Harnack Inequality and Applications for Stochastic Evolution Equations with Monotone Drifts *

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Abstract

As a Generalization to [36] where the Harnack inequality and the strong Feller property are studied for stochastic porous media equations, this paper presents analogous results for a large class of stochastic evolution equations with general monotone drifts. Some ergodicity, compactness and contractivity properties are established for the associated transition semigroups. Moreover, the exponential convergence of the transition semigroups to invariant measure and the existence of a spectral gap are also derived. As examples, the main results can be applied to many concrete SPDEs such as stochastic reaction-diffusion equations, stochastic porous media equations and stochastic p-Laplace equation in Hilbert space.

Keywords: stochastic evolution equation; Harnack inequality; strong Feller property; ergodicity; hyperbounded; ultrabounded; spectral gap; p-Laplace equation; porous media equation.
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1 Introduction and Main results

The dimension-free Harnack inequality has been a very efficient tool for the study of diffusion semigroups in recent years. It was first introduced by Wang in [32] for diffusions on Riemannian manifolds, then this infinite dimensional version of Harnack inequality has been applied and extended intensively later, see e.g. [33, 35, 28, 29] for applications to functional inequalities; [1, 2, 16] for the study of short time behavior of infinite-dimensional

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diffusions; [14, 34] for the estimate of high order eigenvalues, and [5] for applications to the transportation-cost inequality and [13] for heat kernel estimates.

Recently, the dimension-free Harnack inequality was established in [36] for stochastic porous media equations and in [20] for the stochastic fast-diffusion equations. As applications, the strong Feller property, estimates of the transition density and some contractive properties were obtained for the associated transition semigroups. The approach used in [20, 36] is based on a coupling argument developed in [3], where the Harnack inequality was studied for diffusion semigroups on Riemannian manifolds with unbounded curvatures from below. The advantage of this approach is one can avoid the assumption on the lower bounds of curvature, which used essentially in previous works (cf.[1, 2, 5, 28, 29]) and would be very hard to verify in the present framework of non-linear SPDE.

The aim of this paper is to establish the analogous results for general stochastic evolution equations within the variational framework. More precisely, we mainly deal with the stochastic evolution equations with monotone drifts in Hilbert space, whick cover many important types of SPDE such as stochastic reaction-diffusion equations, stochastic porous media equations and stochastic p-Laplace equation (cf.[26, 18, 38]). We first establish the Harnack inequality and the strong Feller property for the associated transition semigroups, then it has been used to derive some ergodicity and contractivity properties for the corresponding transition semigroups. In particular, we give a very easy proof for the (topological) irreduciblity by using the Harnack inequality in Theorem 1.4. Hence we can obtain the uniqueness of invariant measures for the transition semigroups without assuming the strict monotonicity of the drift, which has been required in many earlier works [26, 36, 20, 27, 8]. And we also obtain the convergence rate of the transition semigroups to its equilibrium. This result implies the estimate of decay of the solutions to the deterministic evolution equations (e.g. p-Laplace equation, porous medium equation), which coincide with some well-known results in PDE theory. Moreover, the existence of a spectral gap are also investigated for the corresponding Kolmogorov operator.

Now we describe our framework for SPDE in details. There exist three main different approaches to analyze stochastic partial differential equations in the literature. The "martingale measure approach" was initiated by J. Walsh in [31]. The "variational approach" was first used by Pardoux [25] to study SPDE, then this approach was further developed by Krylov and Rozovoskii [18] and applied to non-linear filtering. Concerning the "semigroup approach" we can refer to the classical book by Da Prato and Zabcyzk [9]. In this paper we will use the variational approach because we mainly treat nonlinear SPDE of evolutionary type. All kinds of dynamics with stochastic influence in nature or man-made complex systems can be modelled by such equations. This type of SPDE has been studied intensively in recent years, we refer to [8, 12, 19, 27, 17, 26, 38](and references therein) for various generalizations and applications.

Let H be a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle_H$ and H^* its dual. Let V be a reflexive and separable Banach space such that $V \subset H$ continuously and densely. Then for its dual space V^* it follows that $H^* \subset V^*$ continuously and densely. Identifying H and

H^* via the Riesz isomorphism we know that

$$V \subset H \equiv H^* \subset V^*$$

is a Gelfand triple. If the dualization between V^* and V is denoted by ${}_{V^*}\langle \cdot, \cdot \rangle_V$ we have

$${}_{V^*}\langle u, v \rangle_V = \langle u, v \rangle_H \ \text{ for all } u \in H, v \in V.$$

Suppose W_t is a cylindrical Wiener process on a separable Hilbert space U w.r.t a complete filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbf{P})$, and $(L_2(U; H), \| \cdot \|_2)$ is the space of all Hilbert-Schmidt operators from U to H. Now we consider the following stochastic evolution equation

$$\mathrm{d}X_t = A(t, X_t)\mathrm{d}t + B_t \mathrm{d}W_t, \ X_0 = x \in H, \tag{1.1}$$

where

$$A : [0, T] \times V \times \Omega \to V^*; \ B : [0, T] \times \Omega \to L_2(U, H)$$

are progressively measurable. We first recall the classical result in [18] for the existence and uniqueness of strong solution. For more general results we refer to [12, 27, 38].

Lemma 1.1. ([18] Theorems II.2.1, II.2.2) *Consider the general stochastic evolution equation*

$$\mathrm{d}X_t = A(t, X_t)\mathrm{d}t + B(t, X_t)\mathrm{d}W_t \tag{1.2}$$

where

$$A : [0, T] \times V \times \Omega \to V^*; \quad B : [0, T] \times V \times \Omega \to L_2(U; H)$$

are progressively measurable. Suppose for a fixed $\alpha > 1$ there exist constants $\theta > 0$, K and a positive adapted process $f \in L^1([0, T] \times \Omega; dt \times \mathbf{P})$ such that the following conditions hold for all $v, v_1, v_2 \in V$ and $(t, \omega) \in [0, T] \times \Omega$.

(A1) *Hemicontinuity of A: The map*

$$\lambda \mapsto {}_{V^*}\langle A(t, v_1 + \lambda v_2), v \rangle_V$$

is continuous on $\mathbb{R}$.

(A2) *Monotonicity of (A, B):*

$$2{}_{V^*}\langle A(t, v_1) - A(t, v_2), v_1 - v_2 \rangle_V + \|B(t, v_1) - B(t, v_2)\|_2^2 \le K\|v_1 - v_2\|_H^2.$$

(A3) *Coercivity of (A, B):*

$$2{}_{V^*}\langle A(t, v), v \rangle_V + \|B(t, v)\|_2^2 + \theta\|v\|_V^\alpha \le f_t + K\|v\|_H^2.$$

(A4) *Boundedness of A:*

$$\|A(t, v)\|_{V^*} \le f_t^{\alpha/(\alpha-1)} + K\|v\|_V^{\alpha-1}.$$

Then for any $X_0 \in L^2(\Omega \to H; \mathcal{F}_0; \mathbf{P})$, (1.2) *has a unique solution* $\{X_t\}_{t\in[0,T]}$ *which is an adapted continuous process on* H *such that* $\mathbf{E}\int_0^T \|X_t\|_V^\alpha dt < \infty$ *and*

$$\langle X_t, v\rangle_H = \langle X_0, v\rangle_H + \int_0^t {}_{V^*}\langle A(s, X_s), v\rangle_V ds + \int_0^t \langle B(s, X_s)dW_s, v\rangle_H$$

hold for all $v \in V$ *and* $(t,\omega) \in [0,T] \times \Omega$.

