Existence results for Fokker–Planck equations in Hilbert spaces

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Abstract

We consider a stochastic differential equation in a Hilbert space with time-dependent coefficients for which no general existence result

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is known. We prove, under suitable assumptions, existence of a measure valued solution, for the corresponding Fokker–Planck equation.

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1 Introduction

Let us consider a stochastic differential equation on a separable Hilbert space H (with norm $|\cdot|$ and inner product $\langle\cdot,\cdot\rangle$) of the form

$$\begin{cases} dX(t) = [AX(t) + F(t, X(t))]dt + \sqrt{C}dW(t), \\ \\ X(0) = x, \end{cases} \tag{1.1}$$

where $A : D(A) \subset H \to H$ is a self-adjoint operator, $C : H \to H$ is linear self-adjoint and nonnegative, $F(t,\cdot) : Y \subset H \to H$ (where Y is a subspace of H), $t \in [0,T]$, form a family of non linear mappings and $W(t)$ is a cylindrical Wiener process in H defined on a stochastic basis $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$.

The Kolmogorov operator L_0 corresponding to (1.1) reads as follows

$$L_0u(t,x) = D_tu(t,x) + \frac{1}{2}\,\mathrm{Tr}\,[CD_x^2u(t,x)] + \langle x, AD_xu(t,x)\rangle + \langle F(t,x), D_xu(t,x)\rangle. \tag{1.2}$$

The operator L_0 is defined on the space $D(L_0) := \mathscr{E}_A([0,T]\times H)$, the linear span of all real parts of functions $u_{\phi,h}$ of the form

$$u_{\phi,h}(t,x) = \phi(t)e^{i\langle x,h(t)\rangle}, \quad t \in [0,T],\ x \in H,$$

where $\phi \in C^1([0,T])$, $h \in C^1([0,T];D(A))$ and $\phi(T) = 0$.

We are interested in the following Fokker–Planck equation

$$\begin{cases} \dfrac{d}{dt}\displaystyle\int_H u(t,x)\mu_t(dx) = \int_H L_0u(t,x)\mu_t(dx) \text{ for } dt\text{-a.e.} \quad t \in (0,T],\ \forall\, u \in D(L_0) \\ \\ \displaystyle\lim_{t\to 0}\int_H \varphi(x)\mu_t(dx) = \int_H \varphi(x)\zeta(dx), \quad \forall\, \varphi \in \mathscr{E}_A(H), \end{cases} \tag{1.3}$$

where $\frac{d}{dt}$ denotes the weak derivative on $[0,T]$. Here $\mathscr{E}_A(H)$ is the linear span of all real parts of functions of the form

$$\varphi(x) = e^{i\langle x,h\rangle}, \qquad x \in H,\ h \in D(A),$$

and, as in (1.4) and (1.5) (see also (2.4) below), we always implicitly assume that

$$\int_{[0,T]\times H} (|x| + |F(t,x)|)\mu_t(dx)dt < \infty,$$

so that $L_0 u \in L^1([0,T]\times H, \mu)$ for all $u \in D(L_0)$, where $\mu(dt,dx) = \mu_t(dx)dt$.

Furthermore, $\zeta \in \mathscr{P}(H)$ is given and $\mu_t(dx)$, $t \in [0,T]$, is a kernel of probability measure (shortly probability kernel) [(1)] from $(H, \mathscr{B}(H))$ to $([0,T],\ \mathscr{B}([0,T]))$, in particular the mapping $t \mapsto \int_H u(t,x)\mu_t(dx)$ is measurable for any bounded measurable function u. By $\mathscr{P}(H)$ we mean the set of all Borel probability measures on H.

We can also write equation (1.3) in the integral form

$$\int_H u(t,x)\mu_t(dx) = \int_H u(0,x)\zeta(dx) + \int_0^t ds \int_H L_0 u(s,x)\mu_s(dx),$$
$$\text{for } dt\text{-a.e. } t \in [0,T],\ \forall\ u \in D(L_0), \quad (1.4)$$

or also, setting $t = T$ as,

$$\int_{[0,T]\times H} L_0 u(s,x)\mu(ds,dx) = -\int_H u(0,x)\zeta(dx), \quad \forall\ u \in D(L_0). \quad (1.5)$$

Let us set our assumptions. Concerning the linear operators A and C we shall assume that

Hypothesis 1.1 *(i) A is self-adjoint.*

(ii) C is bounded, symmetric, nonnegative and such that $C^{-1} \in L(H)$.

(iii) There exists $\delta \in (0,1/2)$ such that $(-A)^{-2\delta}$ is of trace class.

