DIFFERENCE OPERATORS AND DETERMINANTAL POINT PROCESSES

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Dedicated to I. M. Gelfand on the occasion of his 95th birthday

0. Introduction

A point process is an ensemble of random locally finite point configurations in a space $\mathfrak{X}$. A convenient tool for dealing with point processes is provided by the correlation functions (they are similar to the moments of random variables [19]). The correlation functions are indexed by the natural numbers $n = 1, 2, \ldots$, and the nth function depends on n variables ranging over $\mathfrak{X}$. A point process is said to be determinantal [5], [26] if its correlation functions of all orders can be represented as the principal minors of a correlation kernel $K(x, y)$, a function on $\mathfrak{X} \times \mathfrak{X}$. Then the whole information about the point process is contained in this single function in two variables.

Concrete examples of determinantal point processes arise in random matrix theory, in some problems of representation theory, in combinatorial models of mathematical physics, and in other domains. A remarkable fact is that the correlation kernels of determinantal point processes from various sources reveal a similar structure.

We will be interested in limit transitions for point processes depending on a parameter. For instance, a typical problem in random matrix theory consists in the study of limit properties of the spectra of random matrices of order N as the parameter N goes to infinity. In the models of point processes arising in representation theory, the limit transitions may be related to the approximation of an infinite–dimensional group by finite–dimensional ones [23], [6].

The language of correlation functions is well adapted to studying limit transitions: the convergence of processes is controlled by the convergence of correlation functions (just as the convergence of random variables is controlled by the convergence of their moments). For determinantal processes, the situation is simplified, because the convergence of the correlation functions is ensured by the convergence of the correlation kernels. In the present note we will study just limit transitions for kernels.

Certain correlation kernels $K(x, y)$, which arise in many concrete examples from various domains, share the following common properties:

- The space $\mathfrak{X}$, on which the kernel is defined, is a subset of $\mathbb{R}$.
- The kernel can be written in the so–called integrable form, [1] which in the simplest variant looks as follows:

[1] In the sense of Its, Izergin, Korepin, and Slavnov, see [11].

$$(0.1) \qquad K(x,y) = \frac{A(x)B(y) - B(x)A(y)}{x - y} \,,$$

where A and B are certain functions on $\mathfrak{X}$.

In such a situation, the convergence of kernels depending on a parameter is often extracted from the asymptotics of the functions A and B with respect to the parameter. This obvious way, however, may require hard computations if A and B are sufficiently complex special functions. [2] In the present note, we discuss another approach, which exploits one more property of the kernels, which also holds in many concrete examples:

- The operator K in the Hilbert space $L^2(\mathfrak{X})$ (with respect to a natural measure), given by the kernel $K(x,y)$, is a projection operator. Moreover, K can be realized as a spectral projection $\mathcal{P}(\Delta)$ for a certain selfadjoint operator D acting in $L^2(\mathfrak{X})$. Here Δ is a certain part of the spectrum of the operator D, and this operator is determined by a difference or differential operator on $\mathfrak{X}$.

As soon as there is a natural link between K and D, the following simple idea arises: to deduce the convergence of the correlation kernels from that of the corresponding selfadjoint operators D. This idea was stated in [10] and then employed in [13] and [3]. The aim of the present note is to demonstrate the efficiency of such an approach on a number of other examples. I consider only rather simple examples where the point processes live on the one–dimensional lattice and not on a continuous space, and the limit transition does not require a scaling. I think, however, that the method can be useful in more complex situations, too. Some indication on this is contained in §3.4 where it is explained how one can guess, in a very simple way, the scaling leading to the Airy kernel.

The structure of the work is as follows. In §1 we recall general definitions related to point processes (for more detail, see [19], [26]). In §2 we introduce the main model: a 3–parameter family of probability measures on the Young diagrams [3]). There it is also explained how to pass from measures on the Young diagrams to point processes on the lattice and where our main object, the "hypergeometric" difference operator on the lattice, comes from. In §3 we discuss limit transitions related to the degeneration of the z–measures to the Plancherel measure. In §4 we study another limit regime that leads to the so–called Gamma kernel. This interesting kernel (it arose in [7]) describes the asymptotics of the fluctuations of the boundary of a random Young diagram near the point of intersection with the diagonal (in another language, limit properties of the lowest Frobenius coordinates).

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[2]For instance, a solution of the problem of harmonic analysis on the infinite–dimensional unitary group [6] required computing the asymptotics of of Hahn type orthogonal polynomials, which are expressed through the hypergeometric series $_3F_2$.

[3]These measures, called the z–measures, were introduced in [5] for solving the problem of harmonic analysis on the infinite symmetric group. See also the survey paper [24].

1. DEFINITIONS

Let $\mathfrak{X}$ be a topological space. By a *configuration* in $\mathfrak{X}$ we mean a subset $X \subset \mathfrak{X}$ without accumulation points. [4] The space of all configurations in $\mathfrak{X}$ is denoted as $\mathrm{Conf}(\mathfrak{X})$. If $\mathfrak{X}$ is discrete then $\mathrm{Conf}(\mathfrak{X})$ is simply the set $2^{\mathfrak{X}}$ of all subsets of $\mathfrak{X}$. We will be dealing with probability Borel measures on $\mathrm{Conf}(\mathfrak{X})$. Given such a measure P, one may speak about an ensemble of *random* configurations or a *random point process*.

In what follows the space $\mathfrak{X}$ will be discrete, so that all the next definitions will be given for this simple case. About the general case, see, e.g., [19].

We assign to P a sequence of functions on $\mathfrak{X}$, $\mathfrak{X} \times \mathfrak{X}$, $\mathfrak{X} \times \mathfrak{X} \times \mathfrak{X}$, ..., called the *correlation functions*. Here, by definition, the nth correlation function $\rho^{(n)}$ is obtained in the following way: its value $\rho^{(n)}(x_1, \ldots, x_n)$ at an n–tuple of points equals 0 if among these points there are repetitions; otherwise the value is equal to the probability of the event that the random configuration X contains all the points $x_1, \ldots, x_n$.

