Invariance of Subspaces under the Solution Flow of SPDE [*]

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Abstract

In this paper we investigate some regularity property for the solution to SPDE. Under certain assumptions we prove that the solution of an SPDE takes values in some subspace of the original state space if the initial condition does. As examples, the main results are applied to different types of SPDE such as stochastic reaction-diffusion equations, the stochastic p-Laplace equation, stochastic porous media and fast diffusion equations in Hilbert space.

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1 Introduction

The variational approach has been used intensively for analyzing SPDE in recent years. It gives a unified framework to deal with a large class of nonlinear SPDE of evolutionary type, which model all kinds of dynamics with random influence. This approach was first used by Bensoussan and Temam in [5, 6] to study SPDE with additive noise, later this approach was further developed in the works of Pardoux [19], Krylov and Rozovoskii [14] for more general case. For more general result on the existence and uniqueness of solution to SPDE we refer to [11, 21, 27]. Within this framework different types of properties has been established by many authors, e.g. see [9, 15, 24, 22] for the small noise asymptotic property (*i.e.* large deviation principle), [16, 17, 18, 26] for the Harnack inequality and consequent ergodicity, compactness and contractivity for the associated transition semigroup. As one typical example of SPDE in this framework, stochastic porous media equations have been intensively investigated by Kim [13] and Röckner *et al* in [1, 2, 3, 4, 10, 23].

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Let
$$V \subset H \equiv H^* \subset V^*$$

be a Gelfand triple, *i.e.* $(H, \langle \cdot, \cdot \rangle_H)$ is a separable Hilbert space and identified with its dual space H^* by the Riesz isomorphism, V is a reflexive and separable Banach space such that it is continuously and densely embedded into H. $_{V^*}\langle \cdot, \cdot \rangle_V$ denotes the dualization between V and its dual space V^*. $\{W_t\}$ is a cylindrical Wiener process on a separable Hilbert space U w.r.t. a complete filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbf{P})$ and $L_2(U; H)$ is the space of all Hilbert-Schmidt operators from U to H. Consider the following stochastic evolution equation

(1.1)
$$\mathrm{d}X_t = A(t, X_t)\mathrm{d}t + B(t, X_t)\mathrm{d}W_t,$$

where $A : [0, T] \times V \times \Omega \to V^*$ and $B : [0, T] \times V \times \Omega \to L_2(U, H)$ are progressively measurable. By assuming the coefficients A, B satisfy the standard monotone and coercive conditions (see Theorem 3.5 in the Appendix) we know (1.1) has a unique strong solution $\{X_t\}_{t \in [0,T]}$, which is a H-valued continuous process and satisfies

$$\mathbf{E}\left(\sup_{t \in [0,T]} \|X_t\|_H^2 + \int_0^T \|X_t\|_V^\alpha \mathrm{d}t \right) < \infty.$$

Recently, Röckner and Wang proved in [23] the L^2-invariance of the solution for the stochastic porous media equations $(r > 1)$

$$\mathrm{d}X_t = \Delta(|X_t|^{r-1}X_t)\mathrm{d}t + B(t, X_t)\mathrm{d}W_t,$$

i.e. the solution takes values in the L^2 space (note that the original state space is $W^{-1,2}$) and has right continuous paths in L^2 (almost surely). This type of regularity is very useful for the further study of corresponding random dynamical systems. For example, it was used crucially to investigate the existence of the random attractor for stochastic porous media equations (cf.[7]).

In this work we establish this type of regularity properties for a large class of SPDE within the variational framework. Assume $(S, \|\cdot\|_S)$ is a subspace of H and T_n are positive definite self-adjoint operators on H such that

$$\langle x, y \rangle_n := \langle x, T_n y \rangle_H, \ x, y \in H$$

are a sequence of new inner products on H. Suppose the induced norms $\|\cdot\|_n$ are all equivalent to $\|\cdot\|_H$ and
$$\forall x \in S, \ \|x\|_n \uparrow \|x\|_S \ (n \to \infty).$$

Let $H_n := (H, \langle \cdot, \cdot \rangle_n)$, then we get a sequence of new Gelfand triples

$$V \subseteq H_n \equiv H_n^* \subseteq V^*.$$

Now we formulate the main result of this paper.

