NONLINEAR EVOLUTION EQUATIONS FOR MEASURES ON INFINITE DIMENSIONAL SPACES

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The goal of this work is to prove the existence of a solution to the following nonlinear evolution equation for probability measures on a separable Hilbert space X with an orthonormal basis $\{e_i\}$:

$$\partial_t\mu_t + \sum_{i=1}^{\infty} \partial_{e_i}(b^i(\mu,\,\cdot\,,\,\cdot\,)\mu_t) = 0, \quad \mu_0 = \nu, \tag{1}$$

where

$$b^i\colon\ \mathcal{P}(X\times[0,1])\times X\times[0,1]\to\mathbb{R}^1,\ i\in\mathbb{N},$$

are Borel functions and $\mathcal{P}(X\times[0,1])$ is the space of Borel probability measures on $X\times[0,1]$ equipped with the weak topology (in Theorem 2 we consider the weak topology on the space of measures corresponding to the weak topology w on X, i.e., we deal with (X,w), and in Theorems 3 and 4 we are concerned with the usual weak topology on the space of measures).

We shall say that a family $\mu := (\mu_t)_{t\in[0,1]}$ of Borel probability measures on X (regarded also as a measure on $X\times[0,1]$ through the identification $\mu := \mu_t\,dt$) satisfies equation (1) if

$$b^i(\mu,\,\cdot\,,\,\cdot\,)\in L^1(\mu)$$

and, for all $t\in[0,1]$, one has

$$\int_X \varphi\,d\mu_t - \int_X \varphi\,d\nu = \int_0^t\int_X \sum_{i=1}^{m} b^i(\mu,x,s)\partial_{e_i}\varphi(x)\,\mu_s(dx)\,ds \tag{2}$$

for every function φ of the form

$$\varphi(x) = \varphi_0(x_1,\dots,x_m),\ x_i=(x,e_i), \quad \varphi_0\in C_b^\infty(\mathbb{R}^m),\ m\in\mathbb{N}.$$

The class of all such functions will be denoted by $\mathcal{FC}^\infty(\{e_i\})$.

The finite dimensional case has been studied in [9]. We recall the main result of [9]. In the finite dimensional case we deal with the equation

$$\partial_t\mu_t + \mathrm{div}_x(b(\mu,\,\cdot\,,\,\cdot\,)\mu_t) = 0, \quad \mu_0 = \nu, \tag{3}$$

where

$$b = (b^i)_{i=1}^d\colon\ \mathcal{P}(\mathbb{R}^d\times[0,1])\times\mathbb{R}^d\times[0,1]\to\mathbb{R}^d$$

is a Borel mapping. A family $\mu := (\mu_t)_{t\in[0,1]}$ of Borel probability measures on $\mathbb{R}^d$ (regarded also as a measure on $\mathbb{R}^d\times[0,1]$ as above) satisfies equation (3) if

$$b^i(\mu,\,\cdot\,,\,\cdot\,)\in L^1(S\times[0,1],\mu_t(dx)\,dt)$$

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for every compact set $S \subset \mathbb{R}^d$, that is, the function $(x,t) \mapsto |b(\mu,x,t)|$ is integrable with respect to $|\mu|$ on every compact set in $\mathbb{R}^d \times [0,1]$, and the following identity holds for all $t \in [0,1]$:

$$\int_{\mathbb{R}^d} \varphi\, d\mu_t - \int_{\mathbb{R}^d} \varphi\, d\nu = \int_0^t \int_{\mathbb{R}^d} (b(\mu,\,\cdot\,,\,\cdot\,), \nabla\varphi)\, d\mu_s\, ds \quad \forall\, \varphi \in C_0^\infty(\mathbb{R}^d), \tag{4}$$

where $(\,\cdot\,,\,\cdot\,)$ denotes the Euclidean inner product on $\mathbb{R}^d$.

This equation has been an object of intensive studies over the past decade. For recent surveys, see [1], [2], [11], and [12]. In particular, the nonlinear equation has been considered in [11]. A typical condition on b in the linear case is the inclusion $b \in L^1([0,1], W^{1,\infty}(\mathbb{R}^d))$ or the requirement that b is a BV function with respect to x (see [1], [2]). In [9], the existence has been established assuming only some conditions on the growth of b. Namely, the following result has been proved.

