Quasi sure q-variation of a type of continuous two-parameter Gaussian process

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Abstract: In this article, we prove that for a type of continuous two-parameter Gaussian process X_z, the quasi sure limit of the form $\sum_{i=0}^{2^n-1}\sum_{j=0}^{2^n-1}\left|X((z_{i,j}^n\wedge z, z_{i+1,j+1}^n\wedge z])\right|^q$ is zero on certain conditions, where $z=(s,t)$, $z_{i,j}^n=(i2^{-n}, j2^{-n})$.

1 Introduction

Let $\{X_z : z\in[0,1]^2\}$ be a continuous two-parameter Gaussian process such that there are some constants $\alpha,\beta\in(0,1]$ and $C>0$, $\forall z=(s,t), z'=(s',t')$, the following quasi-helix property in the sense of Kahane [7, 8]

$$\|X_{z'}-X_z\|_2\leqslant C(|s'-s|^\alpha+|t'-t|^\beta) \tag{1.1}$$

and

$$\|X((z,z'])\|_2\leqslant C|s'-s|^\alpha|t'-t|^\beta \tag{1.2}$$

hold, where

$$X\left((z,z']\right):=X_{z'}-X_{(s,t')}-X_{(s',t)}+X_z.$$

An elementary example is the bifractional Brownian sheet introduced by Houdré-Villa in 2003.

Given constants $H_1,K_1,H_2,K_2\in(0,1]$, the bifractional Brownian sheet with Hurst parameters (H_1,K_1) and (H_2,K_2) is a centered continuous two-parameter Gaussian process $\{B_z : z\in\mathbb{R}^2\}$ with the covariance

$$\begin{aligned}R(z,z'):=\mathbb{E}(B_zB_{z'}) &= \frac{1}{2^{K_1+K_2}}\Big((|s|^{2H_1}+|s'|^{2H_1})^{K_1}-|s'-s|^{2H_1K_1}\Big)\\ &\quad\cdot\Big((|t|^{2H_2}+|t'|^{2H_2})^{K_2}-|t'-t|^{2H_2K_2}\Big).\end{aligned}$$

The quasi-helix property of the one-parameter bifractional Brownian motion is proved by Houdré and Villa [5]. Later, Russo-Tudor [17] and Tudor-Xiao [18] have established some properties concerning the strong variation, the local time and the stochastic calculus of bifractional Brownian motion. In 2007, using the Malliavin calculus with respect to the Gaussian process, Es-sebaiy-Tudor [3] have derived Itô's and Tanaka's formulas for the multidimensional bifractional Brownian motion. Recently, Kruk-Russo-Tudor [9] have developed a Malliavin calculus with respect to the process having a covariance measure structure which includes the bifractional Brownian motion as a special case.

Noth that if $K_1 = K_2 = 1$, B_z coincides with the ordinary fractional Brownian sheet with Hurst parameters H_1 and H_2. In this case, Cao-He [1, 2] have proved that if $H_1+H_2 > 1$ and $q > \frac{2}{H_1+H_2}$, then the quasi sure limit of the form

$$\sum_{i=0}^{2^n-1}\sum_{j=0}^{2^n-1}\left|B((z_{i,j}^n\wedge z, z_{i+1,j+1}^n\wedge z])\right|^q$$

is zero, where $z_{i,j}^n = (i2^{-n}, j2^{-n})$. The similar results for the Brownian sheet can be found in Ren [15], Liang [10] and Liu-Ren [11].

Motivated by Liu-Ren and Cao-He's works, in this paper we generalize the results in [1, 2] to the continuous two-parameter Gaussian process with the quasi-helix property. The paper is arranged as follows: in section 2, we give some necessary notations and correlative stochastic calculus, and in section 3, after giving some lemmas, we prove our main result, finally in section 4 we state the example of bifractional Brownian sheet.

2 Preliminaries

For $z = (s,t), z' = (s',t') \in \mathbb{R}^2$, we write

$$z\leqslant z' \text{ iff } s\leqslant s' \text{ and } t\leqslant t', \qquad z < z' \text{ iff } s < s' \text{ and } t < t',$$
$$z\lesseqgtr z' \text{ iff } s\leqslant s' \text{ and } t\geqslant t', \qquad z \lneq z' \text{ iff } s < s' \text{ and } t > t'.$$

We shall adopt the notations

$$z\otimes z' = (s,t'), \quad z\wedge z' = (s\wedge s', t\wedge t') \quad \text{and} \quad z\vee z' = (s\vee s', t\vee t').$$

We also use the notation

$$(z,z'] = \{\xi\in\mathbb{R}^2 : z < \xi \leqslant z'\}$$

when $z < z'$, similarly for $[z,z')$, (z,z') and $[z,z']$.