Note that in order to using the coupling method, here we only consider equation (1.1) where the noise is the additive type. We intend to establish Harnack inequality for the associate transition semigroup

$$P_t F(x) := \mathbf{E} F(X_t(x)),\ t \geq 0,\ x \in H$$

where F is a bounded measurable function on H. To define the intrinsic metric induced by B_t, we need to assume $B_t(\omega)$ is non-degenerate for $t > 0$ and $\omega \in \Omega$; that is, $B_t(\omega)y = 0$ implies $y = 0$. Then for $u \in V$

$$\|u\|_{B_t} := \begin{cases} \|y\|_U, & \text{if } y \in U,\ B_t y = u; \\ \infty, & \text{otherwise.} \end{cases}$$

Theorem 1.2. *Suppose* (A1) − (A4) *hold for* (1.1) *with the coercivity exponent* α. *If there exist a constant* $\sigma \geq 2, \sigma > \alpha - 2$ *and continuous functions* $\delta, \gamma, \xi \in C[0,\infty)$ *such that for any* $t \geq 0, \omega \in \Omega$ *and* $u, v \in V$ *we have*

$$2_{V^*}\langle A(t,u) - A(t,v), u - v\rangle_V \leq -\delta_t N(u-v) + \gamma_t \|u-v\|_H^2, \tag{1.3}$$

$$N(u) \geq \xi_t \|u\|_{B_t}^\sigma \|u\|_H^{\alpha-\sigma}, \tag{1.4}$$

where N *is a positive real function on* V *and* ξ, δ *are strictly positive on* $[0,\infty)$, *then* P_t *is strong Feller operator for* $t > 0$, *and for any* $p > 1$ *and positive measurable function* F *on* H *we have*

$$(P_t F)^p(y) \leq P_t F^p(x) \exp\left[\frac{p}{p-1} C(t,\sigma) \|x-y\|_H^{2+\frac{2(2-\alpha)}{\sigma}}\right],\ x, y \in H, \tag{1.5}$$

where

$$C(t,\sigma) = \frac{2t^{\frac{\sigma-2}{\sigma}}(\sigma+2)^{2+\frac{2}{\sigma}}}{(\sigma+2-\alpha)^{2+\frac{2}{\sigma}}\left[\int_0^t (\delta_s\xi_s)^{\frac{1}{\sigma}} \exp(\frac{\alpha-2-\sigma}{2\sigma}\int_0^s \gamma_u du) ds\right]^2}.$$

In particular, if δ, ξ *are time-independent, then*

$$C(t,\sigma) = \frac{2(\sigma+2)^{2+\frac{2}{\sigma}}}{(\sigma+2-\alpha)^{2+\frac{2}{\sigma}}(\delta\xi)^{\frac{2}{\sigma}} t^{\frac{\sigma+2}{\sigma}}}.$$

Remark 1.1. (1) Notice $(A1) - (A4)$ are assumed in Theorem 1.2 only for the existence and uniqueness of the strong solution to (1.1). One can replace those conditions by more general ones in [27, 38] and obtain a similar result.

(2) This theorem covers the main result in [36] if we take $N(u) = \|u\|_V^{r+1}$ for the stochastic porous media equations. Moreover, if we take $N(u) = \mathbf{m}(\mathbf{g}(u))$ for some Young function $\mathbf{g}$, then this theorem can also be applied to stochastic generalized porous media equations [27] in the framework of Orlicz space.

(3) This theorem can also be applied to many other types of SPDE in [26, 18] which satisfy the strongly dissipative condition (1.3)(see Section 3). For concrete examples in this paper we can consider $N(u) = \|u\|_V^\alpha$ for simplicity. In this case (1.3) implies $(A2)$ and $(A3)$. Under (1.3) we have established a stronger version of large deviation principle in [19] for general SPDE with small multiplicative noise.

(4) Note (1.4) implies that V is contained in the range of B_t (as a operator from U to H) for fixed t and ω. If we assume $N(u) = \|u\|_V^\alpha$ and $V \equiv H$, then we know B_t is a bijection map and its inverse operator is also continuous from H to U. Since B_t is a Hilbert-Schmidt operator, then H and U has to be finite dimensional space. In this case (1.4) holds provided B_t are invertible.

(5) The stochastic fast diffusion equations in [27] does not satisfy the assumption (1.3), but we have also obtained the Harnack inequality, strong Feller property and heat kernel estimate in [20] by using more delicate estimate. But we haven't obtained strong contractive property (e.g. hyperbounded) for the associated transition semigroups in [20] because of the weaker dissipativity of the drift.

To apply Theorem 1.2 to obtain the heat kernel estimates, ergodicity and contractive properties of P_t, we only consider the deterministic and time-homogenous case from now on. We first establish some properties for invariant measure.

Theorem 1.3. *Suppose coefficients A, B in* (1.1) *are deterministic and time-independent such that* $(A1)$ *and* $(A4)$ *hold. Assume* (1.3) *hold for $N(\cdot) = \|\cdot\|_V^\alpha$ and the embedding $V \subseteq H$ is compact.*

(i) If $\gamma \le 0$ also holds in the case $\alpha \le 2$, then the Markov semigroup $\{P_t\}$ has an invariant probability measure μ, which satisfies $\mu\left(\|\cdot\|_V^\alpha + e^{\varepsilon_0\|\cdot\|_H^\alpha}\right) < \infty$ for some $\varepsilon_0 > 0$.

(ii) If $\alpha = 2$, then for any $x, y \in H$ we have

$$\|X_t(x) - X_t(y)\|_H^2 \le e^{(\gamma - c_0\delta)t}\|x - y\|_H^2,\ t \ge 0,$$

where c_0 is the constant such that $\|\cdot\|_V^2 \ge c_0\|\cdot\|_H^2$ hold.

Moreover, if $\gamma < c_0\delta$, then there exists a unique invariant measure μ of $\{P_t\}$ and for any Lipschitz continuous function F on H we have

$$|P_tF(x) - \mu(F)| \le Lip(F)e^{-(c_0\delta-\gamma)t/2}(\|x\|_H + C),\ x \in H, \tag{1.6}$$

where $C>0$ is a constant and $Lip(F)$ is the Lipschitz constant of F.

(iii) If $\alpha>2$ and $\gamma\le 0$, then there exists a constant C such that

$$\|X_t(x)-X_t(y)\|_H^2 \le \|x-y\|_H^2 \wedge \left\{Ct^{-\frac{2}{\alpha-2}}\right\},\ t>0,$$

where $X_t(y)$ is the solution to (1.1) with starting point y.

Therefore, $\{P_t\}$ has a unique invariant measure μ and for any Lipschitz continuous function F on H we have

$$\sup_{x\in H}|P_tF(x)-\mu(F)| \le C Lip(F) t^{-\frac{1}{\alpha-2}},\ t>0. \tag{1.7}$$

Remark 1.2. (1.7) describes the algebraically convergence rate of the transition semigroup to the equilibrium. In particular, if $B=0$ and Dirac measure at 0 is the unique invariant measure of $\{P_t\}$, then we can take $F(x)=\|x\|_H$ in (1.7) and have

$$\sup_{x\in H}\|X_t(x)\|_H \le Ct^{-\frac{1}{\alpha-2}},\ t>0.$$

Hence it give the decay estimate of the solution to a large class of deterministic evolution equations. These results coincide with some well-known decay estimates in PDE theory, e.g. the optimal decay of the solution to the classical porous medium equation in [4, 8]. We refer to Section 3 for more examples.

We recall that $\{P_t\}$ is called (topologically) irreducible if $P_t 1_M(\cdot)>0$ on H for any $t>0$ and nonempty open set M. If $\{P_t\}$ is a semigroup defined on $L^2(\mu)$, then $\{P_t\}$ is called hyperbounded semigroup if $\|P_t\|_{L^2(\mu)\to L^4(\mu)}<\infty$ for some $t>0$; $\{P_t\}$ is called ultrabounded semigroup if $\|P_t\|_{L^2(\mu)\to L^\infty(\mu)}<\infty$ for any $t>0$.

Theorem 1.4. *Suppose coefficients A,B in (1.1) are deterministic and time-independent such that all assumptions in Theorem 1.2 hold for $N(\cdot)=\|\cdot\|_V^\alpha$.*

(i) $\{P_t\}$ is irreducible and has a unique invariant measure μ with full support on H. Moreover, μ is strong mixing and for any probability measure ν on H we have

$$\lim_{t\to\infty}\| P_t^*\nu-\mu \|_{var}=0,$$

where $\|\cdot\|_{var}$ is the total variation norm and P_t^* is the adjoint operator of P_t.

(ii) For any $x\in H$, $t>0$ and $p>1$, the transition density $p_t(x,y)$ of P_t w.r.t μ satisfies

$$\|p_t(x,\cdot)\|_{L^p(\mu)} \le \left\{\int_H \exp\left[-pC(t,\sigma)\|x-y\|_H^{2+\frac{2(2-\alpha)}{\sigma}}\right]\mu(dy)\right\}^{-\frac{p-1}{p}}.$$

(iii) If $\alpha=2$, then P_t is hyperbounded and compact on $L^2(\mu)$ for some $t>0$.