The initial measure P is determined by its correlation functions uniquely.

A point process P is said to be *determinantal* [5], [26], if there exists a function $K(x, y)$ on $\mathfrak{X} \times \mathfrak{X}$ such that for any $n = 1, 2, \ldots$

$$\rho^{(n)}(x_1, \ldots, x_n) = \det[K(x_i, x_j)]_{i,j=1}^n.$$

Let us call $K(x, y)$ the *correlation kernel* of the process P. The operator in the coordinate Hilbert space $\ell^2(\mathfrak{X})$ with the matrix $[K(x, y)]$ will be called the *correlation operator* and denoted by K.

The whole information on the determinantal point process P is contained in its correlation kernel (equivalently, correlation operator).

A simple but important example of determinantal point processes is afforded by the *orthogonal polynomial ensembles*. Let us give the definition for the discrete case we are interested in (for more details, see the survey paper [18]).

Let $\mathfrak{X} \subset \mathbb{R}$ be a discrete subset and $W(x) > 0$ be a weight function defined on $\mathfrak{X}$. Fix $N = 1, 2, \ldots$. By definition, the ensemble in question is formed by the N–point configurations $X = \{x_1, \ldots, x_N\} \subset \mathfrak{X}$, whose probabilities are given by the formula

$$(1.1) \qquad \mathrm{Prob}(X) = \mathrm{const} \cdot \prod_{i=1}^N W(x_i) \cdot \prod_{1 \le i < j \le N} (x_i - x_j)^2,$$

where "const" is a suitable normalizing factor.

It is well known (see, e. g., [18]) that such an ensemble is a determinantal process and its correlation kernel $K(x, y)$ is expressed through the orthogonal polynomials $p_0 = 1, p_1, p_2, \ldots$ with the weight function $W(x)$, as follows:

$$K(x, y) = \sum_{i=0}^{N-1} \widetilde{p}_i(x) \widetilde{p}_i(y), \qquad x, y \in \mathfrak{X},$$

where

$$\widetilde{p}_i(x) = \sqrt{W(x)}\, \frac{p_i(x)}{\|p_i\|_W}\,, \qquad x \in \mathfrak{X}, \quad i = 0, 1, \ldots,$$

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[4]More precisely, what we termed a configuration should be called a simple locally finite configuration. Here "simple" means that we exclude multiple points which are allowed in a more general definition.

and $\|\cdot\|_W$ stands for the norm in the weighted Hilbert space $\ell^2(\mathfrak{X}, W)$. Observe that the functions $\widetilde{p}_i(x)$ form an orthonormal system of vectors in $\ell^2(\mathfrak{X})$. Thus, the kernel $K(x, y)$ is the matrix of a finite–dimensional projection in $\ell^2(\mathfrak{X})$: specifically, of the projection onto the subspace of polynomials of degree at most $N-1$, multiplied by $\sqrt{W(x)}$.

Since $K(x, y)$ is equal to the product of $\sqrt{W(x)W(y)}$ with the Nth Christoffel–Darboux kernel for our system of orthogonal polynomials, the kernel $K(x, y)$ can be written in the form

$$K(x, y) = \text{const} \cdot \frac{\widetilde{p}_N(x)\widetilde{p}_{N-1}(y) - \widetilde{p}_{N-1}(x)\widetilde{p}_N(y)}{x - y}$$

and hence can be represented in the integrable form (0.1).

2. Z–measures and discrete orthogonal polynomial ensembles

2.1. **Z–measures** [5], [9]**.** Denote by $\mathbb{Y}$ the set of all partitions $\lambda = (\lambda_1, \lambda_2, \dots)$, which we identify with the corresponding Young diagrams. The *z–measure* with parameters

$$z \in \mathbb{C}, \qquad z' \in \mathbb{C}, \qquad \xi \in \mathbb{C} \setminus [1, +\infty)$$

is the (complex) measure $M_{z,z',\xi}$ on the set $\mathbb{Y}$ assigning to a diagram $\lambda \in \mathbb{Y}$ the weight

$$(2.1) \qquad M_{z,z',\xi}(\lambda) = (1 - \xi)^{zz'} \xi^{|\lambda|} (z)_\lambda (z')_\lambda \left(\frac{\dim \lambda}{|\lambda|!} \right)^2.$$

Here $|\lambda| := \sum_i \lambda_i$ (equivalently, $|\lambda|$ is the number of boxes in the diagram λ);

$$(2.2) \qquad (z)_\lambda = \prod_{i=1}^{\ell(\lambda)} (z - i + 1)_{\lambda_i} = \prod_{(i,j)\in\lambda} (z + j - i),$$

where $(a)_m = a(a+1)\dots(a+m-1)$ is the Pochhammer symbol; $\ell(\lambda)$ is the number of nonzero coordinates λ_i; the second product in (2.2) is taken over all boxes (i, j) of the diagram λ, where i and j denote the numbers of the row and the column containing the box; finally, $\dim \lambda$ denotes the number of standard tableaux of shape λ (equivalently, the dimension of the irreducible representation of the symmetric group of degree $|\lambda|$ indexed by the diagram λ).

The following summation formula holds:

$$\sum_{\lambda \in \mathbb{Y}} M_{z,z',\xi}(\lambda) = 1.$$

Here we assume that either $|\xi| < 1$ (then the series absolutely converges) or the parameters z, z' are integers of opposite sign (then the series terminates).

Note that the z–measures are a particular case of the Schur measures, see [22]).

Note also two properties of the z–measures:

$$(2.3) \qquad M_{z,z',\xi}(\lambda) = M_{z',z,\xi}(\lambda) \qquad M_{z,z',\xi}(\lambda') = M_{-z,-z',\xi}(\lambda),$$

where λ' denotes the transposed diagram.

In what follows we consider only the z–measures with real nonnegative weights: $M_{z,z',\xi}(\lambda) \geq 0$ for all $\lambda \in \mathbb{Y}$. These are probability measures on $\mathbb{Y}$. They fall into the following 4 series.