Let $\Omega = C([0,1]^2)$ be the space of the continuous function on $[0,1]^2$, there is a probability measure $\mathbb{P}$ on $(\Omega,\mathcal{F})$, where $\mathcal{F}$ is the complete σ-algebra generated by the coordinate process such that on the probability space $(\Omega,\mathcal{F},\mathbb{P})$, the coordinate process

$$\omega(z) =: X_z(\omega)$$

is a continuous two-parameter Gaussian process satisfying (1.1) and (1.2). Suppose that H is a separate Hilbert space such that $(\Omega,\mathcal{F},\mathbb{P};H)$ forms an abstract Wiener space, and $\{W_h : h\in H\}$ denotes the isonormal Gaussian process, namely, W_h is a centered Gaussian family of random variables such that for all $h,g\in H$,

$$E(W_hW_g) = (h,g)_H.$$

We briefly recall some basic elements of the stochastic calculus of variations (cf. Nualart [13] and Malliavin [12]). Let $\mathcal{S}$ be the class of smooth and cylindrical random variables of the form

$$F = f(W_{h_1}, \cdots, W_{h_n}),$$

where $h_1, \cdots, h_n \in H$ and $f \in C_b^\infty(\mathbb{R}^n)$. We define its derivative as the H-valued random variable given by

$$DF = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(W_{h_1}, \cdots, W_{h_n}) h_i.$$

Denote by $D^{r,p}$ the Sobolev space of order r and of power p over Ω. The norm in $D^{r,p}$ is defined by

$$\|F\|_{r,p} = \left\| (I-L)^{\frac{r}{2}} F \right\|_p,$$

where $L = -D^* D$ is the Ornstein-Uhlenbeck operator. For $m \in \mathbb{N}$, by Meyer's inequality, the norm $\|F\|_{m,p}$ is equivalent to

$$\left(\sum_{i=0}^{m} \mathbb{E} \left\| D^i F \right\|_{H^{\otimes i}}^p \right)^{\frac{1}{p}}.$$

Given an open set O of Ω, its (r,p)-capacity ($p > 1$) is defined by

$$C_{r,p}(O) = \inf \Big\{ \|F\|_{r,p} : F \geqslant 0 \text{ and } F \geqslant 1 \ \mathbb{P}\text{-a.s. on } O \Big\},$$

and for any subset $A \subset \Omega$, by

$$C_{r,p}(A) = \inf \Big\{ C_{r,p}(O) : \ O \text{ is open and } O \supset A \Big\}.$$

3 Main result

By (1.1) and (1.2) we can conclude easily that for every $p > 0$, there exists a constant $C(p) > 0$ such that

$$\|X_{z'} - X_z\|_p \leqslant C(p)(|s'-s|^\alpha + |t'-t|^\beta), \tag{3.1}$$

$$\|X((z,z'])\|_p \leqslant C(p)|s'-s|^\alpha |t'-t|^\beta. \tag{3.2}$$

Now we give an assumption which is the extensions of (3.2):

For every $m \in \mathbb{N}$ and $p > 0$, there exists a constant $C(m,p) > 0$ such that

$$\|X((z,z'])\|_{m,p} \leqslant C(m,p)|s'-s|^\alpha |t'-t|^\beta. \tag{3.3}$$

We can now state our main result.

Theorem 3.1. *Suppose that the continuous two-parameter Gaussian process $\{X_z : z \in [0,1]^2\}$ satisfies* (1.1) *and* (3.3)*. If $\alpha+\beta > 1$ and $q > \frac{2}{\alpha+\beta}$, then the convergence*

$$\lim_{n\to\infty} \sum_{i=0}^{2^n-1}\sum_{j=0}^{2^n-1} \left| X((z_{i,j}^n \wedge z, z_{i+1,j+1}^n \wedge z]) \right|^q = 0$$

holds uniformly in $z \in [0,1]^2$, $([q],p)$-q.s. for every $p > 1$, where $z_{i,j}^n = (i2^{-n}, j2^{-n})$, $i,j = 0,1,\cdots,2^n$. Thus, in particular, it holds uniformly in $z \in [0,1]^2$, $(\infty,2)$-q.s..

We need to prepare a series of lemmas for the proof. The following Lemma comes from Ren [14], He-Ren [4] and Cao-He [1, 2].

