Nonlinear evolution and transport equations for measures

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The goal of this work is to prove the existence of a solution to the following transport equation:

$$\partial_t\mu_t + \mathrm{div}_x(b(\mu,\,\cdot\,,\,\cdot\,)\mu_t) = 0, \quad \mu_0 = \nu, \tag{1}$$

where

$$b = (b^i)_{i=1}^d\colon\ \mathcal{M}_0(\mathbb{R}^d\times[0,1])\times\mathbb{R}^d\times[0,1]\to\mathbb{R}^d$$

is a Borel mapping and $\mathcal{M}_0(\mathbb{R}^d\times[0,1])$ is the space of finite Borel measures on $\mathbb{R}^d\times[0,1]$ equipped with the weak topology.

We shall say that a family $\mu := (\mu_t)_{t\in[0,1]}$ of finite Borel measures (regarded also as the measure $\mu_t(dx)\,dt$ on $\mathbb{R}^d\times[0,1]$) satisfies equation (1) if $b(\mu,\,\cdot\,,\,\cdot\,)\in L^1(S\times[0,1],\mu_t(dx)\,dt)$ for every compact set $S\subset\mathbb{R}^d$, that is, the function $|b(\mu,\,\cdot\,,\,\cdot\,)|$ is integrable with respect to $|\mu|$ on every compact set in $\mathbb{R}^d\times[0,1]$, and for all $t\in[0,1]$ the following identity holds:

$$\int_{\mathbb{R}^d}\varphi(x)\,\mu_t(dx) - \int_{\mathbb{R}^d}\varphi(x)\,\nu(dx) =$$

$$= \int_0^t\int_{\mathbb{R}^d}(b(\mu,x,s),\nabla\varphi(x))\,\mu_s(dx)\,ds \quad \forall\,\varphi\in C_0^\infty(\mathbb{R}^d). \tag{2}$$

This equation has been an object of intensive studies over the past decade. For recent surveys, see [1], [2], [3], and [4]; in particular, nonlinear equations are considered in [4]. A typical condition on b in the linear case is the inclusion $b\in L^1([0,1],W^{1,\infty}(\mathbb{R}^d))$ or the requirement that b is a BV function with respect to x (see [1], [2]). Our aim is to prove the existence assuming only some conditions on the growth of b. In this paper we only consider probability measures. The spaces of probability measures on $\mathbb{R}^d\times[0,1]$ and $\mathbb{R}^d$ equipped with the weak topology will be denoted by $\mathcal{P}(\mathbb{R}^d\times[0,1])$ and $\mathcal{P}(\mathbb{R}^d)$, respectively. Our main result is the following theorem.

Theorem 1. *Let ν be a probability measure on $\mathbb{R}^d$. Suppose that*

(i) *for every fixed measure $\mu\in\mathcal{P}(\mathbb{R}^d\times[0,1])$, the mapping $x\mapsto b(\mu,x,t)$ is continuous for almost every t and one has uniform convergence $b(\mu_j,x,t)\to b(\mu,x,t)$ on compact sets whenever $\mu_j\to\mu$ weakly*;

(ii) *there exist numbers $c\in(0,\infty)$ and $k\in\mathbb{N}$ such that for all $(x,t)\in\mathbb{R}^d\times[0,1]$ and all $\mu\in\mathcal{P}(\mathbb{R}^d\times[0,1])$ one has*

$$(b(\mu,x,t),x)\le c(1+|x|^2),$$

$$|b(\mu,x,t)|\le c(1+|x|^{2k}), \qquad \int_{\mathbb{R}^d}|x|^{2k}\,\nu(dx)<\infty.$$

Then there exists a family $\mu=(\mu_t)_{t\in[0,1]}$ of probability measures satisfying (1).

Under condition (ii), condition (i) can be reformulated as follows: for every fixed measure μ, the mapping $x\mapsto b(\mu,x,t)$ is continuous for a.e. t and for each compact set $S\subset\mathbb{R}^d$, the mapping b generates a continuous mapping

$$F\colon\ \mathcal{P}(\mathbb{R}^d\times[0,1])\to L^\infty([0,1],C(S))$$

defined by $F(\mu)(t)(x) := b(\mu,x,t)$.