(iv) If $\alpha>2$, then P_t is ultrabounded and compact on $L^2(\mu)$ for any $t>0$. Moreover, there exists a constant $C>0$ such that

$$\|P_t\|_{L^2(\mu)\to L^\infty(\mu)} \le \exp\left[C(1+t^{-\frac{\alpha}{\alpha-2}})\right],\ t>0.$$

Remark 1.3. Based on the Harnack inequality, the irreducibility can be obtained very easily for the transition semigroup. Then according to Doob's theorem (see [22, 15]) one can derive the uniqueness of invariant measures and some ergodic properties for the transition semigroup. Comparing with the uniqueness result for invariant measure in Theorem 1.3, we do not need to assume $\gamma \le 0$ or $\gamma < c_0\delta$ in this case.

Let L_p be the generator of the semigroup $\{P_t\}$ in $L^p(\mu)$. We say that L_p has the spectral gap in $L^p(\mu)$ if there exists $\gamma > 0$ such that

$$\sigma(L_p) \cap \{\lambda : Re\lambda > -\gamma\} = \{0\}$$

where $\sigma(L_p)$ is the spectrum of L_p. The largest constant γ with this property is denoted by $gap(L_p)$.

Theorem 1.5. *Suppose all assumptions in Theorem 1.4 hold and μ denotes the unique invariant measure of $\{P_t\}$.*

(i) If $\alpha = 2$ and $\gamma < c_0\delta$, then the Markov semigroup $\{P_t\}$ is V-uniformly ergodic, i.e. there exist $C, \eta > 0$ such that for all $t \ge 0$ and $x \in H$

$$\sup_{\|F\|_V \le 1} |P_t F(x) - \mu(F)| \le C V(x) e^{-\eta t},$$

where we can take $V(x) = 1 + \|x\|_H^2$ and $V(x) = e^{\varepsilon_0 \|x\|_H^2}$ for some small constant $\varepsilon_0 > 0$,

$$\|F\|_V := \sup_{x \in H} \frac{|F(x)|}{V(x)} < \infty.$$

(ii) If $\alpha > 2$, then the Markov semigroup $\{P_t\}$ is uniformly exponential ergodic, i.e. there exist $C, \eta > 0$ such that for all $t \ge 0$ and $x \in H$

$$\sup_{\|F\|_\infty \le 1} |P_t F(x) - \mu(F)| \le C e^{-\eta t}.$$

Moreover, for each $p \in (1, \infty)$ we have

$$gap(L_p) \ge \frac{\eta}{p},$$

and for each $F \in L^p(\mu)$

$$\|P_t F - \mu(F)\|_p \le C_p e^{-\eta t/p} \|F\|_p, \ t \ge 0.$$

Remark 1.4. Let $\|\mu\|_V$ denote the so-called V-variation

$$\|\mu\|_V = \sup_{\|F\|_V \le 1} |\mu(F)| = \|V \circ \mu\|_{var},$$

where $V \circ \mu$ denotes the measure $V d\mu$. Hence the uniformly exponential ergodicity is equivalent to for any probability measure ν on H

$$\|P_t^* \nu - \mu\|_{var} \le C e^{-\eta t}, t \ge 0.$$

Since $\|\cdot\|_{var} \le \|\cdot\|_V$, the V-uniformly ergodicity implies that

$$\|P_t^* \nu - \mu\|_{var} \le C \nu(V) e^{-\eta t}, t \ge 0.$$

The theorems will be proved in Section 2. To apply the main results, one has to verify condition (1.3) and (1.4). For this purpose a crucial inequality is proved as a lemma in Section 3. Then some concrete examples are discussed as applications.

2 Proofs of the Main Theorems

2.1 Proof of Theorem 1.2

The main techniques in the proof are a coupling argument and Girsanov transformation in infinite dimensional space (cf.[20, 36]). The coupling method dates back to Doeblin's work [10] on Markov chains and it is one of the main tools in particle systems (cf.[6]). The first use of coupling for SPDE up to our knowledge is due to Mueller [24], who used this technique to prove the uniqueness of invariant measures for the stochastic heat equation. We refer to some review papers [23, 21, 15] on this subject for more references.

The coupling we used here, which only depends on the natural distance between two marginal processes, is a modification of the argument in [3]. Such a stronger Harnack inequality (the estimate only depending on the usual norm) will provide more information such as the strong Feller property and the hyper- or ultrabounded property of the transition semigroups. In order to make the proof easier to understand, we first describe the main ideas and steps.

To prove the Harnack inequality for the transition semigroup $\{P_t\}$, it suffices to construct a coupling (X_t, Y_t), which is a continuous adapted process on $H \times H$ such that
(i) X_t solves (1.1) with $X_0 = x$;
(ii) Y_t solves the following equation

$$\mathrm{d}Y_t = A(t, Y_t)\mathrm{d}t + B_t \mathrm{d}\tilde{W}_t,\ Y_0 = y$$

for another cylindrical Brownian motion $\tilde{W}_t$ on U under a weighted probability measure $R\mathbf{P}$, where $\tilde{W}_t$ as well as the density R will be constructed by a Girsanov transformation;
(iii) $X_T = Y_T, a.s.$

As soon as (i)-(iii) are satisfied, then we have

$$\begin{aligned} P_T F(y) &= \mathbf{E} RF(Y_T) = \mathbf{E} RF(X_T) \\ &\le (\mathbf{E} R^{p/(p-1)})^{(p-1)/p} (\mathbf{E} F^p(X_T))^{1/p} \\ &= (\mathbf{E} R^{p/(p-1)})^{(p-1)/p} (P_T F^p(x))^{1/p}, \end{aligned} \tag{2.1}$$

which implies the desired Harnack inequality provided $\mathbf{E}R^{p/(p-1)} < \infty$.

Now we constract the coupling process Y_t. We first take $\varepsilon \in (0,1),\ \beta \in \mathbf{C}([0,\infty); \mathbb{R}_+)$ and consider the equation

$$\mathrm{d}Y_t = \left(A(t, Y_t) + \frac{\beta_t (X_t - Y_t)}{\|X_t - Y_t\|_H^{\varepsilon}} \mathbf{1}_{\{t<\tau\}} \right) \mathrm{d}t + B_t \mathrm{d}W_t,\ Y_0 = y, \tag{2.2}$$

where $X_t := X_t(x)$ and $\tau := \inf\{t \geq 0 : X_t = Y_t\}$ is the coupling time.

According to Lemma 1.1 we can prove that (2.2) also has a unique strong solution $Y_t(y)$ by using a similar argument in [36, Theorem A.2] (in fact, one can prove the added drift is also monotone). Then by (1.3) we have

$$\|X_t - Y_t\|_H^2 \leq \|X_s - Y_s\|_H^2 + \int_s^t \left(-\delta_u N(X_u - Y_u) + \gamma_u \|X_u - Y_u\|_H^2 - \beta_u \|X_u - Y_u\|_H^{2-\varepsilon} \mathbf{1}_{\{u<\tau\}}\right) \mathrm{d}u$$

for all $0 \leq s \leq t$. Hence we have $X_t = Y_t$ for $t \geq \tau$ by using Gronwall's lemma.

And it is easy to show that

$$e^{-\int_0^t \gamma_s \mathrm{d}s} \|X_t - Y_t\|_H^2 \leq \|x - y\|_H^2 - \int_0^t e^{-\int_0^u \gamma_s ds} \Big(\delta_u N(X_u - Y_u) + \beta_u \|X_u - Y_u\|_H^{2-\varepsilon} \mathbf{1}_{\{u<\tau\}}\Big) \mathrm{d}u. \tag{2.3}$$

First, we will prove the coupling time $\tau \leq T$ *a.s.* by choosing β_t appropriately in (2.2).