Lemma 3.2. *(1) $\forall m \in \mathbb{N}$, $m \geqslant k \in \mathbb{N}$ and $p \geqslant 1$, there exists a constant $C(m,k,p) > 0$ such that*

$$\|F^m\|_{k,p} \leqslant C(m,k,p)\|F\|_{mp}^{m-k}\|F\|_{k,mp}^k. \tag{3.4}$$

(2) $\forall q \geqslant 1$, $[q] \geqslant k \in \mathbb{N}$ and $p \geqslant 2$, there exists a constant $C(q,k,p) > 0$ such that

$$\left\| |F|^q \right\|_{k,p} \leqslant C(q,k,p)\|F\|_{qp}^{q-k}\|F\|_{k,qp}^k. \tag{3.5}$$

In the following, we fix $q > \frac{2}{\alpha+\beta}$, $p > 0$, and C denotes a positive constant whose values may change from line to line.

Set

$$Y_{0,z} = 0, \qquad Y_{2^{-n},z} = \sum_{i=0}^{2^n-1}\sum_{j=0}^{2^n-1} \left| X((z_{i,j}^n \wedge z, z_{i+1,j+1}^n \wedge z]) \right|^q,$$

and if $2^{-(n+1)} < \sigma < 2^{-n}$, then we define

$$Y_{\sigma,z} \;=\; Y_{2^{-n},z} + 2^{n+1}(\sigma - 2^{-n})\Big(Y_{2^{-n},z} - Y_{2^{-(n+1)},z}\Big).$$

Lemma 3.3. *There exists a constant $C(q,p) > 0$ such that $\forall n \in \mathbb{N}$,*

$$\sup_z \left\| Y_{2^{-n},z} - Y_{0,z} \right\|_{[q],p} \leqslant C(q,p)2^{-n(q(\alpha+\beta)-2)}. \tag{3.6}$$

Proof. By (3.5) and the assumption (3.3), we have

$$\begin{aligned}
&\left\| Y_{2^{-n},z} - Y_{0,z} \right\|_{[q],p} \\
=\;& \left\| \sum_{i=0}^{2^n-1}\sum_{j=0}^{2^n-1} \left| X((z_{i,j}^n \wedge z, z_{i+1,j+1}^n \wedge z]) \right|^q \right\|_{[q],p} \\
\leqslant\;& C \sum_{i=0}^{2^n-1}\sum_{j=0}^{2^n-1} \left\| X((z_{i,j}^n \wedge z, z_{i+1,j+1}^n \wedge z]) \right\|_{[q],qp}^q \\
\leqslant\;& C \sum_{i=0}^{2^n-1}\sum_{j=0}^{2^n-1} \left| s_{i+1}^n \wedge s - s_i^n \wedge s \right|^{q\alpha} \left| t_{j+1}^n \wedge t - t_j^n \wedge t \right|^{q\beta}
\end{aligned}$$

$$
\begin{aligned}
&\leqslant C2^{-n(q\alpha-1)}2^{-n(q\beta-1)}\left(\sum_{i=0}^{2^n-1}\left|s_{i+1}^n\wedge s-s_i^n\wedge s\right|\right)^{q\alpha\wedge1}\left(\sum_{j=0}^{2^n-1}\left|t_{j+1}^n\wedge t-t_j^n\wedge t\right|\right)^{q\beta\wedge1}\\
&= C2^{-n(q(\alpha+\beta)-2)}s^{q\alpha\wedge1}t^{q\beta\wedge1}\\
&\leqslant C2^{-n(q(\alpha+\beta)-2)},
\end{aligned}
$$

this proves (3.6). □

Apparently, we have

$$
\sup_{\sigma,z}\left\|Y_{\sigma,z}\right\|_{[q],p}\leqslant C(q,p) \tag{3.7}
$$

by Lemma 3.3.

Lemma 3.4. *There exists a constant $C(q,p)>0$ such that*

$$
\sup_n\left\|Y_{2^{-n},z'}-Y_{2^{-n},z}\right\|_p\leqslant C(q,p)\Big(|s'-s|^{\alpha+\beta-1}+|t'-t|^{\alpha+\beta-1}\Big). \tag{3.8}
$$

Proof. We need only to consider two cases $z\leqslant z'$ and $z\lesseqgtr z'$. Since the methods are similar, we only prove (3.8) in the case of $z<z'$. Put

$$
z_n=(s_n,t_n)=\left(\frac{[2^ns]}{2^n},\frac{[2^nt]}{2^n}\right)\ ,\ z_n^+=(s_n^+,t_n^+)=\left(\frac{[2^ns]+1}{2^n},\frac{[2^nt]+1}{2^n}\right),
$$

and set $z_n'=(s_n',t_n')$, ${z_n'}^+=({s_n'}^+,{t_n'}^+)$, where $[t]$ denotes the maximal integer not bigger than t. We will discuss in the following two cases as we usually do (see Liang [10] and Liu-Ren [11]).