Let us notice that from (iii) it follows that the embedding $D(A) \subset H$ is compact.

[(1)] We recall that a probability kernel is a family $\mu_t, t \in [0,T]$, of probability measures on $(H, \mathscr{B}(H))$ such that for all $A \in \mathscr{B}(H)$ the map $t \mapsto \mu_t(A)$ is $\mathscr{B}([0,T])$-measurable.

Remark 1.2 (i) Since we have used this also in our previous papers, let us explain in detail in what precise sense (1.3),(1.4) and (1.5) are really equivalent. So, let $\mu_t(dx),\ t \in [0,T]$, be a probability kernel as above and let $\mu(dt,dx) = \mu_t(dx)dt$ be the corresponding measure on $([0,T] \times H, \mathscr{B}([0,T] \times H))$. Then by definition μ solves (1.3) if the first equation in (1.3) holds and after a possible change of the map $t \mapsto \mu_t(dx)$ on a set of dt-measure zero also the second equation in (1.3) holds. In this case, obviously, $\mu(dt,dx) = \mu_t(dx)dt$ solves (1.4) and (1.5), and such a μ obviously solves (1.4) if and only if it satisfies (1.5). Much more subtle is the fact that if such a μ solves (1.4) (equivalently (1.5)), it also solves the second equation in (1.3) in the above sense. The reason is that the above dt modification of $t \mapsto \mu_t(dx)$ cannot be obtained from (1.4) by just defining it so that $\int_H \varphi d\mu_t$ is equal to the right hand side of (1.4) for $\varphi \in \mathscr{E}_A(H)$ (since then the second equation in (1.3) trivially holds), because the dt-zero set would firstly depend on φ (and there are uncountably many of them) and secondly the right hand side of (1.4) does not per se define a positive measure acting on φ. So, a more involved argument is required.

To this end we fix μ as above solving (1.4). Then clearly the first equation in (1.3) holds. Let us prove that the second holds for a dt-modification of $t \mapsto \mu_t(dx)$. By Hypothesis 1.1(iii), there exists an eigenbasis $\{e_k : k \in \mathbb{N}\}$ of H for A. Define

$$\mathscr{F}C_b^\infty(\{e_k\}) = \{g(\langle e_1,\cdot\rangle,...,\langle e_N,\cdot\rangle) :\ N \in \mathbb{N},\ g \in C_b^\infty(\mathbb{R}^N)$$

and

$$\mathscr{F}C_0^\infty(\{e_k\}) = \text{linear span}\ \{g(\langle e_1,\cdot\rangle,...,\langle e_N,\cdot\rangle) :\ N \in \mathbb{N},\ g \in C_0^\infty(\mathbb{R}^N),$$

where $C_b^\infty(\mathbb{R}^N)$, $C_0^\infty(\mathbb{R}^N)$ denote the set of all bounded smooth real valued functions on $\mathbb{R}^N$ with all partial derivatives bounded, respectively of compact support.

Claim *There exist $\varphi_n \in \mathscr{F}C_0^\infty(\{e_k\})$, $n \in \mathbb{N}$, such that μ satisfies* (1.4) *with φ_n replacing $u \in D(L_0)$ for every $n \in \mathbb{N}$, and such that if μ satisfies* (1.4) *with φ_n replacing $u \in D(L_0)$ for a fixed $t \in [0,T]$ for all $n \in \mathbb{N}$ then μ satisfies* (1.4) *for this t with φ replacing $u \in D(L_0)$ for all $\varphi \in \mathscr{E}_A(H)$.*

Proof. Let $\varphi = g(\langle e_1,\cdot\rangle,...,\langle e_N,\cdot\rangle) \in \mathscr{F}C_0^\infty(\{e_k\})$. Writing its base function $g \in C_0^\infty(\mathbb{R}^N)$ as the Fourier transform of a Schwartz test function and discretizing the Fourier integral, one sees by taking the limit in (1.4) that μ satisfies (1.4) with φ replacing $u \in D(L_0)$. But $C_0^\infty(\mathbb{R}^N)$ is separable with respect to the norm

$$\|g\|_{\infty,2} := \|g\|_\infty + \|Dg\|_\infty + \|D^2 g\|_\infty, \quad g \in C_0^\infty(\mathbb{R}^N).$$