• *Principal series*: The parameters z and z' are complex conjugate and nonreal while the parameter ξ is real and $0 < \xi < 1$.

• *Complementary series*: The both parameters z and z' are real and contained in one and the same open interval of the form $(m, m+1)$ where $m \in \mathbb{Z}$, while the parameter ξ is the same as above $(0 < \xi < 1)$.

• *First degenerate series*: One of the parameters z, z' (say, z) is a nonzero integer, while the other parameter (then it will be z') is a real number of the same sign such that $|z'| > |z| - 1$. Again, $0 < \xi < 1$.

• *Second degenerate series*: The parameters z and z' are integers of opposite sign. The parameter ξ is now subject to another condition: $\xi < 0$.

If a z–measure belongs to the principal or complementary series then $(z+k)(z'+k) > 0$ for all integers $k \in \mathbb{Z}$, which implies that the weights of all diagrams are strictly positive. For the z–measures of the degenerate series some of the weights vanish but all nonzero weights are strictly positive.

2.2. More details on degenerate z–measures. The Meixner and Krawtchouk ensembles.
Consider the first degenerate series. By virtue of the symmetry relation (2.3) it suffices to take $z = N$ and $z' = N + c - 1$, where $N = 1, 2, \ldots$ and $c > 0$. Then the weight of a diagram λ vanishes precisely when $\ell(\lambda) > N$. Thus, the support of the z–measure is the set of the diagrams contained in the horizontal strip of width N. Such diagrams λ are in a one–to–one correspondence with the N–point configurations L on $\mathbb{Z}_+ := \{0, 1, 2, \ldots\}$:

$$(2.4) \qquad \lambda = (\lambda_1, \ldots, \lambda_N, 0, 0, \ldots) \longleftrightarrow L = (l_1, \ldots, l_N),$$

where

$$l_i = \lambda_i + N - i, \quad 1 \leq i \leq N.$$

It is readily checked that in terms of the correspondence (2.4), the weight $M_{N,N+c-1,\xi}(\lambda)$ can be written in the form (1.1), where we have to set $x_i = l_i$ and take as W the weight function for the *Meixner polynomials* [17] on the set $\mathfrak{X} = \mathbb{Z}_+$:

$$W^{\text{Meixner}}(l) = \frac{(c)_l \xi^l}{l!}, \qquad l \in \mathbb{Z}_+.$$

Thus, the correspondence $\lambda \to L$ converts the z–measure of the first degenerate series with parameters $z = N$, $z' = N+c-1$, ξ into the N–point Meixner orthogonal polynomial ensemble with parameters c and ξ.

A similar fact holds for the second degenerate series. Let $z = N$ and $z' = -N'$ where N and N' are two positive integers, and assume $\xi < 0$. Then the weight of a diagram λ does not vanish if and only if $\ell(\lambda) \leq N$ and $\ell(\lambda') \leq N'$, that is, λ has to be contained in the rectangular $\square_{N,N'}$ with N rows and N' columns. The support of such a measure is finite.

The same correspondence $\lambda \leftrightarrow L$ as in (2.4) gives a bijection between the diagrams $\lambda \subseteq \square_{N,N'}$ and the N–point configurations L on the finite set $\{0, 1, \ldots, \widetilde{N}\}$, where $\widetilde{N} = N + N' - 1$. Then the z–measure of the second degenerate series turns into the N–point orthogonal polynomial ensemble on $\mathfrak{X} = \{0, 1, \ldots, \widetilde{N}\}$ that is

determined by the weight function

$$W^{\text{Krawtchouk}}(l) = \binom{\widetilde{N}}{x} p^l (1-p)^{\widetilde{N}-l}, \qquad l = 0, \ldots, \widetilde{N},$$

for the *Krawtchouk orthogonal polynomials* [17] with the parameters

$$p := \frac{\xi}{\xi - 1}, \qquad \widetilde{N} := N + N' - 1.$$

Note that the condition $\xi < 0$ guarantees that $0 < p < 1$.

Thus, the Meixner and Krawtchouk ensembles can be extracted from the z–measures as a rather particular case. As we will see, it is also helpful to use the inverse transition:

Thesis 2.1. *The z–measures can be obtained from the Meixner or Krawtchouk ensembles via analytic continuation (interpolation) with respect to the parameters.*

Let us explain this statement (for more detail, see [9] and [8]). Note that the support of a degenerate measure enlarges as the parameter N grows (or the two parameters N and N' grow), so that any given diagram $\lambda \in \mathbb{Y}$ can be covered. Further, as seen from (2.1), for any fixed diagram λ, its weight $M_{z,z',\xi}(\lambda)$ is an analytic function in ξ whose Taylor coefficients at $\xi = 0$ are given by polynomial functions in z and z'. This allows one to extrapolate results about the degenerate series to the general case, because such functions in z, z', and ξ are uniquely determined by their restriction to the subset of the values of the parameters corresponding to the degenerate series.

As we will see, Thesis 2.1 explains the origin of the difference operator (2.6) introduced below.

2.3. **The hypergeometric deifference operator.** Consider now the principal and complementary series. We cannot use the correspondence (2.4) anymore. Instead of it we will introduce another correspondence, which will take *arbitrary* partitions $\lambda \in \mathbb{Y}$ to *semi–infinite* point configurations $\underline{X}$ on the lattice $\mathbb{Z}' = \mathbb{Z} + \frac{1}{2}$ of half–integers:

$$(2.5) \qquad\qquad \lambda \longleftrightarrow \underline{X} = \{\lambda_i - i + \tfrac{1}{2} \mid i = 1, 2, \ldots\}.$$

(We call the configuration $\underline{X}$ semi–infinite because it contains all the lattice points that are sufficiently far to the left of zero and does not contain points sufficiently far to the right of zero.)