Theorem 1. *Let ν be a probability measure on $\mathbb{R}^d$. Suppose that*

(A1) *for every fixed measure $\mu \in \mathcal{P}(\mathbb{R}^d \times [0,1])$, the mapping $x \mapsto b(\mu,x,t)$ is continuous for a.e. t and one has uniform convergence $b(\mu_j,\,\cdot\,,\,\cdot\,) \to b(\mu,\,\cdot\,,\,\cdot\,)$ on compact sets whenever $\mu_j \to \mu$ weakly;*

(B1) *there exist numbers $c \in (0,\infty)$ and $\kappa \geq 2$ such that for all $(x,t) \in \mathbb{R}^d \times [0,1]$ and $\mu \in \mathcal{P}(\mathbb{R}^d \times [0,1])$ one has*

$$(b(\mu,x,t),x) \leq c(1+|x|^2),$$

$$|b(\mu,x,t)| \leq c(1+|x|^\kappa), \qquad \int_{\mathbb{R}^d} |x|^\kappa\, \nu(dx) < \infty.$$

Then there exists a family $\mu = (\mu_t)_{t\in[0,1]}$ of probability measures satisfying (1)*. Moreover,*

$$\sup_{t\in[0,1]} \int_{\mathbb{R}^d} |x|^\kappa\, \mu_t(dx) \leq M < \infty,$$

where M depends only on c and the moment of ν of order κ.

Note that, under (B1), condition (A1) can be reformulated as follows: for every fixed measure μ, the mapping $x \mapsto b(\mu,x,t)$ is continuous for a.e. t and for each compact set $S \subset \mathbb{R}^d$, the mapping b generates a continuous mapping F from the space $\mathcal{P}(\mathbb{R}^d \times [0,1])$ to $L^\infty([0,1], C(S))$ defined by $F(\mu)(t)(x) := b(\mu,x,t)$.

The approach in [9] is based on the well-known method of "vanishing viscosity" (see, e.g., [10]). One considers the parabolic equation

$$\partial_t\mu_t - \varepsilon\Delta\mu_t + \mathrm{div}_x(b(\mu,\,\cdot\,,\,\cdot\,)\mu_t) = 0, \quad \mu_0 = \nu, \tag{5}$$

understood as the integral identity

$$\int_{\mathbb{R}^d} \varphi\, d\mu_t - \int_{\mathbb{R}^d} \varphi\, d\nu = \int_0^t \int_{\mathbb{R}^d} [\varepsilon\Delta\varphi + (b,\nabla\varphi)]\, d\mu_s\, ds \quad \forall\, \varphi \in C_0^\infty(\mathbb{R}^d),\ t \in [0,1]. \tag{6}$$

According to [5], [4], under the stated assumptions this equation has a unique solution for any given mapping b independent of μ. Hence, for any given μ, we obtain the corresponding drift $b(\mu,\,\cdot\,,\,\cdot\,)$ generating a solution to (6) with that $b(\mu,\,\cdot\,,\,\cdot\,)$. By using this result and Schauder's fixed point theorem, it has been shown in [9] that (5) is solvable in the presence of dependence on b on the required solution μ. Finally, letting $\varepsilon \to 0$, one obtains a solution to (4) in the limit.

Let us proceed to the infinite-dimensional case.

Let $\{e_i\}$ be an orthonormal basis in X. Set

$$P_n x := \sum_{i=1}^{n} x_i e_i, \quad X_n := P_n(X).$$

We shall identify X_n with $\mathbb{R}^n$.

Let us introduce the following two conditions on b.

(A2) for every fixed measure $\mu \in \mathcal{P}(X \times [0,1])$ and every fixed i, the functions

$$x \mapsto b^i(\mu, x, t)$$

are weakly continuous on balls for a.e. t and one has uniform convergence

$$b^i(\mu_j, x, t) \to b^i(\mu, x, t)$$

on bounded sets in $\mathbb{R}^d \times [0,1]$ whenever $\mu_j \to \mu$ weakly with respect to the weak topology on X;

(B2) there exist numbers $\alpha > 0$, $c_i > 0$ and $\kappa \geq 2$ such that for all $(x,t) \in X \times [0,1]$ and $\mu \in \mathcal{P}(X \times [0,1])$ one has

$$\sum_{i=1}^{n} b^i(\mu, x, t) x_i \leq \alpha(1 + |x|^2) \quad \forall\, x \in X_n,\, n \in \mathbb{N},$$

$$|b^i(\mu, x, t)| \leq c_i(1 + |x|^\kappa).$$

The first inequality in (B2) actually means that we use the Lyapunov function $V(x) = (x,x)$ (or its power) in order to get an estimate $LV \leq c + cV$, where L is the first-order operator generated by b. Such estimates ensure certain a priori bounds for solutions, on which our existence results are based (see [5] and [8] concerning techniques of Lyapunov functions).