Our approach is based on the well-known method of "vanishing viscosity" (see, e.g., [5, Theorem 4]) combined with the Schauder theorem. We replace equation (1) by the parabolic equation

$$\partial_t\mu_t - \varepsilon\Delta\mu_t + \mathrm{div}_x(b(\mu,\,\cdot\,,\,\cdot\,)\mu_t) = 0, \quad \mu_0=\nu, \tag{3}$$

understood as the following integral identity for all $t \in [0,1]$:

$$\int_{\mathbb{R}^d} \varphi \, d\mu_t - \int_{\mathbb{R}^d} \varphi \, d\nu = \int_0^t \int_{\mathbb{R}^d} [\varepsilon \Delta \varphi + (b, \nabla \varphi)] \, d\mu_s \, ds \quad \forall \, \varphi \in C_0^\infty(\mathbb{R}^d). \tag{4}$$

Under our assumptions on b in the case where b is independent of μ, the next proposition follows from the results established in [6] and [7] (see [6, Corollary 3.3, Lemma 2.1, Lemma 2.2] and [7, Theorem 3.3]).

Proposition 1. *Suppose that a probability measure ν on $\mathbb{R}^d$ has a density $\varrho_0 \in C_0^\infty(\mathbb{R}^d)$, the coefficient b does not depend on μ, and there exist numbers $c \in (0, \infty)$ and $k \geq 1$ such that for all $(x,t) \in \mathbb{R}^d \times [0,1]$ one has*

$$(b(x,t), x) \leq c(1 + |x|^2),$$

$$|b(x,t)| \leq c(1 + |x|^{2k}), \quad \int_{\mathbb{R}^d} |x|^{2k} \, \nu(dx) < \infty.$$

Then there exists a unique family $\mu = (\mu_t)_{t \in [0,1]}$ of probability measures on $\mathbb{R}^d$ solving equation (3)*. Moreover, there exists a number N depending only on c, k and $\int_{\mathbb{R}^d} |x|^{2k} \, d\nu$ such that*

$$\sup_{t \in [0,1]} \int_{\mathbb{R}^d} |x|^{2k} \, d\mu_t < N.$$

Certainly, the same is true for any interval $[0,T]$ in place of $[0,1]$.

First we prove our main result in the case of b independent of μ by letting $\varepsilon \to 0$. This case is simpler than the general one, but the proof can be extended to the infinite-dimensional case. So we assume that b does not depend on μ, the mapping $x \mapsto b(x,t)$ is continuous for almost every t, and condition (ii) of Theorem 1 is fulfilled.

Let us fix a Borel probability measure ν on $\mathbb{R}^d$ and a sequence of probability measures $\nu^n = \varrho^n(x) \, dx$, where $\varrho^n \in C_0^\infty(\mathbb{R}^d)$, such that $\{\nu^n\}$ converges weakly to ν and one has

$$\sup_n \int_{\mathbb{R}^d} |x|^{2k} \, d\nu^n < \infty.$$

According to Proposition 1, for each n, there exists a unique family $\mu^n = (\mu^n_t)_{t \in [0,1]}$ of probability measures on $\mathbb{R}^d$ satisfying the equation

$$\partial_t \mu^n_t - n^{-1} \Delta \mu^n_t + \operatorname{div}_x(b \mu^n_t) = 0, \quad \mu^n_0 = \nu^n$$

in the sense that for all $\varphi \in C_0^\infty(\mathbb{R}^d)$ and $t \in [0,1]$ one has

$$\int_{\mathbb{R}^d} \varphi \, d\mu^n_t - \int_{\mathbb{R}^d} \varphi \, d\nu^n = \int_0^t \int_{\mathbb{R}^d} [n^{-1} \Delta \varphi + (b, \nabla \varphi)] \, d\mu^n_s \, ds. \tag{5}$$

