu_3(\infty,0,k)=I$.

2.5. **Spectral functions.** The matrix structure of (2.9) implies that the columns of μ_1 and μ_3 are analytic and bounded in domains where the corresponding exponential factors decay to 0 [7]. These domains are bounded by the curves $\operatorname{Im}\Omega(k)=0$ and $\operatorname{Im}k=0$, and by the branch cuts (if they are not already included in the curves). The columns of μ_2 are entire functions.

2.5.1. *The x-spectral map.* The spectral mapping

$$\{q_0(x)\}\mapsto\{a_s(k),b_s(k)\}$$

is defined in terms of the eigenfunctions μ_2 and μ_3 for $t=0$, with $q(x,0)$ in (2.9) replaced by $q_0(x)$. It has the standard form of the scattering problem for the Dirac equation (2.2a) with potential supported on the half-line $x\geq 0$, see, e.g. [16]. Namely, the scattering matrix

$$s(k)\equiv\begin{pmatrix}\bar{a}_s(\bar{k}) & b_s(k)\\ -\bar{b}_s(\bar{k}) & a_s(k)\end{pmatrix},\quad \operatorname{Im}k=0,$$

is defined as the matrix relating Φ_2 and Φ_3:

$$\Phi_3(x,0,k)=\Phi_2(x,0,k)\begin{pmatrix}\bar{a}_s(\bar{k}) & b_s(k)\\ -\bar{b}_s(\bar{k}) & a_s(k)\end{pmatrix} \qquad (2.11)$$

and thus $s(k)=\Phi_3(0,0,k)=\mu_3(0,0,k)$.

The spectral functions $a_s(k)$ and $b_s(k)$ have the following properties:

- They are analytic in $\mathbb{C}_+=\{k\mid\operatorname{Im}k>0\}$.
- They are continuous in $\bar{\mathbb{C}}_+$.
- Moreover, $a_s(k)=1+\mathrm{O}(1/k)$ and $b_s(k)=\mathrm{O}(1/k)$ for $k\to\infty$.
- For $k\in\mathbb{R}$, $|a_s(k)|^2+|b_s(k)|^2=1$.

Genericity Assumption. We will assume that $q_0(x)$ is *generic* so that $a_s(k)$ has only a finite number of zeros in $\mathbb{C}_+$, each of multiplicity one.

The inverse mapping $\{a_s(k), b_s(k)\} \mapsto \{q_0(x)\}$ can be described in terms of the solution of an associated RH problem [16].

2.5.2. *The t-spectral map.* The spectral mapping

$$\{g_0(t), g_1(t)\} \mapsto \{A(k), B(k)\}$$

is defined in terms of the eigenfunctions μ_1 and μ_2 for $x = 0$, with $q(0,t)$ and $q_x(0,t)$ in (2.9) replaced by $g_0(t)$ and $g_1(t)$, respectively. The scattering matrix

$$S(k) \equiv \begin{pmatrix} \bar{A}(\bar{k}) & B(k) \\ -\bar{B}(\bar{k}) & A(k) \end{pmatrix}, \quad \operatorname{Im}\Omega(k) = 0,$$

is defined as the matrix relating Φ_1 and Φ_2:

$$\Phi_1(0,t,k) = \Phi_2(0,t,k) \begin{pmatrix} \bar{A}(\bar{k}) & B(k) \\ -\bar{B}(\bar{k}) & A(k) \end{pmatrix} \tag{2.12}$$

and thus, due to the normalizations, $S(k) = \Phi_1(0,0,k) = \mu_1(0,0,k)$.

Considering $g_0(t)$ and $g_1(t)$, $t \in [0,\infty)$ as *arbitrary* smooth functions satisfying $g_0(t) - a\mathrm{e}^{2\mathrm{i}\omega t} \to 0$, $g_1(t) - c\,\mathrm{e}^{2\mathrm{i}\omega t} \to 0$ when $t \to +\infty$ fast enough, the integral equations (2.9a) and (2.9b) yield the following analytical properties for A and B, cf. [7]:

- $A(k)$ and $B(k)$ are analytic in $\{k \mid \operatorname{Im}\Omega(k) > 0\}$ and continuous up to the boundary except possibly at points of the set
$$P = \Big\{\text{zeros of } \Omega(k), \frac{\mathrm{i}\bar{c}}{2a}, \frac{-\mathrm{i}c}{2a}\Big\}. \tag{2.13}$$
- $A(k) = 1 + \mathrm{O}(1/k)$ and $B(k) = \mathrm{O}(1/k)$ for $k \to \infty$.
- $A(k)\bar{A}(\bar{k}) + B(k)\bar{B}(\bar{k}) = 1$ for $\operatorname{Im}\Omega(k) = 0$.
  If, in addition, k is on a branch cut, this equality is to be understood for the limiting values from each side of the cut, i.e., $A_\pm(k)\bar{A}_\pm(\bar{k}) + B_\pm(k)\bar{B}_\pm(\bar{k}) = 1$.

2.6. **A non-vanishing lemma.**

Definition 1. The contour Σ is defined as the boundary of the domain $\{k \mid \operatorname{Im}\Omega(k) > 0\}$, i.e.,

$$\Sigma = \{k \mid \operatorname{Im}\Omega(k) = 0\} \cup \{\text{branch cuts}\}, \tag{2.14}$$

where the "branch cuts" are finite arcs connecting the roots of $\Omega^2(k)$ and the points $\{\frac{\mathrm{i}\bar{c}}{2a}, -\frac{\mathrm{i}c}{2a}\}$. We also define

$$\Sigma_0 = \{k \mid \operatorname{Im}\Omega(k) = 0 \text{ and } k \text{ is on a branch cut}\} \subset \Sigma.$$

Thus, for $k \in \Sigma_0$ we have $\operatorname{Im}\Omega(k) = 0$ whereas for all k near (but away from) Σ_0, we have $\operatorname{Im}\Omega(k) > 0$, see Figure 1.

The following property of the ratio of B to A plays an important role in our further considerations.

Lemma 1. *If $\Sigma_0 \neq \varnothing$, then the difference $\big(\frac{B(k)}{A(k)}\big)_+ - \big(\frac{B(k)}{A(k)}\big)_-$ of the limiting values of $\frac{B}{A}$ along Σ_0 cannot identically vanish on Σ_0.*

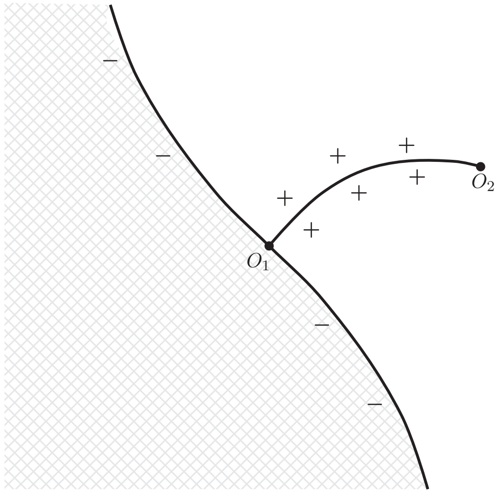

FIGURE 1. A fragment of Σ, where the arc (O_1, O_2) is a part of Σ_0, and the signs $+$ and $-$ mark domains with $\operatorname{Im}\Omega(k) > 0$ and $\operatorname{Im}\Omega(k) < 0$, respectively.