Lemma 2.1. *If β satisfies $\int_0^T \beta_t e^{-\frac{\varepsilon}{2}\int_0^t \gamma_s \mathrm{d}s} \mathrm{d}t \geq \frac{2}{\varepsilon} \|x - y\|_H^\varepsilon$, then $X_T = Y_T$, a.s.*

Proof. By (2.3) and the chain rule we have

$$\left\{e^{-\int_0^t \gamma_s \mathrm{d}s} \|X_t - Y_t\|_H^2\right\}^{\varepsilon/2} \leq \|x - y\|_H^\varepsilon - \frac{\varepsilon}{2} \int_0^t \beta_s e^{-\frac{\varepsilon}{2}\int_0^s \gamma_u \mathrm{d}u} \mathrm{d}s, \ t \leq \tau \wedge T.$$

If $T < \tau(\omega_0)$ for some $\omega_0 \in \Omega$, then by taking $t = T$ and using the assumption we have

$$e^{-\frac{\varepsilon}{2}\int_0^T \gamma_s \mathrm{d}s} \|X_T(\omega_0) - Y_T(\omega_0)\|_H^\varepsilon \leq \|x - y\|_H^\varepsilon - \frac{\varepsilon}{2} \int_0^T \beta_t e^{-\frac{\varepsilon}{2}\int_0^t \gamma_s \mathrm{d}s} \mathrm{d}t \leq 0.$$

This implies $X_T(\omega_0) = Y_T(\omega_0)$, which contradicts with the assumption $T < \tau(\omega_0)$.

Hence $\tau \leqslant T$, *a.s.* The proof is complete. □

Proof of Theorem 1.2 : Let $\varepsilon = 1 - \frac{\alpha}{\sigma+2} \in (0, 1)$, then by (2.3) and (1.4) we have

$$\begin{aligned} \mathrm{d}\left\{\|X_t - Y_t\|_H^2 e^{-\int_0^t \gamma_s \mathrm{d}s}\right\}^\varepsilon &\leq -\varepsilon \delta_t e^{-\varepsilon \int_0^t \gamma_s \mathrm{d}s} \|X_t - Y_t\|_H^{2(\varepsilon-1)} N(X_t - Y_t) \mathrm{d}t \\ &\leq -\varepsilon \delta_t \xi_t e^{-\varepsilon \int_0^t \gamma_s \mathrm{d}s} \frac{\|X_t - Y_t\|_{B_t}^\sigma}{\|X_t - Y_t\|_H^{2+\sigma-\alpha-2\varepsilon}} \mathrm{d}t \\ &= -\varepsilon \delta_t \xi_t e^{-\varepsilon \int_0^t \gamma_s \mathrm{d}s} \frac{\|X_t - Y_t\|_{B_t}^\sigma}{\|X_t - Y_t\|_H^{\sigma\varepsilon}} \mathrm{d}t \\ &= -\frac{\beta_t^\sigma \|X_t - Y_t\|_{B_t}^\sigma}{c^\sigma \|X_t - Y_t\|_H^{\sigma\varepsilon}} \mathrm{d}t, \end{aligned} \tag{2.4}$$

where

$$\beta_t^\sigma = c^\sigma \varepsilon \delta_t \xi_t e^{-\varepsilon \int_0^t \gamma_s \mathrm{d}s}, \quad c = \frac{2\|x - y\|_H^\varepsilon}{\varepsilon \int_0^T (\varepsilon \delta_t \xi_t)^{\frac{1}{\sigma}} e^{-(\frac{1}{2}+\frac{1}{\sigma})\varepsilon \int_0^t \gamma_s \mathrm{d}s} \mathrm{d}t}.$$

Let

$$\zeta_t := \frac{\beta_t B_t^{-1}(X_t - Y_t)}{\|X_t - Y_t\|_H^{\varepsilon}} \mathbf{1}_{\{t<\tau\}}.$$

By using Hölder's inequality and (2.4) we obtain

$$\begin{aligned}\int_0^T \|\zeta_t\|_U^2 \mathrm{d}t &= \int_0^T \frac{\beta_t^2 \|X_t - Y_t\|_{B_t}^2}{\|X_t - Y_t\|_H^{2\varepsilon}} \mathrm{d}t \\ &\le T^{\frac{\sigma-2}{\sigma}} \Big(\int_0^T \frac{\beta_t^{\sigma} \|X_t - Y_t\|_{B_t}^{\sigma}}{\|X_t - Y_t\|_H^{\sigma\varepsilon}} \mathrm{d}t \Big)^{\frac{2}{\sigma}} \\ &\le T^{\frac{\sigma-2}{\sigma}} \Big(c^{\sigma} \|x - y\|_H^{2\varepsilon} \Big)^{\frac{2}{\sigma}}.\end{aligned} \tag{2.5}$$

Hence we have

$$\mathbf{E} \exp\Big[\frac{1}{2} \int_0^T \|\zeta_t\|_U^2 \mathrm{d}t\Big] < \infty. \tag{2.6}$$

Therefore, we can rewrite (2.2) as

$$\mathrm{d}Y_t = A(t, Y_t)\mathrm{d}t + B_t \mathrm{d}\tilde{W}_t, \; Y_0 = y$$

where

$$\tilde{W}_t := W_t + \int_0^t \zeta_s \mathrm{d}s.$$

By (2.6) and the Girsanov theorem (e.g.[9, Th 10.14, Prop.10.17]) we know that $\{\tilde{W}_t\}$ is a cylindrical Brownian motion on U under the weighted probability measure $R\mathbf{P}$, where

$$R := \exp\Big[\int_0^T \langle \zeta_t, \mathrm{d}W_t \rangle - \frac{1}{2} \int_0^T \|\zeta_t\|_U^2 \mathrm{d}t \Big].$$

Therefore, the distribution of $\{Y_t(y)\}_{t\in[0,T]}$ under $R\mathbf{P}$ is same with the distribution of $\{X_t(y)\}_{t\in[0,T]}$ under $\mathbf{P}$.

Let $p' = \frac{p}{p-1}$, then for any $q > 1$

$$\begin{aligned}\mathbf{E}R^{p'} &= \exp\Big[p' \int_0^T \langle \zeta_t, \mathrm{d}W_t \rangle - \frac{p'}{2} \int_0^T \|\zeta_t\|_U^2 \mathrm{d}t\Big] \\ &\le \Big[\mathbf{E} \exp(qp' \int_0^T \langle \zeta_t, \mathrm{d}W_t \rangle - \frac{q^2 (p')^2}{2} \int_0^T \|\zeta_t\|_U^2 \mathrm{d}t)\Big]^{\frac{1}{q}} \\ &\quad \cdot \Big[\mathbf{E} \exp(\frac{qp'(qp'-1)}{2(q-1)} \int_0^T \|\zeta_t\|_U^2 \mathrm{d}t)\Big]^{\frac{q-1}{q}} \\ &\le \Big[\mathbf{E} \exp(\frac{qp'(qp'-1)}{2(q-1)} \int_0^T \|\zeta_t\|_U^2 \mathrm{d}t)\Big]^{\frac{q-1}{q}} \\ &\le \exp\Big[\frac{p'(qp'-1)}{2} T^{\frac{\sigma-2}{\sigma}} \Big(c^{\sigma} \|x - y\|_H^{2\varepsilon} \Big)^{\frac{2}{\sigma}} \Big].\end{aligned} \tag{2.7}$$

By taking $q \downarrow 1$ we have

$$(P_T)^p(y) \le P_T F^p(x)(\mathbf{E}R^{p'})^{p'-1} \le P_T F^p(x) \exp\Big[\frac{p}{p-1} C(t,\sigma)\|x-y\|_H^{2+\frac{2(2-\alpha)}{\sigma}}\Big], \tag{2.8}$$

where

$$C(t,\sigma) = \frac{2t^{\frac{\sigma-2}{\sigma}}(\sigma+2)^{2+\frac{2}{\sigma}}}{(\sigma+2-\alpha)^{2+\frac{2}{\sigma}}\Big[\int_0^t (\delta_s\xi_s)^{\frac{1}{\sigma}}\exp(\frac{\alpha-2-\sigma}{2\sigma}\int_0^s \gamma_u \mathrm{d}u)\mathrm{d}s\Big]^2}.$$

From (2.7) we know that R is uniformly integrable, then by the dominated convergence theorem we have

$$\lim_{y\to x}\mathbf{E}|R-1| = \mathbf{E}\lim_{y\to x}|R-1| = 0.$$

Hence

$$|P_T F(y) - P_T F(x)| = |\mathbf{E}RF(X_T) - \mathbf{E}F(X_T)| \le \|F\|_\infty \mathbf{E}|R-1| \to 0 (y\to x).$$

This implies $P_T F \in C_b(H)$. Therefore, P_T is strong Feller operator. □

2.2 Proof of Theorem 1.3

(i) In the present case, $\{P_t\}$ is a Markov semigroup (cf.[18, 26]). The existence of an invariant measure μ can be proved by the standard Krylov-Bogoliubov procedure (cf.[26, 36]). Let

$$\mu_n := \frac{1}{n}\int_0^n \delta_0 P_t dt,\ n \ge 1,$$

where δ_0 is the Dirac measure at 0. Recall $X_t(y)$ is the solution to (1.1) with starting point y, then by (1.3) and the Gronwall Lemma

$$\|X_t(x) - X_t(y)\|_H^2 \le e^{\gamma t}\|x-y\|_H^2, \quad \forall x, y \in H.$$

This implies that P_t is a Feller semigroup.