Theorem 1.1. *Suppose the assumptions $(H1) - (H4)$ in Theorem 3.5 hold, $T_n : V \to V$ is continuous and there exist a constant C and an adapted process $f \in L^1([0,T] \times \Omega; \mathrm{d}t \times \mathbf{P})$ such that for $n \geq 1$*

$$(1.2) \quad 2_{V^*}\langle A(t,v), T_n v\rangle_V + \|B(t,v)\|^2_{L_2(U,H_n)} \leq C\|v\|^2_n + f_t, \ v \in V, \ 0 \leq t \leq T, \ \mathbf{P} - a.s..$$

(i) If $\mathbf{E}\|X_0\|^2_S < \infty$, then for any $p \in [1,2)$ we have

$$\mathbf{E} \sup_{t \in [0,T]} \|X_t\|^p_S < \infty.$$

(ii) If $\mathbf{E}\|X_0\|^p_S < \infty$ for some $p \geq 2$ and

$$(1.3) \qquad \|B(t,v)\|^2_{L_2(U,H_n)} \leq C\|v\|^2_n + f_t, \ v \in V, \ 0 \leq t \leq T, \ \mathbf{P} - a.s.,$$

where $f \in L^{\frac{p}{2}}([0,T] \times \Omega; \mathrm{d}t \times \mathbf{P})$, then there exists a constant C_p such that

$$\mathbf{E} \sup_{t \in [0,T]} \|X_t\|^p_S \leq C_p \left(\mathbf{E}\|X_0\|^p_S + \mathbf{E} \int_0^T f_t^{p/2} \mathrm{d}s \right).$$

Moreover, $\{X_t\}_{t \in [0,T]}$ is right continuous in S.

Remark 1.1. (1) This type of regularity has been required in [12] for establishing the convergence rate of implicit approximations for stochastic evolution equations and in [9] for deriving the LDP for semilinear SPDE. Hence this result gives a sufficient condition for verifying this type of regularity. In section 3 we apply this result to many examples such as stochastic reaction-diffusion equations, the stochastic p-Laplace equation, stochastic porous media and fast diffusion equations in Hilbert space.

(2) The idea of using equivalent norms $\|\cdot\|_n$ to approximate $\|\cdot\|_S$ has been used in [23] for establishing the L^2-invariance of the solution to stochastic porous media equations. In order to apply Itô's formula to the equation on different Gelfand triples, we introduce the continuous operator I_n in the proof to transfer the equation between different triples. Hence our proof is much simpler by avoiding some technical lemmas (see section 2.2) used in [23].

2 Proof of Theorem 1.1

Firstly, we give two lemmas as the preparations for the proof of the main result. Note that we use different Riesz maps i_n to identify $H_n \equiv H_n^*$ in the new Gelfand triples, and i denotes the Riesz map for identifying $H \equiv H^*$.

Lemma 2.1. *If $T_n : V \to V$ is continuous, then $i_n \circ i^{-1} : H^* \to H_n^*$ is continuous w.r.t. $\|\cdot\|_{V^*}$. Therefore, there exists a unique continuous extension I_n of $i_n \circ i^{-1}$ on V^* such that*

$$(2.1) \qquad \qquad {}_{V^*}\langle I_n f, v\rangle_V = {}_{V^*}\langle f, T_n v\rangle_V, \quad f \in V^*, \ v \in V.$$

Proof. For any $f \in H^* \subset V^*$, we know $i_n \circ i^{-1} f \in H_n^*$ and

$$
\begin{aligned}
\|i_n \circ i^{-1} f\|_{V^*} &= \sup_{v \in V, \|v\|_V = 1} |_{V^*}\langle i_n \circ i^{-1} f, v \rangle_V| = \sup_{v \in V, \|v\|_V = 1} |\langle i^{-1} f, v \rangle_n| \\
&= \sup_{v \in V, \|v\|_V = 1} |\langle i^{-1} f, T_n v \rangle_H| = \sup_{v \in V, \|v\|_V = 1} |_{V^*}\langle f, T_n v \rangle_V| \\
&\leq \sup_{v \in V, \|v\|_V \leq c_n} |_{V^*}\langle f, v \rangle_V| \leq c_n \|f\|_{V^*},
\end{aligned}
$$

(2.2)

where c_n is the operator norm of T_n from V to V. Obviously we also have

$$
{}_{V^*}\langle i_n \circ i^{-1} f, v \rangle_V = {}_{V^*}\langle f, T_n v \rangle_V, \quad f \in H^*, \ v \in V.
$$

Then it is well known that $i_n \circ i^{-1}$ can be uniquely extended to a continuous operator on V^* such that (2.1) holds. $\qquad\square$

Since we want to apply the Itô formula to the solution of (1.1) in different Gelfand triples, we need to write down the Itô formula for the square norm of the solution in a more precise way by involving the corresponding Riesz map explicitly.