There is a simple link between (2.4) and (2.5). Given a diagram $\lambda \in \mathbb{Y}$, let N be so large that $N \geq \ell(\lambda)$, so that the N–point configuration L determined by (2.4) exists. Then $\underline{X}$ is obtained from L by shifting it to the left by $N - \frac{1}{2}$ and next adding the left–infinite "tail" $\{\frac{1}{2} - i \mid i = N + 1, N + 2, \ldots\}$.

Let us assume the parameters (z, z') lie in the principal or complementary series. Consider the following difference operator on the lattice $\mathbb{Z}'$:

$$(2.6) \quad \mathcal{D}_{z,z',\xi} f(x) = \sqrt{\xi(z + x + \tfrac{1}{2})(z' + x + \tfrac{1}{2})}\, f(x+1) - (x + \xi(z + z' + x)) f(x)$$
$$+ \sqrt{\xi(z + x - \tfrac{1}{2})(z' + x - \tfrac{1}{2})}\, f(x-1).$$

This difference operator was introduced in the papers [8] and [9] as the result of analytic continuation, in the spirit of Thesis 2.1, of the Meixner difference operator. In more details, the analytic continuation procedure is as follows:

1. Consider the Meixner difference operator on $\mathbb{Z}_+$:

$$\mathcal{D}_{c,\xi}^{\text{Meixner}} f(x) = \xi(x+c)f(x+1) - (x+\xi(x+c))f(x) + xf(x-1).$$

It multiplies the nth Meixner polynomial by $-(1-\xi)n$, see [17], formula (1.9.5). Add to $\mathcal{D}_{c,\xi}^{\text{Meixner}}$ the constant term $(1-\xi)(N-\frac{1}{2})$ and observe that the first N Meixner polynomials are just those eigenvectors of the resulting operator

$$(2.7) \qquad \mathcal{D}_{c,\xi}^{\text{Meixner}} + (1-\xi)(N-\tfrac{1}{2})$$

that correspond to nonnegative eigenvalues.

2. Pass from the weight ℓ^2–space $\ell^2(\mathbb{Z}_+, W^{\text{Meixner}})$ to the ordinary space $\ell^2(\mathbb{Z}_+)$. This means that (2.7) should be replaced by the composition of operators

$$(2.8) \qquad (W^{\text{Meixner}})^{\frac{1}{2}} \circ \left(\mathcal{D}_{c,\xi}^{\text{Meixner}} + (1-\xi)(N-\tfrac{1}{2})\right) \circ (W^{\text{Meixner}})^{-\frac{1}{2}}.$$

3. Make in (2.8) the change of the variable $l \to x := l - (N - \frac{1}{2})$, which means that the support $\mathbb{Z}_+$ of the weight function is transformed to the subset $\{-(N-\frac{1}{2}) + m \mid m = 0, 1, 2, \dots\} \subset \mathbb{Z}'$. Note that this subset grows together with N and in the limit it exhausts the whole lattice $\mathbb{Z}'$.

4. Formally substitute $N = z$ and $c = z' - z + 1$ into the coefficients of the resulting operator.

Then we get $\mathcal{D}_{z,z',\xi}$. Likewise, $\mathcal{D}_{z,z',\xi}$ can be obtained from the difference operator connected to the Krawtchouk polynomials.

Denote by $\{e_x\}$, $x \in \mathbb{Z}'$, the natural orthonormal basis in the Hilbert space $\ell^2(\mathbb{Z}')$ and let $\ell_0^2(\mathbb{Z}') \subset \ell^2(\mathbb{Z}')$ stand for the algebraic subspace formed by linear combinations of the basis elements. We will consider $\mathcal{D}_{z,z',\xi}$ (see (2.6)) as a symmetric operator in $\ell^2(\mathbb{Z}')$ with domain $\ell_0^2(\mathbb{Z}')$.

Proposition 2.2. *The operator $\mathcal{D}_{z,z',\xi}$ defined in this way is essentially selfadjoint.*

Proof. The standard way is to check that the eigenfunctions of the difference operator (2.6) with nonreal eigenvalues do not belong to $\ell^2(\mathbb{Z}')$. This can be done by computing explicitly the eigenfunctions (they are expressed through the Gauss hypergeometric function, see [9]). The computation can be simplified if one replaces (2.6) with the difference operator

$$\sqrt{\xi}\,|x+\tfrac{1}{2}|\,f(x+1) - (x+\xi(z+z'+x))f(x) + \sqrt{\xi}\,|x-\tfrac{1}{2}|\,f(x-1),$$

which differs from (2.6) by a bounded operator (adding a bounded selfadjoint operator preserves essential selfadjointness).

I will sketch now another argument. It is perhaps less elementary but it requires no computations and builds a bridge to representations of the group $SL(2)$, which turns out to be useful in other questions related to the z–measures.

Consider the complex Lie algebra $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}(2, \mathbb{C})$ with the basis

$$\mathbf{E} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{H} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

and the commutation relations

$$[\mathbf{H}, \mathbf{E}] = 2\mathbf{E}, \quad [\mathbf{H}, \mathbf{F}] = -2\mathbf{F}, \quad [\mathbf{E}, \mathbf{F}] = \mathbf{H}.$$

The real linear span of the elements $\mathbf{E} + \mathbf{F}$, $i(\mathbf{E} - \mathbf{F})$, and $i\mathbf{H}$ is a real form $\mathfrak{g} \subset \mathfrak{g}_{\mathbb{C}}$: this is the Lie algebra $\mathfrak{su}(1, 1)$.

Define a representation $S_{z,z'}$ of the Lie algebra $\mathfrak{g}_{\mathbb{C}}$ in the algebraic space $\ell_0^2(\mathbb{Z}') \subset \ell^2(\mathbb{Z}')$ by the following formulas for the basis elements

$$
\begin{aligned}
(2.9) \qquad S_{z,z'}(\mathbf{E})\, e_x &= \sqrt{(z + x + \tfrac{1}{2})(z' + x + \tfrac{1}{2})}\; e_{x+1} \\
S_{z,z'}(\mathbf{F})\, e_x &= -\sqrt{(z + x - \tfrac{1}{2})(z' + x - \tfrac{1}{2})}\; e_{x-1} \\
S_{z,z'}(\mathbf{H})\, e_x &= (z + z' + 2x)e_x
\end{aligned}
$$

It is readily checked that the above operators satisfy the commutation relations and hence do determine a representation.