Hence for the existence of an invariant measure, it is well-known that one only needs to verify the tightness of $\{\mu_n : n \ge 1\}$.

Since $\gamma \le 0$ in the case $\alpha \le 2$, then by (1.3) and $(A4)$ we have

$$2_{V^*} < A(x), x >_V \le -\delta\|x\|_V^\alpha + \gamma\|x\|_H^2 + 2\ {}_{V^*}\langle A(0), x\rangle_V \le \theta_2 - \theta_1\|x\|_V^\alpha \tag{2.9}$$

holds for some constant $\theta_1, \theta_2 > 0$. By using the Itô formula we have

$$\|X_t\|_H^2 \le \|x\|_H^2 + \int_0^t (c - \theta_1\|X_s\|_V^\alpha)ds + 2\int_0^t \langle X_s, BdW_s\rangle_H, \tag{2.10}$$

where $c > 0$ is some constant which may change from line to line.

Note that $M_t := \int_0^t \langle X_s, BdW_s\rangle_H$ is a martingale, then (2.10) implies that

$$\mu_n(\|\cdot\|_V^\alpha) = \frac{1}{n}\int_0^n \mathbf{E}\|X_t(0)\|_V^\alpha dt \le \frac{c}{\theta_1},\ n \ge 1. \tag{2.11}$$

Since the embedding $V \subseteq H$ is compact, then for any constant K the set $\{x \in H:\ \|x\|_V \le K\}$ is relatively compact in H. Therefore, (2.11) implies that $\{\mu_n\}$ is tight, hence the limit of a convergent subsequence provides an invariant measure μ of $\{P_t\}$.

Now we need to prove the concentration property of μ. If ε_0 is small enough, then by (2.10) and Itô's formula

$$\begin{aligned}
e^{\varepsilon_0\|X_t\|_H^\alpha} \le & e^{\varepsilon_0\|x\|_H^\alpha} + \int_0^t \left(c - \theta_1\|X_s\|_V^\alpha + \alpha\varepsilon_0\|B\|_2^2\|X_s\|_H^\alpha\right)\frac{\alpha\varepsilon_0}{2}\|X_s\|_H^{\alpha-2}e^{\varepsilon_0\|X_s\|_H^\alpha}\mathrm{d}s \\
& + \alpha\varepsilon_0\int_0^t \|X_s\|_H^{\alpha-2}e^{\varepsilon_0\|X_s\|_H^\alpha}\langle X_s, BdW_s\rangle_H \\
\le & e^{\varepsilon_0\|x\|_H^\alpha} + \int_0^t \left(c - c_1\|X_s\|_H^\alpha\right)\frac{\alpha\varepsilon_0}{2}\|X_s\|_H^{\alpha-2}e^{\varepsilon_0\|X_s\|_H^\alpha}\mathrm{d}s \\
& + \alpha\varepsilon_0\int_0^t \|X_s\|_H^{\alpha-2}e^{\varepsilon_0\|X_s\|_H^q}\langle X_s, BdW_s\rangle_H \\
\le & e^{\varepsilon_0\|x\|_H^\alpha} + \int_0^t \left(c_2 - c_3e^{\varepsilon_0\|X_s\|_H^\alpha}\right)\mathrm{d}s + \alpha\varepsilon_0\int_0^t \|X_s\|_H^{\alpha-2}e^{\varepsilon_0\|X_s\|_H^\alpha}\langle X_s, BdW_s\rangle_H
\end{aligned} \tag{2.12}$$

holds for some positive constants c, c_1, c_2 and c_3. Therefore

$$\mu_n(e^{\varepsilon_0\|\cdot\|_H^\alpha}) = \frac{1}{n}\int_0^n \mathbf{E}e^{\varepsilon_0\|X_t(0)\|_H^\alpha}\mathrm{d}t \le \frac{1}{c_3 n} + \frac{c_2}{c_3},\ n \ge 1.$$

Hence we have $\mu(e^{\varepsilon_0\|\cdot\|_H^\alpha}) < \infty$ for some $\varepsilon_0 > 0$. In particular, this implies $\mu(\|\cdot\|_H^2) < \infty$.

By (2.10) there also exists a constant C such that

$$\mathbf{E}\int_0^1 \|X_t(x)\|_V^\alpha dt \le C(1 + \|x\|_H^2),\ \forall x \in H.$$

Therefore

$$\mu(\|\cdot\|_V^\alpha) = \int_H \mu(dx)\int_0^1 \mathbf{E}(\|X_t(x)\|_V^\alpha)dt \le C + C\int_H \|x\|_H^2\mu(dx) < \infty.$$

(ii) If $\alpha = 2$, then for any $x, y \in H$

$$\|X_t(x) - X_t(y)\|_H^2 \le \|x - y\|_H^2 + \int_0^t \left(-\delta\|X_s(x) - X_s(y)\|_V^2 + \gamma\|X_s(x) - X_s(y)\|_H^2\right) ds.$$

By the Gronwall lemma we have

$$\|X_t(x) - X_t(y)\|_H^2 \le e^{(\gamma - c_0\delta)t}\|x - y\|_H^2,\ \forall x, y \in H.$$

If $\gamma < c_0\delta$, then (2.9) still holds. Hence $\{P_t\}$ has an invariant measure by repeating the argument in (i). And we also have

$$\lim_{t\to\infty} \|X_t(x) - X_t(y)\|_H = 0,\ \forall x, y \in H.$$

By the dominated convergence theorem we know for any invariant measure μ and for any bounded continuous function F

$$|P_tF(x) - \mu(F)| \le \int_H \mathbf{E}|F(X_t(x)) - F(X_t(y))|\mu(dy) \to 0(t \to \infty).$$

This implies the uniqueness of invariant measures.

We denote the invariant measure by μ. By (i) we know $\mu(\|\cdot\|_H^2) < \infty$, hence for any bounded Lipschitz function F on H we have

$$\begin{aligned}|P_tF(x) - \mu(F)| &\le \int_H \mathbf{E}|F(X_t(x)) - F(X_t(y))|\mu(dy)\\ &\le \mathrm{Lip}(F)e^{(\gamma-c_0\delta)t/2}\int_H \|x-y\|_H\mu(dy)\\ &\le \mathrm{Lip}(F)e^{(\gamma-c_0\delta)t/2}\left(\|x\|_H + C\right),\ x \in H,\end{aligned}$$

where $C > 0$ is a constant.

(iii) If $\alpha > 2$ and $\gamma \le 0$, then there exists a constant $c > 0$ such that

$$\|X_t(x) - X_t(y)\|_H^2 \le \|x-y\|_H^2 - c\int_0^t \|X_s(x) - X_s(y)\|_H^\alpha ds,\ t \ge 0.$$

Suppose h_t solves the equation

$$h_t' = -ch_t^{\frac{\alpha}{2}},\ h_0 = (\|x-y\|_H + \varepsilon)^2, \tag{2.13}$$

where ε is a positive constant. Then by a standard comparison argument we have

$$\|X_t(x) - X_t(y)\|_H^2 \le h_t \le Ct^{-\frac{2}{\alpha-2}}, \tag{2.14}$$

where $C > 0$ is a constant. In fact, we can define

$$\varphi_t := h_t - \|X_t(x) - X_t(y)\|_H^2,\quad \tau := \inf\{t \ge 0 :\ \varphi_t < 0\}.$$

If $\tau < \infty$, then we know $\varphi_\tau \le 0$ by the continuity.