We recall that the requirement of weak continuity on balls simply means that $b^i(\mu, x^j, t) \to b^i(\mu, x, t)$ whenever $x^j \to x$ weakly.

We also recall that weak convergence $\nu_j \to \nu$ in $\mathcal{P}(X)$ with respect to the weak topology on X is weaker than the usual weak convergence (associated with the norm topology on X) and is equivalent to the following two conditions: for every bounded continuous function f of finitely many variables the integrals of f with respect to ν_j converge to the integral with respect to ν and, for every $\varepsilon > 0$, there is a ball $U_\varepsilon \subset X$ with

$$\nu_j(X \backslash U_\varepsilon) < \varepsilon \quad \text{for all } j.$$

The situation is similar for weak convergence $\mu_j \to \mu$ in $\mathcal{P}(X \times [0,1])$ with respect to the weak topology on X, but the last condition is replaced by

$$\mu_j((X \backslash U_\varepsilon) \times [0,1]) < \varepsilon \quad \text{for all } j.$$

It should be noted that we do not assume that $b = (b^i)$ corresponds to a vector field in X: it is merely a collection of scalar functions b^i.

Theorem 2. *Let ν be a Borel probability measure on X such that for some $p > \kappa$ one has*

$$\int_X |x|^p \, \nu(dx) < \infty.$$

Let the collection $b = (b^i)$ have properties (A2) *and* (B2) *above. Then there exists a family $\mu = (\mu_t)_{t\in[0,1]}$ of probability measures satisfying* (1)*. Moreover, one has*

$$\sup_{t\in[0,1]} \int_X |x|^p \, \mu_t(dx) < \infty.$$

Proof. It follows from our assumptions that, for every fixed n, the mapping

$$B_n\colon\ (\mu, x, t) \mapsto (b^1(\mu, x, t), \dots, b^n(\mu, x, t)), \quad \mathcal{P}(X_n \times [0,1]) \times X_n \times [0,1] \to X_n,$$

satisfies the hypotheses of Theorem 1. Therefore, for every Borel probability measure ν_n on X_n with finite moment of order κ there exists a solution $\mu^n = (\mu^n_t)_{t\in[0,1]}$ of equation (1) with $\mu^n_0 = \nu_n$. In particular, this is true if we take $\nu^n = \nu \circ P_n^{-1}$. We shall prove that there is a subsequence $\{n_j\}$ such that, for each $t \in [0,1]$, the measures $\mu^{n_j}_t$ converge weakly on (X, w) to a Borel probability measure μ_t and the obtained family $\mu = (\mu_t)_{t\in[0,1]}$ satisfies equation (1). First we observe that there is a number M independent of n such that for all n one has

$$\sup_{t\in[0,1]} \int_{X_n} |x|^p \, \mu^n_t(dx) \le M. \tag{7}$$

Therefore, there is a subsequence $\{n_j\}$ such that, for every rational number $t \in [0,1]$, the sequence of measures $\mu^{n_j}_t$ converges weakly on (X, w) to a Borel probability measure μ_t. Clearly, one has

$$\int_X |x|^p \, \mu_t(dx) \le M. \tag{8}$$

Let us prove that along with (7) this yields weak convergence on (X, w) for each $t \in [0,1]$. Let $b_j = b(\mu^{n_j}, \,\cdot\,, \,\cdot\,)$ and $b = b(\mu, \,\cdot\,, \,\cdot\,)$. Let $\varphi \in \mathcal{FC}^\infty$. The function φ depends on $x_1, \dots, x_m$. For each j we have

$$\psi_j(t) := \int_X \varphi \, d\mu^{n_j}_t - \int_X \varphi \, d\nu^{n_j} = \int_0^t \int_X (b_j, \nabla\varphi) \, d\mu^{n_j}_s \, ds. \tag{9}$$