Moreover, there exists a number C independent of n such that

$$\sup_{t \in [0,1]} \int_{\mathbb{R}^d} |x|^{2k} \, d\mu^n_t < C,$$

where k is the number from condition (ii) in Theorem 1. Therefore, the set of measures

$$\{\mu^n_t \colon \ t \in [0,1], n \in \mathbb{N}\}$$

is uniformly tight on $\mathbb{R}^d$. Now we fix a countable dense set $\mathcal{F} \in C_0^\infty(\mathbb{R}^d)$ (the latter space is equipped with the topology of uniform convergence of all derivatives on compact sets) and take a countable dense set $T \subset [0,1]$. We can find a subsequence $\{\mu^{n_k}_t\}$ which converges weakly to a probability measure μ_t for each $t \in T$. Let us prove that

$$\lim_{n \to \infty} \int_{\mathbb{R}^d} \varphi(x) \, d\mu^{n_k}_t = \int_{\mathbb{R}^d} \varphi(x) \, d\mu_t \tag{6}$$

for all $t\in[0,1]$ and all $\varphi\in\mathcal{F}$. Let us fix a function $\varphi\in\mathcal{F}$. Then there is a number B such that $|b(x,t)|\le B$ for all $x\in\operatorname{supp}\varphi$ and $t\in[0,1]$, so we have

$$\Bigl|\int_{\mathbb{R}^d}\varphi\,d\mu_t^{\varepsilon_k}-\int_{\mathbb{R}^d}\varphi\,d\mu_s^{\varepsilon_k}\Bigr|\le(\|\Delta\varphi\|_\infty+B\|\nabla\varphi\|_\infty)|t-s|.$$

Let us fix a point $t\in[0,1]$. For any $\varepsilon>0$ there are a number $r\in T$ and a natural number N such that for all numbers $k>N$ one has

$$\Bigl|\int_{\mathbb{R}^d}\varphi\,d\mu_t^{n_k}-\int_{\mathbb{R}^d}\varphi\,d\mu_r^{n_k}\Bigr|\le\varepsilon/3,\quad \Bigl|\int_{\mathbb{R}^d}\varphi\,d\mu_r^{n_k}-\int_{\mathbb{R}^d}\varphi\,d\mu_r\Bigr|\le\varepsilon/6.$$

Then, for all $k,l>N$, we have

$$\begin{aligned}&\Bigl|\int_{\mathbb{R}^d}\varphi\,d\mu_t^{n_k}-\int_{\mathbb{R}^d}\varphi\,d\mu_t^{n_l}\Bigr|\\ &\le\Bigl|\int_{\mathbb{R}^d}\varphi\,d\mu_t^{n_k}-\int_{\mathbb{R}^d}\varphi\,d\mu_r^{n_k}\Bigr|+\Bigl|\int_{\mathbb{R}^d}\varphi\,d\mu_r^{n_k}-\int_{\mathbb{R}^d}\varphi\,d\mu_r^{n_l}\Bigr|+\Bigl|\int_{\mathbb{R}^d}\varphi\,d\mu_r^{n_l}-\int_{\mathbb{R}^d}\varphi\,d\mu_t^{n_l}\Bigr|\le\varepsilon.\end{aligned}$$

Hence the sequence $\mu_t^{n_k}$ on $\mathbb{R}^d$ is weakly fundamental and uniformly tight. Therefore, we obtain (6) for all t and all $\varphi\in\mathcal{F}$. Then equality (6) holds for all continuous functions φ with compact support. Letting $k\to\infty$ in (5) we obtain the equality

$$\int_{\mathbb{R}^d}\varphi\,d\mu_t-\int_{\mathbb{R}^d}\varphi\,d\nu=\int_0^t\int_{\mathbb{R}^d}(b,\nabla\varphi)\,d\mu_s\,ds$$

because, for almost every fixed s, by the continuity of $x\mapsto b(x,s)$, we have

$$\int_{\mathbb{R}^d}(b(x,s),\nabla\varphi(x))\,d\mu_s^{n_k}\to\int_{\mathbb{R}^d}(b(x,s),\nabla\varphi(x))\,d\mu_s$$

and the left-hand side is uniformly bounded, which enables us to integrate in s and obtain the aforementioned equality. This gives Theorem 1 in the linear case.