Proof. Clearly, it is the jumps of $E(k)$ across Σ_0 that yield the jumps of A and B. Now observe that the limiting values $(\mu_1)_\pm(0,t,k)$ can be expressed as follows:

$$(\mu_1)_\pm(0,t,k) = \nu(t,k)\mathcal{E}_\pm(t,k), \tag{2.15}$$

where $\nu(t,k)$, $k \in \Sigma_0$ is the solution of the integral equation

$$\nu(t,k) = I - \int_t^\infty \Psi(t,k)\Psi^{-1}(\tau,k)(\tilde{Q} - \tilde{Q}_0)(0,\tau,k)\nu(\tau,k)\Psi(\tau,k)\Psi^{-1}(t,k)\mathrm{d}\tau. \tag{2.16}$$

Notice that since $\tilde{Q}_0$ is a polynomial in k, $\Psi(t,k)\Psi^{-1}(\tau,k)$ and its inverse are entire functions in k. Thus we have

$$\begin{pmatrix} B(k) \\ A(k) \end{pmatrix}_\pm = \nu(0,k)\begin{pmatrix} E_{12}(k) \\ E_{22}(k) \end{pmatrix}_\pm$$

and hence

$$\left(\frac{B(k)}{A(k)}\right)_\pm = \frac{\nu_{11}H_\pm + \nu_{12}(2\mathrm{i}ak + \bar{c})}{\nu_{21}H_\pm + \nu_{22}(2\mathrm{i}ak + \bar{c})}. \tag{2.17}$$

Taking into account that

$$H_+ - H_- = \Omega_+ - \Omega_- = 2\Omega_+$$

and that $\det\nu \equiv 1$, it follows from (2.17) that

$$\left(\frac{B(k)}{A(k)}\right)_+ - \left(\frac{B(k)}{A(k)}\right)_- = \frac{2(2\mathrm{i}ak + \bar{c})\Omega_+(k)}{(\nu_{21}H_+ + \nu_{22}(2\mathrm{i}ak + \bar{c}))(\nu_{21}H_- + \nu_{22}(2\mathrm{i}ak + \bar{c}))} \tag{2.18}$$

and thus $\left(\frac{B(k)}{A(k)}\right)_+ - \left(\frac{B(k)}{A(k)}\right)_- \not\equiv 0$ for $k \in \Sigma_0$. $\square$

In view of the specific form of $E(k)$, the spectral functions $A(k)$ and $B(k)$ may have various singular behaviors at $k \in P$, see (2.13). Instead of describing all possible cases corresponding to arbitrary $\{a, \omega, c\}$, we will return to this point latter, in Section 5, after we have studied the restrictions on the set of parameters imposed by the fact that $\{q_0, g_0, g_1\}$ must be consistent.

Similarly, one can, in principle, formulate and study the inverse map $\{A, B\} \mapsto \{g_0, g_1\}$ in the general case, but we will do this also only for the value of parameters "selected" by the consistency condition in the form of the global relation.

3. Admissible sets of parameters

3.1. **A contraint on Σ_0 coming from the global relation.** The global relation is the characterization, in terms of the spectral functions, of the fact that the set of functions $\{q_0, g_0, g_1\}$ can indeed be realized as initial and boundary values of some solution of the NLS equation in the quarter plane $x > 0$, $t > 0$. The derivation and the form of this relation repeat those in the case of decaying boundary values [16]. Assuming that $q(x,t)$ satisfies the NLS equation, observe that the scattering relations (2.11) and (2.12) are valid for all (x,t):

$$\Phi_3(x,t,k) = \Phi_2(x,t,k) \begin{pmatrix} \bar{a}_s(\bar{k}) & b_s(k) \\ -\bar{b}_s(\bar{k}) & a_s(k) \end{pmatrix}, \tag{3.1a}$$

$$\Phi_1(x,t,k) = \Phi_2(x,t,k) \begin{pmatrix} \bar{A}(\bar{k}) & B(k) \\ -\bar{B}(\bar{k}) & A(k) \end{pmatrix}. \tag{3.1b}$$

Evaluating Φ_3 at $x = 0$, $t = T$ using the scattering relations (3.1) gives

$$\mu_3(0,T,k)\mathrm{e}^{-2\mathrm{i}k^2T\sigma_3} = \mu_1(0,T,k)\mathrm{e}^{-\mathrm{i}(\Omega(k)-\omega)T\sigma_3}S^{-1}(k)s(k).$$

Passing to the limit $T \to +\infty$ we obtain, in particular, that the (1,2) entry of $S^{-1}(k)s(k)$ must vanish for all k such that $\operatorname{Im} k > 0$ and $\operatorname{Im}\Omega(k) > 0$:

$$A(k)b_s(k) - a_s(k)B(k) = 0, \quad \text{if } \operatorname{Im} k > 0 \text{ and } \operatorname{Im}\Omega(k) > 0. \tag{3.2}$$

As we noticed in Section 1, the *global relation* (3.2) can be used, in principle, to construct directly $q_x(0,t)$ from a given pair of functions $q(x,0)$ and $q(0,t)$, but this construction is complicated, so that, in particular, it is unclear how it can be used for studying the long time asymptotics of $q_x(0,t)$. However, it turns out that certain simple *consequences* of (3.2) may be helpful in this study.

Proposition 1. *Let Σ be the contour* (2.14) *associated with the spectral functions corresponding to the set $\{q(x,0), q(0,t), q_x(0,t)\}$ of initial and boundary data defined by some solution $q(x,t)$ of the NLS$_+$ equation in the first quadrant $x > 0$, $t > 0$. Then $\Sigma_0 \cap \{k \mid \operatorname{Im} k > 0\} = \varnothing$.*

Proof. The proof follows immediately from Lemma 1, equation (3.2), and from the observation that $b_s(k)$ and $a_s(k)$ are continuous in the upper halfplane $\operatorname{Im} k > 0$. □

Remark. Since $\Omega(k) \sim 2k^2 + \omega$ for $k \to \infty$, both regions $\operatorname{Im}\Omega(k) > 0$ and $\operatorname{Im} k^2 > 0$ have "asymptotically" (for large k) the same structure. Thus the domain $\{k \mid \operatorname{Im} k > 0,\ \operatorname{Im}\Omega(k) > 0\}$ can be viewed as a "deformation" of the first quadrant $\{k \mid \operatorname{Re} k > 0,\ \operatorname{Im} k > 0\}$ in the finite plane.

3.2. **Admissible parameter sets.** Now let us analyze how Proposition 1 can help in selecting admissible parameter sets $\{a, \omega, c\}$. In view of (2.5) it is useful to represent the set $\operatorname{Im}\Omega(k) = 0$ as the intersection of the sets defined by $\operatorname{Im}\Omega^2(k) = 0$ and $\operatorname{Re}\Omega^2(k) \geq 0$. Indeed, the equality $\operatorname{Im}\Omega^2(k) = 0$ gives (infinite) curves whereas the inequality $\operatorname{Re}\Omega^2(k) \geq 0$ selects parts of these curves, eventually leading to the formation of finite arcs in the resulting contour. From (2.5) it follows that the set $\{k \mid \operatorname{Im}\Omega^2(k) = 0\}$ consists of several infinite arcs:

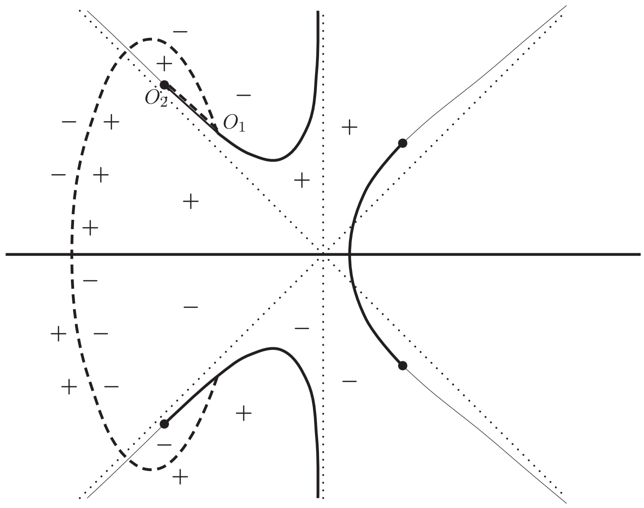

FIGURE 2. Case $c_2 > 0$: The bold dots are the zeros of $\Omega^2(k)$. The contour $\operatorname{Im}\Omega(k) = 0$ (bold full lines) is continuated by thin dashed lines, where $\operatorname{Im}\Omega^2(k) = 0$ and $\operatorname{Re}\Omega^2(k) < 0$. The part of the contour $\operatorname{Im}\Omega(k) = 0$ in $\{k \mid \operatorname{Re} k > 0\}$ may degenerate to a point. A part of the branch cut (bold dashed arcs) connecting zeros of $\Omega^2(k)$ with $\operatorname{Re} k < 0$ is chosen in such a way that $(O_1, O_2) \in \Sigma_0 \cap \{k \mid \operatorname{Im} k > 0\}$ is nonempty.

one of them is always the real axis $\{k \mid k_2 = 0\}$ (here we adopt the notation $k = k_1 + \mathrm{i}k_2$, $k_j \in \mathbb{R}$), the others are described by the equation

$$16k_1(k_1^2 - k_2^2) + 8k_1\omega + \beta = 0. \tag{3.3}$$

The set $\{k \mid \operatorname{Re}\Omega^2(k) \geq 0\}$ is characterized by the inequality

$$4(k_1^2 - k_2^2)^2 - 16k_1^2k_2^2 + 4\omega(k_1^2 - k_2^2) + \beta k_1 + \gamma \geq 0. \tag{3.4}$$

From (3.3) we see that the form of the curves is determined only by β and ω (mostly, by their signs). This requires considering the following cases.