Lemma 2.2. ([21], Theorem A.2) *Let $K := L^\alpha([0, T] \times \Omega \to V; \mathrm{d}t \times \mathbf{P})(\alpha > 1)$ and $X_0 \in L^2(\Omega \to H; \mathcal{F}_0; \mathbf{P})$. Suppose we have a H-valued process X_t which satisfies*

$$
iX_t = iX_0 + \int_0^t Y_s \mathrm{d}s + i\left(\int_0^t Z_s \mathrm{d}W_s\right), \quad t \in [0, T],
$$

where $Y \in K^* = L^{\alpha/(\alpha-1)}([0, T] \times \Omega \to V^*; \mathrm{d}t \times \mathbf{P})$ and $Z \in L^2([0, T] \times \Omega \to L_2(U; H); \mathrm{d}t \times \mathbf{P})$ are two adapted processes. If there exists an element $\bar{X}$ in K such that $X = \bar{X} \ \mathrm{d}t \times \mathbf{P}, a.s.,$ then X_t is a continuous adapted process on H such that $\mathbf{E} \sup_{t \in [0,T]} \|X_t\|_H^2 < \infty$ and

(2.3) $\qquad \|X_t\|_H^2 = \|X_0\|_H^2 + \int_0^t (2_{V^*}\langle Y_s, \bar{X}_s \rangle_V + \|Z_s\|_{L_2(U;H)}^2) \mathrm{d}s + 2 \int_0^t \langle X_s, Z_s \mathrm{d}W_s \rangle_H$

holds $\mathbf{P}$-a.s. for all $t \in [0, T]$. We can replace $\bar{X}_s$ by X_s in (2.3) if we set $_{V^*}\langle Y_s, X_s \rangle_V = 0$ for $X_s \notin V$.

Proof of Theorem 1.1: (i) It follows from the definition that the solution X_t to (1.1) satisfies

(2.4) $\qquad iX_t = iX_0 + \int_0^t A(s, X_s) \mathrm{d}s + i\left(\int_0^t B(s, X_s) \mathrm{d}W_s\right), \ t \in [0, T].$

According to Lemma 2.1, by applying the continuous operator I_n to (2.4) we have

$$
i_n X_t = i_n X_0 + \int_0^t I_n A(s, X_s) \mathrm{d}s + i_n \left(\int_0^t B(s, X_s) \mathrm{d}W_s\right), \ t \in [0, T].
$$

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By Lemma 2.2 we can apply the Itô formula on the new Gelfand triple $V \subseteq H_n \equiv H_n^* \subseteq V^*$ to obtain

$$\|X_t\|_n^2 = \|X_0\|_n^2 + \int_0^t \left(2 \,_{V^*}\langle I_n A(s, X_s), X_s \rangle_V + \|B(s, X_s)\|_{L_2(U;H_n)}^2 \right) \mathrm{d}s$$

(2.5)
$$+ 2 \int_0^t \langle X_s, B(s, X_s)\mathrm{d}W_s \rangle_n$$

$$\leq \|X_0\|_n^2 + \int_0^t \left(C\|X_s\|_n^2 + f_s \right) \mathrm{d}s + 2 \int_0^t \langle X_s, B(s, X_s)\mathrm{d}W_s \rangle_n.$$

Hence

(2.6) $\qquad \mathrm{e}^{-Ct}\|X_t\|_n^2 \leq \|X_0\|_n^2 + \int_0^t \mathrm{e}^{-Cs} f_s \mathrm{d}s + 2 \int_0^t \mathrm{e}^{-Cs}\langle X_s, B(s, X_s)\mathrm{d}W_s \rangle_n =: N_t.$

It is easy to show that N_t is a local submartingale, *i.e.* the sum of an increasing process and a local martingale. Hence by a standard localization argument we know for any $p \in [1, 2)$

(2.7)
$$\mathbf{P}\big(\sup_{t \in [0,T]} \|X_t\|_n^p \geq r \big) = \mathbf{P}\big(\sup_{t \in [0,T]} \|X_t\|_n^2 \geq r^{2/p} \big)$$

$$\leq \mathbf{P}\big(\sup_{t \in [0,T]} N_t \geq \mathrm{e}^{-CT} r^{2/p} \big) \leq r^{-2/p}\mathrm{e}^{CT}\mathbf{E}N_T < \infty,$$

since $\mathbf{E}N_T \leq \mathbf{E}\|X_0\|_S^2 + \mathbf{E}\int_0^T \mathrm{e}^{-Cs} f_s \mathrm{d}s < \infty$. Then

$$\mathbf{E}\sup_{t \in [0,T]} \|X_t\|_n^p = \int_0^\infty \mathbf{P}\big(\sup_{t \in [0,T]} \|X_t\|_n^p \geq r \big) \mathrm{d}r$$

$$\leq \int_0^1 \mathbf{P}\big(\sup_{t \in [0,T]} \|X_t\|_n^p \geq r \big) \mathrm{d}r + \int_1^\infty \mathbf{P}\big(\sup_{t \in [0,T]} \|X_t\|_n^p \geq r \big) \mathrm{d}r$$

$$\leq 1 + \int_1^\infty r^{-2/p}\mathrm{e}^{CT}\mathbf{E}N_T \mathrm{d}r = 1 + \frac{p}{2-p}\mathrm{e}^{CT}\mathbf{E}N_T.$$