(1) The case $z_n=z_n'$.

In this case, we have

$$
\begin{aligned}
&Y_{2^{-n},z'}-Y_{2^{-n},z}\\
=\ &\left(\left|X\Big((z_n,z']\Big)\right|^q-\left|X\Big((z_n,z]\Big)\right|^q\right)\\
&+\sum_{j=0}^{2^nt_n-1}\left(\left|X\Big((z_n\otimes z_{i,j}^n,z'\otimes z_{i+1,j+1}^n]\Big)\right|^q-\left|X\Big((z_n\otimes z_{i,j}^n,z\otimes z_{i+1,j+1}^n]\Big)\right|^q\right)\\
&+\sum_{i=0}^{2^ns_n-1}\left(\left|X\Big((z_{i,j}^n\otimes z_n,z_{i+1,j+1}^n\otimes z']\Big)\right|^q-\left|X\Big((z_{i,j}^n\otimes z_n,z_{i+1,j+1}^n\otimes z]\Big)\right|^q\right)\\
=:\ &J_1+J_2+J_3.
\end{aligned}
$$

By

$$
|a^q-b^q|\leqslant q(a\vee b)^{q-1}|a-b|\ (a,b\geqslant0,q\geqslant1),
$$

Hölder inequality, (3.1) and (3.2) we have

$$
\|J_1\|_p\ \leqslant\ C\left\|\left|X\Big((z_n,z']\Big)\right|\vee\left|X\Big((z_n,z]\Big)\right|\right\|_{(q-1)2p}^{q-1}
$$

$$\cdot\Big\|X\Big((z_n,z']\Big)-X\Big((z_n,z]\Big)\Big\|_{2p}$$
$$\leqslant\ C\left(|s'-s|^{\alpha}+|t'-t|^{\beta}\right),$$

and

$$\begin{aligned}
\|J_2\|_p &\leqslant \sum_{j=0}^{2^n t_n-1}\left\|\Big|X\Big((z_n\otimes z^n_{i,j},z'\otimes z^n_{i+1,j+1}]\Big)\Big|^q-\Big|X\Big((z_n\otimes z^n_{i,j},z\otimes z^n_{i+1,j+1}]\Big)\Big|^q\right\|_p\\
&\leqslant C\sum_{j=0}^{2^n t_n-1}\left\|X\Big((z_n\otimes z^n_{i,j},z'\otimes z^n_{i+1,j+1}]\Big)-X\Big((z_n\otimes z^n_{i,j},z\otimes z^n_{i+1,j+1}]\Big)\right\|_{2p}\\
&= C\sum_{j=0}^{2^n t_n-1}\left\|X\Big((z\otimes z^n_{i,j},z'\otimes z^n_{i+1,j+1}]\Big)\right\|_{2p}\\
&\leqslant C|s'-s|^{\alpha}\sum_{j=0}^{2^n t_n-1}|t^n_{j+1}-t^n_j|^{\beta}\\
&\leqslant C|s'-s|^{\alpha+\beta-1}|s'-s|^{1-\beta}(2^n t_n)^{1-\beta}\left(\sum_{j=0}^{2^n t_n-1}|t^n_{j+1}-t^n_j|\right)^{\beta}\\
&\leqslant C|s'-s|^{\alpha+\beta-1}.
\end{aligned}$$

Similarly, we have

$$\|J_3\|_p\leqslant C|t'-t|^{\alpha+\beta-1}.$$

So

$$\Big\|Y_{2^{-n},z'}-Y_{2^{-n},z}\Big\|_p\leqslant C\Big(|s'-s|^{\alpha+\beta-1}+|t'-t|^{\alpha+\beta-1}\Big).$$