Hence we can find $\{\varphi_k : k \in \mathbb{N}\} \in \mathscr{F}C_0^\infty(\{e_k\})$ such that if μ satisfies (1.4) for some $t \in [0,T]$ with φ_k replacing $u \in D(L_0)$ for all $k \in \mathbb{N}$, then it does so for this t and all $\varphi \in \mathscr{F}C_0^\infty(\{e_k\})$ replacing $u \in D(L_0)$, and by an easy localization argument it does so also for all $\varphi \in \mathscr{F}C_b^\infty(\{e_k\})$. A further easy approximation then proves the Claim. $\Box$

Now we can easily define the required modification of $t \mapsto \mu_t(dx)$. Let

$$M := \{t \in [0,T] : \text{ (1.4) holds for } t \text{ and } \varphi_k \text{ replacing } u \in D(L_0) \text{ for all } k \in \mathbb{N}\},$$

where $\varphi_k,\ k \in \mathbb{N}$, are as in the Claim. Define

$$\widetilde{\mu}_t(dx) = \begin{cases} \mu_t(dx) & \text{if } t \in M \\ \zeta & \text{if } t \in [0,T] \setminus M. \end{cases}$$

Then by the Claim (1.4) holds with $\widetilde{\mu}_t$ replacing μ_t for all $\varphi \in \mathscr{E}_A(H)$ replacing $u \in D(L_0)$ and all $t \in M$. Hence the second equation in (1.3) holds for the dt-modification $\widetilde{\mu}_t(dx),\ t \in [0,T]$, since it is equal to ζ on $[0,T] \setminus M$.

(ii) We note that applying (1.4) to a countable subset of functions $\phi \in C^1([0,T])$ replacing $u \in D(L_0)$ with $\phi(T) = 0$, which is dense with respect to $\|\cdot\|_\infty$, it follows that $\mu_t(H) = 1$ for dt-a.e. $t \in [0,T]$. Hence by e.g. setting $\mu_t = \zeta$ for those t for which this does not hold, we see that the requirement that for a solution $\mu = \mu_t(dx)dt$ of (1.4) the $\mu_t(dx)$ are all probability measures automatically holds after a dt-modification of the map $t \mapsto \mu_t(dx)$.

It is well known that, under Hypothesis 1.1(iii) the stochastic convolution

$$W_A(t) = \int_0^t e^{(t-s)A}\sqrt{C}dW(s), \quad t \geq 0,$$

is a well defined mean square continuous process in H with values in $D((-A)^\delta)$ and that

$$\sup_{t\in[0,T]} \mathbb{E}|(-A)^\delta W_A(t)|^2 \leq \|C\| \operatorname{Tr}[(-A)^{-2\delta}] := c_\delta. \tag{1.6}$$

Concerning the nonlinear operators $F(t,\cdot),\ t \in [0,T]$, we shall assume that

Hypothesis 1.3 *(i) There exists a measurable mapping $a : Y \to \mathbb{R}$ and $c > 0$ such that*

$$\langle F(t, y + z), y\rangle \le a(z)|y| + c|y|^2, \quad \forall\, y, z \in Y,\ t \in [0, T]. \tag{1.7}$$

(ii) There exists $\kappa > 0$ such that setting $a := \infty$ on $H\backslash Y$ we have

$$\mathbb{E}\left[a(W_A(t))^2 + |W_A(t)|^2\right] \le \kappa \quad \forall\, t \in [0, T]. \tag{1.8}$$

(iii) For each $\alpha > 0$ there exists a continuous mapping $F_\alpha : [0, T] \times H \to H$, such that for all $t \in [0, T],\ x \in H$,

$$\lim_{\alpha \to 0} F_\alpha(t, x) = F(t, x), \tag{1.9}$$

$$|F_\alpha(t, x)| \le |F(t, x)|, \tag{1.10}$$

$$|F(t, x) - F_\alpha(t, x)| \le \alpha |F(t, x)|^2. \tag{1.11}$$

Example 1.4 Let $H = L^2(0, 1)$, $Ax = D^2x$ for all $x \in H^2(0, 1)$ such that $x(0) = x(1) = 0$, $C = I$. Moreover, let p be a polynomial of odd degree $d > 1$ and such that

$$p'(\xi) \le \beta, \quad \forall\, \xi \in \mathbb{R},$$

where $\beta \in \mathbb{R}$. Finally, let $h : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be bounded and continuous. Then set

$$F(t, x)(\xi) = p(x(\xi)) + h(t, x(\xi)), \quad x \in L^{2d}(0, 1),\ \xi \in [0, 1],$$

and $Y = L^{2d}(0, 1)$. It is easy to see that Hypotheses 1.1 and 1.3 are fulfilled with

$$a(z) = |p(z)| + \sup_{(t,s) \in \mathbb{R} \times \mathbb{R}} |h(t, s)|, \quad \forall\, z \in Y$$

and $c = \beta$ (cf. Section 3 for details).