3.2.1. *Admissible sets $\{a, \omega, c\}$ with $\omega > 0$.*

1. $\beta < 0$ (this corresponds to $c_2 > 0$). In this case, for any c_1, there are always two distinct zeros of $\Omega(k)$ with negative real part, and we can always choose the branch cut connecting these zeros in such a way that there is a nonempty arc $(O_1, O_2) \in \Sigma_0 \cap \{k \mid \operatorname{Im} k > 0\}$, see Figure 2. Therefore, in view of Proposition 1, this case is non-admissible.

   Remark. Notice that Figure 2 shows another such arc — provided that the two zeros of $\Omega(k)$ with positive real part are different and that another part of a branch cut is made along the part of the curve $\operatorname{Im}\Omega(k) = 0$ connecting these zeros. However, these zeros may collide to a double, real zero and in that case no branch cut appears in the quadrant $\{k \mid \operatorname{Im} k > 0, \operatorname{Re} k > 0\}$.

2. $\beta > 0$ ($c_2 < 0$). In this case, the contour $\operatorname{Im}\Omega^2(k) = 0$ is symmetric to the contour of the previous case (with $\beta < 0$) with respect to the imaginary axis. Thus one can always

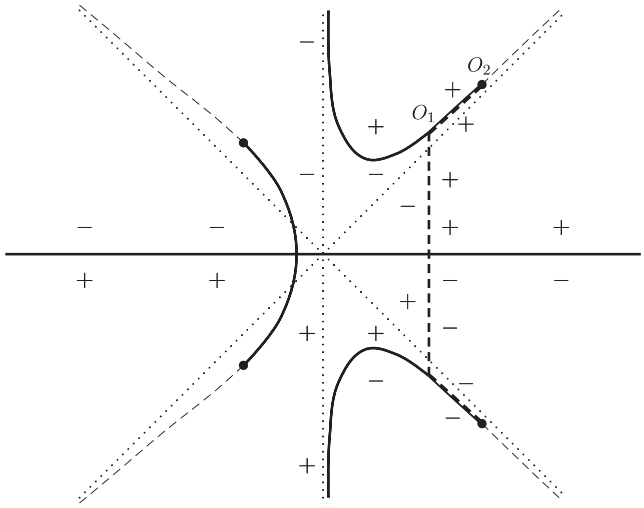

FIGURE 3. Case $c_2 < 0$: The bold dots are the zeros of $\Omega^2(k)$. The contour $\operatorname{Im}\Omega(k) = 0$ (bold full lines) is continuated by thin dashed lines, where $\operatorname{Im}\Omega^2(k) = 0$ and $\operatorname{Re}\Omega^2(k) < 0$. A part of the branch cut (bold dashed arcs) connecting zeros of $\Omega^2(k)$ with $\operatorname{Re} k > 0$ is chosen in such a way that $(O_1, O_2) \in \Sigma_0 \cap \{k \mid \operatorname{Im} k > 0\}$ is nonempty.

choose a branch cut connecting the (distinct) zeros (now with $\operatorname{Re} k > 0$) in such a way that there is a nonempty arc $(O_1, O_2) \in \Sigma_0 \cap \{k \mid \operatorname{Im} k > 0\}$, see Figure 3. Therefore, in view of Proposition 1, this case is non-admissible.

3. $\beta = 0$ ($c_2 = 0$). In this case, the contour $\operatorname{Im}\Omega^2(k) = 0$ consists of the real axis, the imaginary axis, and the branches of the hyperbola $k_1^2 - k_2^2 = -\frac{\omega}{2}$.
   (a) For $0 < \omega < \frac{a^2}{2}$, a non-admissible arc (O_1, O_2) (a part of the hyperbola lying in the first quadrant) is present for all c_1, see Figure 4.

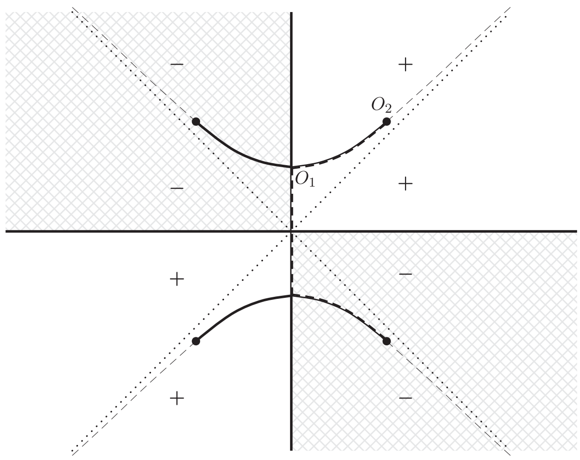

FIGURE 4. Case $c_2 = 0$, $0 < \omega < \frac{a^2}{2}$.

(b) For $\omega = \frac{a^2}{2}$, the only admissible case is $c_1 = 0$, when the zeros of $\Omega^2(k)$ with positive imaginary part collide to a point on the imaginary axis.

(c) For $\omega > \frac{a^2}{2}$, the arcs on the hyperbola are absent for all c_1 such that $c_1^2 \le a^2(2\omega - a^2)$. However, for $c_1^2 < a^2(2\omega - a^2)$, there are four distinct zeros of $\Omega^2(k)$ on the imaginary axis, and thus there are two gaps in the set $\operatorname{Im}\Omega(k) = 0$ on the imaginary axis. In this case, we can choose the branch cuts connecting these zeros as shown in Figure 5, and thus obtain non-admissible arcs. The only admissible case appears to be that with $c_1^2 = a^2(2\omega - a^2)$, where the zeros collide to the points $\pm \mathrm{i}\sqrt{\frac{\omega}{2}}$, which are the double roots of Ω^2, and thus $\Sigma = \{k \mid \operatorname{Im} k^2 = 0\}$.

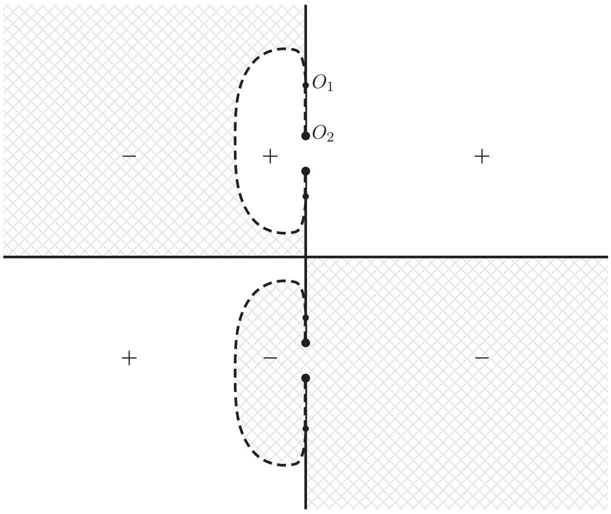

FIGURE 5. Contour Σ in the case $c_2 = 0$, $c_1^2 < a^2(2\omega - a^2)$.