We know that the functions ψ_j converge at all rational points of $[0,1]$. Let us show that they converge at every point. To this end, it suffices to observe that they are uniformly Lipschitzian on $[0,1]$. This is seen from the estimate

$$\Bigl| \int_X (b_j, \nabla\varphi) \, d\mu^{n_j}_s \Bigr| \le C_1(\varphi) \int_X c(1 + |x|^\kappa) \, d\mu^{n_j}_s \le C_2(\varphi),$$

where $C_1(\varphi)$ and $C_2(\varphi)$ are independent of j (these numbers depend only on the number m, the constants $c_1, \dots, c_m$ from condition (ii), and $\sup_x |\nabla\varphi(x)|$). Furthermore, we have

$$\Bigl|(b_j(x,t), \nabla\varphi(x))\Bigr| \le K_1 + K_2 \|x\|^\kappa,$$

$$\Bigl|(b(x,t), \nabla\varphi(x))\Bigr| \le K_1 + K_2 \|x\|^\kappa,$$

where K_1 and K_2 are independent of j, t, x. Since the function $x \mapsto (b(x,s), \nabla\varphi(x))$ is weakly continuous on bounded sets in X for almost every $s \in [0,1]$, we obtain

$$\int_X (b(x,t), \nabla\varphi(x)) \, \mu_s(dx) = \lim_{j\to\infty} \int_X (b(x,t), \nabla\varphi(x)) \, \mu^{n_j}_s(dx)$$

for almost every $s \in [0,1]$ (see, e.g., [3, Lemma 8.4.3]). Along with the previous estimates and (7) this yields that

$$\int_0^t \int_X (b(x,s), \nabla\varphi(x))\, \mu_s(dx) = \lim_{j\to\infty} \int_0^t \int_X (b(x,s), \nabla\varphi(x))\, \mu_s^{n_j}(dx).$$

Therefore,

$$\int_0^t \int_X (b(x,t), \nabla\varphi(x))\, \mu_s(dx) = \lim_{j\to\infty} \int_0^t \int_X (b_j(x,t), \nabla\varphi(x))\, \mu_s^{n_j}(dx).$$

Indeed, for any fixed ball U in X we have

$$\Bigl| \int_0^t \int_X (b(x,s), \nabla\varphi(x))\, \mu_s^{n_j}(dx) - \int_0^t \int_X (b_j(x,s), \nabla\varphi(x))\, \mu_s^{n_j}(dx) \Bigr|$$

$$\le \sup_{(x,s)\in U\times[0,t]} |(b(x,s), \nabla\varphi(x)) - (b_j(x,s), \nabla\varphi(x))| + \int_0^t \int_{X\setminus U} [K_1 + K_2\|x\|^\kappa]\, \mu_s^{n_j}(dx).$$

Given $\varepsilon > 0$, we apply (7) to find a ball U such that the second integral on the right is less than $\varepsilon/2$ for all j. Then we find j_0 such that for all $j \ge j_0$ one has

$$\sup_{(x,s)\in U\times[0,t]} |(b(x,s), \nabla\varphi(x)) - (b_j(x,s), \nabla\varphi(x))| \le \varepsilon/2.$$

Therefore, μ satisfies (2). □

A possible disadvantage of our hypotheses can be the requirement of weak continuity of the functions b^i on balls. For example, this excludes functions depending on the norm since the latter is not weakly sequentially continuous. We have not yet clarified whether this hypothesis is really needed, but it is used in our proof in the last step where we verify that the obtained weak limit satisfies the desired equation. In addition, this hypothesis is naturally connected with the other assumption that b^i is weakly continuous in μ with respect to the weak topology, which again is stronger than the continuity associated with the norm topology. Our second main result relaxes these two assumptions to probably more natural continuities associated with the norm topology at the expense of certain stronger dissipativity of the drift.

Let us consider the Borel function

$$V(x) = \sum_{n=1}^\infty \lambda_i x_i^2, \quad \text{where } \lambda_i > 0 \text{ and } \lambda_i \to +\infty,$$

defined on the compactly embedded weighted Hilbert space X_V of sequences $x = (x_i)$ with finite norm $V^{1/2}$.