Remark 1.5 Under Hypotheses 1.1 and 1.3 we do not know whether equation (1.1) has a solution or not. Notice that (1.7) is a weaker condition than quasi-monotonicity of $F(t, \cdot)$.

In a series of papers [1], [2], [3] and [4] we considered parabolic equations for measures on $\mathbb{R}^d$. In [5] and [6] (see also [9] for the case when F is independent of t) we were concerned with similar problems in infinite dimensions. Here we present a different existence result.

2 Existence

It is convenient to introduce a family of approximating stochastic equations

$$\begin{cases} dX_\alpha(t) = [AX_\alpha(t) + F_\alpha(t, X_\alpha(t))]dt + \sqrt{C}dW(t), \\ \\ X_\alpha(0) = x. \end{cases} \tag{2.1}$$

For each $\alpha \in (0,1]$, $F_\alpha : [0,T] \times H$ is well defined and continuous by Hypothesis 1.3(iii).

Since $C^{-1} \in L(H)$, by Girsanov's theorem it follows that equation (2.1) has a unique weak solution which we denote by $X_\alpha(\cdot, x)$. Let us introduce the transition evolution operator

$$P_{0,t}^\alpha \varphi(x) = \mathbb{E}[\varphi(X_\alpha(t,x))], \quad t > 0,\ \varphi \in B_b(H). \tag{2.2}$$

The Kolmogorov operator L_α corresponding to (2.1) is for $u \in D(L_0)$ given by

$$L_\alpha u(t,x) = D_t u(t,x) + \frac{1}{2}\,\mathrm{Tr}\,[CD_x^2 u(t,x)] + \langle x, A^* D_x u(t,x)\rangle + \langle F_\alpha(t,x), D_x u(t,x)\rangle. \tag{2.3}$$

and the Fokker–Planck equation looks like

$$\int_H u(t,x)\mu_t^\alpha(dx) = \int_H u(0,x)\zeta(dx) + \int_0^t ds \int_H L_\alpha u(s,x)\mu_s^\alpha(dx), \quad \text{for all } t \in [0,T],\ \forall\, u \in D(L_0), \tag{2.4}$$

or

$$\int_0^T ds \int_{[0,T]\times H} L_\alpha u(s,x)\mu^\alpha(dt,dx) = -\int_H u(0,x)\zeta(dx), \quad \forall\, u \in D(L_0), \tag{2.5}$$

where $\mu^\alpha(dt,dx) = \mu_t^\alpha(dx)dt$.

We need a further assumption.

Hypothesis 2.1 *There exist $K > 0$ and a lower semicontinuous function $\tilde{F} : [0,T] \times H \to [0,\infty]$ such that $|F| + |x| \le \tilde{F}$ on $[0,T] \times H$, where $|F| := \infty$ on $[0,T] \times (H \setminus Y)$, and*

$$\mathbb{E}|\tilde{F}(t, X_\alpha(t,x))|^2 \le K(1 + |\tilde{F}(t,x)|^2), \quad \forall\, x \in Y,\ \alpha \in (0,1],\ t \in [0,T]. \tag{2.6}$$

Define

$$c_1(t) := \int_0^t \int_H |\tilde{F}(s,x)|^2 ds \zeta(dx), \quad t \in [0,T]. \tag{2.7}$$

Arguing as in [4], [6] one can show that if $\zeta \in \mathscr{P}(H)$ is such that

$$c_1(T) < +\infty,$$

then equation (2.5) has a solution μ_t^α which is determined by the identity

$$\int_H \varphi(x)\mu_t^\alpha(dx) = \int_H P_{0,t}^\alpha \varphi(x)\zeta(dx), \quad \forall\, \varphi \in \mathscr{E}_A(H). \tag{2.8}$$

Lemma 2.2 *Assume that Hypothesis* 2.1 *is fulfilled. Then we have*

$$\int_0^t \int_H |\tilde{F}(s,x)|^2 \mu^\alpha(ds,dx) \le K(t + c_1(t)), \quad \forall\, \alpha \in (0,1],\ t \in [0,T]. \tag{2.9}$$