(2) The case $z_n<z'_n$.
In this case, we have

$$\begin{aligned}
&Y_{2^{-n},z'}-Y_{2^{-n},z}\\
=\ &\Big|X\Big((z_n,z_n^+]\Big)\Big|^q-\Big|X\Big((z_n,z]\Big)\Big|^q+\Big|X\Big((z'_n,z']\Big)\Big|^q\\
+\ &\sum_{j=0}^{2^n t_n-1}\left(\Big|X\Big((z_n\otimes z^n_{i,j},z_n^+\otimes z^n_{i+1,j+1}]\Big)\Big|^q-\Big|X\Big((z_n\otimes z^n_{i,j},z\otimes z^n_{i+1,j+1}]\Big)\Big|^q\right)\\
+\ &\sum_{i=0}^{2^n s_n-1}\left(\Big|X\Big((z^n_{i,j}\otimes z_n,z^n_{i+1,j+1}\otimes z_n^+]\Big)\Big|^q-\Big|X\Big((z^n_{i,j}\otimes z_n,z^n_{i+1,j+1}\otimes z]\Big)\Big|^q\right)\\
+\ &\sum_{i=2^n s_n^+}^{2^n s'_n-1}\Big|X\Big((z^n_{i,j}\otimes z_n,z^n_{i+1,j+1}\otimes z_n^+]\Big)\Big|^q+\sum_{j=2^n t_n^+}^{2^n t'_n-1}\Big|X\Big((z_n\otimes z^n_{i,j},z_n^+\otimes z^n_{i+1,j+1}]\Big)\Big|^q\\
+\ &\sum_{i=2^n s_n^+}^{2^n s'_n-1}\sum_{j=0}^{2^n t_n-1}\Big|X\Big((z^n_{i,j},z^n_{i+1,j+1}]\Big)\Big|^q+\sum_{i=0}^{2^n s_n-1}\sum_{j=2^n t_n^+}^{2^n t'_n-1}\Big|X\Big((z^n_{i,j},z^n_{i+1,j+1}]\Big)\Big|^q
\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{i=2^n s_n^+}^{2^n s_n'-1} \sum_{j=2^n t_n^+}^{2^n t_n'-1} \left| X\Big((z_{i,j}^n, z_{i+1,j+1}^n] \Big) \right|^q \\
&+ \sum_{j=0}^{2^n t_n'-1} \left| X\Big((z_n' \otimes z_{i,j}^n, z' \otimes z_{i+1,j+1}^n] \Big) \right|^q + \sum_{i=0}^{2^n s_n'-1} \left| X\Big((z_{i,j}^n \otimes z_n', z_{i+1,j+1}^n \otimes z'] \Big) \right|^q \\
=:&\ I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7,
\end{aligned}
$$

where $I_k, k = 1, \cdots, 7$ denote the k-th line of the RHS of the equation. Similarity to J_1, J_2, J_3 we have

$$
\begin{aligned}
\|I_1\|_p &\leqslant C\Big(|s_n^+ - s|^\alpha + |t_n^+ - t|^\beta + |s' - s_n'|^{q\alpha} |t' - t_n'|^{q\beta} \Big) \\
&\leqslant C\Big(|s_n^+ - s|^\alpha + |t_n^+ - t|^\beta + |s' - s_n'|^{2q\alpha} + |t' - t_n'|^{2q\beta} \Big),
\end{aligned}
$$

and

$$
\|I_2\|_p \leqslant C |s_n^+ - s|^{\alpha+\beta-1}, \quad \|I_3\|_p \leqslant C |t_n^+ - t|^{\alpha+\beta-1}.
$$

By the same way, we obtain

$$
\begin{aligned}
&\left\| \sum_{i=2^n s_n^+}^{2^n s_n'-1} \left| X\Big((z_{i,j}^n \otimes z_n, z_{i+1,j+1}^n \otimes z_n^+] \Big) \right|^q \right\|_p \\
\leqslant& \sum_{i=2^n s_n^+}^{2^n s_n'-1} \left\| X\Big((z_{i,j}^n \otimes z_n, z_{i+1,j+1}^n \otimes z_n^+] \Big) \right\|_{pq}^q \\
\leqslant&\ C \sum_{i=2^n s_n^+}^{2^n s_n'-1} |s_{i+1}^n - s_i^n|^{q\alpha} |t_n^+ - t_n|^{q\beta} \\
\leqslant&\ C |s_n' - s_n^+|^{q\alpha \vee 1},
\end{aligned}
$$

and

$$
\left\| \sum_{j=2^n t_n^+}^{2^n t_n'-1} \left| X\Big((z_n \otimes z_{i,j}^n, z_n^+ \otimes z_{i+1,j+1}^n] \Big) \right|^q \right\|_p \leqslant C |t_n' - t_n^+|^{q\beta \vee 1},
$$

we deduce

$$
\|I_4\|_p \leqslant C(|s_n' - s_n^+|^{q\alpha \vee 1} + |t_n' - t_n^+|^{q\beta \vee 1}).
$$