Let us proceed to the general case where b may depend on μ. We construct a solution to (1) as a weak limit of solutions to approximating nondegenerate parabolic equations with the extra terms $-\varepsilon\Delta\mu_t$, where the coefficient b satisfies condition (ii) of Theorem 1, but in place of condition (i) we impose the following much weaker condition:

(i)$'$ the mapping b is defined on the space $\mathcal{P}_{\rm abs}\times\mathbb{R}^d\times[0,1]$, where the space $\mathcal{P}_{\rm abs}$ consists of all absolutely continuous probability measures on $\mathbb{R}^d\times[0,1]$ and is equipped with variation distance, and for Lebesgue a.e. (x,t), the mapping $\mu\mapsto b(\mu,x,t)$ is continuous in the variation distance.

In fact, we need even less: it suffices that b be defined only for μ from the subset in $\mathcal{P}_{\rm abs}$ consisting of measures of the form $\varrho(x,t)\,dx\,dt$ such that $x\mapsto\varrho(x,t)$ is a probability density for each $t\in[0,1]$.

Let us consider the following nonlinear parabolic equation:

$$\partial_t\mu_t-\varepsilon\Delta\mu_t+\operatorname{div}_x(b(\mu,\,\cdot\,,\,\cdot\,)\mu_t)=0,\quad \mu_0=\nu,\tag{7}$$

where $\nu=\varrho_0(x)\,dx$ and $\varrho_0\in C_0^\infty(\mathbb{R}^d)$.

First we prove the existence of a solution to equation (7). Suppose that we are given real numbers $\alpha>0$ and $c_1>0$ and that, for each closed interval $J_r=[r,1-r]$ and each closed ball $B_R=\{x\in\mathbb{R}^d\colon\ |x|\le R\}$, where $r,R>0$ we are given a number $c_2(r,R)>0$. Let k be the number from condition (ii) and let $C^\alpha(E)$ denote the Banach space of α-Hölder functions on E with its natural norm.

Let us consider the set $\mathcal{K}\subset L^1(\mathbb{R}^d\times[0,1])$ of all functions ϱ satisfying the following conditions:

$$\varrho\ge 0,\ \int_{\mathbb{R}}\varrho(x,t)\,dx=1,\ \int_{\mathbb{R}^d}|x|^{2k}\varrho(x,t)\,dx\le c_1\quad\forall\,t\in[0,1],$$

$$\|\varrho\|_{C^\alpha(J_r\times B_R)} \le c_2(r,R) \quad \forall\, r,R>0,$$

$\varrho(x,0)=\varrho_0(x)$ a.e. and for each $\varphi\in C_0^\infty(\mathbb{R}^d)$ the function

$$t\mapsto \int_{\mathbb{R}^d}\varphi(x)\varrho(x,t)\,dx$$

is Lipschitzian with some constant $C(\varphi)$.

Lemma 1. *The set $\mathcal{K}$ is convex and compact in the Banach space $L^1([0,1]\times\mathbb{R}^d)$.*

Proof. Indeed, given a sequence in $\mathcal{K}$, by a diagonal argument we choose a subsequence $\{\varrho_n\}$ that converges uniformly on compact sets in $\mathbb{R}^d\times(0,1)$ (here we use the bounds on the Hölder norms). Since

$$\int_{\mathbb{R}^d}|x|^{2k}\varrho(x,t)\,dx\le c_1 \quad \forall\, t\in[0,1],$$

the set of probability measures $\varrho(x,t)\,dx$ on $\mathbb{R}^d$, where $\varrho\in\mathcal{K}$ and $t\in[0,1]$, is uniformly tight. Hence, for each fixed $t\in[0,1]$, the measures $\varrho_n(x,t)\,dx$ on $\mathbb{R}^d$ converge weakly to a probability measure μ_t on $\mathbb{R}^d$, where $\mu_0=\nu$. Locally uniform convergence of densities shows that $\mu=\mu_t\,dt$ has a density ϱ, which is locally Hölder continuous on $\mathbb{R}^d\times(0,1)$ and satisfies the equality $\mu_t=\varrho(x,t)\,dx$. For each fixed $\varphi\in C_0^\infty(\mathbb{R}^d)$ we have

$$\int_{\mathbb{R}^d}\varphi(x)\,\varrho(x,t)\,dx=\lim_{n\to\infty}\int_{\mathbb{R}^d}\varphi(x)\,\varrho_n(x,t)\,dx, \quad t\in[0,1],$$

hence the left-hand side is Lipschitzian with constant $C(\varphi)$. Therefore, $\varrho\in\mathcal{K}$. Hence $\varrho_n\to\varrho$ in the norm of $L^1(\mathbb{R}^d\times[0,1])$; we recall that pointwise convergence of probability densities to a probability density yields convergence in mean, see [8, Theorem 2.8.9]. Obviously, $\mathcal{K}$ is convex. □