The representation $S_{z,z'}$ equips $\ell_0^2(\mathbb{Z}')$ with the structure of an irreducible Harish–Chandra module over $\mathfrak{g}$ such that this algebra acts by skew symmetric operators. It follows that $S_{z,z'}$ gives rise to an irreducible unitary representation of the group $SU(1,1)^{\sim}$, the simply connected covering group of the matrix group $SU(1,1) = SL(2, \mathbb{R})$ (see [12], p. 14, where a proof is given for a concrete example but the argument holds in the general case). Moreover, according to a well–known theorem by Harish–Chandra, all vectors from $\ell_0^2(\mathbb{Z}')$ are analytic vectors of this representation.

On the other hand, as seen from (2.6) and (2.9),

$$(2.10) \qquad \mathcal{D}_{z,z',\xi} = S_{z,z'}\left(\sqrt{\xi}\,\mathbf{E} - \sqrt{\xi}\,\mathbf{F} - \tfrac{1+\xi}{2}\,\mathbf{H}\right) + \tfrac{1-\xi}{2}(z + z')\mathbf{1},$$

where $\mathbf{1}$ denotes the identity operator. Therefore, all vectors from $\ell_0^2(\mathbb{Z}')$ are analytic vectors of the operator $\mathcal{D}_{z,z',\xi}$, which implies ([25], X.39) that it is essentially selfadjoint. $\square$

Remark 2.3. The use of the representation $S_{z,z'}$ of the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ is suggested by Okounkov's work [21]. Note that the corresponding unitary representation of the group $SU(1,1)^{\sim}$ belongs to the principal or complementary series, in exact agreement with the terminology for the z–measures (which was introduced prior to [21], relying on a purely formal analogy).

Let us agree to denote by $D_{z,z',\xi}$ the selfadjoint operator in $\ell^2(\mathbb{Z}')$ that is the closure of the operator $\mathcal{D}_{z,z',\xi}$. It follows from Proposition 2.2 that the domain of the operator $D_{z,z',\xi}$ consists of all those functions $f \in \ell^2(\mathbb{Z}')$ that remain in $\ell^2(\mathbb{Z}')$ after application of the difference operator (2.6).

Proposition 2.4. *The selfadjoint operator $D_{z,z',\xi}$ has simple, pure discrete spectrum filling the subset $(1 - \xi)\mathbb{Z}' \subset \mathbb{R}$.*

Proof. The matrix $\sqrt{\xi}\,\mathbf{E} - \sqrt{\xi}\,\mathbf{F} - \tfrac{1+\xi}{2}\,\mathbf{H}$ that enters (2.10) is conjugated with the matrix $-\tfrac{1-\xi}{2}\mathbf{H}$ by means of an element of $SU(1, 1)$. Therefore, $D_{z,z',\xi}$ is unitarily equivalent to the closure of the operator $S_{z,z'}(-\tfrac{1-\xi}{2}\mathbf{H}) + \tfrac{1-\xi}{2}(z + z')\mathbf{1}$. This operator is the diagonal operator in the basis $\{e_x\}$ acting as multiplication of e_x by $-(1-\xi)x$. This makes the claim evident. $\square$

Thus, for any triple of parameters (z, z', ξ) from the principal or complementary series there exists an orthonormal basis $\{\psi_{a;z,z',\xi}\}$ in $\ell^2(\mathbb{Z}')$, indexed by points $a \in \mathbb{Z}'$ and such that

$$\mathcal{D}_{z,z',\xi}\psi_{a;z,z',\xi} = (1 - \xi)a\,\psi_{a;z,z',\xi}, \qquad a \in \mathbb{Z}'.$$

As shown in [9], the eigenvectors $\psi_{a;z,z',\xi}$ can be written down explicitly: for them there exists a convenient contour integral representation and also an expression through the Gauss hypergeometric function.

In view of the connection of the functions $\psi_{a;z,z',\xi}$ with the hypergeometric function I will call $\mathcal{D}_{z,z',\xi}$ the *hypergeometric difference operator*.

If A is a selfadjoint operator for which 0 is not a point of discrete spectrum, then we will denote by $\mathrm{Proj}_+(A)$ the spectral projection corresponding to the positive part of the spectrum of A. In particular, as the spectrum of $D_{z,z',\xi}$ does not contain 0, we may form the projection $\mathrm{Proj}_+(D_{z,z',\xi})$.

The image of the z–measure $M_{z,z',\xi}$ under the correspondence $\lambda \mapsto \underline{X}$ introduced in (2.5) determines a point process on $\mathbb{Z}'$, which we denote as $P_{z,z',\xi}$.

Theorem 2.5. $P_{z,z',\xi}$ *is a determinantal process. Its correlation kernel, denoted as* $\underline{K}_{z,z',\xi}(x, y)$, *is the matrix of the spectral projection* $\mathrm{Proj}_+(D_{z,z',\xi})$. *That is,*

$$\underline{K}_{z,z',\xi}(x, y) = \sum_{a \in \mathbb{Z}'_+} \psi_{a;z,z',\xi}(x)\psi_{a;z,z',\xi}(y), \qquad x, y \in \mathbb{Z}'.$$

(We do not put the bar over $\psi_{a;z,z',\xi}(y)$ because the eigenfunctions are real–valued, see formula (2.1) in [9].)

In the above formulation, the result is contained in [9]. There are also references to previous works. We call $K_{z,z',\xi}(x, y)$ the *discrete hypergeometric kernel*. It can be written in the integrable form (0.1), see [9], Prop. 3.10.