Let $n \to \infty$, by the monotone convergence theorem and Fatou's lemma we have

$$\mathbf{E}\sup_{t \in [0,T]} \|X_t\|_S^p = \mathbf{E}\lim_{n \to \infty} \sup_{t \in [0,T]} \|X_t\|_n^p$$

$$\leq \liminf_{n \to \infty} \mathbf{E}\sup_{t \in [0,T]} \|X_t\|_n^p \leq 1 + \frac{p}{2-p}\mathrm{e}^{CT}\mathbf{E}N_T < \infty.$$

(ii) By Itô's formula and Young's inequality we have

(2.8)
$$\|X_t\|_n^p = \|X_0\|_n^p + \frac{p}{2}\int_0^t \|X_s\|_n^{p-2} \cdot 2 \,_{V^*}\langle I_n A(s, X_s), X_s \rangle_V \mathrm{d}s$$

$$+ p \int_0^t \|X_s\|_n^{p-2}\langle X_s, B(x, X_s)\mathrm{d}W_s \rangle_n + p(\frac{p}{2} - 1)\int_0^t \|X_s\|_n^{p-4}\|X_s \circ B(t, X_s)\|_{L_2(U,H_n)}^2 \mathrm{d}t$$

$$\leq \|X_0\|_n^p + \frac{p}{2}\int_0^t C(\|X_s\|_n^p + f_s\|X_s\|_n^{p-2})\mathrm{d}s + p \int_0^t \|X_s\|_n^{p-2}\langle X_s, B(x, X_s)\mathrm{d}W_s \rangle_n$$

$$\leq \|X_0\|_n^p + C \int_0^t (\|X_s\|_n^p + f_s^{p/2})\mathrm{d}s + p \int_0^t \|X_s\|_n^{p-2}\langle X_s, B(x, X_s)\mathrm{d}W_s \rangle_n,$$

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where C is a constant which may change from line to line.

Then by the Burkhölder-Davis-Gundy inequality and (1.3) we have

$$\mathbf{E} \sup_{u \in [0,t]} \left| \int_0^u \|X_s\|_n^{p-2} \langle X_s, B(s, X_s) \mathrm{d}W_s \rangle_n \right|$$

$$\leq 3\mathbf{E} \left(\int_0^t \|X_s\|_n^{2p-2} \|B(s, X_s)\|_{L_2(U;H_n)}^2 \mathrm{d}s \right)^{1/2}$$

(2.9)
$$\leq 3\mathbf{E} \left(\sup_{s \in [0,t]} \|X_s\|_n^{2p-2} \int_0^t (\|X_s\|_n^2 + f_s) \mathrm{d}s \right)^{1/2}$$

$$\leq 3\mathbf{E} \left[\varepsilon \sup_{s \in [0,t]} \|X_s\|_n^p + C_\varepsilon \left(\int_0^t (C\|X_s\|_n^2 + f_s) \mathrm{d}s \right)^{p/2} \right]$$

$$\leq 3\varepsilon \mathbf{E} \sup_{s \in [0,t]} \|X_s\|_n^p + 3 \cdot 2^{p/2} C_\varepsilon \mathbf{E} \int_0^t \left(\|X_s\|_n^p + f_s^{p/2} \right) \mathrm{d}s,$$

where $\varepsilon > 0$ is a small constant and C_ε comes from Young's inequality.

Then combining with (2.8) and Gronwall's lemma we have for any stopping time $\tau \leq T$

$$\mathbf{E} \sup_{t \in [0,\tau]} \|X_t\|_n^p \leq C \left(\mathbf{E} \|X_0\|_n^p + \mathbf{E} \int_0^T f_s^{p/2} \mathrm{d}s \right),$$

where C is a constant independent of n.