It should be noted that we obtain a convex compact set even if we omit the last condition in the definition of $\mathcal{K}$ regulating the behavior in t and specifying the value at $t=0$. However, for subsequent applications to parabolic equations the introduced class turns out to be more convenient. The probability measure with density $\varrho\in\mathcal{K}$ on $\mathbb{R}^d\times[0,1]$ will be denoted by the same symbol ϱ and ϱ_t will denote both the probability density $x\mapsto\varrho(x,t)$ on $\mathbb{R}^d$ and the measure with this density.

Now we define a mapping $T\colon\,\mathcal{K}\to\mathcal{K}$ as follows:

$$\chi=T(\varrho) \quad\Longleftrightarrow\quad \partial_t\chi_t-\varepsilon\Delta\chi_t+\operatorname{div}_x(b(\varrho,\,\cdot\,,\,\cdot\,)\chi_t)=0, \quad \chi_0=\nu.$$

According to Proposition 1 and [9, Corollary 3.9], the mapping T is well-defined. Note that the Lipschitzness of the integral of $\varphi\in C_0^\infty(\mathbb{R}^d)$ with respect to $\chi_t(dx)$ follows from (4) due to the uniform boundedness of $b(\varrho,x,t)$ on $\operatorname{supp}\varphi\times[0,1]$.

Lemma 2. *The mapping T is continuous.*

Proof. Let $\varrho^n,\varrho\in\mathcal{K}$, $\|\varrho^n-\varrho\|_{L^1}\to0$ and $\chi^n=T(\varrho^n)$. Since $\mathcal{K}$ is compact, we can find a convergent subsequence $\{\chi^{n_k}\}$. We prove that $\chi:=\lim\limits_{k\to\infty}\chi^{n_k}$ satisfies equation (7) with $b=b(\varrho,\,\cdot\,,\,\cdot\,)$. For every $\varphi\in C_0^\infty(\mathbb{R}^d)$ we have the identity

$$\int_{\mathbb{R}^d}\varphi(x)\chi^{n_k}(x,t)\,dx-\int_{\mathbb{R}^d}\varphi(x)\,\nu(dx)=\varepsilon\int_0^t\int_{\mathbb{R}^d}\Delta\varphi(x)\chi^{n_k}(x,s)\,dx\,ds+$$
$$+\int_0^t\int_{\mathbb{R}^d}(b(\varrho^{n_k},x,s),\nabla\varphi(x))\chi^{n_k}(x,s)\,dx\,ds. \tag{8}$$

Set $S:=\operatorname{supp}\varphi$. Since $|b(\varrho^{n_k},x,t)|$ is uniformly bounded on $S\times[0,1]$ and

$$\|\chi^{n_k}-\chi\|_{L^1}\to0, \quad \|\varrho^{n_k}-\varrho\|_{L^1}\to0, \quad |b(\varrho^{n_k},x,t)-b(\varrho,x,t)|\to0 \quad \text{a.e.},$$

we can let $k \to \infty$ in (8) and obtain for all $t \in [0,1]$

$$\int_{\mathbb{R}^d} \varphi(x)\chi(x,t)\,dx - \int_{\mathbb{R}^d} \varphi(x)\,\nu(dx) = \varepsilon \int_0^t \int_{\mathbb{R}^d} \Delta\varphi(x)\chi(x,s)\,dx\,ds + \\ + \int_0^t \int_{\mathbb{R}^d} (b(\varrho,x,s), \nabla\varphi(x))\chi(x,s)\,dx\,ds.$$

This shows that $\chi = T(\varrho)$. According to the uniqueness assertion in Proposition 1 we conclude that each subsequence in $\{\chi^n\}$ contains a subsequence convergent to χ. This yields that $\chi^n \to \chi$, hence T is continuous. □

Applying Schauder's fixed point theorem we conclude that there exists $\varrho \in \mathcal{K}$ such that $\varrho = T(\varrho)$ and the family of measures $\mu_t = \varrho(x,t)\,dx$ satisfies equation (7). Hence we arrive at the following assertion.