3. Plancherel measure, dicrete Bessel kernel, and discrete sine kernel

3.1. Poissonized Plancherel measure. Consider the limit regime

$$(3.1) \qquad\qquad \xi \to 0, \quad z \to \infty, \quad z' \to \infty, \quad \xi zz' \to \theta,$$

where $\theta > 0$ is a new parameter. Then the z–measures $M_{z,z',\xi}$ converge to a probability measure M_θ on $\mathbb{Y}$:

$$M_\theta(\lambda) = \lim_{zz'\xi \to \theta} M_{z,z',\xi}(\lambda) = e^{-\theta}\theta^{|\lambda|}\left(\frac{\dim \lambda}{|\lambda|!}\right)^2.$$

The measure M_θ arises in the result of poissonization of the sequence the Plancherel measures $M^{(n)}$, $n = 1, 2, \ldots$, where

$$M^{(n)}(\lambda) = \frac{(\dim \lambda)^2}{n!}, \qquad |\lambda| = n.$$

For this reason M_θ is called the *poissonized Plancherel measure* with parameter θ.

For more details about the Plancherel measures $M^{(n)}$ see the papers [20] and [27]; see also [16] and [14]. The poissonized version M_θ was first considered in [1]; further results were obtained in [4] and [15].

As in §2, we use the correspondence $\lambda \mapsto \underline{X}$ to pass from the measure M_θ on $\mathbb{Y}$ to a point process on $\mathbb{Z}'$; let us denote this point process as P_θ. It is determinantal; its correlation kernel is described in §3.2 below.

3.2. **From discrete hypergeometric kernel to discrete Bessel kernel.** In the regime (3.1), the formal limit of the difference operator $\mathcal{D}_{z,z',\xi}$ is the difference operator $\mathcal{D}_\theta^{\mathrm{Bessel}}$ on the lattice $\mathbb{Z}'$, acting according to formula

$$\mathcal{D}_\theta^{\mathrm{Bessel}} f(x) = \sqrt{\theta} f(x+1) - x f(x) + \sqrt{\theta} f(x-1).$$

Proposition 3.1. *Consider $\mathcal{D}_\theta^{Bessel}$ as a symmetric operator in $\ell^2(\mathbb{Z}')$ with domain $\ell_0^2(\mathbb{Z}')$. Then this is an essentially selfadjoint operator.*

Proof. Indeed, $\mathcal{D}_\theta^{\mathrm{Bessel}}$ is the sum of the diagonal operator $f \mapsto -xf$ and a bounded operator. $\qquad\square$

Alike the hypergeometric operator (2.6), the operator $\mathcal{D}_\theta^{\mathrm{Bessel}}$ is connected with a group representation, only the group is different, namely, it is the universal covering group $\widetilde{G}$ for the group G of motions of the plane $\mathbb{R}^2$. Let us discuss this connection shortly.

Let $\mathfrak{g}$ be the Lie algebra of the group G and $\mathfrak{g}_\mathbb{C}$ be the complexification of $\mathfrak{g}$. In $\mathfrak{g}_\mathbb{C}$, there is a basis $\{\mathbf{E}, \mathbf{F}, \mathbf{H}\}$ with the commutation relations

$$[\mathbf{H}, \mathbf{E}] = \mathbf{E}, \quad [\mathbf{H}, \mathbf{F}] = -\mathbf{F}, \quad [\mathbf{E}, \mathbf{F}] = 0.$$

Consider the representation S of the Lie algebra $\mathfrak{g}_\mathbb{C}$ in the dense subspace $\ell_0^2(\mathbb{Z}') \subset \ell^2(\mathbb{Z}')$ defined on the basis elements by the formulas

$$S(\mathbf{E})\, e_x = e_{x+1}, \quad S(\mathbf{F})\, e_x = -e_{x-1}, \quad S(\mathbf{H}) e_x = x e_x,$$

It is directly checked that all vectors from $\ell_0^2(\mathbb{Z}')$ are analytic vectors for S and $\mathfrak{g}$ acts by skew symmetric operators. Therefore, S gives rise to a unitary representation of the group $\widetilde{G}$. We have

$$\mathcal{D}_\theta^{\mathrm{Bessel}} = S\left(\sqrt{\theta}\,\mathbf{E} - \sqrt{\theta}\,\mathbf{F} - \mathbf{H}\right),$$

which is an analog of equality (2.10).

Denote by $D_\theta^{\mathrm{Bessel}}$ the closure of the symmetric operator $\mathcal{D}_\theta^{\mathrm{Bessel}}$. According to Proposition 3.1, D_θ is selfadjoint.

Set

$$\psi_{a;\theta}(x) := J_{x+a}(2\sqrt{\theta}), \qquad x \in \mathbb{Z}',$$

where $a \in \mathbb{Z}'$ is a parameter and $J_m(\,\cdot\,)$ is the Bessel function of index m.

Proposition 3.2. *For any fixed $\theta > 0$, the functions $\psi_{a;\theta}$ form an orthonormal basis in $\ell^2(\mathbb{Z}')$ and*

$$D_\theta^{Bessel} \psi_{a;\theta} = a\, \psi_{a;\theta}, \qquad a \in \mathbb{Z}'.$$

Thus, D_θ^{Bessel} has pure discrete, simple spectrum filling the lattice $\mathbb{Z}' \subset \mathbb{R}$.

Proof. The element $\sqrt{\theta}\,\mathbf{E} - \sqrt{\theta}\,\mathbf{F} - \mathbf{H} \in i\mathfrak{g}$ is conjugated with $-\mathbf{H}$ under the adjoint action of the group G. It follows that the selfadjoint operators $\overline{S(\sqrt{\theta}\,\mathbf{E} - \sqrt{\theta}\,\mathbf{F} - \mathbf{H})}$ and $\overline{S(-\mathbf{H})}$ (where the bar means closure) are conjugated by means of a unitary operator corresponding to an element of the group $\widetilde{G}$. Since $D_\theta^{\mathrm{Bessel}}$ coincides with $\overline{S(\sqrt{\theta}\,\mathbf{E} - \sqrt{\theta}\,\mathbf{F} - \mathbf{H})}$, its spectrum is the same as that of the diagonal operator $\overline{S(-\mathbf{H})}$. Thus, we see that the spectrum is indeed the same as was stated.