By the mean-value theorem we have

$$\begin{aligned}\varphi_t &\ge \varphi_0 - c\int_0^t \left(h_s^{\frac{\alpha}{2}} - \|X_s(x) - X_s(y)\|_H^\alpha\right) ds\\ &\ge \varepsilon^2 - K\int_0^t \varphi_s ds,\ 0 \le t \le \tau,\end{aligned}$$

where $K>0$ is some constant. Then by the Gronwall lemma we have

$$\varphi_\tau \geq \varepsilon^2 e^{-K\tau} > 0,$$

which is contradict to $\varphi_\tau \leq 0$. Hence (2.14) holds.

Therefore, for any $x \in H$ and bounded Lipschitz function F on H, we have

$$|P_t F(x) - \mu(F)| \leq \int_H \mathbf{E}|F(X_t(x) - F(X_t(y)))|\mu(dy) \leq C\mathrm{Lip}(F) t^{-\frac{1}{\alpha-2}}.$$

Hence (1.7) holds and the uniqueness of invariant measures also follows. □

2.3 Proof of Theorem 1.4

(i) By the definition of $\|\cdot\|_B$ and (1.4), for any constant K there exists $\overline{K}>0$ such that

$$\begin{aligned}&\{x \in H: \ \|x\|_B \leq K\} \subseteq \{Bu: \ u \in U; \ \|u\|_U \leq \overline{K}\};\\&\{x \in H: \ \|x\|_V \leq K\} \subseteq \{x \in H: \|x\|_B \leq \overline{K}\}.\end{aligned}$$

Since B is a Hilbert-Schmidt (hence compact) operator, then the following set

$$\{x \in H: \ \|x\|_V \leq K\}$$

is relatively compact in H for any constant K, *i.e.* the embedding $V \subseteq H$ is compact. Hence $\{P_t\}$ has an invariant measure according to Theorem 1.3.

Suppose μ is an invariant measure of P_t, then by taking $p=2$ in (1.5) we have

$$\begin{aligned}&(P_t 1_M(x))^2 \int_H e^{-2C(t,\sigma)\|x-y\|_H^{2+\frac{2(2-\alpha)}{\sigma}}} \mu(dy)\\&\leq \int_H P_t 1_M(y)\mu(dy) = \mu(M),\end{aligned} \tag{2.15}$$

where M is a Borel set on H. Hence the transition kernel $P_t(x, dy)$ is absolutely continuous w.r.t. μ, and we denote the density by $p_t(x,y)$.

If μ does not have full support on H, this means there exist $x_0 \in H$ and $r>0$ such that

$$B(x_0; r) := \{y \in H: \|y - x_0\|_H \leq r\}$$

is a null set of μ. Then (2.15) implies that $P_t(x_0, B(x_0;r)) = 0$, *i.e.*

$$\mathbf{P}\left(X_t(x_0) \in B(x_0;r)\right) = 0, \quad t>0.$$

Since $X_t(x_0)$ is a continuous process on H, we have $\mathbf{P}\left(X_0 \in B(x_0;r)\right) = 0$, which is contradict with $X_0 = x_0$.

Therefore, μ has full support on H.

According to the Harnack inequality (1.5) we have

$$(P_t 1_M)^p(x_0) \le P_t 1_M(x) \exp\Big[\frac{p}{p-1} C(t,\sigma)\|x-x_0\|_H^{2+\frac{2(2-\alpha)}{\sigma}}\Big],\ x, x_0 \in H.$$

Therefore, to prove the irreducibility, one only has to show for any given nonempty open set M and $t>0$, there exists $x_0 \in H$ such that $P_t 1_M(x_0) > 0$.

Note that the full support property of μ implies

$$\int_H P_t 1_M(x)\mu(\mathrm{d}x) = \int_H 1_M(x)\mu(\mathrm{d}x) = \mu(M) > 0.$$

So $P_t 1_M(\cdot)$ cannot be the zero function. Therefore $\{P_t\}$ is irreducible.

Since $\{P_t\}$ have also the strong Feller property, then the uniqueness of invariant measures follows from the classical Doob theorem [7] (or See [15, Th 2.1]).

Note that the solution has continuous paths on H, then the other assertions follow from the general result in the ergodic theory (cf.[30, Th 2.2 and Prop 2.5], [22]).

(ii) For any $p>1$ and nonnegative measurable function f with $\mu(f^{p/(p-1)}) \le 1$, by replacing p with $p/(p-1)$ in (1.5) we have

$$\big(P_t f(x)\big)^{p/(p-1)} \le \big(P_t f^{p/(p-1)}(y)\big) \exp\big[pC(t,\sigma)\|x-y\|_H^{2+\frac{2(2-\alpha)}{\sigma}}\big], \quad x, y \in H.$$

Taking integration w.r.t. $\mu(dy)$ on both sides we have

$$\big(P_t f(x)\big)^{p/(p-1)} \int_H e^{-pC(t,\sigma)\|x-y\|_H^{2+\frac{2(2-\alpha)}{\sigma}}} \mu(dy) \le \mu(f^{p/(p-1)}) \le 1.$$

This implies that

$$P_t f(x) \le \Big(\int_H e^{-pC(t,\sigma)\|x-y\|_H^{2+\frac{2(2-\alpha)}{\sigma}}} \mu(dy)\Big)^{-(p-1)/p}.$$

Note that

$$P_t f(x) = \int_H f(y) P_t(x, dy) = \int_H f(y) p_t(x,y)\mu(dy),$$

hence

$$\|p_t(x,\cdot)\|_{L^p(\mu)} = \sup_{\|f\|_{L^q(\mu)}\le 1} \Big|\int_H f(y)p_t(x,y)\mu(dy)\Big| \le \Big(\int_H e^{-pC(t,\sigma)\|x-y\|_H^{2+\frac{2(2-\alpha)}{\sigma}}} \mu(dy)\Big)^{-(p-1)/p},$$

where $q = p/(p-1)$.

(iii) By (1.5) there exists a constant $c>0$ such that

$$(P_t f)^2(x) \exp\Big[-\frac{c\|x-y\|_H^{2+\frac{2(2-\alpha)}{\sigma}}}{t^{\frac{\sigma+2}{\sigma}}}\Big] \le P_t f^2(y), \quad x, y \in H,\ t>0. \tag{2.16}$$

Integrating on both sides w.r.t. $\mu(dy)$, for $f \in L^2(\mu)$ with $\mu(f^2)=1$ we have

$$(P_t f)^2(x) \le \frac{1}{\mu(B(0,1))} \exp\Big[\frac{c(\|x\|_H+1)^{2+\frac{2(2-\alpha)}{\sigma}}}{t^{\frac{\sigma+2}{\sigma}}}\Big], \quad x\in H, t>0, \tag{2.17}$$

where $B(0;1):=\{y\in H: \|y\|_H\le 1\}$ and $\mu\left(B(0;1)\right)>0$.

If $\alpha=2$, then there exists $C>0$ such that

$$\int_H (P_t f)^4(x)\mu(dx) \le \frac{C}{\mu(B(0,1))}\int_H \exp\Big[\frac{C\|x\|_H^2}{t^{\frac{\sigma+2}{\sigma}}}\Big]\mu(dx)<\infty$$

holds for sufficiently large $t>0$, since $\mu(e^{\varepsilon_0\|\cdot\|_H^2})$ is finite according to Theorem 1.3(i).

Hence P_t is hyperbounded operator for sufficient large $t>0$. Since P_t has a density w.r.t. μ, P_t is also compact in $L^2(\mu)$ for large $t>0$ according to [37, Theorem 2.3].