Proof. Taking into account (2.8) and (2.6) we have for all $\alpha \in (0,1]$, $t \in [0,T]$,

$$\begin{aligned}\int_0^t \int_H |\tilde{F}(s,x)|^2 \mu^\alpha(ds,dx) &= \int_0^t \int_H P_{0,t}^\alpha(|\tilde{F}(s,\cdot)|^2)(x)\zeta(dx)ds \\ &= \int_0^t \int_H \mathbb{E}|\tilde{F}(s,X_\alpha(s,x))|^2\zeta(dx)ds \\ &\le \int_{[0,T]\times H} K(1+|\tilde{F}(s,x)|^2)\zeta(dx)ds \le K(t+c_1(t)),\end{aligned}$$

so that (2.9) follows. □

We note that indeed $L_\alpha u \in L^1([0,T]\times H, \mu^\alpha)$ for all $u \in D(L_0)$ by Lemma 2.2. Furthermore, by (2.8) the map $t \mapsto \int_H u(t,x)\mu_t^\alpha(dx)$ is continuous on $[0,T]$ for all $u \in D(L_0)$. Hence since the right hand side of (2.4) is so, too, we have (2.4) for all $t \in [0,T]$ in this case.

Our aim is to pass to the limit as $\alpha \to 0$ in (2.5), proving existence for the Fokker–Planck equation (1.5). This will be done in the following two steps showing that

Step 1 $\{\mu^\alpha\}_{\alpha>0}$ is tight.

Step 2 If μ is a cluster point of $\{\mu^\alpha\}_{\alpha>0}$ there exists $\alpha_k \downarrow 0$ such that

$$\lim_{k\to\infty} \int_{[0,T]\times H} L_{\alpha_k} u \, d\mu^{\alpha_k} = \int_{[0,T]\times H} L_0 u \, d\mu, \quad \forall\, u \in \mathscr{E}_A([0,T]\times H), \tag{2.10}$$

and $\mu(dt, dx) = \mu_t(dx)dt$.

We note that Step 2 and Remark 1.2(i) imply that μ satisfies (1.4), hence by Remark 1.2(ii) after a possible modification each μ_t is a probability measure.

Let us first prove tightness of $\{\mu^\alpha\}_{\alpha>0}$.

Proposition 2.3 *Assume that Hypotheses* 1.1 *and* 1.3 *are fulfilled. Let* $\zeta \in \mathscr{P}(H)$ *such that* $\int_H |x|^2 d\zeta < \infty$. *Then* $(\mu^\alpha)_{\alpha\in(0,1]}$ *is tight.*

Proof. Set $Y_\alpha(t) = X_\alpha(t) - W_A(t)$. Then (in the mild sense)

$$\frac{d}{dt}\, Y_\alpha(t) = AY_\alpha(t) + F_\alpha(t, X_\alpha(t)), \quad t \ge 0.$$

Multiplying both sides by $Y_\alpha(t)$, yields

$$\frac{1}{2}\,\frac{d}{dt}\, |Y_\alpha(t)|^2 + |(-A)^{1/2} Y_\alpha(t)|^2 = \langle F_\alpha(t, Y_\alpha(t) + W_A(t)), Y_\alpha(t)\rangle.$$

By (1.10) and (1.7) we obtain

$$\frac{1}{2}\,\frac{d}{dt}\, |Y_\alpha(t)|^2 + |(-A)^{1/2} Y_\alpha(t)|^2 \le a(W_A(t))\; |Y_\alpha(t)| + c|Y_\alpha(t)|^2$$

which yields

$$\frac{d}{dt}\, |Y_\alpha(t)|^2 + 2|(-A)^{1/2} Y_\alpha(t)|^2 \le (1+c)|Y_\alpha(t)|^2 + |a(W_A(t))|^2. \tag{2.11}$$

It follows that

$$|Y_\alpha(t)|^2 \le e^{(1+c)t}|x|^2 + \int_0^t e^{(1+c)(t-s)} |a(W_A(t))|^2 ds$$

from which, taking expectation and recalling (1.8),

$$\mathbb{E}|Y_\alpha(t)|^2 \le e^{(1+c)T}(|x|^2 + \kappa). \tag{2.12}$$

Consequently,

$$\begin{aligned}\mathbb{E}|X_\alpha(t,x)|^2 &\le 2e^{(1+c)T}(|x|^2+\kappa) + 2\mathbb{E}|W_A(t)|^2 \\ &\le 2e^{(1+c)T}(|x|^2+\kappa) + 2\kappa =: \kappa_1(|x|^2+1). \quad (2.13)\end{aligned}$$

This is equivalent to

$$P^\alpha_{0,t}(|x|^2) \le \kappa_1(|x|^2+1).$$

By (2.8) it follows that

$$\int_H |x|^2 \mu^\alpha_t(dx) = \int_H P^\alpha_{0,t}(|x|^2)\zeta(dx) \le \kappa_1 \int_H |x|^2\zeta(dx) + \kappa_1. \quad (2.14)$$