The fact that the functions $\psi_{a;\theta}(x)$ are eigenfunctions of the operator $\mathcal{D}_\theta^{\text{Bessel}}$ follows from the well–known recurrence relations for the Bessel functions ([2],7.2.8 (56)). Finally, the relation $\|\psi_{a;\theta}\|^2 = 1$ can be proved in the same way as Proposition 2.4 in [9], by using the contour integral representation of the Bessel functions. $\square$

Proposition 3.3. *In the regime* (3.1), *the operators $D_{z,z',\xi}$ converge to the operator D_θ^{Bessel} in the strong resolvent sense.*

About the notion of strong resolvent convergence see the textbook [25], Section VIII.7.

Proof. Obviously, $D_{z,z',\xi} \to D_\theta^{\text{Bessel}}$ on the subspace $\ell_0^2(\mathbb{Z}')$. By virtue of Propositions 2.2 and 3.1, $\ell_0^2(\mathbb{Z}')$ is a common essential domain for all selfadjoint operators under consideration. Then the claim follows from a well–known general theorem ([25], Theorem VIII.25). $\square$

Corollary 3.4. *The spectral projection operators $\text{Proj}_+(D_{z,z',\xi})$ strongly converge to the spectral projection operator $\text{Proj}_+(D_\theta^{\text{Bessel}})$.*

Proof. This is a direct consequence of Proposition 3.3 and another general theorem ([25], Theorem VIII.24 (b)). For applicability of that theorem it is important that 0 not be a point of discrete spectrum of the operators $D_{z,z',\xi}$ and D_θ^{Bessel}, and this fact follows from the description of their spectra, see Propositions 2.4 and 3.2. $\square$

Corollary 3.4 shows that $\text{Proj}_+(D_\theta^{\text{Bessel}})$ serves as the correlation operator for P_θ. The kernel of the projection $\text{Proj}_+(D_\theta^{\text{Bessel}})$ can be written in the form

$$K_\theta^{\text{Bessel}}(x,y) = \sum_{a \in \mathbb{Z}_+'} \psi_{a;\theta}(x)\psi_{a;\theta}(y) = \sqrt{\theta}\,\frac{\psi_{-\frac{1}{2};\theta}(x)\psi_{\frac{1}{2};\theta}(y) - \psi_{\frac{1}{2};\theta}(x)\psi_{-\frac{1}{2};\theta}(y)}{x - y}$$

This kernel is called the *discrete Bessel kernel*; it was independently derived in [15] and (in somewhat different form) in [4].

3.3. **From discrete Bessel kernel to discrete sine kernel.** Consider the following limit regime depending on the parameter $c \in (-1,1)$:

$$(3.2) \qquad\qquad \theta \to \infty, \qquad x \approx 2c\sqrt{\theta} + \widetilde{x}, \qquad \widetilde{x} \in \mathbb{Z}.$$

This means that as θ goes to ∞, we shift the lattice $\mathbb{Z}'$ so that to focus on a neighborhood of the point $2c\sqrt{\theta}$. The following formal limit holds

$$\left(\frac{1}{\sqrt{\theta}}\,\mathcal{D}_\theta^{\text{Bessel}}\right)_{x \leadsto \widetilde{x}} \to \mathcal{D}_c^{\text{sine}}$$

with the following difference operator in the right–hand side:

$$\mathcal{D}_c^{\text{sine}} f(\widetilde{x}) = f(\widetilde{x}+1) - 2cf(\widetilde{x}) + f(\widetilde{x}-1), \qquad \widetilde{x} \in \mathbb{Z}.$$

Proposition 3.5. *Consider $\mathcal{D}_c^{\text{sine}}$ as a symmetric operator in $\ell^2(\mathbb{Z})$ with domain $\ell_0^2(\mathbb{Z})$. Then this is an essentially selfadjoint operator.*

Its closure, denoted as D_c^{sine}, has pure discrete spectrum of multiplicity 2, filling the interval $(-2-2c, 2-2c)$.

Proof. The difference operator $\mathcal{D}_c^{\mathrm{sine}}$ is invariant under shifts of $\mathbb{Z}$, which makes its study quite simple. Indeed, pass from the lattice $\mathbb{Z}$ to the unit circle $|\zeta| = 1$ by means of the Fourier transform. Then our operator will become the operator of multiplication by the function

$$\zeta \ \to \ 2(\Re\zeta - c),$$

which easily implies the claim about the spectrum. $\square$

Proposition 3.6. *In the regime* (3.2), *the operators $D_\theta^{\mathrm{Bessel}}$ converge to the operator D_c^{sine} in the strong resolvent sense.*

The proof is the same as in Proposition 3.3. As in Corollary 3.4, we deduce from this the strong convergence $\mathrm{Proj}_+(D_\theta^{\mathrm{Bessel}}) \to \mathrm{Proj}_+(D_c^{\mathrm{sine}})$. The kernel of the projection $\mathrm{Proj}_+(D_c^{\mathrm{sine}})$ is readily computed: it has the form

$$K_c^{\mathrm{sine}}(x, y) = \frac{\sin((\arccos c)(x - y))}{\pi(x - y)}, \qquad x, y \in \mathbb{Z}.$$

and is called the *discrete sine kernel* [4].

3.4. **From discrete Bessel kernel to Airy kernel.** The Airy kernel first emerged in random matrix theory: under suitable conditions on the matrix ensemble, this kernel describes the asymptotics of the spectrum "at the edge", as the order of matrices goes to infinity. The Airy kernel is defined on the whole real line and is expressed through the classical Airy function $Ai(u)$ and its derivative:

$$K^{\mathrm{Airy}}(u, v) = \int_0^{+\infty} Ai(u+s)Ai(v+s)ds = \frac{Ai(u)Ai'(v) - Ai'(u)Ai(v)}{u - v}, \qquad u, v \in \mathbb{R}.$$

As shown in [4] and [15], the discrete Bessel kernel $K_\theta^{\mathrm{Bessel}}(x, y)$ converges to the Airy kernel $K^{\mathrm{Airy}}(u, v)$ if $\theta \to +\infty$ and the variable $x \in \mathbb{Z}'$ is related to the variable $u \in \mathbb{R}$ by the scaling $x = 2\theta^{1/2} + \theta^{1/6} \cdot u$. Below we present a heuristic derivation of this claim.