(iv) If $\alpha>2$, then by (2.12) we have for small enough $\varepsilon_0>0$

$$de^{\varepsilon_0\|X_t\|_H^\alpha} \le (c-\theta\|X_t\|_H^{2\alpha-2}e^{\varepsilon_0\|X_t\|_H^\alpha})dt + \alpha\varepsilon_0\|X_t\|_H^{\alpha-2}e^{\varepsilon_0\|X_t\|_H^\alpha}\langle X_t, BdW_t\rangle_H, \tag{2.18}$$

where $c,\theta>0$ are some constants. By Jensen's inequality we have

$$\mathbf{E}e^{\varepsilon_0\|X_t\|_H^\alpha} \le e^{\varepsilon_0\|x\|_H^\alpha} + ct - \theta\varepsilon_0^{-(2\alpha-2)/\alpha}\int_0^t \mathbf{E}e^{\varepsilon_0\|X_u\|_H^\alpha}\left(\log \mathbf{E}e^{\varepsilon_0\|X_u\|_H^\alpha}\right)^{\frac{2\alpha-2}{\alpha}} du.$$

Let $h(t)$ solve the equation

$$h'(t) = c-\theta\varepsilon_0^{-(2\alpha-2)/\alpha}h(t)\big\{\log h(t)\big\}^{(2\alpha-2)/\alpha}, \quad h(0)=\exp\left[\varepsilon_0\left(\|x\|_H^\alpha+c\right)\right]. \tag{2.19}$$

Then by a standard comparison argument we know

$$\mathbf{E}e^{\varepsilon_0\|X_t(x)\|_H^\alpha} \le h(t) \le \exp\Big[c_0\big(1+t^{-\alpha/(\alpha-2)}\big)\Big], \quad t>0, x\in H \tag{2.20}$$

hold for a constant $c_0>0$. By using (2.17) we have

$$\begin{aligned}\|P_t f\|_\infty &= \|P_{t/2}P_{t/2}f\|_\infty \\ &\le c_1\sup_{x\in H}\mathbf{E}\exp\Big[\frac{c_1}{t^{(\sigma+2)/\sigma}}\Big(1+\|X_{\frac{t}{2}}(x)\|_H\Big)^{2+\frac{2(2-\alpha)}{\sigma}}\Big], \quad t>0,\end{aligned} \tag{2.21}$$

where $c_1>0$ is a constant. By Young's inequality there exists $c_2>0$ such that

$$\frac{c_1}{t^{\frac{\sigma+2}{\sigma}}}(1+u)^{2+\frac{2(2-\alpha)}{\sigma}} \le \varepsilon_0(1+u^\alpha)+c_2t^{-\alpha/(\alpha-2)}, \quad u,t>0.$$

Therefore, there exists a constant $C>0$ such that

$$\|P_t\|_{L^2(\mu)\to L^\infty(\mu)} \le \exp[C(1+t^{-\frac{\alpha}{\alpha-2}})],\ t>0.$$

The compactness of P_t also follows from the [37]. □

2.4 Proof of Theorem 1.5

The proof is based on [11, Theorem 2.5-2.7]. Since the proof of (i) and (ii) are very similar, here we only give the detailed proof for (ii) which is more difficult. According to Theorem 1.4, we know $\{P_t\}$ is strong Feller and irreducible. Now we only need to verify the following properties:

(1) For each $r > 0$ there exist $t_0 > 0$ and a compact set $M \subset H$ such that

$$\inf_{x \in B_r} P_{t_0} \mathbf{1}_M(x) > 0,$$

where $B_r = \{y \in H : \ \|y\|_H \le r\}$.

(2) There exist constants $K < \infty$ and $t_1 > 0$ such that

$$\mathbf{E}\|X_t(x)\|_H^2 \le K, \ x \in H, \ t \ge t_1.$$

By using the Itô formula we have

$$\|X_t\|_H^2 \le \|x\|_H^2 + \int_0^t \left(c - \frac{\delta}{2}\|X_s\|_V^\alpha + \gamma \|X_s\|_H^2 \right) ds + \int_0^t \langle X_s, BW_s \rangle_H.$$

Since $\alpha > 2$, there exists a constant $c_1 > 0$

$$\|X_t\|_H^2 \le \|x\|_H^2 + \int_0^t \left(c_1 - \frac{\delta}{4}\|X_s\|_V^\alpha \right) ds + \int_0^t \langle X_s, BW_s \rangle_H.$$

This implies that there exists $C > 0$ such that

$$\mathbf{E}\int_0^t \|X_s\|_V^\alpha ds \le C(t + \|x\|_H^2), \ t \ge 0. \tag{2.22}$$

And by using Jensen's inequality

$$\mathbf{E}\|X_t\|_H^2 \le \|x\|_H^2 + \int_0^t \left[C_1 - C_2 \left(\mathbf{E}\|X_s\|_H^2 \right)^{\alpha/2} \right] ds.$$

Then by a standard comparison estimate we get

$$\mathbf{E}\|X_t(x)\|_H^2 \le C(1 + t^{-\frac{2}{\alpha-2}}), \ x \in H, \ t > 0.$$

Hence property (2) holds.

According to (1.5), for the property (1) it is enough to show that there exist t_0 and a compact set M in H such that $P_{t_0}\mathbf{1}_M(x) > 0$ for some $x \in B_r$.

By a simple contradiction argument, (2.22) implies that there exists $t_0 > 0$ such that $P_{t_0}\mathbf{1}_M(x) > 0$ for the compact set $M := \left\{ y \in H : \|y\|_V \le [C(1+r^2)]^{1/\alpha} \right\}$ and any $x \in B_r$.

If $\alpha = 2$ and $\gamma < c_0\delta$, then we can prove

$$\mathbf{E}\|X_t(x)\|_H^2 \le e^{-\beta t}\|x\|_H^2 + C, \ t \ge 0, \ x \in H,$$

hold for some constants $\beta > 0$ and C.

Then the conclusions follow from the [11, Theorem 2.5, 2.7].

3 Examples

To apply our main results, one has to verify condition (1.3) and (1.4). To this end, we present some simple sufficient conditions for (1.3) and (1.4). We first establish the following inequality, which is crucial for verifying (1.3) in concrete examples.

Lemma 3.1. *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space and $\|\cdot\|$ denote its norm, then for any $r \geq 0$ we have*

$$\langle \|a\|^r a - \|b\|^r b, a-b\rangle \geq 2^{-r}\|a-b\|^{r+2}, \ \ a,b \in H. \tag{3.1}$$

Proof. (i) If $\|a\| = \|b\|$, then (3.1) holds obviously.
(ii) If $\|a\| \neq \|b\|$, we may assume $\|a\| > \|b\|$ without lost of generality. Then we have

$$\begin{aligned} &\langle \|a\|^r a - \|b\|^r b, a-b\rangle \\ =&\|b\|^r\|a-b\|^2 + (\|a\|^r - \|b\|^r)\langle a, a-b\rangle \\ =&\|b\|^r\|a-b\|^2 + \frac{1}{2}(\|a\|^r - \|b\|^r)(\|a\|^2 + \|a-b\|^2 - \|b\|^2) \\ >&\|b\|^r\|a-b\|^2 + \frac{1}{2}(\|a\|^r - \|b\|^r)\|a-b\|^2 \\ =&\frac{1}{2}(\|a\|^r + \|b\|^r)\|a-b\|^2 \\ \geq&2^{-r}\|a-b\|^{r+2}. \end{aligned} \tag{3.2}$$

Hence the proof is complete. □

Remark 3.1. If $r < 0$, then (3.1) does not hold in general. Hence the assumption (1.3) in Theorem 1.2 does not hold for the stochastic fast diffusion equations. For more details we refer to [20].

In the following examples $L(Y,Z)$ denotes the space of all bounded linear operators from Y to Z and $\mathbf{Ran}(B)$ denotes the range of operator B.

Example 3.2. *(Stochastic reaction-diffusion equation)*
Let Λ be an open bounded domain in $\mathbb{R}^d$ with smooth boundary and Δ be the Laplace operator on $L^2(\Lambda)$ with Dirichlet boundary condition. Consider the following triple

$$W_0^{1,2}(\Lambda) \subseteq L^2(\Lambda) \subseteq \left(W_0^{1,2}(\Lambda)\right)^*$$

and the stochastic reaction-diffusion equation

$$\mathrm{d}X_t = (\Delta X_t - c|X_t|^{p-2}X_t)\mathrm{d}t + B\mathrm{d}W_t, \ \ X_0 = x \in L^2(\Lambda) \tag{3.3}$$

where $1 < p \leq 2$ and $c \geq 0$, B is a Hilbert-Schmidt operator and W_t is a cylindrical Wiener process on $L^2(\Lambda)$, then $(A1)-(A4)$ and (1.3) hold with $N(u) = \|u\|_{1,2}^2$ (Sobolev norm). Hence the assertions in **Theorem 1.3** *hold for* (3.3).