Similarly,

$$
\begin{aligned}
&\left\| \sum_{i=2^n s_n^+}^{2^n s_n'-1} \sum_{j=0}^{2^n t_n-1} \left| X\Big((z_{i,j}^n, z_{i+1,j+1}^n] \Big) \right|^q \right\|_p \\
\leqslant& \sum_{i=2^n s_n^+}^{2^n s_n'-1} \sum_{j=0}^{2^n t_n-1} \left\| X\Big((z_{i,j}^n, z_{i+1,j+1}^n] \Big) \right\|_{pq}^q
\end{aligned}
$$

$$\leqslant C \sum_{i=2^n s_n^+}^{2^n s_n'-1} \sum_{j=0}^{2^n t_n-1} |s_{i+1}^n - s_i^n|^{q\alpha} |t_{j+1}^n - t_j^n|^{q\beta}$$
$$\leqslant C|s_n' - s_n^+|,$$

and

$$\left\| \sum_{i=0}^{2^n s_n-1} \sum_{j=2^n t_n^+}^{2^n t_n'-1} \left| X\Big((z_{i,j}^n, z_{i+1,j+1}^n] \Big) \right|^q \right\|_p \leqslant C|t_n' - t_n^+|,$$

so that

$$\|I_5\|_p \leqslant C(|s_n' - s_n^+| + |t_n' - t_n^+|).$$

Trivially,

$$\begin{aligned} \|I_6\|_p &\leqslant \sum_{i=2^n s_n^+}^{2^n s_n'-1} \sum_{j=2^n t_n^+}^{2^n t_n'-1} \left\| X\Big((z_{i,j}^n, z_{i+1,j+1}^n] \Big) \right|_{pq}^q \\ &\leqslant C \sum_{i=2^n s_n^+}^{2^n s_n'-1} \sum_{j=2^n t_n^+}^{2^n t_n'-1} |s_{i+1}^n - s_i^n|^{q\alpha} |t_{j+1}^n - t_j^n|^{q\beta} \\ &\leqslant C|s_n' - s_n^+||t_n' - t_n^+| \\ &\leqslant C(|s_n' - s_n^+|^2 + |t_n' - t_n^+|^2). \end{aligned}$$

At last,

$$\begin{aligned} &\left\| \sum_{j=0}^{2^n t_n'-1} \left| X\Big((z_n' \otimes z_{i,j}^n, z' \otimes z_{i+1,j+1}^n] \Big) \right|^q \right\|_p \\ \leqslant & \sum_{j=0}^{2^n t_n'-1} \left\| X\Big((z_n' \otimes z_{i,j}^n, z' \otimes z_{i+1,j+1}^n] \Big) \right\|_{pq}^q \\ \leqslant & \; C \sum_{j=0}^{2^n t_n'-1} |s' - s_n'|^{q\alpha} |t_{j+1}^n - t_j^n|^{q\beta} \\ \leqslant & \; C|s' - s_n'|^{q\alpha + (q\beta - 1)\wedge 0}, \end{aligned}$$

and

$$\left\| \sum_{i=0}^{2^n s_n'-1} \left| X\Big((z_{i,j}^n \otimes z_n', z_{i+1,j+1}^n \otimes z'] \Big) \right|^q \right\|_p \leqslant C|t' - t_n'|^{q\beta + (q\alpha - 1)\wedge 0},$$

we obtain

$$\|I_7\|_p \leqslant C(|s' - s_n'|^{q\alpha + (q\beta - 1)\wedge 0} + |t' - t_n'|^{q\beta + (q\alpha - 1)\wedge 0}).$$

So (3.8) holds. The cases $s_n = s_n'$ but $t_n < t_n'$ or $s_n < s_n'$ but $t_n = t_n'$ are similar. □

Note that Y is piecewise linear in σ, by (3.6), (3.8) and Kolmogorov's criterion we can conclude that there exists a constant $C(q,p) > 0$ such that

$$\left\|Y_{\sigma',z'} - Y_{\sigma,z}\right\|_p \leqslant C(q,p)\Big(|\sigma'-\sigma|^{q(\alpha+\beta)-2} + |z'-z|^{\alpha+\beta-1}\Big). \tag{3.9}$$

So $(\sigma, z) \mapsto Y_{\sigma,z}$ admits a continuous modification and the convergence

$$\lim_{n\to\infty} \sum_{i=0}^{2^n-1} \sum_{j=0}^{2^n-1} \Big|X((z_{i,j}^n \wedge z, z_{i+1,j+1}^n \wedge z])\Big|^q = 0$$

holds uniformly in $z \in [0,1]^2$, a.s..