3.2.2. *Admissible sets $\{a, \omega, c\}$ with $\omega < 0$.*

1. $\beta < 0$. In contrast with the case $\omega > 0$, three (qualitatively different) forms of the curves $\operatorname{Im}\Omega^2(k) = 0$ may be realized, depending on the relation between ω and β.

   (a) $-\frac{3}{4\sqrt[3]{2}}|\beta|^{\frac{2}{3}} < \omega < 0$. In this case, the form of the set $\operatorname{Im}\Omega(k) = 0$ is similar to that in the case $\omega > 0$, $\beta < 0$ and thus this case is non-admissible for the same reasons.

   (b) $\omega \le -\frac{3}{4\sqrt[3]{2}}|\beta|^{\frac{2}{3}}$ (at $\omega = -\frac{3}{4\sqrt[3]{2}}|\beta|^{\frac{2}{3}}$, the semi-infinite arcs (in $\operatorname{Re} k < 0$; cf. Figure 2)) touch the real axis). The zeros with negative real part (and thus the branch cut connecting these zeros) are now in the "safe" regions of the k-plane, see Figure 6 (which corresponds to $\omega < -\frac{3}{4\sqrt[3]{2}}|\beta|^{\frac{2}{3}}$); in these cases, it is the absence/presence of the arc in $\operatorname{Im}\Omega(k) = 0$ connecting the other two zeros of $\Omega(k)$ having positive real part (similar to that in Figure 2) that distinguishes admissible/nonadmissible cases. More precisely, the case may be admissible only if $\Omega^2(k)$ has a double, positive zero; the analysis of (3.4) shows that this is the case only if $c_1 = 0$ and $c_2 = 2a\sqrt{\frac{a^2-\omega}{2}}$. Then the corresponding range of admissible ω becomes $\omega \le -3a^2$, and we arrive at the parameters adopted, by *assumption*, in [3, 4].

2. $\beta > 0$. Similarly to the case $\omega > 0$, the contours $\operatorname{Im}\Omega(k) = 0$ are symmetric to those considered above (with $\omega < 0$, $\beta < 0$) and thus all these cases are non-admissible, for all c_1, due to the presence of a non-admissible arc connecting distinct zeros of $\Omega(k)$ with positive real parts.
3. $\beta = 0$. In this case, in contrast with the case $\omega > 0$, $\beta = 0$, for any c_1 there are four distinct zeros of $\Omega^2(k) = 0$ and thus there is always a non-admissible arc (a part of the hyperbola $k_1^2 - k_2^2 = -\frac{\omega}{2}$ lying in the quadrant $k_1 > 0$, $k_2 > 0$).

3.2.3. *Admissible sets* $\{a, \omega, c\}$ *with* $\omega = 0$.

1. For $\beta \neq 0$, the curves $\operatorname{Im}\Omega(k) = 0$ have similar structures to those in the case $\omega < 0$ and thus are non-admissible (for all c_1).
2. For $\beta = 0$, similar to the case $\omega < 0$, $\beta = 0$, for any c_1 there are four distinct zeros of $\Omega^2(k) = 0$ and thus there is always a non-admissible arc (a part of the cross $k_1^2 - k_2^2 = 0$ lying in the quadrant $k_1 > 0$, $k_2 > 0$).

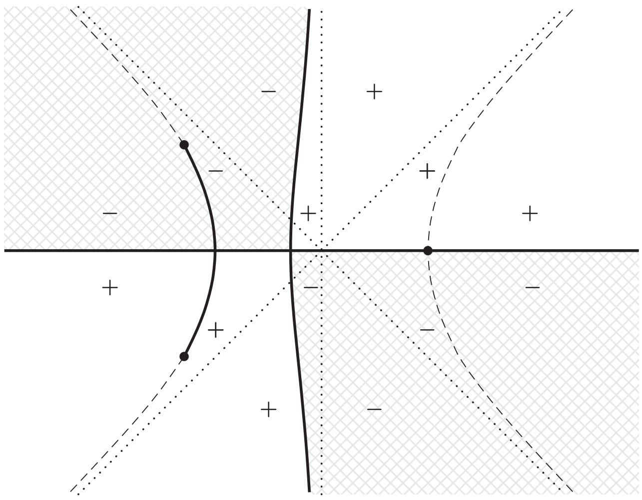

FIGURE 6. Contour $\operatorname{Im}\Omega(k) = 0$ in the case $c_2 > 0$, $\omega < -3a^2$.

3.2.4. *Admissible sets* $\{a, \omega, c\}$. The above considerations lead us to the following theorem.

Theorem 2 (admissible parameter sets)**.** *Let* $q(x,t)$ *be the solution of the IBV problem* (1.3), *where*

(a) $g_0(t) - a\mathrm{e}^{2\mathrm{i}\omega t} \to 0$ *when* $t \to +\infty$ *sufficiently fast, with* $a > 0$ *and* $\omega \in \mathbb{R}$,

under the assumption that

(b) $q_x(0,t) - c\,\mathrm{e}^{2\mathrm{i}\omega t} \to 0$ *when* $t \to +\infty$, *for some* $c \in \mathbb{C}$.

Then the admissible values of $\{a, \omega, c\}$ *are described as follows:*

$\omega \leq -3a^2$	$\omega \geq \frac{a^2}{2}$
$c = 2\mathrm{i}a\sqrt{\frac{a^2-\omega}{2}}$	$c = \pm a\sqrt{2\omega - a^2}$

Proof. Actually, so far we have proven that the parameter values other than those described in Theorem 2 are non-admissible. To complete the proof, i.e., to prove that the described values are indeed admissible, we notice again that the global relation (3.2) is a *characterization* of admissible triples $\{q_0, g_0, g_1\}$. Hence, the required $q(x,t)$ can be constructed starting from the spectral functions satisfying the analytic properties stated in Section 2 *plus* the global relation (3.2) (obviously, there are many of such spectral functions) and proceeding by constructing (and solving) the master Riemann–Hilbert problem, see Section 6, the solution of which gives $q(x,t)$. □

4. Asymptotics in the range $\omega \geq a^2/2$

In what follows, we will concentrate on the admissible cases with $\omega \geq \frac{a^2}{2}$.

The contour Σ for the associated RH problem is given by $\operatorname{Im} k^2 = 0$, i.e., it is the same as in the case of decaying boundary values [16].

On the other hand, specific features related to non-decaying boundary values are reflected in the jump conditions and singularities on the imaginary axis $\operatorname{Re} k = 0$ which is a part of the contour.

In the next sections we proceed as follows:

- First, we describe the inverse spectral mapping for the t-equation (2.2b) in terms of the associated RH problem.
- Then, we present the construction of the RH problem whose solution gives the solution $q(x,t)$ of the IBV problem. Finally, we study the consequences of this construction for the long-time asymptotics of $q(x,t)$.

5. Inverse t-map

Now we can specify all spectral objects introduced above by fixing cuts and branches. The branches of roots are fixed by the large-k behavior (2.7), and cuts are intervals of the imaginary axis connecting the zeros of $\Omega(k)$ and the points $\pm\frac{\mathrm{i}c}{2a}$, where $c = \pm a\sqrt{2\omega - a^2}$. One can distinguish the following cases.

1. $\omega = \frac{a^2}{2}$, $c = 0$.
   In this case, $\Omega(k) = 2k^2 + \frac{a^2}{2}$, $H = a^2$, the points $\pm\frac{\mathrm{i}c}{2a}$ collide to 0, $\sqrt{2\Omega - H} = 2k$, and $\sqrt{\Omega}$ is defined on the plane with the cut connecting $-\frac{\mathrm{i}a}{2}$ and $\frac{\mathrm{i}a}{2}$.
2. $\omega > \frac{a^2}{2}$, $c = \pm a\sqrt{2\omega - a^2}$.
   In this case, $\Omega(k) = 2k^2 + \omega$, $H = a^2$, $2\Omega - H = 4k^2 + 2\omega - a^2$, the points $\pm\frac{\mathrm{i}c}{2a}$ are zeros of $2\Omega - H$, and the cuts are from $-\mathrm{i}\sqrt{\frac{\omega}{2}}$ to $-\mathrm{i}\sqrt{\frac{\omega}{2} - \frac{a^2}{4}}$ and from $\mathrm{i}\sqrt{\frac{\omega}{2}}$ to $\mathrm{i}\sqrt{\frac{\omega}{2} - \frac{a^2}{4}}$.