Therefore, by using a standard localization argument we have

$$\mathbf{E} \sup_{t \in [0,T]} \|X_t\|_S^p = \sup_{n \geq 1} \mathbf{E} \sup_{t \in [0,T]} \|X_t\|_n^p \leq C \left(\mathbf{E} \|X_0\|_S^p + \mathbf{E} \int_0^T f_s^{p/2} \mathrm{d}s \right).$$

The right continuity of $\{X_t\}$ in S can be derived by using the same argument in [23, Theorem 2.8]. $\qquad \square$

3 Applications to concrete SPDEs

In this section, we show that (1.2) and (1.3) can be verified for many concrete SPDE in Hilbert space, hence Theorem 1.1 can be applied to those examples. For simplicity, we mainly consider the additive type noise (e.g. $B \in L^2([0,T] \times \Omega, L_2(U, S))$) in the examples. Then it is obvious to see that (1.3) holds. A simple multiplicative noise example such that (1.3) holds is

$$B(t, v)u := B_0 + \sum_{i=1}^N \eta_i(t) \langle u, u_i \rangle_U v, \ u \in U, v \in V,$$

where $B_0 \in L^2([0,T] \times \Omega, L_2(U, S))$ is progressively measurable, $u_i \in U$, $\eta_i : [0,T] \times \Omega \to \mathbb{R}$ is progressively measurable and bounded for $1 \leq i \leq N$.

Example 3.1. *Let Λ be an open bounded domain in $\mathbb{R}^d$ and $L^p := L^p(\Lambda)$. Consider the following triple*

$$L^p \subseteq L^2 \subseteq (L^p)^* \equiv L^{\frac{p}{p-1}}$$

and the stochastic equation

$$(3.1) \qquad \mathrm{d}X_t = \left(-|X_t|^{p-2}X_t + \eta_t X_t\right)\mathrm{d}t + B_t\mathrm{d}W_t, \ t \in [0, T],$$

where $p \geq 2$, η is a bounded process and W_t is a cylindrical Wiener process on L^2. If $S = W_0^{1,2}$, $X_0 \in L^2(\Omega, S)$ and $B \in L^2([0, T] \times \Omega, L_2(L^2, S))$, then there exists a constant C such that

$$\mathbf{E} \sup_{t \in [0,T]} \|X_t\|_S^2 \leq C\left(\mathbf{E}\|X_0\|_S^2 + \mathbf{E}\int_0^T \|B_t\|_{L_2}^2\mathrm{d}t\right).$$

Proof. Note that $S = W_0^{1,2} = \mathcal{D}(\sqrt{-\Delta})$, where Δ is the Laplace operator on L^2 with the Dirichlet boundary condition. Then we define $T_n = -\Delta(1 - \frac{\Delta}{n})^{-1}$ which is the Yosida approximation of Δ. It is well known that the heat semigroup $\{P_t\}_{t\geq 0}$ (generated by Δ) is a contractive semigroup and T_n are continuous operators on L^p. Therefore, by using the Hölder inequality and the contraction property of P_t on L^p we have

$$_{V^*}\langle A(t,u), T_n u\rangle_V = {}_{V^*}\langle -|u|^{p-2}u, -\Delta(1 - \frac{\Delta}{n})^{-1}u\rangle_V + \eta_t\|u\|_n^2$$

$$= {}_{V^*}\langle -|u|^{p-2}u, nu - n(1 - \frac{\Delta}{n})^{-1}u\rangle_V + \eta_t\|u\|_n^2$$

$$(3.2) \qquad = -n\int_0^\infty e^{-t}{}_{V^*}\langle |u|^{p-2}u, u - P_{\frac{t}{n}}u\rangle_V\mathrm{d}t + \eta_t\|u\|_n^2$$

$$\leq -n\int_0^\infty e^{-t}\left[\int_\Lambda (|u|^p - |u|^{p-2}uP_{\frac{t}{n}}u)\mathrm{d}\xi\right]\mathrm{d}t$$

$$\leq C\|u\|_n^2, \ u \in L^p,$$

where C is a constant.

Hence (1.2) holds and the conclusion follows from Theorem 1.1. $\qquad\square$

Example 3.2. *(Stochastic reaction-diffusion equation)*
Let Λ be an open bounded domain in $\mathbb{R}^d$. We consider the following triple

$$W_0^{1,2}(\Lambda) \cap L^p(\Lambda) \subseteq L^2(\Lambda) \subseteq (W_0^{1,2}(\Lambda) \cap L^p(\Lambda))^*$$

and the stochastic reaction-diffusion equation

$$(3.3) \qquad \mathrm{d}X_t = (\Delta X_t - |X_t|^{p-2}X_t + \eta_t X_t)\mathrm{d}t + B_t\mathrm{d}W_t,$$

where $p \geq 2$, η is a bounded process and W_t is a cylindrical Wiener process on $L^2(\Lambda)$. If $S = W_0^{1,2}(\Lambda)$, $X_0 \in L^2(\Omega, S)$ and $B \in L^2([0, T] \times \Omega, L_2(L^2(\Lambda), S))$, then there exists a constant C such that

$$\mathbf{E} \sup_{t \in [0,T]} \|X_t\|_S^2 \leq C\left(\mathbf{E}\|X_0\|_S^2 + \mathbf{E}\int_0^T \|B_t\|_{L_2}^2\mathrm{d}t\right).$$