Moreover, by (2.11) we get

$$2\int_0^T |(-A)^{1/2}Y_\alpha(t)|^2 dt \le |x|^2 + (1+c)\int_0^T |Y_\alpha(t)|^2 dt + \int_0^T |a(W_A(t))|^2 dt,$$

which implies

$$\begin{aligned}&\int_0^T |(-A)^\delta Y_\alpha(t)|^2 dt \\ &\le \|(-A)^{-1/2+\delta}\| \left(|x|^2 + (1+c)\int_0^T |Y_\alpha(t)|^2 dt + \int_0^T |a(W_A(t))|^2 dt\right)\end{aligned}$$

and then, taking expectation by (1.6) we obtain

$$\begin{aligned}&\int_0^T \mathbb{E}|(-A)^\delta X_\alpha(t,x)|^2 dt \\ &\le 2\|(-A)^{-1/2+\delta}\| \left(|x|^2 + (1+c)\int_0^T \mathbb{E}|Y_\alpha(t)|^2 dt + \int_0^T \mathbb{E}|a(W_A(t))|^2 dt\right) + 2c_\delta T.\end{aligned}$$

Now (1.8) and (2.12) imply

$$\begin{aligned}&\int_0^T \mathbb{E}|(-A)^\delta X_\alpha(t,x)|^2 dt \\ &\le 2\|(-A)^{-1/2+\delta}\| \left(|x|^2 + (1+c)\int_0^T (e^{(1+c)T}(|x|^2+\kappa)) dt + T\kappa\right) + 2c_\delta T \\ &=: \kappa_2(1+|x|^2).\end{aligned}$$

Consequently,

$$\int_0^T P^\alpha_{0,t}(|(-A)^\delta x|^2)dt \le \kappa_2(1+|x|^2).$$

Again by(2.8) follows that

$$\int_{[0,T]\times H} |(-A)^\delta x|^2 \mu^\alpha(dt,dx) = \int_{[0,T]\times H} P^\alpha_{0,t}(|(-A)^\delta x|^2)dt\zeta(dx)$$
$$\le \kappa_2\left(\int_H |x|^2\zeta(dx)+1\right).$$

Since $(-A)^{-\delta}$ is compact, the tightness of $(\mu_\alpha)_{\alpha\in(0,1]}$ follows by a standard argument. □

We are now ready to prove

Theorem 2.4 *Assume that Hypotheses* 1.1, 1.3 *and* 2.1 *hold and that*

$$c_1(T) = \int_0^T dt \int_H (|x|^2 + |F(t,x)|^2)\zeta(dx) < \infty.$$

Let μ be a cluster point of $(\mu^\alpha)_{\alpha\in(0,1]}$. Then μ is a solution of the Fokker–Planck equation (1.5).

Proof. Let $\alpha_k \downarrow 0$ such that (μ^{α_k}) weakly converges to μ. Since $\tilde F$ is lower semicontinuous it follows by (2.9) that

$$\int_{[0,T]\times H} |\tilde F(t,x)|^2 \mu(dt,dx) \le K(T + c_1(T)),$$

in particular, $\mu([0,T]\times Y) = 1$, because $\tilde F = \infty$ on $H \setminus Y$.

Since

$$\int_0^T ds \int_H L_{\alpha_k} u(s,x)\mu^{\alpha_k}_s(dx) = -\int_H u(0,x)\zeta(dx), \quad \forall\, u \in D(L_0),$$

it is enough to show that

$$\lim_{k\to\infty} \int_{[0,T]\times H} \langle F_{\alpha_k}(s,x), D_x u(s,x)\rangle \mu^{\alpha_k}(ds,dx)$$
$$= \int_{[0,T]\times H} \langle F(s,x), D_x u(s,x)\rangle \mu(ds,dx), \quad \forall\, u \in D(L_0). \quad (2.15)$$

and that

$$\lim_{k\to\infty}\int_{[0,T]\times H}\langle x, D_x u(s,x)\rangle \mu^{\alpha_k}(ds,dx)$$
$$= \int_{[0,T]\times H}\langle x, D_x u(s,x)\rangle \mu(ds,dx), \quad \forall\, u\in D(L_0). \quad (2.16)$$