Let us modify our previous assumptions (A2) and (B2) as follows.

(A3) for every fixed measure $\mu \in \mathcal{P}(X \times [0,1])$ and every fixed i, the functions

$$x \mapsto b^i(\mu, x, t)$$

are defined and continuous on the compacts sets $\{V \le R\}$ with respect to the norm on X for a.e. t and one has uniform convergence

$$b^i(\mu_j, x, t) \to b(\mu, x, t)$$

on the sets $\{V \le R\} \times [0,1]$ whenever $\mu_j \to \mu$ weakly with respect to the norm topology on X;

(B3) there exist numbers $\alpha > 0$, $c_i > 0$, and $\kappa \geq 1$ such that for all $(x,t) \in X_V \times [0,1]$ and $\mu \in \mathcal{P}(X \times [0,1])$ one has

$$\sum_{i=1}^{n} \lambda_i b^i(\mu, x, t) x_i \leq \alpha(1 + V(x)) \quad \forall\, x \in X_n,\ n \in \mathbb{N},$$

$$|b^i(\mu, x, t)| \leq c_i(1 + V(x)^{\kappa}).$$

Theorem 3. *Let ν be a Borel probability measure on X such that for some $p > \kappa$ one has*

$$\int_X V(x)^p\, \nu(dx) < \infty.$$

Then, under assumptions (A3) *and* (B3)*, there exists a family $\mu = (\mu_t)_{t\in[0,1]}$ of probability measures satisfying* (1)*. Moreover, one has $\mu_t(X_V) = 1$ for all t,*

$$\sup_{t\in[0,1]} \int_X V(x)^p\, \mu_t(dx) < \infty.$$

Proof. The only difference with the previous theorem is that now we want to construct a sequence of finite-dimensional solutions μ^j which would converge weakly with respect to the norm topology on X. To this end, we have to ensure the uniform tightness of the constructed measures with respect to the norm topology. Due to our assumption that $\lambda_i \to +\infty$ the sets $\{x\colon V(x) \leq R\}$ are compact in X with respect to the norm topology. Hence it suffices to establish a uniform estimate

$$\sup_{t\in[0,1]} \int_X V(x)^p\, \mu_t^n(dx) \leq M_p < \infty. \tag{10}$$

This estimate holds indeed due to [5, Lemma 2.2] since, for every n and the differential operator

$$L = \sum_{i=1}^{n} b^i \partial_{x_i},$$

we have the following inequality on $\mathbb{R}^n$ for the function $\Psi(x) := \Bigl(\sum_{i=1}^{n} \lambda_i x_i^2\Bigr)^p$:

$$L\Psi(x) = 2p \sum_{i=1}^{n} \lambda_i x_i b^i(\mu, x, t) \Bigl(\sum_{i=1}^{n} \lambda_i x_i^2\Bigr)^{p-1} \leq 4p\alpha(1 + \Psi(x)).$$

It follows that $\mu_t(X_V) = 1$ for every $t \in [0,1]$. Clearly, estimate (10) remains valid for the limiting measures μ_t. In order to justify (1) for μ we use the continuity of the functions b^i on the sets $\{V \leq R\}$ and [3, Lemma 8.4.3]. $\square$

In some situations (as in the example below), assumption (B3) is less convenient than (B2) in its first part since the Lyapunov function behind is not the inner product in X but the inner product in X_V. However, it is possible to modify this part of (B3) as follows.

(B4) There exist numbers $\alpha > 0$, $c_i > 0$, and $\kappa \geq 1$ such that for all $(x,t) \in X_V \times [0,1]$ and $\mu \in \mathcal{P}(X \times [0,1])$ one has

$$\sum_{i=1}^{n} b^i(\mu, x, t) x_i \leq \alpha - \alpha V(x) \quad \forall\, x \in X_n,\ n \in \mathbb{N},$$

$$|b^i(\mu, x, t)| \leq c_i(1 + V(x)\|x\|^{\kappa}).$$

Theorem 4. *Let ν be a Borel probability measure on X such that for some $p > \kappa$ one has*

$$\int_X V(x)\|x\|^{2p}\,\nu(dx) < \infty.$$

Then, under assumptions (A3) *and* (B4)*, there exists a family $\mu = (\mu_t)_{t\in[0,1]}$ of probability measures satisfying* (1)*. Moreover, one has $\mu_t(X_V) = 1$ for all t,*

$$\sup_{t\in[0,1]} \int_X V(x)\|x\|^{2p}\,\mu_t(dx) < \infty.$$

Proof. Essentially the same reasoning as in the previous theorem applies, but in place of (10) we order obtain the estimate

$$\sup_{t\in[0,1]} \int_X V(x)\|x\|^{2p}\,\mu_t^n(dx) \le M_p < \infty. \tag{11}$$