At last, we can now give

Proof of Theorem 3.1: By (3.7), (3.9) and (3.4), we have $\forall m \geqslant [q]$, there exists a constant $C(m,q,p) > 0$ such that

$$\begin{aligned}
&\left\|\Big(Y_{\sigma',z'} - Y_{\sigma,z}\Big)^m\right\|_{[q],p} \\
\leqslant\ & C(m,q,p)\left\|Y_{\sigma',z'} - Y_{\sigma,z}\right\|_{mp}^{m-[q]} \left\|Y_{\sigma',z'} - Y_{\sigma,z}\right\|_{[q],mp}^{[q]} \\
\leqslant\ & C(m,q,p)\Big(|\sigma'-\sigma|^{(m-[q])(q(\alpha+\beta)-2)} + |z'-z|^{(m-[q])(\alpha+\beta-1)}\Big),
\end{aligned}$$

Choosing m sufficiently big such that $\Big(m-[q]\Big)\Big[(q(\alpha+\beta)-2)\wedge(\alpha+\beta-1)\Big] > 3$, we conclude that $(\sigma, z) \mapsto Y_{\sigma,z}$ admits a $([q],p)$-modification by the quasi sure version of Kolmogorov's criterion (see Ren [14, Theorem 3.1]) and the first part of the theorem is therefore proved. Combining the first part and Hu-Ren [6, Theorem 2.19] (or Ren-Zhang [16, Theorem 3.7]) gives the second part and the proof is thus finished. □

4 Example: bifractional Brownian sheet

From now on we consider the bifractional Brownian sheet B_z with Hurst parameters (H_1, K_1) and (H_2, K_2). Let

$$\begin{aligned}
R_{H,K}(t,t') &:= \frac{1}{2^K}((|t|^{2H} + |t'|^{2H})^K - |t'-t|^{2HK}), \\
Q_{H,K}(t,t') &:= |t|^{2HK} + |t'|^{2HK} - 2R_{H,K}(t,t'),
\end{aligned}$$

where $H, K \in (0,1]$. By Houdré and Villa (see [5, Proposition 2.1]), we have

$$2^{-K}|t'-t|^{2HK} \leqslant Q_{H,K}(t,t') \leqslant 2^{1-K}|t'-t|^{2HK},$$

and we can prove the following Lemma by using this conclusion.

Lemma 4.1. *There is a constant $C > 0$ such that*

$$\begin{aligned}
\|B_{z'} - B_z\|_2 &\leqslant C(|s'-s|^\alpha + |t'-t|^\beta), \\
\|B((z,z'])\|_2 &\leqslant C|s'-s|^\alpha |t'-t|^\beta,
\end{aligned}$$

where $\alpha = H_1K_1$ and $\beta = H_2K_2$.

Proof. We can easily deduce

$$\begin{aligned}
\mathbb{E}(B_{z'}-B_z)^2 &= \frac{1}{2^{K_1+K_2-1}}\Bigg(|s'-s|^{2\alpha}(|t|^{2H_2}+|t'|^{2H_2})^{K_2}+|t'-t|^{2\beta}(|s|^{2H_1}+|s'|^{2H_1})^{K_1} \\
&\quad -|s'-s|^{2\alpha}|t'-t|^{2\beta}+2^{K_1+K_2-1}(|s|^{2\alpha}|t|^{2\beta}+|s'|^{2\alpha}|t'|^{2\beta}) \\
&\quad -(|s|^{2H_1}+|s'|^{2H_1})^{K_1}(|t|^{2H_2}+|t'|^{2H_2})^{K_2}\Bigg) \\
&\leqslant 2^{1-K_1}|s'-s|^{2\alpha}+2^{1-K_2}|t'-t|^{2\beta}+\frac{1}{2}(|s'|^{2\alpha}-|s|^{2\alpha})(|t'|^{2\beta}-|t|^{2\beta}) \\
&\quad +\frac{1}{2}(|s|^{2\alpha}+|s'|^{2\alpha})(|t|^{2\beta}+|t'|^{2\beta}) \\
&\quad -\frac{1}{2^{K_1+K_2-1}}(|s|^{2H_1}+|s'|^{2H_1})^{K_1}(|t|^{2H_2}+|t'|^{2H_2})^{K_2} \\
&\leqslant 2^{1-K_1}|s'-s|^{2\alpha}+2^{1-K_2}|t'-t|^{2\beta}+\frac{1}{4}(|s'^2-s^2|^{2\alpha}+|t'^2-t^2|^{2\beta}) \\
&\quad +\left(\frac{1}{2}-\frac{1}{2^{K_1+K_2-1}}\right)(|s|^{2H_1}+|s'|^{2H_1})^{K_1}(|t|^{2H_2}+|t'|^{2H_2})^{K_2}.
\end{aligned}$$

The second inequality is established by

$$\mathbb{E}(B((z,z']))^2 = Q_{H_1,K_1}(s,s')Q_{H_2,K_2}(t,t').$$

□

The reproducing kernel Hilbert space $\tilde{H}$ is defined as the closure of the linear span of the indicator functions $\{I_{[0,z]} : z \in [0,1]^2\}$ with respect to the scalar product

$$(I_{[0,z]}, I_{[0,z']})_{\tilde{H}} = R(z,z').$$

Thus $(\Omega, \mathcal{F}, \mathbb{P}; \tilde{H})$ forms a Gaussian probability space such that

$$W_{I_{[0,z]}} = B_z,$$

and we don't need to found the abstract Wiener space because H and $\tilde{H}$ is isomorphic. Obviously, $DB_z = I_{[0,z]}$ and $\forall m \geqslant 2$, $D^m B_z = 0$.