The formulation of the inverse scattering mapping as a RH problem follows the same lines as in the case of decaying boundary values [16]. It is motivated by the conjugation conditions, across Σ, for particularly chosen eigenvalues.

Genericity Assumptions. We assume the following genericity conditions:

(i) $A(k)$ has at most a finite number of zeros λ_j, $j = 1, \dots, l$ in $\mathbb{C} \setminus \Sigma$.
(ii) If $c \neq 0$, we assume that $A(k)\sqrt{\frac{2\Omega(k)}{2\Omega(k) - H(k)}}$ has limits $\neq 0, \infty$ at all $k \in \Sigma$, $k \neq \frac{\mathrm{i}c}{2a}$.
(iii) If $c = 0$, we assume also that $A(0) \neq 0$.

Introduce, in the framework of the direct problem, a piecewise meromorphic function

$$M^{(t)}(t,k)=\begin{cases}\begin{pmatrix}\Phi_1^{(1)} & \frac{\Phi_2^{(2)}}{\bar A}\end{pmatrix}\mathrm{e}^{-\mathrm{i}(\Omega(k)-\omega)t\sigma_3}, & \text{for } \operatorname{Im}k^2<0,\\ \begin{pmatrix}\frac{\Phi_2^{(1)}}{A} & \Phi_1^{(2)}\end{pmatrix}\mathrm{e}^{-\mathrm{i}(\Omega(k)-\omega)t\sigma_3}, & \text{for } \operatorname{Im}k^2>0.\end{cases} \tag{5.1}$$

Then direct calculations show that $M^{(t)}$ satisfies the following jump on $\Sigma=\{k\mid \operatorname{Im}k^2=0\}$:

$$M_-^{(t)}(t,k)=M_+^{(t)}(t,k)J^{(t)}(t,k), \tag{5.2}$$

where the subscripts $+$ and $-$ denote the limiting values from respectively $\operatorname{Im}k^2>0$ and $\operatorname{Im}k^2<0$, and the jump matrix $J^{(t)}(t,k)$ is defined as follows:

(i) for all $k\in\mathbb{R}$ and all $k\in\mathrm{i}\mathbb{R}\setminus\left\{\left[-\mathrm{i}\sqrt{\frac{\omega}{2}},-\mathrm{i}\sqrt{\frac{\omega}{2}-\frac{a^2}{4}}\right]\cup\left[\mathrm{i}\sqrt{\frac{\omega}{2}-\frac{a^2}{4}},\mathrm{i}\sqrt{\frac{\omega}{2}}\right]\right\}$,

$$J^{(t)}(t,k)=\begin{pmatrix}1 & -\frac{B(k)}{\bar A(k)}\mathrm{e}^{-2\mathrm{i}(\Omega(k)-\omega)t}\\ -\frac{\bar B(\bar k)}{A(k)}\mathrm{e}^{2\mathrm{i}(\Omega(k)-\omega)t} & \frac{1}{A(k)\bar A(\bar k)}\end{pmatrix}; \tag{5.3}$$

(ii) for $k\in\left(-\mathrm{i}\sqrt{\frac{\omega}{2}},-\mathrm{i}\sqrt{\frac{\omega}{2}-\frac{a^2}{4}}\right)\cup\left(\mathrm{i}\sqrt{\frac{\omega}{2}-\frac{a^2}{4}},\mathrm{i}\sqrt{\frac{\omega}{2}}\right)$,

$$J^{(t)}(t,k)=-\begin{pmatrix}1 & -\frac{B_-(k)}{\bar A_-(k)}\mathrm{e}^{-2\mathrm{i}(\Omega(k)-\omega)t}\\ -\frac{\bar B_+(\bar k)}{A_+(k)}\mathrm{e}^{2\mathrm{i}(\Omega(k)-\omega)t} & \frac{1}{A_+(k)\bar A_-(\bar k)}\end{pmatrix}. \tag{5.4}$$

The properties of the eigenfunctions and spectral functions of the direct spectral problem $\{g_0,g_1\}\mapsto\{A,B\}$ suggest defining the inverse map $\{A,B\}\mapsto\{g_0,g_1\}$ in terms of the following RH problem: given $A(k)$ and $B(k)$ such that A satisfies the genericity assumptions on its zeros as above, find a 2×2 matrix-valued function $M^{(t)}(t,k)$ such that:

- $M^{(t)}(t,k)$ is meromorphic in $\mathbb{C}\setminus\Sigma$, $\Sigma=\{k\mid\operatorname{Im}k^2>0\}$ and continuous up to the boundary for $k\in\Sigma\setminus P$, where $P=\{\mathrm{i}\sqrt{\frac{\omega}{2}},-\mathrm{i}\sqrt{\frac{\omega}{2}},\mathrm{i}\sqrt{\frac{\omega}{2}-\frac{a^2}{4}},-\mathrm{i}\sqrt{\frac{\omega}{2}-\frac{a^2}{4}}\}$;
- for $k\in\Sigma\setminus P$, the limiting values $M_+^{(t)}(t,k)$ and $M_-^{(t)}(t,k)$ as k is approached from $\{k\mid \operatorname{Im}k^2>0\}$ and $\{k\mid\operatorname{Im}k^2<0\}$ respectively, are related by (5.2), where the jump matrix $J^{(t)}$ is given by (5.3)–(5.4);
- $M^{(t)}(t,k)$ has simple poles at $k=\lambda_j$ with $\operatorname{Im}\lambda_j^2>0$ and at $k=\bar\lambda_j$, $j=1,\dots,l$ satisfying the residue conditions (cf. [16])

$$\operatorname{res}_{k=\bar\lambda_j}[M^{(t)}(t,k)]^{(1)}=-\mathrm{e}^{-2\mathrm{i}(\Omega(\bar\lambda_j)-\omega)t}\frac{1}{\bar B(\bar\lambda_j)\dot{\bar A}(\bar\lambda_j)}[M^{(t)}(t,\bar\lambda_j)]^{(2)},$$
$$\operatorname{res}_{k=\lambda_j}[M^{(t)}(t,k)]^{(2)}=\mathrm{e}^{2\mathrm{i}(\Omega(\lambda_j)-\omega)t}\frac{1}{B(\lambda_j)\dot A(\lambda_j)}[M^{(t)}(t,\lambda_j)]^{(1)},$$

  where the dot denotes the derivative with respect to k;
- $M^{(t)}(t,k)\to I$ for $k\to\infty$;
- as k approaches a point from P, $M^{(t)}(t,k)$ satisfies the following boundedness conditions:

1. $\omega = \frac{a^2}{2}$, $c = 0$.
   In this case, we have generically $A(k) \sim A_\pm(k^2+\frac{a^2}{4})^{-\frac{1}{2}}$ for $k \to \pm\frac{\mathrm{i}a}{2}$, with some $A_\pm \neq 0$,
$$M^{(t)}(t,k)\begin{pmatrix}(k^2+\frac{a^2}{4})^{-\frac{1}{2}} & 0\\ 0 & (k^2+\frac{a^2}{4})^{\frac{1}{2}}\end{pmatrix} \text{ is bounded for } k \to \pm\frac{\mathrm{i}a}{2},\ \pm\operatorname{Im}k^2 > 0, \tag{5.5}$$
$$M^{(t)}(t,k)\begin{pmatrix}(k^2+\frac{a^2}{4})^{\frac{1}{2}} & 0\\ 0 & (k^2+\frac{a^2}{4})^{-\frac{1}{2}}\end{pmatrix} \text{ is bounded for } k \to \pm\frac{\mathrm{i}a}{2},\ \mp\operatorname{Im}k^2 > 0. \tag{5.6}$$
   Also, $M^{(t)}(t,k)$ is bounded as $k \to 0$.
2. $\omega > \frac{a^2}{2}$, $c = \pm a\sqrt{2\omega - a^2}$.
   In this case, we have generically $A(k) \sim A_\pm(k^2+\frac{\omega}{2})^{-\frac{1}{2}}$ for $k \to \pm\mathrm{i}\sqrt{\frac{\omega}{2}}$, with some $A_\pm \neq 0$,
$$M^{(t)}(t,k)\begin{pmatrix}(k^2+\frac{\omega}{2})^{-\frac{1}{2}} & 0\\ 0 & (k^2+\frac{\omega}{2})^{\frac{1}{2}}\end{pmatrix} \text{ is bounded for } k \to \pm\mathrm{i}\sqrt{\frac{\omega}{2}},\ \pm\operatorname{Im}k^2 > 0, \tag{5.7}$$
$$M^{(t)}(t,k)\begin{pmatrix}(k^2+\frac{\omega}{2})^{\frac{1}{2}} & 0\\ 0 & (k^2+\frac{\omega}{2})^{-\frac{1}{2}}\end{pmatrix} \text{ is bounded for } k \to \pm\mathrm{i}\sqrt{\frac{\omega}{2}},\ \mp\operatorname{Im}k^2 > 0. \tag{5.8}$$
   Also, $A(k) \sim A_-(k+\frac{\mathrm{i}c}{2a})^{1/2}$ for $k \to -\frac{\mathrm{i}c}{2a}$ with some $A_- \neq 0$, and
$$M^{(t)}(t,k)\begin{pmatrix}(k+\frac{\mathrm{i}c}{2a})^{\frac{1}{2}} & 0\\ 0 & (k+\frac{\mathrm{i}c}{2a})^{-\frac{1}{2}}\end{pmatrix} \text{ is bounded for } k \to -\frac{\mathrm{i}c}{2a} \tag{5.9}$$
   with $\operatorname{sign} c \cdot \operatorname{Re} k < 0$ and $\operatorname{Im} k^2 > 0$,
$$M^{(t)}(t,k)\begin{pmatrix}(k-\frac{\mathrm{i}c}{2a})^{-\frac{1}{2}} & 0\\ 0 & (k-\frac{\mathrm{i}c}{2a})^{\frac{1}{2}}\end{pmatrix} \text{ is bounded for } k \to \frac{\mathrm{i}c}{2a} \tag{5.10}$$
   with $\operatorname{sign} c \cdot \operatorname{Re} k < 0$ and $\operatorname{Im} k^2 < 0$.

Let $M^{(t)}(t,k)$ be the unique solution of this RH problem. Then $g_0(t)$ and $g_1(t)$ are given as follows:
$$\begin{aligned} g_0(t) &= 2\mathrm{i}\lim_{k\to\infty} kM^{(t)}_{12}(t,k),\\ g_1(t) &= \lim_{k\to\infty}[4k^2M^{(t)}_{12}(t,k) + 2\mathrm{i}g_0(t)kM^{(t)}_{22}(t,k)]. \end{aligned}$$

Remark. The uniqueness of the solution of the RH problem is guaranteed by the boundedness conditions stated above. Notice that the conditions of type (5.5)–(5.10) imply that the sought solution is not in $L^2(\Sigma)$, which is a standard functional class for boundary values for RH problems in question (see, e.g., [10]). Nevertheless, these conditions secure the ratio $M_1M_2^{-1}$ of two possible solutions to be bounded at P, so that standard arguments allow concluding that $M_1M_2^{-1} \equiv I$.

6. RH problem for $q(x,t)$

The construction of the RH problem, whose solution is directly used to express the solution of the NLS equation, is motivated, similarly to the cases of the x and t-equations, by the analytical properties of the associated eigenfunctions for all (x,t).

Assume that $q(x,t)$ is given, define $d(k)$ in terms of the associated spectral functions:

$$d(k) := a_s(k)\bar{A}(\bar{k}) + b_s(k)\bar{B}(\bar{k}), \qquad k \in \bar{D}_2, \tag{6.1}$$

where $D_2 = \{k \mid \operatorname{Im} k^2 < 0, \operatorname{Im} k > 0\}$, and assume that $d(k)$ has at most a finite number of simple zeros $\mu_j \in D_2$, $j = 1, \dots, m$ (genericity assumption). Define a piecewise meromorphic function (cf. [16])

$$M(x,t,k) := \tilde{M}(x,t,k)\mathrm{e}^{\mathrm{i}(kx+2k^2t)\sigma_3}, \tag{6.2}$$

where

$$\tilde{M}(x,t,k) = \begin{cases} \begin{pmatrix} \dfrac{\Phi_2^{(1)}}{a_s} & \Phi_3^{(2)} \end{pmatrix}, & \text{for } \operatorname{Im} k > 0, \operatorname{Im} k^2 > 0 \\ \begin{pmatrix} \dfrac{\Phi_1^{(1)}}{d} & \Phi_3^{(2)} \end{pmatrix}, & \text{for } \operatorname{Im} k > 0,\ \operatorname{Im} k^2 < 0 \\ \begin{pmatrix} \Phi_3^{(1)} & \dfrac{\Phi_1^{(2)}}{\bar{d}} \end{pmatrix}, & \text{for } \operatorname{Im} k < 0,\ \operatorname{Im} k^2 > 0 \\ \begin{pmatrix} \Phi_3^{(1)} & \dfrac{\Phi_2^{(2)}}{\bar{a}_s} \end{pmatrix}, & \text{for } \operatorname{Im} k < 0,\ \operatorname{Im} k^2 < 0 \end{cases} \tag{6.3}$$

Then the scattering relations (3.1a) and (3.1b) imply that the limiting values of M on $\operatorname{Im} k^2 = 0$ satisfy the jump relations

$$M_-(x,t,k) = M_+(x,t,k)\mathrm{e}^{-\mathrm{i}(kx+2k^2t)\sigma_3}J_0(k)\mathrm{e}^{\mathrm{i}(kx+2k^2t)\sigma_3}, \tag{6.4}$$

where

$$J_0(k) = \begin{cases} \begin{pmatrix} 1 & -\bar{r}(k) \\ -r(k) & 1+|r(k)|^2 \end{pmatrix}, & \text{for } k > 0, \\ \begin{pmatrix} 1 & 0 \\ C_-(k) & 1 \end{pmatrix}, & \text{for } k \in \mathrm{i}\,\mathbb{R}_+, \\ \begin{pmatrix} 1 & \bar{C}_+(\bar{k}) \\ 0 & 1 \end{pmatrix}, & \text{for } k \in \mathrm{i}\,\mathbb{R}_-, \\ \begin{pmatrix} 1+|r(k)+C(k)|^2 & \bar{r}(k)+\bar{C}(k) \\ r(k)+C(k) & 1 \end{pmatrix}, & \text{for } k < 0, \end{cases} \tag{6.5}$$

where

$$r(k) = \frac{\bar{b}_s(\bar{k})}{a_s(k)}, \tag{6.6a}$$

$$C(k) = -\frac{\bar{B}(\bar{k})}{a_s(k)d(k)} \qquad \text{for } k \in \{k \mid \operatorname{Im} k \geq 0,\ \operatorname{Re} k < 0\}, \tag{6.6b}$$

$$C_-(k) = -\frac{\bar{B}_-(\bar{k})}{a_s(k)d_-(k)}, \tag{6.6c}$$

$$\bar{C}_+(\bar{k}) = -\frac{B_+(k)}{\bar{a}_s(\bar{k})\bar{d}_+(\bar{k})}. \tag{6.6d}$$

The jump relation suggests the definition of the following RH problem.