Proof. Let Δ be the Laplace operator on $L^2(\Lambda)$ with the Dirichlet boundary condition, then we define $T_n = -\Delta(1 - \frac{\Delta}{n})^{-1}$, $\{P_t\}_{t \geq 0}$ and $\mathcal{E}$ denote the corresponding semigroup and Dirichlet form of Δ. It is easy to show that T_n are continuous operators on $W_0^{1,2}(\Lambda)$ since

$$T_n = n \left(I - \left(I - \frac{\Delta}{n} \right)^{-1} \right).$$

Then we have

$$_{V^*}\langle \Delta u, -\Delta(1 - \frac{\Delta}{n})^{-1} u \rangle_V$$
$$= \; _{V^*}\langle \Delta u, nu - n(1 - \frac{\Delta}{n})^{-1} u \rangle_V$$
$$= \; -n \int_0^\infty e^{-t} \langle \nabla u, \nabla u - \nabla P_{\frac{t}{n}} u \rangle_{L^2(\Lambda)} \mathrm{d}t$$
$$\leq \; -n \int_0^\infty e^{-t} (\mathcal{E}(u, u) - \mathcal{E}(u, P_{\frac{t}{n}} u)) \mathrm{d}t$$
$$\leq \; 0,$$

where the last step follows from the contraction property of the Dirichlet form $\mathcal{E}$.

Therefore, combining with (3.2) we know that (1.2) holds and the conclusion follows from Theorem 1.1.

$\square$

Remark 3.1. (1) This regularity property is used in [12] (see assumption (T3)) for establishing the convergence rate of the implicit approximations for stochastic evolution equations.

(2) In the above example one can replace Δ by a more general negative definite self-adjoint operator L and obtain a similar result for $S = \mathcal{D}(\sqrt{-L})$. This type of regularity has been used in [9, Lemma 3.2] for establishing the large deviation principle for semilinear SPDEs.

Example 3.3. *(stochastic porous media and fast diffusion equation)*
Let Λ be an open bounded domain in $\mathbb{R}^d$. For $r > 0, r \geq \frac{d-2}{d+2}$ we consider the following triple

$$V := L^{r+1}(\Lambda) \subseteq H := (W_0^1(\Lambda))^* \subseteq V^*$$

and the stochastic porous media(or fast diffusion) equation

(3.4) $$\mathrm{d}X_t = \left(\Delta(|X_t|^{r-1} X_t) + \eta_t X_t \right) \mathrm{d}t + B_t \mathrm{d}W_t,$$

where W_t is a cylindrical Wiener process on $L^2(\Lambda)$ and η is a bounded process. If $S = L^2(\Lambda)$, $X_0 \in L^2(\Omega, S)$ and $B \in L^2([0, T] \times \Omega, L_2(L^2(\Lambda)))$, then there exists a constant C such that

$$\mathbf{E} \sup_{t \in [0,T]} \|X_t\|_S^2 \leq C \left(\mathbf{E}\|X_0\|_S^2 + \mathbf{E} \int_0^T \|B_t\|_{L_2}^2 \mathrm{d}t \right).$$

Proof. According to [20, Example 4.1.11;Remark 4.1.15] we know the conditions $(H1)-(H4)$ in Theorem 3.5 hold for $r > 0, r \geq \frac{d-2}{d+2}$. Hence we only need to verify (1.2) in Theorem 1.1 here.

It is well known that the heat semigroup $\{P_t\}$ is contractive on $L^p(\Lambda)$ for any $p > 1$. Now we define the Yosida approximation operator

$$T_n = -\Delta(I - \frac{\Delta}{n})^{-1} = n\left(I - (I - \frac{\Delta}{n})^{-1}\right),$$

it's easy to show that T_n are continuous operators on $L^{r+1}(\Lambda)$ by using the formula

$$(I - \frac{\Delta}{n})^{-1}u = \int_0^\infty e^{-t}P_{\frac{t}{n}}u\mathrm{d}t.$$