We have in fact

$$\begin{aligned}
&\Big|\int_{[0,T]\times H}\langle F_{\alpha_k}(t,x), D_x u(s,x)\rangle\ \mu^{\alpha_k}(ds,dx)\\
&-\int_{[0,T]\times H}\langle F(s,x), D_x u(s,x)\rangle \mu(ds,dx)\Big|\\
&\le \Big|\int_{[0,T]\times H}\langle (F_{\alpha_k}(s,x)-F(s,x), D_x u(s,x)\rangle\ \mu^{\alpha_k}(ds,dx)\Big|\\
&+\Big|\int_{[0,T]\times H}\langle F(s,x), D_x u(s,x)\rangle \mu^{\alpha_k}(ds,dx)\\
&-\int_{[0,T]\times H}\langle F(s,x), D_x u(s,x)\rangle \mu(ds,dx)\Big|\\
&=: I_1+I_2.
\end{aligned} \quad (2.17)$$

In view of (1.11), (2.9) we have

$$\begin{aligned}
I_1 &\le \sup|D_x u|\int_{[0,T]\times H}|F_{\alpha_k}(s,x)-F(s,x)|\ \mu^{\alpha_k}(ds,dx)\\
&\le \alpha_k \sup|D_x u|\int_{[0,T]\times H}|\tilde F(s,x)|^2\ \mu^{\alpha_k}(ds,dx)\le K(T+c_1(T))\alpha_k\sup|D_x u|.
\end{aligned} \quad (2.18)$$

Moreover, for any $\epsilon > 0$,

$$
\begin{aligned}
I_2 \le & \Big| \int_{[0,T]\times H} \langle F_\epsilon(t,x), D_x u(t,x)\rangle \mu^{\alpha_k}(dt,dx) \\
& - \int_{[0,T]\times H} \langle F_\epsilon(t,x), D_x u(t,x)\rangle \mu(dt,dx) \Big| \\
& + \epsilon \sup |D_x u| \Big(\int_{[0,T]\times H} |F(t,x)|^2 d\mu^{\alpha_k}(dt,dx) \\
& + \int_{[0,T]\times H} |F(t,x)|^2 d\mu(dt,dx)| \Big) \\
\le & \Big| \int_{[0,T]\times H} \langle F_\epsilon(t,x), D_x u(t,x)\rangle d\mu^{\alpha_k}(dt,dx) \\
& - \int_{[0,T]\times H} \langle F_\epsilon(t,x), D_x u(t,x)\rangle d\mu(dt,dx) \Big| \\
& + 2K(T + c_1(T))\epsilon \sup |D_x u|.
\end{aligned} \tag{2.19}
$$

Now the equation (2.15) follows letting $k \to \infty$ and then $\epsilon \to 0$. (2.16) is proved analogously.

It remains to prove that $\mu(dt,dx) = \mu_t(dx)dt$. But the projection of μ onto $([0,T], \mathscr{B}([0,T]))$ is Lebesgue measure since it is the weak limit of the corresponding projections of μ^{α_k} which are all Lebesgue measure. Hence μ disintegrates as

$$\mu(dt,dx) = \mu_t(dx)dt$$

where $\mu_t(dx),\ t \in [0,T]$, are kernels. $\Box$

3 An application

Let $H = L^2(0,1)$, $A : D(A) \subset H \to H$ be defined by

$$Ax(\xi) = \partial_\xi^2 x(\xi), \quad D(A) = H^2(0,1) \cap H_0^1(0,1), \quad \xi \in [0,1].$$

Let

$$F(t,x)(\xi) = p(x(\xi)) + h(t, x(\xi)), \quad x \in L^{2m}(0,1),\ \xi \in [0,1],$$

where p is a polynomial of odd degree $m > 1$ such that $p' \le c$ and $h : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is bounded and continuous. Under these assumptions we do not know whether the stochastic equation (1.1) has a solution. Finally, let $C, C^{-1} \in L(H)$, C symmetric nonnegative

We set $Y = L^{2m}(0,1)$ and prove that Hypotheses 1.1, 1.3 and 2.1 are fulfilled.

First, Hypothesis 1.1 holds with $\omega = \pi^2$ because A^{-1} is of trace class. Let us check Hypothesis 1.3. Since the polynomial p is decreasing we have for each $y, z \in Y$

$$\begin{aligned} p(y+z)y + h(t,y+z)y &= (p(y+z) - p(z))y + p(z)y + h(t,y+z)y \\ &\le c|y|^2 + |p(z)||y| + \|h\|_\infty |y| \le c|y|^2 + c_1(1+|z|^m)|y|, \quad (3.1) \end{aligned}$$

where $c_1 > 0$. Consequently

$$\langle F(t,y+z), y\rangle \le c|y|^2 + c_1(1 + |z|^m_{L^{2m}(0,1)})\, |y|. \quad (3.2)$$

So, (1.7) holds. Moreover, (1.8) is proved in [8]. For $\alpha \in (0,1]$ define

$$F_\alpha(t,x)(\xi) = \frac{F(t,x)(\xi)}{1+\alpha|F(t,x)(\xi)|}, \quad \xi \in (0,1).$$

Hence also Hypothesis 1.3 holds since (iii) is obviously true for F_α. Finally, Hypothesis 2.1 follows from the proposition below for $\tilde F(t,x) := C(1 + |x|^m_{L^{2m}(0,1)})$ and C a large enough constant.