Moreover, if B is a one-to-one operator such that

$$W_0^{1,2}(\Lambda) \subseteq \mathbf{Ran}(B), \quad B^{-1} \in L(W_0^{1,2}(\Lambda); L^2(\Lambda)),$$

then (1.4) *also holds. In particular, if $d = 1$ and $B := (-\Delta)^{-\theta}$ with $\theta \in (\frac{1}{4}, \frac{1}{2}]$, then B is a Hilbert-Schmidt operator and* (1.4) *holds. Hence the assertions in* **Theorem 1.2, 1.4** *and* **1.5** *also holds for* (3.3)*. Particularly, the associated transition semigroup of* (3.3) *is hyperbounded.*

Remark 3.2. Suppose that

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots$$

are the eigenvalues of $-\Delta$ and the corresponding eigenvectors $\{e_i\}_{i\geq 1}$ form an orthonormal basis on $L^2(\Lambda)$. If $Be_i := b_i e_i$ and there exists a positive constant C such that

$$\sum_i b_i^2 < +\infty; \quad b_i \geq \frac{C}{\sqrt{\lambda_i}}, \quad i \geq 1,$$

then B is a Hilbert-Schmidt operator and (1.4) holds.

On the other hand, by the Sobolev inequality (see [34],Corollary 1.1 and 3.1) we know that

$$\lambda_i \geq c i^{2/d}, \quad i \geq 1,$$

hold for some constant $c > 0$. This implies that the space dimension d is less than 2. However, if we consider a general negative definite self-adjoint operator L instead of Δ in (3.3), e.g. $L := -(-\Delta)^q, q > 0$. Then, by the spectral representation theorem, our results can apply to examples on $\mathbf{R}^d$ with $d \geq 2$. For more details we refer to [20, 36].

Example 3.3. *(Stochastic p-Laplace equation)*
Let Λ be an open bounded domain in $\mathbb{R}^d$ with smooth boundary. Consider the triple

$$W_0^{1,p}(\Lambda) \subseteq L^2(\Lambda) \subseteq (W_0^{1,p}(\Lambda))^*$$

and the stochastic p-Laplace equation

$$\mathrm{d}X_t = \left[\mathbf{div}(|\nabla X_t|^{p-2}\nabla X_t) - c|X_t|^{\tilde{p}-2}X_t\right]\mathrm{d}t + B\mathrm{d}W_t, X_0 = x, \tag{3.4}$$

where $c \geq 0$, $2 \leq p < \infty, 1 \leq \tilde{p} \leq p$, B is a Hilbert-Schmidt operator and W_t is a cylindrical Wiener process on $L^2(\Lambda)$, then the assertions in **Theorem 1.3** *hold for* (3.4)*.*

Moreover, if $d = 1$ and $B := (-\Delta)^{-\theta}$ with $\theta \in (\frac{1}{4}, \frac{1}{2}]$, then (1.4) *also holds. Therefore the assertions in* **Theorem 1.2, 1.4** *and* **1.5** *also hold for* (3.4)*. In particular, the associated transition semigroup of* (3.4) *is ultrabounded and compact provided $p > 2$.*

Proof. According to [26, Example 4.1.9], $(A1)-(A4)$ hold for (3.4). Hence we only need to verify (1.3) for $N(u)=\|u\|_{1,p}^p$ under our assumptions. By using Lemma 3.1 and the Poincaré inequality we have

$$\begin{aligned}
&{}_{V^*}\langle \mathbf{div}(|\nabla u|^{p-2}\nabla u)-\mathbf{div}(|\nabla v|^{p-2}\nabla v), u-v\rangle_V \\
&= -\int_\Lambda \langle\ |\nabla u(x)|^{p-2}\nabla u(x)-|\nabla v(x)|^{p-2}\nabla v(x), \nabla u(x)-\nabla v(x)\rangle_{\mathbb{R}^d} dx \\
&\le -2^{p-2}\int_\Lambda |\nabla u(x)-\nabla v(x)|^p dx \\
&\le -C\|u-v\|_{1,p}^p,\ u,v\in W_0^{1,p}(\Lambda),
\end{aligned}$$

where $C>0$ is a constant. And it is easy to show that

$${}_{V^*}\langle |u|^{\tilde{p}-2}u-|v|^{\tilde{p}-2}v, u-v\rangle_V \ge 0.$$

Hence (1.3) holds.

If $d=1$ and $B:=(-\Delta)^{-\theta}$ with $\theta\in(\frac{1}{4},\frac{1}{2}]$, then there exists a constant $c>0$ such that (see the remark above)

$$\|u\|_{1,2}\ge c\|u\|_B,\ u\in W_0^{1,p}(\Lambda).$$

This implies (1.4) holds. □

Remark 3.3. (1) The Harnack inequality and some consequent properties still hold if one also add some locally bounded linear (or order less than p) perturbation in the drift. Only for certain properties (e.g.hyperbounded or ultrabounded) we need to require the drift is dissipative (i.e.$\gamma\le 0$).

(2) If we assume $B=0$ in (3.4), then by (iii) of Theorem 1.3 we can get the following decay of the solution to the classical p-Laplace equation

$$\sup_{x\in L^2(\Lambda)}\|X_t(x)\|_{L^2}\le Ct^{-\frac{1}{p-2}},\ t>0.$$

The following SPDE has been studied in [18, 19], in which the main part of drift in the equation is a high order generalization of the Laplace operator.

Example 3.4. *Let Λ be an open bounded domain in $\mathbb{R}^1$ and $m\in\mathbb{N}_+$. Consider the following triple*

$$W_0^{m,p}(\Lambda)\subseteq L^2(\Lambda)\subseteq (W_0^{m,p}(\Lambda))^*$$

and the stochastic evolution equation

(3.5)

$$\mathrm{d}X_t(x)=\left[(-1)^{m+1}\frac{\partial^m}{\partial x^m}\left(\left|\frac{\partial^m}{\partial x^m}X_t(x)\right|^{p-2}\frac{\partial^m}{\partial x^m}X_t(x)\right)-c|X_t(x)|^{\tilde{p}-2}X_t(x)\right]\mathrm{d}t+B\mathrm{d}W_t,$$

where $c\ge 0$, $2\le p<\infty, 1\le\tilde{p}\le p$, $B\in L_2(L^2(\Lambda))$ and W_t is a cylindrical Wiener process on $L^2(\Lambda)$, then the assertions in **Theorem1.3** *hold for* (3.5).

Moreover, if B is also a one-to-one operator such that $B^{-1} \in L(W_0^{m,p}(\Lambda); L^2(\Lambda))$, then (1.4) *is also satisfied. Hence the assertions in* **Theorem 1.2, 1.4** *and* **1.5** *hold for (3.5). In particular, the associate transition semigroup of the solution is ultrabounded if $p > 2$ and hyperbounded if $p = 2$.*

Proof. Take $N(u) = \|u\|_{m,p}^p$, then the proof is similar to the argument in Example 3.3. □

Remark 3.4. (i) If we assume $p > 2$ and $B = 0$ in (3.5), then by Theorem 1.3 we also obtain the decay of the solution to the deterministic evolution equation, i.e.

$$\sup_{f \in L^2(\Lambda)} \|X_t^f\|_{L^2} \le C t^{-\frac{1}{p-2}}, \ t > 0,$$

where X_t^f denote the solution to the following equation

$$\frac{\mathrm{d}X_t(x)}{\mathrm{d}t} = (-1)^{m+1} \frac{\partial^m}{\partial x^m} \left(\left| \frac{\partial^m}{\partial x^m} X_t(x) \right|^{p-2} \frac{\partial^m}{\partial x^m} X_t(x) \right) - c|X_t(x)|^{\tilde{p}-2} X_t(x), \ X_0 = f \in L^2(\Lambda).$$

(ii) Assume that

$$0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \le \cdots$$

are the eigenvalues of a positive definite self-adjoint operator L where $\mathcal{D}(\sqrt{L}) = W_0^{m,2}(\Lambda)$, the corresponding eigenvector $\{e_i\}_{i \ge 1}$ is an ONB of $L^2(\Lambda)$. Suppose $Be_i := b_i e_i$ and there exists a constant $C > 0$ such that

$$\sum_i b_i^2 < +\infty; \quad b_i \ge \frac{C}{\sqrt{\lambda_i}}, \quad i \ge 1,$$

then B is a Hilbert-Schmidt operator on $L^2(\Lambda)$ and (1.4) is satisfied.

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