Then by the Hölder inequality and the contractivity of $\{P_t\}$ on $L^{r+1}(\Lambda)$ we have

$$\begin{aligned}
{}_{V^*}\langle\Delta(|u|^{r-1}u), -\Delta(1 - \frac{\Delta}{n})^{-1}u\rangle_V \\
= \quad \langle|u|^{r-1}u, nu - n(1 - \frac{\Delta}{n})^{-1}u\rangle_{L^2} \\
= \quad -n\int_0^\infty e^{-t}\left(\int_\Lambda |u|^{r+1}\mathrm{d}x - \int_\Lambda |u|^{r-1}u \cdot P_{\frac{t}{n}}u\mathrm{d}x\right)\mathrm{d}t \\
\leq \quad 0.
\end{aligned}$$

Hence the conclusion follows from the Theorem 1.1. □

Remark 3.2. Note that if $r > 1$, this result has been obtained in [23, Theorem 2.8] where more general stochastic porous media equations were studied. But under the present framework our proof is much simpler and the result here also holds for stochastic fast diffusion equations (*i.e.* $r < 1$). In the example we assume $r \geq \frac{d-2}{d+2}$ such that the embedding $L^{r+1}(\Lambda) \subseteq (W_0^1(\Lambda))^*$ is dense and continuous. For $0 < r < \frac{d-2}{d+2}$ one should use the Gelfand triple studied in [21].

Example 3.4. (*Stochastic p-Laplace equation*)
Let Λ be an open bounded domain in $\mathbb{R}^d$ with convex and smooth boundary. We consider the following triple

$$W^{1,p}(\Lambda) \subseteq L^2(\Lambda) \subseteq (W^{1,p}(\Lambda))^*$$

and the stochastic p-Laplace equation

(3.5) $$\mathrm{d}X_t = \left[\mathbf{div}(|\nabla X_t|^{p-2}\nabla X_t) - \eta_t|X_t|^{\tilde{p}-2}X_t\right]\mathrm{d}t + B_t\mathrm{d}W_t,$$

where $2 \leq p < \infty, 1 \leq \tilde{p} \leq p$, W_t is a cylindrical Wiener process on $L^2(\Lambda)$ and η is a positive bounded process. If $S = W^{1,2}(\Lambda)$, $X_0 \in L^2(\Omega, S)$ and $B \in L^2([0,T] \times \Omega, L_2(L^2, S))$, then there exists a constant C such that

$$\mathbf{E}\sup_{t\in[0,T]}\|X_t\|_S^2 \leq C\left(\mathbf{E}\|X_0\|_S^2 + \mathbf{E}\int_0^T \|B_t\|_{L_2}^2\mathrm{d}t\right).$$

Proof. According to the results in [20] (e.g. Example 4.1.9), we only need to verify the assumption (1.2) in Theorem 1.1. Since $S = W^{1,2}(\Lambda) = \mathcal{D}(\sqrt{-\Delta})$, where Δ is the Laplace operator on $L^2(\Lambda)$ with the Neumann boundary condition. It is well known that the corresponding semigroup $\{P_t\}$ is the Neumann heat semigroup (*i.e.* the corresponding Markov process is the Brownian Motion with reflecting boundary) and $P_t : L^2(\Lambda) \to W^{1,2}(\Lambda)$. Moreover, we know that P_t maps $L^p(\Lambda)$ into $W^{1,p}(\Lambda)$ continuously (see [8, section 2] for more general results). Hence for all $t \geq 0$, $P_t : W^{1,p}(\Lambda) \to W^{1,p}(\Lambda)$ is continuous.

Now we define

$$T_n = -\Delta(I - \frac{\Delta}{n})^{-1} = n\left(I - (I - \frac{\Delta}{n})^{-1}\right).$$

It is easy to show that T_n are also continuous operators on $W^{1,p}(\Lambda)$ since

$$(I - \frac{\Delta}{n})^{-1}u = \int_0^\infty e^{-t}P_{\frac{t}{n}}u\mathrm{d}t.$$

Moreover, since the boundary of the domain is convex and smooth, we have the following gradient estimate (cf.[25, Theorem 2.5.1])

$$|\nabla P_t u| \leq P_t|\nabla u|, \quad u \in W^{1,p}(\Lambda).$$

Since $\{P_t\}$ is a contractive semigroup on $L^p(\Lambda)$, it is easy to see that $\{P_t\}$ is a contractive semigroup on $W^{1,p}(\Lambda)$. Therefore,