Let $g(u)$ be a smooth function on $\mathbb{R}$. Assign to it the function $f(x)$ on $\mathbb{Z}'$ by setting $f(x) = g(u)$, with the understanding that x and u are related to each other as above. Then we get

$$f(x \pm 1) = g(u \pm \theta^{-1/6}) \approx g(u) \pm \theta^{-1/6} \cdot g'(u) + \tfrac{1}{2}\theta^{-1/3} \cdot g''(u),$$

so that

$$\mathcal{D}_\theta^{\mathrm{Bessel}}f(x) = \theta^{1/2}(f(x+1) - x\theta^{-1/2}f(x) + f(x-1)) \approx \theta^{1/6}(g''(u) - ug(u)).$$

This simple computation explains the origin of the factor 2 and the exponents $1/2$ and $1/6$.

Next, the eigenvalue equation

$$\mathcal{D}_\theta^{\mathrm{Bessel}}\psi = a\psi, \quad a \in \mathbb{Z}',$$

turns, after the renormalization $a = \theta^{1/6}s$, into the equation

$$\mathcal{D}^{\mathrm{Airy}}\psi = s\psi, \quad s \in \mathbb{R},$$

where

$$\mathcal{D}^{\mathrm{Airy}}g(u) = g''(u) - ug(u)$$

is the Airy operator. Its eigenfunctions have the form

$$\psi_s(u) = Ai(u + s), \qquad s \in \mathbb{R},$$

and the Airy kernel corresponds to the spectral projection onto the positive part of the spectrum.

It would be interesting to make this argument rigorous by using the notion of Mosco convergence.

4. From discrete hypergeometric kernel to Gamma kernel

Let us fix the parameters (z, z') from the principal or complementary series and let ξ go to 1. As shown in [7], in this regime, there exists a limit for the correlation kernels. Below we prove this fact in a new way (Corollary 4.4).

The convergence of the kernels implies that the images of the measures $M_{z,z',\xi}$ under the correspondence (2.5) converge to a probability measure on $\mathrm{Conf}(\mathbb{Z}')$ generating a determinantal point process. Note that for the initial z–measures on $\mathbb{Y}$ the picture is different: they converge to 0. Of course, this cannot happen on the compact space $\mathrm{Conf}(\mathbb{Z}') = 2^{\mathbb{Z}'}$.

The first indication on the existence of a limit is the fact that the hypergeometric difference operator (2.6) still makes a sense under the formal substitution $\xi = 1$: then we get the difference operator

$$(4.1) \quad \mathcal{D}_{z,z'}^{\mathrm{gamma}} f(x) = \sqrt{(z + x + \tfrac{1}{2})(z' + x + \tfrac{1}{2})}\, f(x+1) - (z + z' + 2x) f(x)$$
$$+ \sqrt{(z + x - \tfrac{1}{2})(z' + x - \tfrac{1}{2})}\, f(x-1).$$

In the sequel we follow the scheme we have already worked out.

Proposition 4.1. *Consider $\mathcal{D}_{z,z'}^{\mathrm{gamma}}$ as a symmetric operator in $\ell^2(\mathbb{Z}')$ with domain $\ell_0^2(\mathbb{Z}')$. Then this is an essentially selfadjoint operator.*

Proof. The both arguments given in the proof of Proposition 2.2 hold without changes. $\square$

Denote by $D_{z,z'}^{\mathrm{gamma}}$ the selfadjoint operator that is the closure of the operator $\mathcal{D}_{z,z'}^{\mathrm{gamma}}$.

Proposition 4.2. *$D_{z,z'}^{\mathrm{gamma}}$ has pure continuous spectrum.*

Proof. Note that for $\xi = 1$, the relation (2.10) turns into

$$\mathcal{D}_{z,z'}^{\mathrm{gamma}} = S_{z,z'} \left(\mathbf{E} - \mathbf{F} - \mathbf{H} \right).$$

Observe now that $\mathbf{E} - \mathbf{F} - \mathbf{H}$ is a nilpotent element in $i\mathfrak{su}(1,1) \subset \mathfrak{sl}(2,\mathbb{C})$. For any $r > 0$ the matrix $r(\mathbf{E} - \mathbf{F} - \mathbf{H})$ is conjugated to $\mathbf{E} - \mathbf{F} - \mathbf{H}$ by means of an element from $SU(1,1)$. Therefore, $r D_{z,z'}^{\mathrm{gamma}}$ is unitarily equivalent to $D_{z,z'}^{\mathrm{gamma}}$. This implies the claim. $\square$

Actually, one can say more: the spectrum is simple and it fills the whole real axis. This can be deduced, for instance, from a suitable realization of the representation $S_{z,z'}$. The eigenfunctions of the operator $D_{z,z'}^{\mathrm{gamma}}$ constituting the continuous spectrum can be written down explicitly, see [8]; they are expressed through the classical Whittaker function.

Proposition 4.3. *As $\xi \to 1$, the selfadjoint operators $D_{z,z',\xi}$ converge to the self-adjoint operator $D_{z,z'}^{\mathrm{gamma}}$ in the strong resolvent sense.*

The proof is the same as in Proposition 3.3. Just as in Corollary 3.4 we deduce from this

Corollary 4.4. *As $\xi \to 1$, the projection operators $\mathrm{Proj}_+(D_{z,z',\xi})$ strongly converge to the projection operator $\mathrm{Proj}_+(D_{z,z'}^{\mathrm{gamma}})$.*

The kernel of the limit projection operator is explicitly described in [7], [8]. It can be written in the integrable form (0.1), where the functions A and B are expressed through the Gamma function, which explains the origin of the term "Gamma kernel".

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