Proposition 3.1 *Let $\alpha > 0$. Then for any $m \in \mathbb{N}$ there exists $c_m > 0$ such that*

$$\mathbb{E}\left(|X_\alpha(t,x)|^{2m}_{L^{2m}(0,1)}\right) \le c_m(1 + |x|^{2m}_{L^{2m}(0,1)}), \quad t \in [0,T].$$

Proof. Setting $Y_\alpha(t) = X_\alpha(t) - W_A(t)$, (2.1) reduces to

$$\begin{cases} Y'_\alpha(t) = AY_\alpha(t) - F_\alpha(Y_\alpha(t) + W_A(t)), \quad t \in [0,T], \\ \\ Y_\alpha(0) = x. \end{cases}$$

Now, multiplying both sides of the first equation by $(Y_\alpha(t))^{2m-1}$ yields (after integration by parts)

$$\frac{1}{2m}\,\frac{d}{dt}\int_0^1 |Y_\alpha(t)|^{2m} d\xi + (2m-1)\int_0^1 |Y_\alpha(t)|^{2m-2}|\partial_\xi Y_\alpha(t)|^2 d\xi$$

$$= \int_0^1 F_\alpha(Y_\alpha(t) + W_A(t))Y_\alpha(t)^{2m-1} d\xi.$$

Taking into account (3.1) we find

$$\frac{1}{2m}\,\frac{d}{dt}\,\int_0^1 |Y_\alpha(t)|^{2m}d\xi + (2m-1)\int_0^1 |Y_\alpha(t)|^{2m-2}|\partial_\xi Y_\alpha(t)|^2 d\xi$$
$$\leq c\int_0^1 |Y_\alpha(t)|^{2m}d\xi + c_1\int_0^1 (1+|W_A(t)|^m)|Y_\alpha(t)|^{2m-1}d\xi.$$

Moreover,

$$\int_0^1 |Y_\alpha(t)|^{2m-2}|\partial_\xi Y_\alpha(t)|^2 d\xi = m^{-2}\int_0^1 |\partial_\xi (Y_\alpha^m(t))|^2 d\xi \geq 0.$$

Consequently, there exists constants $a_1,\ \tilde{c}>0$ such that

$$\frac{d}{dt}\,\int_0^1 |Y_\alpha(t)|^{2m}d\xi \leq \tilde{c}\int_0^1 |Y_\alpha(t)|^{2m}d\xi + a_1\int_0^1 (1+|W_A(t)|^m)^{2m}d\xi.$$

Consequently,

$$|Y_\alpha(t)|^{2m}_{L^{2m}(0,1)}d \leq e^{\tilde{c}t}\,|x|^{2m}_{L^{2m}(0,1)} + a_1\int_0^t e^{\tilde{c}(t-s)}\int_0^1 (1+|W_A(t)|^m)^{2m}d\xi\,ds,$$

and, for a constant $a_2>0$,

$$|Y_\alpha(t)|^{2m}_{L^{2m}(0,1)} \leq e^{\tilde{c}t}\,|x|^{2m}_{L^{2m}(0,1)} + a_2 \sup_{(s,\xi)\in[0,T]\times[0,1]} (1+|W_A(s,\xi)|^m)^{2m},\ \forall\, t\in[0,T].$$

By [8, Theorem 4.8] there exists $a_3>0$ such that

$$\mathbb{E}\left(|Y_\alpha(t)|^{2m}_{L^{2m}(0,1)}\right) \leq e^{\tilde{c}t}\,|x|^{2m}_{L^{2m}(0,1)} + a_3,\ \forall\, t\in[0,T],$$

and so, there exists $a_4>0$ such that

$$\mathbb{E}\left(|X_\alpha(t,x)|^{2m}_{L^{2m}(0,1)}\right) \leq e^{\tilde{c}t}\,|x|^{2m}_{L^{2m}(0,1)} + a_4,\quad \forall\, t\in[0,T].$$

Now the conclusion follows. □

In conclusion all assumptions of Theorem 2.4 are fulfilled.

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