RH-problem. Given a_s, b_s, A, B satisfying the genericity assumptions and the global relation (3.2), find $M(x,t,k)$, $x \geq 0, t \geq 0$ such that:

- $M(x,t,k)$ is meromorphic in $\mathbb{C} \setminus \Sigma$.
- for $k \in \Sigma \setminus \{\frac{\mathrm{i}c}{2a}, -\frac{\mathrm{i}c}{2a}\}$, $M_\pm$ satisfy the jump relation (6.4), where $M_\pm$ are limiting values of M from $\pm \operatorname{Im} k^2 > 0$.
- $M(x,t,k) \to I$ for $k \to \infty$.
- M has simple poles in $\mathbb{C} \setminus \Sigma$ at $k = \kappa_j$ and $k = \bar{\kappa}_j$ for $\kappa_j \in \{k \mid \operatorname{Im} k > 0, \operatorname{Re} k > 0\}$ and at $k = \mu_j$ and $k = \bar{\mu}_j$ for $\mu_j \in \{\operatorname{Im} k > 0, \operatorname{Re} k < 0\}$, where it satisfies the residue relations

$$\begin{aligned}
\operatorname{res}_{k=\kappa_j} M^{(1)}(x,t,k) &= m_j^1 \mathrm{e}^{2\mathrm{i}(\kappa_j x + 2\kappa_j^2 t)} M^{(2)}(x,t,\kappa_j), \\
\operatorname{res}_{k=\mu_j} M^{(1)}(x,t,k) &= m_j^2 \mathrm{e}^{2\mathrm{i}(\mu_j x + 2\mu_j^2 t)} M^{(2)}(x,t,\mu_j), \\
\operatorname{res}_{k=\bar{\mu}_j} M^{(2)}(x,t,k) &= -\bar{m}_j^2 \mathrm{e}^{-2\mathrm{i}(\bar{\mu}_j x + 2\bar{\mu}_j^2 t)} M^{(2)}(x,t,\bar{\mu}_j), \\
\operatorname{res}_{k=\bar{\kappa}_j} M^{(2)}(x,t,k) &= -\bar{m}_j^1 \mathrm{e}^{-2\mathrm{i}(\bar{\kappa}_j x + 2\bar{\kappa}_j^2 t)} M^{(1)}(x,t,\bar{\kappa}_j),
\end{aligned} \tag{6.7}$$

  with

$$\begin{aligned}
m_j^1 &= (b_s(\kappa_j)\dot{a}_s(\kappa_j))^{-1}, \qquad & m_j^2 &= \operatorname{res}_{k=\mu_j} C(k), \\
\bar{m}_j^1 &= (\bar{b}(\bar{\kappa}_j)\dot{\bar{a}}(\bar{\kappa}_j))^{-1}, \qquad & \bar{m}_j^2 &= \operatorname{res}_{k=\bar{\mu}_j} \bar{C}(\bar{k}).
\end{aligned}$$

- If $c > 0$ and $a_s(k_0) = 0$ with $k_0 = \frac{\mathrm{i}c}{2a}$, then

$$\begin{aligned}
M^{(2)}(x,t,k_0) &= b_s(k_0)\dot{a}_s(k_0) \lim_{\substack{k \to k_0 \\ \operatorname{Re} k > 0}} (k - k_0) M^{(1)}(x,t,k), \\
M^{(1)}(x,t,-k_0) &= -\bar{b}_s(-k_0)\dot{\bar{a}}_s(-k_0) \lim_{\substack{k \to -k_0 \\ \operatorname{Re} k > 0}} (k + k_0) M^{(2)}(x,t,k).
\end{aligned} \tag{6.8}$$

- Otherwise, $M(x,t,k)$ is bounded for $k \to \pm \frac{\mathrm{i}c}{2a}$ for $\operatorname{Re} k > 0$.

Proposition 2. *Let $M(x,t,k)$ be the (unique) solution of this RH problem. Then the solution $q(x,t)$ of the IBV problem* (1.3) *for the NLS equation is given in terms of $M(x,t,k)$ by*

$$q(x,t) = 2\mathrm{i} \lim_{k \to \infty} (kM(x,t,k))_{12}\,. \tag{6.9}$$

Moreover (cf. [16]), the functions $a_s(k)$, $b_s(k)$ are the spectral functions associated with $q(x,0)$ via the direct scattering map for (2.2a) and $A(k)$, $B(k)$ are the spectral functions associated with $\{q(0,t), q_x(0,t)\}$ via the direct scattering map for (2.2b).

Remark. Under the genericity assumptions made above, the global relation (3.2) implies that $a_s(k)$ can have a zero on $\mathrm{i}\mathbb{R}$, $a_s(k_0)=0$, only if $c>0$ and $k_0=\frac{\mathrm{i}c}{2a}$.

7. Long-time asymptotics

The previous section shows that the long-time asymptotic analysis for $q(x,t)$, the solution of the IBV problem (1.3), reduces to the long-time asymptotic analysis of a solution of the associated RH problem. Assume for simplicity that there are no zeros of d and a involved in the residue conditions (6.7). The jump relation written in the form

$$M_- = M_+ \mathrm{e}^{-\mathrm{i}t\theta\sigma_3} J_0(k) \mathrm{e}^{\mathrm{i}t\theta\sigma_3},$$

where $\theta=\theta(k,\xi)=2k^2+\xi k$ with $\xi=\frac{x}{t}$, shows that for $t\to+\infty$, the jump matrix oscillates for $k\in\mathbb{R}$ whereas it decays to the identity matrix for $k\in\mathrm{i}\mathbb{R}\setminus\{0\}$. This indicates that the long-time behavior of the solution of the RH problem should be similar to that of the RH problem associated to the whole line, initial value problem [11]. Indeed, since the problem configuration is essentially the same as in the case of the IBV problem with vanishing boundary values (the contour, $\Sigma=\{k \mid \operatorname{Im} k^2=0\}$, is exactly the same, and the jump matrix has a similar structure), one can proceed in the same way as in [16]. Namely, first, deform the problem by introducing a diagonal factor, in order to have an appropriate triangular factorization of the jump matrix on $\mathbb{R}$ [11]: $M_{(1)} := M\delta^{-\sigma_3}(k,\xi)$, where

$$\delta(k,\xi)=\exp\left\{\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\kappa_0(\xi)}\frac{\log(1+|\rho(s)|^2)\mathrm{d}s}{s-k}\right\}, \qquad k\in\mathbb{C}\setminus(-\infty,\kappa_0(\xi)]$$

with $\rho(k)=r(k)+C(k)$ and $\kappa_0(\xi)=-\xi/4$. Then, using suitable rational approximations $\hat{r}$, $\hat{C}$, $\hat{\rho}$, and $\hat{\rho}_1$ of respectively r, C, ρ, and $\rho_1=\frac{\rho}{1+|\rho|^2}$, transform the problem by introducing

$M_{(2)} := M_{(1)}G(x,t,k)$, where

$$G(x,t,k)=\begin{cases}\begin{pmatrix}1 & 0\\ -\hat r(k)\delta^{-2}(k)\mathrm{e}^{2\mathrm{i}t\theta(k)} & 1\end{pmatrix}, & k\in D_1,\ \arg(k-\kappa_0)\in(0,\pi/4),\\[2ex] \begin{pmatrix}1 & \bar{\hat r}(\bar k)\delta^{2}(k)\mathrm{e}^{-2\mathrm{i}t\theta(k)}\\ 0 & 1\end{pmatrix}, & k\in D_4,\ \arg(k-\kappa_0)\in(7\pi/4,2\pi),\end{cases} \tag{7.1a}$$

$$G(x,t,k)=\begin{cases}\begin{pmatrix}1 & 0\\ \hat C(k)\delta^{-2}(k)\mathrm{e}^{2\mathrm{i}t\theta(k)} & 1\end{pmatrix}, & k\in D_1,\ \arg(k-\kappa_0)\in(\pi/4,\pi/2),\\[2ex] \begin{pmatrix}1 & -\bar{\hat C}(\bar k)\delta^{2}(k)\mathrm{e}^{-2\mathrm{i}t\theta(k)}\\ 0 & 1\end{pmatrix}, & k\in D_4,\ \arg(k-\kappa_0)\in(3\pi/2,7\pi/4),\end{cases} \tag{7.1b}$$

$$G(x,t,k)=\begin{cases}\begin{pmatrix}1 & 0\\ -\hat\rho(k)\delta^{-2}(k)\mathrm{e}^{2\mathrm{i}t\theta(k)} & 1\end{pmatrix}, & k\in D_2,\ \arg(k-\kappa_0)\in(0,\pi/4),\\[2ex] \begin{pmatrix}1 & \bar{\hat\rho}(\bar k)\delta^{2}(k)\mathrm{e}^{-2\mathrm{i}t\theta(k)}\\ 0 & 1\end{pmatrix}, & k\in D_3,\ \arg(k-\kappa_0)\in(7\pi/4,2\pi),\end{cases} \tag{7.1c}$$