Proposition 2. *Suppose that a probability measure ν has a density from $C_0^\infty(\mathbb{R}^d)$ and b and ν satisfy condition* (ii) *of Theorem* 1 *and condition* (i)$'$ *above. Then there exists a family $\mu = (\mu_t)_{t\in[0,1]}$ of probability measures on $\mathbb{R}^d$ satisfying* (7)*. Moreover, there exists a number N depending only on c, k and $\int_{\mathbb{R}^d} |x|^{2k}\,d\nu$ such that*

$$\sup_{t\in[0,1]} \int_{\mathbb{R}^d} |x|^{2k}\,d\mu_t < N.$$

Now we prove Theorem 1. Let us fix a probability measure ν. We can find probability measures $\nu^n = \varrho_n(x)\,dx$ weakly convergent to the measure ν such that $\varrho_n \in L^\infty(\mathbb{R}^d)$ and

$$\sup_n \int_{\mathbb{R}^d} |x|^{2k}\,d\nu^n < \infty.$$

For each $\varepsilon = n^{-1}$ we take the solution $(\mu_t^n)_{t\in[0,1]}$ to equation (7) with $\mu_0^n = \nu^n$. Repeating our reasoning from the linear case we find a sequence $\{n_k\}$ such that $\{\mu_t^{n_k}\}$ converges weakly to μ_t for all $t \in [0,1]$. Denote $b(\mu,\,\cdot\,,\,\cdot\,)$ and $b(\mu^{n_k},\,\cdot\,,\,\cdot\,)$ by b and b_k respectively. We have the identity

$$\int_{\mathbb{R}^d} \varphi\,d\mu_t^{n_k} - \int_{\mathbb{R}^d} \varphi\,d\nu^{n_k} = \int_0^t \int_{\mathbb{R}^d} [n_k^{-1}\Delta\varphi + (b_k, \nabla\varphi)]\,d\mu_s^{n_k}\,ds.$$

Let $S := \operatorname{supp}\varphi$. Note that

$$\Big|\int_0^t \int_{\mathbb{R}^d} (b_k, \nabla\varphi)\,d\mu_s^{n_k}\,ds - \int_0^t \int_{\mathbb{R}^d} (b, \nabla\varphi)\,d\mu_s\,ds\Big| \le \\ \le \|b_k - b\|_{L^\infty(S\times[0,1])} \|\nabla\varphi\|_\infty + \\ + \Big|\int_0^t \int_{\mathbb{R}^d} (b, \nabla\varphi)\,d\mu_s^{n_k}\,ds - \int_0^t \int_{\mathbb{R}^d} (b, \nabla\varphi)\,d\mu_s\,ds\Big|.$$

Letting $k \to \infty$ we obtain

$$\int_{\mathbb{R}^d} \varphi\,d\mu_t - \int_{\mathbb{R}^d} \varphi\,d\nu = \int_0^t \int_{\mathbb{R}^d} (b(\mu,\,\cdot\,,\,\cdot\,), \nabla\varphi)\,d\mu_s\,ds$$

since by our assumption the mapping $x \mapsto b(\mu,x,s)$ is continuous for a.e. $s \in [0,1]$ and the function $(x,s) \mapsto |(b(\mu,x,s), \nabla\varphi(x))|$ is uniformly bounded. This completes the proof of Theorem 1.

It is worth noting that, according to (2), since the function $(b(\mu,x,t), \nabla\varphi(x))$ has bounded support and is uniformly bounded for each $\varphi \in C_0^\infty(\mathbb{R}^d)$, the function

$$t \mapsto \int_{\mathbb{R}^d} \varphi(x)\,\mu_t(dx)$$

is Lipschitzian. This function is continuously differentiable if b is continuous in t.

Finally, we observe that positivity of measures is essential for our a priori estimates employed in the proof. The same techniques apply to much more general second order elliptic operators in place of the Laplacian: we only need the assumptions from [6], [7], and [9]. Extensions to the infinite-dimensional case in the spirit of [10] will be considered in a forthcoming paper.

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