To this end, we employ the Lyapunov function $\Psi(x) = \|x\|^{2p+1}$. We have

$$L\Psi(x) = (2p+1)\|x\|^{2p} \sum_{i=1}^n b^i(\mu, x, t)x_i \le \alpha' - \alpha' V(x)\|x\|^{2p}, \quad x \in X_n.$$

By using this estimate and [5, Lemma 2.2] we arrive at (11), which along with the second inequality in (B4) enables us to show that the limiting measure μ satisfies the desired equation. □

Example 1. Let U be a bounded domain in $\mathbb{R}^2$ with regular boundary, let Δ be the Laplacian with zero boundary condition having an eigenbasis $\{e_i\}$ with the corresponding eigenvalues $\{\lambda_i\}$. Let $X = L^2(U)$ and let

$$V(x) = \int_U |\nabla x(u)|^2\,du.$$

Then X_V is the Sobolev class $W_0^{2,1}(U)$. Finally, let b be given by a heuristic expression

$$b(\mu, x, t) = \Delta x + \alpha_3(\mu, x, t)x^3 + \alpha_2(\mu, x, t)x^2 + \alpha_1(\mu, x, t)x + \alpha_0(\mu, x, t),$$

where the functions $\alpha_3, \dots, \alpha_0$ are Borel measurable, uniformly bounded, continuous in x on balls in $W_0^{2,1}(U)$ with respect to the L^2-norm, and satisfy the estimate

$$\alpha_3(\mu, x, t) \le -M, \quad \text{where } M > 0 \text{ is a constant.}$$

Suppose also that if $\mu_j \to \mu$ weakly, then $\alpha_k(\mu_j, x, t) \to \alpha_k(\mu, x, t)$ uniformly in $t \in [0, 1]$ and x from every fixed ball in $W_0^{2,1}(U)$, $0 \le k \le 3$.

The corresponding functions b^i are defined by

$$b^i(\mu, x, t) := \lambda_i \int_U x(u)e_i(u)\,du + \int_U [\alpha_3(\mu, x, t)x^3(u) + \cdots + \alpha_0(\mu, x, t)]e_i(u)\,du.$$

Then there exists a family $\mu = (\mu_t)_{t\in[0,1]}$ of probability measures satisfying (1).

Let us verify conditions (A3) and (B4). The multiplicative Sobolev inequality provides an estimate

$$\int_U |x(t)|^4\,du \le C \int_U |\nabla x(u)|^2\,du \int_U |x(u)|^2\,du,$$

which shows that the second inequality in (B4) is fulfilled with $\kappa = 2$. The first estimate required in (B4) follows from the equality

$$\int_U [x(u)\Delta x(u) + \alpha_3(\mu, x, t)x(u)^4 + \cdots + \alpha_0(\mu, x, t)x(u)]\, du$$
$$= -\int_U |\nabla x(u)|^2\, du + \int_U [\alpha_3(\mu, x, t)x(u)^4 + \cdots + \alpha_0(\mu, x, t)x(u)]\, du$$

on all X_n, where the right-hand side is majorized by

$$C_1 - \int_U |\nabla x(u)|^2\, du$$

with some constant C_1 since $\alpha_3(\mu, x, t) \le -M$. The continuity of b^i on balls in $W_0^{2,1}(U)$ with respect to the L^2-norm follows from our assumptions and the Sobolev embedding combined with the compactness of the embedding of $W_0^{2,1}(U)$ to $L^2(U)$.

It is essential in this example that the term involving the Sobolev norm appears when we apply the operator L to the inner product from L^2. Theorem 3 does not work here because the growth of $|b^i|$ is not controlled by powers of the L^2-norm, but once we involve bigger functions, we need stronger dissipativity to control their moments. Similar results can be proved for more general equations with second order terms, which will be considered in a separate paper.

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