$$G(x,t,k)=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\qquad k\in\begin{cases}D_2, & \pi/4<\arg(k-\kappa_0)<3\pi/4);\\ D_3, & 7\pi/4>\arg(k-\kappa_0)>5\pi/4,\end{cases} \tag{7.1d}$$

$$G(x,t,k)=\begin{cases}\begin{pmatrix}1 & -\bar{\hat\rho}_1(\bar k)\delta^{2}(k)\mathrm{e}^{-2\mathrm{i}t\theta(k)}\\ 0 & 1\end{pmatrix}, & k\in D_2,\ \arg(k-\kappa_0)\in(3\pi/4,\pi),\\[2ex] \begin{pmatrix}1 & 0\\ \hat\rho_1(k)\delta^{-2}(k)\mathrm{e}^{2\mathrm{i}t\theta(k)} & 1\end{pmatrix}, & k\in D_3,\ \arg(k-\kappa_0)\in(\pi,5\pi/4),\end{cases} \tag{7.1e}$$

where

$$D_1=\{k\mid \operatorname{Im}k>0,\operatorname{Re}k>0\},\qquad D_2=\{k\mid \operatorname{Im}k>0,\operatorname{Re}k<0\},$$
$$D_3=\{k\mid \operatorname{Im}k<0,\operatorname{Re}k<0\},\qquad D_4=\{k\mid \operatorname{Im}k<0,\operatorname{Re}k>0\}.$$

Following this way, we arrive at a RH problem on the cross centered at $k=\kappa_0$ with jump matrix decaying to the identity matrix (but not uniformly near κ_0), which is exactly the same as in [11] for the case of the initial-value problem. The latter fact implies that the asymptotics of $q(x,t)$ has the same form as in the case of the IV problem, i.e., the form of the modulated oscillations of the Zakharov–Manakov type: for all $\varepsilon>0$,

$$q(x,t)=t^{-1/2}\alpha\left(-\frac{x}{4t}\right)\exp\left\{\frac{\mathrm{i}x^2}{4t}+2\mathrm{i}\alpha^2\left(-\frac{x}{4t}\right)\log t+\mathrm{i}\phi\left(-\frac{x}{4t}\right)\right\}+o(t^{-1/2}), \tag{7.2}$$
$$t\to+\infty,\ \frac{x}{t}>\varepsilon,$$

with the amplitude α and the phase ϕ given by

$$\alpha^2(k) = \frac{1}{4\pi}\log\Big(1+|\rho(k)|^2\Big),$$
$$\phi(k) = 6\alpha^2(k)\log 2 + \frac{3\pi}{4} + \arg\rho(k) + \arg\Gamma(-2i\alpha^2(k)) + 4\int_{-\infty}^{k}\log|\mu-k|d\alpha^2(\mu),$$

where $\Gamma(z)$ denotes Euler's gamma-function.

Theorem 3. *Consider the IBV problem* (1.3)*, where $q_0(x)$ and $g_0(t)$ are smooth functions compatible at $x=0, t=0$ and such that $q_0(x)\to 0$ when $x\to+\infty$ and $g_0(t) - a\mathrm{e}^{2\mathrm{i}\omega t}\to 0$ when $t\to+\infty$ fast enough, with $a>0$ and $\omega\ge\frac{a^2}{2}$. Assume that its solution $q(x,t)$ satisfies*

$$q_x(0,t) - c\,\mathrm{e}^{2\mathrm{i}\omega t}\to 0, \qquad t\to+\infty \tag{7.3}$$

fast enough, with $c^2 = a^2(2\omega - a^2)$. Assume also that no zero of the spectral functions is involved in (6.7)*.*

Then for all $\varepsilon>0$, the principal term of the long-time behavior of $q(x,t)$ in the domain $\frac{x}{t}>\varepsilon$ is described by decaying modulated oscillations (7.2) *with parameters that can be expressed in terms of the spectral functions associated to the initial condition $q_0(x)$ and the boundary values $q(0,t)=g_0(t)$ and $q_x(0,t)$.*

Remark. Similarly to the case of the IBV problem with vanishing boundary values, zeros of $d(k)$ in D_2 generate solitons moving to the right and thus being seen in the quarter plane $x>0, t>0$. Consequently, the case with solitons can be treated in the same way as in [15], following the Zakharov–Shabat dressing procedure [18].

8. Concluding remarks

The IBV problems in the parameter range considered in this paper contain a family of exact solutions, in the form of stationary solitons. This family (parametrized by $\omega>0$ and $\varphi_0\in\mathbb{R}$) is given by

$$q(x,t)\equiv q(x,t;\omega,\varphi_0) = \frac{\sqrt{2\omega}}{\cosh(\sqrt{2\omega}x+\varphi_0)}\mathrm{e}^{2\mathrm{i}\omega t}. \tag{8.1}$$

For these solutions we have

$$q(0,t) = \frac{\sqrt{2\omega}}{\cosh\varphi_0}\mathrm{e}^{2\mathrm{i}\omega t}\equiv a\mathrm{e}^{2\mathrm{i}\omega t}, \tag{8.2}$$

where a and ω are related by $a=\sqrt{2\omega}/\cosh\varphi_0$ and thus $\omega\ge a^2/2$. The values of $q_x(x,t)$ for $x=0$ has the form

$$q_x(0,t) = \frac{-2\omega\sinh\varphi_0}{\cosh^2\varphi_0}\mathrm{e}^{2\mathrm{i}\omega t}, \tag{8.3}$$

and thus, in terms of ω and a, $q_x(0,t) = \pm a\sqrt{2\omega - a^2}\mathrm{e}^{2\mathrm{i}\omega t}$, the sign being determined by the sign of φ_0.

We infer from Theorem 3 that the long-time behavior of the Dirichlet data does not uniquely determine the long-time behavior of the Neumann boundary values: depending on a particular choice of this Dirichlet data (approaching a fixed periodic function) and on the (decaying) initial data, one can see (at least) two different asymptotic regimes for the Neumann data.

The stationary solitons provide also another illustration of this phenomenon, in the case of the Neumann boundary value problem, where, instead of the Dirichlet data, one prescribes the values of the normal derivative of the solution on the boundary $x = 0$. Indeed, for $\varphi_0 = 0$ we have $q_x(0, t) \equiv 0$ for a one-parameter family of solutions $q(x, t)$ of the NLS (parametrized by ω), with different Dirichlet values: $q(0, t) = \sqrt{2\omega}\, \mathrm{e}^{2\mathrm{i}\omega t}$.

As we noticed in Section 1, the global relation allows, in principle, to study the long-time behavior of the Neumann values, given the Dirichlet data, via the study of the nonlinear Dirichlet-to-Neumann map, which has been obtained [1] combining analytic properties stated in the global relation with the Marchenko-type integral representations for eigenfunctions of (2.2b).

In this paper, we have shown that the global relation in its simple spectral form (3.2) can be also used very efficiently for selecting admissible behaviors of the Neumann values, even without studying the Dirichlet-to-Neumann map in details. In particular, simple geometric reasons show that for all real ω, the Neumann values $q_x(0, t)$ of the solution of the IBV problem (1.3) with Dirichlet boundary conditions satisfying $g_0(t) \sim a\mathrm{e}^{2\mathrm{i}\omega t}$ when $t \to +\infty$, can (asymptotically) evaluate as $q_x(0, t) \sim c\mathrm{e}^{2\mathrm{i}\omega t}$ only for ω in the range $\omega \in (-\infty, -3a^2] \cup [\frac{a^2}{2}, \infty)$ and only for particular c as function of (a, ω).

The problem of a complete description of all possible variants for the behavior of the Neumann values, having fixed the asymptotics of the Dirichlet data, remains open. A reasonable conjecture is that finite gap solutions should be involved in such a description.

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