Lemma 4.2. *$\forall p > 0$, there is a constant $C(p) > 0$ such that for every $m \in \mathbb{N}$,*

$$\begin{aligned}
\|B_{z'}-B_z\|_{m,p} &\leqslant C(p)(|s'-s|^{\alpha}+|t'-t|^{\beta}), \\
\|B((z,z'])\|_{m,p} &\leqslant C(p)|s'-s|^{\alpha}|t'-t|^{\beta}.
\end{aligned}$$

Proof. Trivially,

$$\|DB_{z'}-DB_z\|_{\tilde{H}} = \|I_{[0,z']}-I_{[0,z]}\|_{\tilde{H}} = \|B_{z'}-B_z\|_2,$$

and

$$\|DB((z,z'])\|_{\tilde{H}} = \|I_{(z,z']}\|_{\tilde{H}} = \|B((z,z'])\|_2.$$

The proof is very easy to obtain by Lemma 4.1. □

So Theorem 3.1 holds for bifractional Brownian sheet B_z by Lemma 4.2 and our theorem implies the results in Cao-He [1, 2]. By (see He-Ren [4, Lemma 2.4])

$$\|F^m\|_{r,p} \leqslant C(m,r,p)\|F\|^m_{r,mp}\ (m\in\mathbb{N})$$

and the quasi sure version of Kolmogorov's criterion, Lemma 4.2 yields that B_z admits a ∞-modification and the sample paths of which are Hölder continuous of order $\gamma\in[0,\alpha\wedge\beta)$, (m,p)-q.s. for any $m\in\mathbb{N}$ and $p>1$.

References

[1] Cao G, He K. Quasi sure p-variation of fractional Brownian motion, Statist.Probab.Lett., Vol.77/5, 2007, 543-548.

[2] Cao G, He K. Quasi sure p-variation of fractional Brownian sheet, Stochastic Anal.Appl., Vol.24/6, 2006, 1223-1238.

[3] Es-sebaiy K, Tudor C. Multidimensional bifractional Brownian motion: Itô and Tanaka formulas. Stoch. Dyn., Vol.7/3(2007), 365-388.

[4] He K, Ren J. Quasi sure quadratic variation of local times of smooth semimartingales, Bull.Sci.Math., Vol.128/2(2004), 91-104.

[5] Houdré C, Villa J. An example of infinite dimensional quasi-helix. Contemp.Math., Amer.Math.Soc., 336 (2003), 195-201.

[6] Hu J, Ren J. Infinite dimensinoal quasi continuity, path continuity and ray continuity of functions with fractional regularity, J.Math.Pures Appl., Vol.80/1(2001), 131-152.

[7] Kahane J. Hélices et quasi-hélices, Adv.Math. Vol.7B(1981), 417-433.

[8] Kahane J. Some random series of functions. Cambridge University Press, 1985.

[9] Kruk I, Russo F, Tudor C. Wiener integrals, Malliavin calculus and covariance measure structure, preprint.

[10] Liang Z. Quasi sure quadratic variations of two-parameter smooth martingales on the wiener space, J. Math.Kyoto Univ., Vol.36/3(1996), 619-640.

[11] Liu J, Ren J. Quasi-sure product variation of two-parameter smooth martingales on the Wiener space, Acta Mathematica Sinica(B), Vol.22/4(2006), 1103-1114.

[12] Malliavin P. Stochastic analysis, Springer, 1997.

[13] Nualart D. The Malliavin calculus and related topics, Springer-Verlag, Berlin, 1995.

[14] Ren J. Analysis quasi sûre des équation différentielles stochastiques, Bull.Sci.Math., Vol.114(1990), 187-213.

[15] Ren J. Quasi sure quadratic variation of smooth martingales, J.Math.Kyoto Univ., Vol.34-1(1994), 191-206.

[16] Ren J, Zhang X. Path continuity of fractional Dirichlet functionals, Bull.Sci.Math., Vol.127/4(2003), 368-378.

[17] Russo F, Tudor C. On bifractional Brownian motion. Stochastic Process.Appl., Vol.116/5(2006), 830-856.

[18] Tudor C, Xiao Y. Sample path properties of bifractional Brownian motion. Bernoulli, Vol.13/4(2007), 1023- 1052.