$$
\begin{aligned}
&{}_{V^*}\langle \mathbf{div}(|\nabla u|^{p-2}\nabla u), -\Delta(1 - \frac{\Delta}{n})^{-1}u\rangle_V \\
&= {}_{V^*}\langle \mathbf{div}(|\nabla u|^{p-2}\nabla u), nu - n(1 - \frac{\Delta}{n})^{-1}u\rangle_V \\
&= n\int_0^\infty e^{-t} {}_{V^*}\langle \mathbf{div}(|\nabla u|^{p-2}\nabla u), u - P_{\frac{t}{n}}u\rangle_V \mathrm{d}t \\
&= -n\int_0^\infty e^{-t}\left(\int_\Lambda |\nabla u|^p\mathrm{d}x - \int_\Lambda |\nabla u|^{p-2}\nabla u \cdot \nabla P_{\frac{t}{n}}u\mathrm{d}x\right)\mathrm{d}t \\
&\leq 0,
\end{aligned}
$$

where in the last step we use the Hölder inequality and the contractivity of $\{P_t\}$ on $W^{1,p}(\Lambda)$ to conclude

$$
\begin{aligned}
&\int_\Lambda |\nabla u|^{p-2}\nabla u \cdot \nabla P_s u\mathrm{d}x \\
&\leq \left(\int_\Lambda |\nabla u|^p\mathrm{d}x\right)^{\frac{p-1}{p}} \cdot \left(\int_\Lambda |\nabla P_s u|^p\mathrm{d}x\right)^{\frac{1}{p}} \\
&\leq \left(\int_\Lambda |\nabla u|^p\right)^{\frac{p-1}{p}} \cdot \left(\int_\Lambda |P_s|\nabla u||^p\mathrm{d}x\right)^{\frac{1}{p}} \\
&\leq \int_\Lambda |\nabla u|^p\mathrm{d}x.
\end{aligned}
$$

Then the conclusion follows from Theorem 1.1. $\qquad\square$

Appendix: The existence and uniqueness of solution

We include the classical existence and uniqueness result in [14] for the reader's convenience. We first recall the definition of the solution to (1.1).

Definition 3.1. A continuous H-valued $(\mathcal{F}_t)$-adapted process $\{X_t\}_{t\in[0,T]}$ is called a solution of (1.1), if for its $\mathrm{d}t \otimes \mathbf{P}$-equivalent class $\bar{X}$ we have

$$\bar{X} \in L^\alpha([0,T] \times \Omega, \mathrm{d}t \otimes \mathbf{P}; V) \cap L^2([0,T] \times \Omega, \mathrm{d}t \otimes \mathbf{P}; H)$$

and $\mathbf{P}$-a.s.

$$X_t = X_0 + \int_0^t A(s, \bar{X}_s)\mathrm{d}s + \int_0^t B(s, \bar{X}_s)\mathrm{d}W_s, \ t \in [0,T].$$

Theorem 3.5. ([14] Theorems II.2.1, II.2.2) *Suppose for a fixed $\alpha > 1$ there exist constants $\theta > 0$, K and a positive adapted process $f \in L^1([0,T] \times \Omega; \mathrm{d}t \times \mathbf{P})$ such that the following conditions hold for all $v, v_1, v_2 \in V$ and $(t, \omega) \in [0,T] \times \Omega$.*

(H1) *(Hemicontinuity) The map $s \mapsto {}_{V^*}\langle A(t, v_1 + sv_2), v \rangle_V$ is continuous on $\mathbb{R}$.*

(H2) *(Monotonicity)*

$$2_{V^*}\langle A(t, v_1) - A(t, v_2), v_1 - v_2 \rangle_V + \|B(t, v_1) - B(t, v_2)\|_{L_2(U;H)}^2 \le K\|v_1 - v_2\|_H^2.$$

(H3) *(Coercivity)*

$$2_{V^*}\langle A(t, v), v \rangle_V + \|B(t, v)\|_{L_2(U;H)}^2 + \theta\|v\|_V^\alpha \le f_t + K\|v\|_H^2.$$

(H4) *(Boundedness)*
$$\|A(t, v)\|_{V^*} \le f_t^{(\alpha-1)/\alpha} + K\|v\|_V^{\alpha-1}.$$

Then for any $X_0 \in L^2(\Omega \to H; \mathcal{F}_0; \mathbf{P})$ (1.1) has a unique solution $\{X_t\}_{t\in[0,T]}$ and satisfies

$$\mathbf{E} \sup_{t\in[0,T]} \|X_t\|_H^2 < \infty.$$

Moreover, we have the following crucial Itô's formula

$$\|X_t\|_H^2 = \|X_0\|_H^2 + \int_0^t \left(2_{V^*}\langle A(s, X_s), X_s \rangle_V + \|B(s, X_s)\|_{L_2(U;H)}^2\right) \mathrm{d}s + 2 \int_0^t \langle X_s, B(s, X_s)\mathrm{d}W_s \rangle_H.$$

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